CS 5291: Stochastic Processes for Networking

HW2: Solution

Problem 1

$$P\{N(1)=0\} = \frac{e^{-\lambda\cdot 1}(\lambda\cdot 1)^0}{0!} = e^{-3} \Rightarrow \lambda = 3$$

$$E[N(4)-N(2)|N(1)=3] = E[N(4)-N(2)] = E[N(2)] = \lambda \cdot 2 = 6$$

Problem 2

(a) Average rate $\lambda = 12$ customers per hour

interval length $\tau = \frac{2}{3}$ hours

Let $X \sim Poisson(\lambda\tau) = Poisson(8)$ be the number of arrivals in an interval

$$P(X=2) = \frac{e^{-\lambda\tau}(\lambda\tau)^2}{2!} = \frac{e^{-8}8^2}{2!} = 0.0107$$

(b) Let I_1 be the interval between 21:00 and 21:40, I_2 be the interval between 21:40 and 22:00

interval length for I_1 is $\tau_1 = \frac{2}{3}$ hours, interval length for I_2 is $\tau_2 = \frac{1}{3}$ hours

Let $X \sim Poisson(\lambda\tau_1) = Poisson(8)$ be the number of arrivals in I_1
Let $Y \sim Poisson(\lambda\tau_2) = Poisson(4)$ be the number of arrivals in I_2

$$P(X=4)\cdot P(Y=6) = \frac{e^{-\lambda\tau_1}(\lambda\tau_1)^4}{4!}\cdot\frac{e^{-\lambda\tau_2}(\lambda\tau_2)^6}{6!} = \frac{e^{-8}(8)^4}{4!}\cdot\frac{e^{-4}(4)^6}{6!} = 0.006$$

Problem 3

$$\mu = np \Rightarrow p = \frac{\mu}{n}$$

$$\lim_{n\to\infty}\binom{n}{k}p^k(1-p)^{n-k} = \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left(\frac{\mu}{n}\right)^k\left(1-\frac{\mu}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left(\frac{\mu}{n}\right)^k\left(1-\frac{\mu}{n}\right)^n\frac{(n-\mu)^{-k}}{n^{-k}}$$

$$= \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n\frac{\mu^k}{k!}\frac{n(n-1)(n-2)\cdots(n-k+1)}{(n-\mu)^k}$$

$$= \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n\cdot\lim_{n\to\infty}\frac{\mu^k}{k!}\cdot\lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-k+1)}{(n-\mu)^k} = e^{-\mu}\frac{\mu^k}{k!}$$

since $\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = e^{-\mu}$

Problem 4

$X_1, X_2, \dots \sim \exp(\lambda)$

$$P\{N(t)=n\} = \int_0^t P(X_{n+1} > t-x \mid S_n = x)\cdot f_{S_n}(x)dx = \int_0^t (1-F_{X_{n+1}}(t-x))\cdot f_{S_n}(x)dx$$

$$= \int_0^t e^{-\lambda(t-x)}\cdot \lambda e^{-\lambda x}\frac{(\lambda x)^{n-1}}{(n-1)!}dx = e^{-\lambda t}\int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!}dx = e^{-\lambda t}\lambda^n \frac{x^n}{n!}\Big|_0^t$$

$$= e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

Problem 5

(a) $P(X+Y=n) = \sum_{i=0}^{n} P(X=i, Y=n-i) = \sum_{i=0}^{n} \frac{e^{-\lambda_1}\lambda_1^i}{i!}\frac{e^{-\lambda_2}\lambda_2^{(n-i)}}{(n-i)!}$

$$= e^{-(\lambda_1+\lambda_2)}\sum_{i=0}^{n}\frac{\lambda_1^i\lambda_2^{(n-i)}}{i!\,(n-i)!}\frac{n!}{n!} = e^{-(\lambda_1+\lambda_2)}\frac{1}{n!}\sum_{i=0}^{n}\frac{n!}{i!\,(n-i)!}\lambda_1^i\lambda_2^{(n-i)}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n \sim Poisson(\lambda_1+\lambda_2)$$

$$P(X=k \mid X+Y=n) = \frac{P(X=k, Y=n-k)}{P(X+Y=n)} = \frac{\frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{(n-k)}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n}$$

$$= \frac{n!}{k!\,(n-k)!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$P(X=k) = \frac{e^{-\lambda_1}\lambda_1^k}{k!} \neq P(X=k \mid X+Y=n)$$

(b) $E[X \mid X+Y=n] = \sum_{x=0}^{\infty} x\cdot P(X=x \mid X+Y=n) = \sum_{x=1}^{n} x\cdot\binom{n}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-x}$

$$= \frac{1}{(\lambda_1+\lambda_2)^n}\sum_{x=1}^{n} x\cdot\frac{n!}{x!\,(n-x)!}\lambda_1^x\lambda_2^{n-x} = \frac{n}{(\lambda_1+\lambda_2)^n}\sum_{x=1}^{n}\frac{(n-1)!}{(x-1)!\,(n-x)!}\lambda_1^x\lambda_2^{n-x}$$

Let $i = x-1 \rightarrow \frac{n}{(\lambda_1+\lambda_2)^n}\sum_{i=0}^{n-1}\frac{(n-1)!}{(i)!(n-i-1)!}\lambda_1^{i+1}\lambda_2^{n-i-1} = \frac{n\lambda_1}{(\lambda_1+\lambda_2)^n}\sum_{i=0}^{n-1}\binom{n-1}{i}\lambda_1^i\lambda_2^{n-1-i}$

$$= \frac{n\lambda_1}{(\lambda_1+\lambda_2)^n}(\lambda_1+\lambda_2)^{n-1} = \frac{n\lambda_1}{\lambda_1+\lambda_2}$$

$E[X] = \lambda_1 \neq E[X \mid X+Y=n]$

(c) $E\big[E[X \mid X+Y]\big] = \sum_n E[X \mid X+Y=n]\cdot P(X+Y=n)$

$$= \sum_n \frac{n\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n = \frac{\lambda_1}{\lambda_1+\lambda_2}\sum_n n\cdot e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot (\lambda_1 + \lambda_2) = \lambda_1$$

$$E[X] = \lambda_1 = E\big[E[X|X+Y]\big]$$

Problem 6

(a) Let T_A and T_B be the random variable that the time before Alice or Bob dies

Let T_k be the time that k-th kidney arrives

$$P(\text{Alice obtains a new kidney}) = P(T_1 < T_A) = \frac{\lambda}{\lambda + \lambda_A}$$

(b) To obtain the probability of Bob obtains a new kideny, we will condition on the first event, i.e. witch happens first: a kidney arrives, Alice dies or Bob dies

$P(\text{Bob obtains a new kidney})$

$$= P(\text{Bob obtains a new kidney}|T_1 = \min\{T_1, T_A, T_B\}) \cdot P(T_1 = \min\{T_1, T_A, T_B\})$$

$$+P(\text{Bob obtains a new kidney}|T_A = \min\{T_1, T_A, T_B\}) \cdot P(T_A = \min\{T_1, T_A, T_B\})$$

$$+P(\text{Bob obtains a new kidney}|T_B = \min\{T_1, T_A, T_B\}) \cdot P(T_B = \min\{T_1, T_A, T_B\})$$

$$= P(T_2 < T_B) \cdot P(T_1 = \min\{T_1, T_A, T_B\}) + P(T_1 < T_B) \cdot P(T_A = \min\{T_1, T_A, T_B\}) + 0$$

$$= \frac{\lambda}{\lambda + \lambda_B} \cdot \frac{\lambda}{\lambda + \lambda_A + \lambda_B} + \frac{\lambda}{\lambda + \lambda_B} \cdot \frac{\lambda_A}{\lambda + \lambda_A + \lambda_B} = \frac{\lambda}{\lambda + \lambda_B} \cdot \frac{\lambda + \lambda_A}{\lambda + \lambda_A + \lambda_B}$$

Problem 7

Let N denote the number of minutes in the cave

Let R be the event that the dog chooses its right

Let L be the event that the dog chooses its left

$$E[N] = \frac{1}{2}E[N|L] + \frac{1}{2}E[N|R] = \frac{1}{2}(5 + E[N]) + \frac{1}{2}\left(\frac{1}{4} \cdot 4 + \frac{3}{4}(7 + E[N])\right) = \frac{7}{8}E[N] + \frac{45}{8}$$

$$\Rightarrow E[N] = 45$$

Problem 8

(a) Average rate $\lambda = 10$ customers per hour

Let $N_1(t) \sim Poisson(\lambda_1 t)$ be the number of customers who arrive and buy something in a period t

Average rate for $N_1(t)$ is $\lambda_1 = \lambda \cdot p = 10 \cdot 0.3 = 3$ customers who arrive and buy something per hour

$$P\{N_1(1) \geq 6\} = 1 - \sum_{j=0}^{5} P\{N_1(1) = j\} = 1 - \sum_{j=0}^{5} e^{-3\cdot 1} \frac{(3 \cdot 1)^j}{j!} = 0.0839$$

(b) Let $X \sim \exp(\lambda_1)$ be the interval times that customer arrive and buy something

Let S_5 be the time of the fifth sale of the day

$E[S_5] = 5 \cdot E[X] = 5 \cdot \frac{1}{\lambda_1} = \frac{5}{3}$ hours or 100 minutes

The expected time of the fifth sale is 10:40 a.m.