Stochastic Processes for Networking

HW3: Solution

Problem 1 (10 %)

Consider this as a renewal reward process.

Define cycles in the process through the times at which the trains leave.

Then the long run average cost is $\frac{E[\text{cost per cycle}]}{E[\text{length per cycle}]}$

$E[\text{cost per cycle}]$

$= E\left[\text{cost from } 1^{st} \text{ passenger} + \text{cost from } 2^{\text{nd}} \text{ passenger} + \cdots + \text{cost from } k^{th} \text{ passenger}\right] + 6$

$= (c \cdot (k-1) \cdot \mu + c \cdot (k-2) \cdot \mu + \cdots + c \cdot 0 \cdot \mu) + 6$

$= c \cdot \mu \cdot \dfrac{(k-1+0)k}{2} + 6$

$= c \cdot \mu \cdot \dfrac{k(k-1)}{2} + 6$

$E[\text{length per cycle}] = k \cdot \mu$

$$\Rightarrow \frac{E[\text{cost per cycle}]}{E[\text{length per cycle}]} = \frac{c \cdot \mu \cdot \frac{k(k-1)}{2} + 6}{k \cdot \mu} = \frac{c(k-1)}{2} + \frac{6}{k \cdot \mu}$$

Problem 2 (5 %, 5 %, 10 %)

(a) Let $X \sim \exp(2.5)$ be the interval time between mistakes

Let T be the life time of an individual

$$E[T] = E\left[\sum_{i=1}^{196} X\right] = 196 \cdot E[X] = 196 \cdot \frac{1}{2.5} = 78.4$$

(b) $Var[T] = Var[\sum_{i=1}^{196} X] = 196 \cdot Var[X] = 196 \cdot \frac{1}{2.5^2} = 31.36$

(c) $P\{T < 67.2\} = P\{\frac{T-E[T]}{\sqrt{Var[T]}} < \frac{67.2-E[T]}{\sqrt{Var[T]}}\} \approx P\left\{Z < \frac{67.2-78.4}{5.6}\right\} = P\{Z < -2\} = 1 - Q(-2) =$ 0.0228 where $Z \sim N(0,1)$

(c) (another solution) $\mu = 0.4, \sigma^2 = 0.16$

$$P\{N(67.2) \geq 196\} = P\left\{\frac{N(67.2) - \frac{67.2}{0.4}}{\sqrt{67.2 \cdot \frac{0.16}{0.4^3}}} \geq \frac{196 - \frac{67.2}{0.4}}{\sqrt{67.2 \cdot \frac{0.16}{0.4^3}}}\right\} = P\left\{\frac{N(67.2) - \frac{67.2}{0.4}}{\sqrt{67.2 \cdot \frac{0.16}{0.4^3}}} \geq 2.16\right\}$$

$= Q(2.16) = 0.0154$

Problem 3 (10 %)

A job completion constitutes a renewal

Let T be the time between renewals

Let W be the time it takes to finish the next job

Let S be the time of the next shock

$$
\begin{aligned}
E[T|W=w] &= \int_0^\infty E[T|W=w,S=s]P\{S=s\}ds \\
&= \int_0^w (s+E[T])\lambda e^{-\lambda s}ds + \int_w^\infty w\lambda e^{-\lambda s}ds \\
&= \int_0^w s\lambda e^{-\lambda s}ds + E[T]\left(1-e^{-\lambda w}\right) + we^{-\lambda w} \\
&= -se^{-\lambda s}\Big|_0^w + \int_0^w e^{-\lambda s}ds + E[T]\left(1-e^{-\lambda w}\right) + we^{-\lambda w} \\
&= -we^{-\lambda w} + \frac{1-e^{-\lambda w}}{\lambda} + E[T]\left(1-e^{-\lambda w}\right) + we^{-\lambda w} \\
&= \left(1-e^{-\lambda w}\right)\left(E[T]+\frac{1}{\lambda}\right)
\end{aligned}
$$

$$
\begin{aligned}
E[T] &= E\big[E[T|W=w]\big] \\
&= \left(1-E\left[e^{-\lambda w}\right]\right)\left(E[T]+\frac{1}{\lambda}\right) \\
&= E[T] + \frac{1}{\lambda} - E\left[e^{-\lambda w}\right]\cdot E[T] - \frac{E\left[e^{-\lambda w}\right]}{\lambda}
\end{aligned}
$$

$$
\Rightarrow E[T] = \frac{\frac{1}{E[e^{-\lambda w}]}-1}{\lambda} = \frac{\frac{1}{\int_0^\infty e^{-\lambda w}f(w)dw}-1}{\lambda}
$$

$$
\Rightarrow rate = \frac{1}{E[T]} = \frac{\lambda}{\frac{1}{\int_0^\infty e^{-\lambda w}f(w)dw}-1} = \frac{\lambda\int_0^\infty e^{-\lambda w}f(w)dw}{1-\int_0^\infty e^{-\lambda w}f(w)dw} \quad \text{(expression of } f(t)\text{)}
$$

$$
= \frac{\lambda\left(e^{-\lambda w}F(w)\Big|_0^\infty + \int_0^\infty \lambda e^{-\lambda w}F(w)dw\right)}{1-\left(e^{-\lambda w}F(w)\Big|_0^\infty + \int_0^\infty \lambda e^{-\lambda w}F(w)dw\right)} = \frac{\lambda\int_0^\infty \lambda e^{-\lambda w}F(w)dw}{1-\int_0^\infty \lambda e^{-\lambda w}F(w)dw}
$$

$$
= \frac{\lambda\int_0^\infty \lambda e^{-\lambda w}F(w)dw}{\int_0^\infty \lambda e^{-\lambda w}1dw - \int_0^\infty \lambda e^{-\lambda w}F(w)dw} = \frac{\lambda\int_0^\infty \lambda e^{-\lambda w}F(w)dw}{\int_0^\infty \lambda e^{-\lambda w}\left(1-F(w)\right)dw}
$$

$$
= \frac{\int_0^\infty \lambda e^{-\lambda w}F(w)dw}{\int_0^\infty e^{-\lambda w}\left(1-F(w)\right)dw} \quad \text{(expression of } F(t)\text{)}
$$

Problem 3 (another solution)

Let Y be the time between shock

Let X be the time it takes to finish the job

A shock or job complete constitutes a renewal

$$rate = \frac{E[\text{value for one cycle}]}{E[\text{length of the cycle}]} = \frac{P\{x < y\}}{E[\min(x, y)]} = \frac{\int_0^\infty F(x)\lambda e^{-\lambda x} dx}{\int_0^\infty E[\min(x, y)]\lambda e^{-\lambda y} dy}$$

$$\int_0^\infty E[\min(x, y)]\lambda e^{-\lambda y} dy = \int_0^\infty \left[\int_0^y x f(x)dx + \int_y^\infty y f(x)dx\right] \lambda e^{-\lambda y} dy$$

$$= \int_0^\infty \left[x \cdot F(x)\Big|_0^y - \int_0^y F(x)dx + y \cdot F(x)\Big|_y^\infty\right] \lambda e^{-\lambda y} dy$$

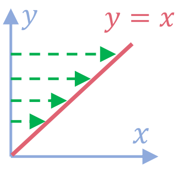
$$= \int_0^\infty \left[y \cdot F(y) - \int_0^y F(x)dx + y \cdot \big(1 - F(y)\big)\right] \lambda e^{-\lambda y} dy$$

$$= \int_0^\infty \left[y - \int_0^y F(x)dx\right] \lambda e^{-\lambda y} dy$$

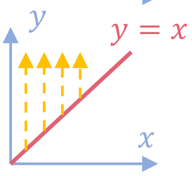
$$= \int_0^\infty \int_0^y \big(1 - F(x)\big)dx\, \lambda e^{-\lambda y} dy$$

$$rate = \frac{\int_0^\infty F(x)\lambda e^{-\lambda x} dx}{\int_0^\infty E[\min(x, y)]\lambda e^{-\lambda y} dy} = \frac{\int_0^\infty F(x)\lambda e^{-\lambda x} dx}{\int_0^\infty \int_0^y \big(1 - F(x)\big)dx\, \lambda e^{-\lambda y} dy}$$

$$= \frac{\int_0^\infty F(x)\lambda e^{-\lambda x} dx}{\int_0^\infty \int_0^y \big(1 - F(x)\big)dx\, \lambda e^{-\lambda y} dy}$$



$$= \frac{\int_0^\infty F(x)\lambda e^{-\lambda x} dx}{\int_0^\infty \int_x^\infty \lambda e^{-\lambda y} dy\, \big(1 - F(x)\big)dx}$$



$$= \frac{\int_0^\infty F(x)\lambda e^{-\lambda x} dx}{\int_0^\infty e^{-\lambda x}\big(1 - F(x)\big)dx}$$

Problem 4 (5 %, 5 %)

(a) $\because \{N_1 = n\}$ depends on $X_1, X_2, \dots, X_n$ only (is independent of $X_{n+1}, X_{n+2} \dots$) $\therefore N_1$ is stopping time

$\because N_2 = \begin{cases} 3, & X_1 = 0 \\ 5, & X_1 = 1 \end{cases}$ depends on X_1 only $\therefore N_2$ is stopping time

$\because N_3 = \begin{cases} 3, & X_4 = 0 \\ 2, & X_4 = 1 \end{cases}$ depends on X_4 $\therefore N_3$ is NOT stopping time

(b) $E[X] = 0 \cdot P\{X_i = 0\} + 1 \cdot P\{X_i = 1\} = p$

For N_1, $E\left[\sum_{i=1}^{N_1} X_i\right] = 5$

by Wald's equation, $E\left[\sum_{i=1}^{N_1} X_i\right] = E[N_1] \cdot E[X]$

$$\Rightarrow 5 = E[N_1] \cdot E[X] = E[N_1] \cdot p \Rightarrow E[N_1] = \frac{5}{p}$$

For N_2, $E[N_2] = 3 \cdot P\{X_1 = 0\} + 5 \cdot P\{X_1 = 1\} = 3(1-p) + 5p = 2p + 3$

by Wald's equation, $E\left[\sum_{i=1}^{N_2} X_i\right] = E[N_2] \cdot E[X] = (2p+3) \cdot p = 2p^2 + 3p$

Problem 5 (5 %, 5 %, 5 %, 5 %)

(a) Let $X_i = \begin{cases} 2, \text{ Door 1 chosen (probability} = \frac{1}{3}) \\ 4, \text{ Door 2 chosen (probability} = \frac{1}{3}) \\ 6, \text{ Door 3 chosen (probability} = \frac{1}{3}) \end{cases}$ be the amount of time the miner has to travel after i-th choice

Let $N = \min\{i | X_i = 2\}$ be the number of choices the miner makes until becoming free

N is a stopping time since the event $N = n$ is determined by the first n observation of X

(b) $E[X] = 2 \cdot P\{X_i = 2\} + 4 \cdot P\{X_i = 4\} + 6 \cdot P\{X_i = 6\} = (2+4+6) \cdot \frac{1}{3} = 4$

$\because N$ follows geometric distribution with $p = \frac{1}{3}$ $\therefore E[N] = \frac{1}{p} = 3$

$$E[T] = E\left[\sum_{i=1}^{N} X_i\right] = E[N] \cdot E[X] = 3 \cdot 4 = 12$$

(c) $E[\sum_{i=1}^{N} X_i \,|N = n] = E[\sum_{i=1}^{N} X_i \,|X_1 \neq 2, X_2 \neq 2, \dots, X_{n-1} \neq 2, X_n = 2]$

$$= 2 + (n-1) \cdot E[X_i | X_i \neq 2] = 2 + (n-1) \cdot \frac{4 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} = 2 + (n-1) \cdot 5 = 5n - 3$$

(d) $E[T] = E[\sum_{i=1}^{N} X_i] = E\left[E[\sum_{i=1}^{N} X_i \,|N = n]\right] = E[5N - 3] = 5 \cdot E[N] - 3 = 5 \cdot 3 - 3 = 12$

Problem 6 (10%)

The mean-value function $m(t) = c \cdot t$ is a Poisson process because a Poisson process has $E[N(t)] = \lambda \cdot t$ and the given renewal process in this problem is a Poisson process with rate $\lambda = \frac{1}{2}$. As a result, $P(N(5) = 0) = e^{-\left(\frac{1}{2}\right) \cdot 5} \frac{\left(\frac{1}{2} \cdot 5\right)^0}{0!} = e^{-\frac{5}{2}}$

Problem 7 (10%)

Let $N(x) = min\{n: U_1 + U_2 + \cdots + U_n > x\}$ and $g(x) = E[N(x)]$.

$$E[N(x)|U_1 = t] = \begin{cases} 1, if\ t > x \\ 1 + g(x-t), if\ t \leq x \end{cases}$$

Consider $x \leq 1$,

$$g(x) = \int_0^1 E[N(x)|U_1 = t]P\{U_1 = t\}dt = \int_0^x \big(1 + g(x-t)\big)dt + \int_x^1 1dt = 1 + \int_0^x g(x-t)dt$$

$$= 1 + \int_0^x g(t)dt$$

$g'(x) = g(x), g(x) = Ae^x$.

$\because g(0) = 1, A = 1. \therefore g(x) = e^x$

$E[N] = g(1) = e$

Problem 8 (10%)

Define "on" as system is working, "off" as system is shut down for repairing machines.

$$E[on] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$E[off] = \sum_{i=1}^{3} E[repair\ time|machine\ i\ fails] \cdot P(machine\ i\ fails)$$

$$= \frac{1}{5} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{0+4}{2} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3}{2} \cdot \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}\left(\frac{1}{5} \cdot \lambda_1 + 2 \cdot \lambda_2 + \frac{3}{2} \cdot \lambda_3\right)$$

Propotion of time the system is working = $\frac{E[on]}{E[off]+E[on]} = \frac{1}{1+\frac{1}{5}\cdot\lambda_1+2\cdot\lambda_2+\frac{3}{2}\cdot\lambda_3}$