CS 5291: Stochastic Processes for Networking

HW2

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with $P\{N(1) = 0\} = e^{-3}$. Let S_n denote the n-th arrival time. Find $E[N(4) - N(2) | N(1) = 3]$.

2. On Friday night, the number of customers arriving at a lounge bar can be modeled as a Poisson process with a rate/intensity of twelve customers per hour.

   (a) Find the probability that there are 2 customers between 21:00 and 21:40.

   (b) Find the probability that there are 4 customers between 21:00 and 21:40 and 6 customers between 21:40 and 22:00.

3. Let X be a binomial random variable with parameters n and p. That is, the probability mass function (pmf) of X is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0,1,\dots,n$$

   Suppose that $\mu = np$ is a finite number. Starting with the above pmf, prove that as $n \to \infty$, the above pmf becomes

$$P(X = k) = e^{-\mu} \frac{\mu^k}{k!}, k = 0,1,\dots$$

4. (In this question, we deal with the key step of "Def 3 $\Rightarrow$ Def 1" for Poisson process.) Consider a counting process whose interarrival times, denoted by $X_1, X_2, \dots$, are i.i.d. exponentially-distributed random variables, each with mean $1/\lambda$. We know that the n-th arrival time (denoted by S_n) of the counting process is Erlang-distributed and the pdf of S_n is

$$f_{S_n}(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}, t \geq 0$$

   Condition on the observed value of S_n and then use the law of total probability to derive $P\{N(t) = n\}$. More precisely, derive $P\{N(t) = n\}$ by starting with

$$P\{N(t) = n\} = \int_0^t P(X_{n+1} > t - x \mid S_n = x) \cdot f_{S_n}(x)\, dx$$

5. Assume X and Y are independent Poisson random variables with means $E[X] = \lambda_1$ and $E[Y] = \lambda_2$. (The probability mass function of Poisson distribution is $\frac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, \dots$, where λ is the mean.)

   (a) Compute the conditional probability mass function $\mathrm{P}\{X = k | X + Y = n\}$. Is it the same as $\mathrm{P}\{X = k\}$?

Hint: You can start by the definition of conditional probability.

(b) Find $E[X \mid X + Y = n]$. Is it the same as $E[X]$?

(c) Find $E[E[X \mid X + Y]]$. Is it the same as $E[X]$?

6. Both Alice and Bob require of kidney transplants. If he/she does not receive a new kidney, Alice will die after an exponential time with mean $1/\lambda_A$, and Bob will die after an exponential time with mean $1/\lambda_B$. New kidneys arrive in accordance with a Poisson process with rate λ. It has been decided that the first kidney will go to Alice (or to Bob if Bob is alive and Alice is not at that time) and the next one to Bob (if Bob is still alive).

   (a) What is the probability that Alice obtains a new kidney?

   (b) What is the probability that Bob obtains a new kidney?

7. A dog is trapped in a cave. It needs to choose one of two directions. If it goes to the left, then it will walk around in the cave for five minutes and will then return to its initial position. If it goes to the right, then with probability $\frac{1}{4}$ it will depart the cave after four minutes of traveling, and with probability $\frac{3}{4}$ it will return to its initial position after seven minutes of traveling. Assuming that the dog is at all times equally likely to go to the left or the right, what is the expected number of minutes that it will be trapped in the cave?

8. Customers arrive to a supermarket according to the Poisson process at a rate of ten arrivals per hour. Suppose that only 30% of customers buys something in the supermarket.

   (a) Find the probability that 6 or more sales are made in a period of one hour.

   (b) If the supermarket opens at 9 A.M. Find the expected time of the fifth sale of the day.