CS 5291: Stochastic Processes for Networking

HW3

1. Suppose that at any time a single train is ready for boarding service in a train station and the passengers get on the train according to a renewal process with a mean interarrival time μ. Each train has with k seats; whenever there are k passengers on board, the train departs immediately (and a next train becomes ready for boarding service immediately). Suppose that whenever there are n passengers waiting on the train, the station incurs a cost at the rate of nc dollars per unit time (which increases linearly with respect to the number of passengers waiting on the train). Suppose that each time a train departs, the station incurs a cost of 6 dollars. What is the average long-run cost per unit time incurred by the station?

2. A certain scientific theory supposes that mistakes in cell division (細胞分裂) occur according to a Poisson process with rate 2.5 per year, and that an individual dies when 196 such mistakes have occurred. Find

   (a) the mean lifetime of an individual

   (b) the variance of the lifetime of an individual

   (c) an approximate of the probability that an individual dies before age 67.2

   Hint: Use the central limit theorem for renewal processes. Suppose the value of the complementary cumulative distribution function of the normal distribution, $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz$, can be known by table lookup.

3. A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having pdf $f(t)$ and cdf $F(t)$ to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinuous working on the present job and starts a new one. In the long run, at what rate are jobs completed [in the expression of either $f(t)$ or $F(t)$]?

4. Let $X_1, X_2, \ldots$ be independent random variables with $\mathrm{P}\{X_i = 1\} = p = 1 - \mathrm{P}\{X_i = 0\}, i \geq 1$. Let the random variables N_1, N_1, N_3 be:

$$N_1 = \min\{n: X_1 + X_2 + \cdots + X_n = 5\}$$

$$N_2 = \begin{cases} 3, & X_1 = 0 \\ 5, & X_1 = 1 \end{cases}$$

$$N_3 = \begin{cases} 3, & X_4 = 0 \\ 2, & X_4 = 1 \end{cases}$$

(a) Among the random variables $N_1, N_2, \ldots$, which are *stopping times* for the sequence $X_1, X_2, \ldots$?

(b) For the stopping times in part (a), derive $\mathrm{E}[N], \mathrm{E}[X]$, and $\mathrm{E}[\sum_{i=1}^{N} X_i]$ by using Wald's equation.

5. Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let T denote the time duration it takes the miner to become free.

   (a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time N such that $T = \sum_{i=1}^{N} X_i$.

   Hint: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.

   (b) Use Wald's equation to find $\mathrm{E}[T]$.

   (c) Compute $\mathrm{E}[\sum_{i=1}^{N} X_i \,|N = n]$, where n is a (given) positive integer.

   (d) Use part (c) for a second derivation of $\mathrm{E}[T]$ by the law of total expectation.

6. If the mean-value function of the renewal process $\{N(t), t \geq 0\}$ is given by $m(t) = \frac{t}{2}, t \geq 0$, What is $P\{N(5) = 0\}$?

7. $U_1, U_2, \ldots$ are uniform random variables in $(0,1)$ that are independent of each other. N is a random variable defined as follows:

$$N = \min\{n: U_1 + U_2 + \cdots + U_n > 1\}$$

   Find $E[N]$.

8. There are three machines (machines 1, 2, and 3) working concurrently, all of which are needed for a system to work. Machine i functions for an exponential time with rate λ_i before it fails, $i = 1,2,3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is

exponential with rate 5; the time to fix machine 2 is uniform on (0,4); and the time to fix machine 3 is a gamma random variable with parameters $n = 3$ and $\lambda = 2$. Once a failed machine is repaired, it is as good as new and all machines restart. What proportion of time is the system working?

Hint: *It is an alternating renewal process, a special case of regenerative processes. We can solve this problem by conditioning on which machine fails.*