CS 5291: Stochastic Processes for Networking

HW1

1. X is an exponentially distributed random variable with parameter λ. $Y = X^2$. Find the mathematical expectation of Y.
2. Suppose X is a non-negative and continuous random variable whose pdf is $f_X(x)$ and whose cdf is $F_X(x)$. Starting from the definition of the mathematical expectation $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$, prove that $E[X] = \int_0^{\infty} \left(1 - F_X(x)\right) dx$.
3. A fair coin is tossed until two tails occur successively. Find the expected number of the tosses required.
   Hint: Let
   $$X = \begin{cases} 0 \text{ if the first toss results in tails} \\ 1 \text{ if the first toss results in heads,} \end{cases}$$
   and condition on X.
4. A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Define events E_1, E_2, E_3, and E_4 as follows:
   $$E_1 = \{\text{the first pile has exactly 1 ace}\},$$
   $$E_2 = \{\text{the second pile has exactly 1 ace}\},$$
   $$E_3 = \{\text{the third pile has exactly 1 ace}\},$$
   $$E_4 = \{\text{the fourth pile has exactly 1 ace}\}.$$
   Find $\mathrm{P}\{E_1, E_2, E_3, E_4\}$, the probability that each pile has an ace.
   Hint:
   $$\mathrm{P}\{E_1 E_2 \ldots E_n\} = \mathrm{P}\{E_1\}\, \mathrm{P}\{E_2|E_1\}\, \mathrm{P}\{E_3|E_1 E_2\} \ldots\, \mathrm{P}\{E_n|E_1 \ldots E_{n-1}\}$$
5. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables, each having a uniform distribution over (0,1). Let $X = \max\,(Y_1, Y_2, \ldots, Y_n)$.
   (a) Show that the cumulative distribution function of X is $F_X(x) = x^n, 0 \leq x \leq 1$.
   (b) What is the probability density function of X?
6. Derive the moment generating functions for each of the following random variables. Then, derive the expected value, second moment, and variance for each of the random variables.
   (a) Uniform distribution that takes on values in [a,b].
   (b) Exponential distribution with pdf $f(x) = \lambda e^{-\lambda x}$.
7. X is a Poisson random variable whose probability mass function is $P_X(n) =$

$\frac{e^{-\lambda}\lambda^n}{n!}, n = 0,1,2,3, \ldots$

(a) Derive the moment generating function of X.

(b) Find the *tightest* Chernoff's bound for X.

8. With $K(t) = \ln(E[e^{tX}])$, show that $K'(0) = E[X], K''(0) = \mathrm{Var}(X)$.