CS 5291: Stochastic Processes for Networking

HW1 Solution

1. (10%) $E[Y] = E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$

$$= x^2(-e^{-\lambda x})\Big|_0^\infty - \int_0^\infty 2x(-e^{-\lambda x})dx = \frac{2}{\lambda}\int_0^\infty x\lambda e^{-\lambda x}dx = \frac{2}{\lambda^2}$$

2. (10%)$E[X] = \int_0^\infty x f(x)dx$

$$= \int_0^\infty (1 - F_X(x))dx = x(1 - F_X(x))\Big|_0^\infty + \int_0^\infty x f(x)dx$$

$$= \int_0^\infty x f(x)dx = E[X]$$

3. (10%)$Let\ N\ be\ the\ number\ of\ tosses\ required;$

$Then\ E[X] = E\big[E[N|X]\big]$

$$= E\big[E[N|X=1]\big]P(X=1) + E\big[E[N|X=0]\big]P(X=0)$$

$$= (1 + E[N]) \times \frac{1}{2} + \left(2 \times \frac{1}{2} + (2 + E[N]) \times \frac{1}{2}\right) \times \frac{1}{2}$$

$$= \frac{3}{4}E[N] + \frac{3}{2}$$

$$E[N] = 6$$

4. (10%)$P\{E_1, E_2, E_3, E_4\} = \frac{C_1^4 C_{12}^{48}}{C_{13}^{52}} \times \frac{C_1^3 C_{12}^{36}}{C_{13}^{39}} \times \frac{C_1^2 C_{12}^{24}}{C_{13}^{26}} \times 1$

5.

(a) (10%) If x is a real number between 0 and 1, then $X < x$ if and only if $Y_k < x$ for all k, and $P(Y_k < x) = x$ since x is uniform on (0,1). (The pdf is equal to 1, so the cdf is $\int_0^\infty 1dx = x$)

Because the Y_k are independent,

$$P(X < x) = P(Y_1 < x) \times P(Y_2 < x) \times \cdots \times P(Y_n < x) = x^n$$

(b) (5%) $The\ PDF\ of\ X\ is\ obtained\ by\ differentiating,$

$$f_X(x) = \frac{F_X(x)}{dx} = nx^{n-1},\ 0 \leq x \leq 1$$

6. (a) (10%)

Uniform distribution: $f_X(x) = 1/(b-a)$

$$\varphi(t) = E[e^{tX}]$$

$$\varphi(t) = \int_a^b e^{tx} * \left(\frac{1}{(b-a)}\right) dx = \left(\frac{1}{(b-a)}\right) * (1/t) \int_a^b te^{tx}\, dx$$

$$= \left(\frac{1}{t(b-a)}\right)\left[e^{tx}|_a^b\right] = \frac{(e^{tb}-e^{ta})}{t(b-a)}$$

$$\varphi'(t) = \left(\frac{1}{(b-a)}\right) * \left(\frac{(be^{tb}-ae^{ta})}{t} - \frac{(e^{tb}-e^{ta})}{t^2}\right)$$

$$\varphi''(t) = \left(\frac{1}{(b-a)}\right) * \left(\frac{2(e^{tb}-e^{ta})}{t^3} - \frac{2(be^{tb}-ae^{ta})}{t^2} + \frac{(b^2e^{tb}-a^2e^{ta})}{t}\right)$$

$$E[X] = \varphi'(0) = \lim_{s\to 0}\left(\frac{1}{(b-a)}\right) * \left(\frac{(be^{tb}-ae^{ta})}{t} - \frac{(e^{tb}-e^{ta})}{t^2}\right)$$

$$= \frac{(a+b)}{2}\,(by\ L'Hospital's\,rule)$$

$$E[X^2] = \varphi''(0) = \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{(b-a)^2}{12}$$

(b) (10%)

Exponential distribution: $f_X(x) = \lambda e^{-\lambda x}$

$$\varphi(t) = E[e^{tX}] = \int_0^\infty e^{tx}\,\lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda-t)x}\, dx$$

$$= \frac{\lambda}{(\lambda-t)}\int_0^\infty (\lambda-t)e^{-(\lambda-t)x}\, dx = \frac{\lambda}{(\lambda-t)}$$

$$\varphi'(t) = \frac{\lambda}{(\lambda-t)^2}$$

$$\varphi''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

$$E[X] = \varphi'(0) = \frac{1}{\lambda}$$

$$E[X^2] = \varphi''(0) = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{1}{\lambda^2}$$

7.

(a) (10%)

$$f_X(x) = \frac{(e^{-\lambda}\lambda^x)}{x!}$$

$$M_X(t) = \sum_{x=0}^{\infty} \frac{e^{tx}(e^{-\lambda}\lambda^x)}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{e^t\lambda} = e^{-\lambda(1-e^t)}$$

(b) (5%)

$$P\{X \geq a\} \leq \min_t\left(e^{-ta}M_X(t)\right) = \min_t\left(e^{-ta-\lambda(1-e^t)}\right)$$

$$Let\ g(t) = e^{-ta-\lambda(1-e^t)}$$

$$To\ find\ the\ minimum\ of\ g(t), we\ need\ to\ solve\ \frac{dg(t)}{dt} = 0$$

$$\frac{dg(t)}{dt} = 0 \Rightarrow (\lambda e^t - a)e^{-ta-\lambda(1-e^t)} = 0 \Rightarrow t = \ln\left(\frac{a}{\lambda}\right)$$

$$P\{X \geq a\} \leq g\left(\ln\left(\frac{a}{\lambda}\right)\right) = e^{-a\ln\left(\frac{a}{\lambda}\right)-\lambda+a} = e^{-\lambda}\left(\frac{\lambda e}{a}\right)^a$$

8. (10%)

$$e^K(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tX} f_X(x)dx$$

differentiating both sides w.r.t. t.

$$\frac{d}{dt}e^{K(t)} = e^K(t)K'(t) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} e^{tX} f_X(x)dx\right) = \int_{-\infty}^{\infty} Xe^{tX} f_X(x)dx$$

$$= E[Xe^{tX}]$$

$$e^{K(t)}K'(t) = E[e^{tX}]K'(t) = E[Xe^{tX}] \Rightarrow K'(t) = \frac{E[Xe^{tX}]}{E[e^{tX}]}$$

$$K'(0) = \frac{E[X]}{1} = E[X]$$

differentiating again w.r.t. t

$$\frac{d}{dt}\left(e^{K(t)}K'(t)\right) = e^{K(t)}K'(t)^2 + e^{K(t)}K''(t) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} Xe^{tX} f_X(x)dx\right)$$

$$= \int_{-\infty}^{\infty} X^2\, e^{tX} f_X(x)dx = E[X^2 e^{tX}]$$

$$e^{K(t)}K'(t)^2 + e^K(t)K''(t) = E[e^{tX}]\left(\frac{E[Xe^{tX}]^2}{E[e^{tX}]^2} + K''(t)\right) = E[X^2 e^{tX}]$$

$$K''(t) = \frac{E[X^2 e^{tX}]}{E[e^{tX}]} - \frac{E[Xe^{tX}]^2}{E[e^{tX}]^2} = \frac{E[X^2 e^{tX}]E[e^{tX}] - E[Xe^{tX}]^2}{E[e^{tX}]^2}$$

$$K''(0) = E[X^2] - E[X]^2 = Var(X)$$