Embedded Systems

~~31/07/2023~~

04/08/2023

Properties of systems.

1) Causal systems
2) Memoryless systems.
3) Linearity & time Inversions.

} Lee & seshia

4) BIBO Stability.
5) Lyapunos stability } Aluv book

Bounded-Input Bounded-output stability

→ A symbol $x : \mathbb{R} \to \mathbb{R}$ is bounded if there is an $a \in \mathbb{R}$ such that $\forall t \; |x(t)| \leq a$

→ A system 'S', which take input signals from x & produce outputs as y

$$S : X \to Y \quad \text{s.t. } x, Y \in \mathbb{R}^{\mathbb{R}}$$

is BIBO stable if for any bounded input signal x the output signal $S(x)$ is also bounded.

Ex :- Helicopter

$T_y(t)$ → Input

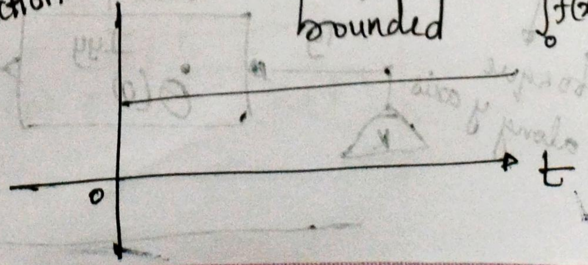
$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t \boxed{T_y(t)} \, dt$$

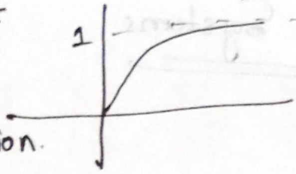
→ output

BIBO stable?

if $f(x)$ is constant function.

but output is not bounded $\int_0^t f(x) \, dt$



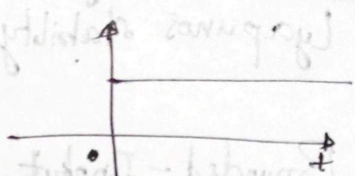
if $f(t) = 1 - e^{-t}$  $t \to \infty$ $f(t) = 1$ (upper bound)

It is bounded function for all 't'

⟹ Helicopter system is not stable (as there is no lower & upper bound)

Consider the input $T_y(t) = u(t)$ ⟶ unit step function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



But output will be $u(t) \times t$ (increases with 't')

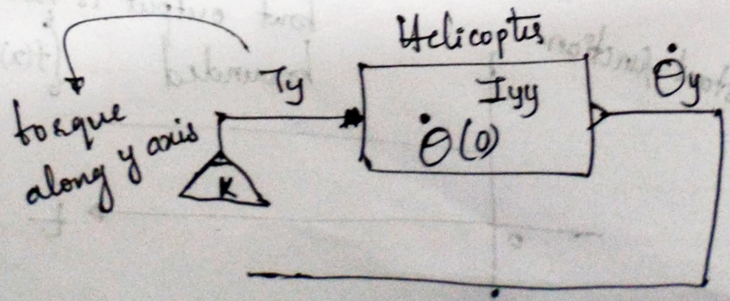
$$\boxed{\int_0^t T_y(t)\,dt}$$

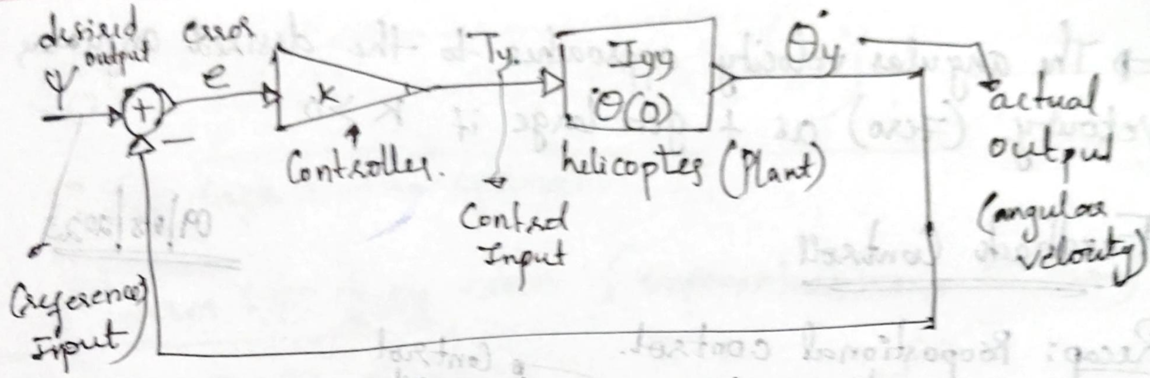
⟹ To stabilise the system, we will provide appropriate inputs such that system becomes stable

Feedback control

⟹ System with feedback is output fed back to input.
↳ directed cycles. ~~cycles~~

⟹ Control system
- measure and error e
  - discrepancy between desired and actual output
  - and use the error to provide control input
  - to the system.





Proportional Controller

→ Generally we call it controlling a plant (in this case it is helicopter)

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(t)\,dt$$

substituted by

ke

$= k(\psi - \dot{\theta}_y)$

$v = u + at$

$$\Rightarrow \dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{k}{I_{yy}} \int_0^t (\psi(t) - \dot{\theta}_y(t))\,dt$$

This equation can be solved by Laplace transformation (not needed for this course)

We assume that, the desired output

$\psi(t) = 0$ for all t, [We don't want helicopter to rotate)

Intially, it is spinning with some angular velocity, we want to stop it so we make angular velocity '0'

$$\boxed{\dot{\theta}_y(t) = \dot{\theta}_y(0) - \frac{k}{I_{yy}} \int_0^t \dot{\theta}_y(\tau)\,d\tau} \rightarrow (1)$$

Calculus $\int_0^t a e^{a\tau} d\tau = e^{at} u(t) - 1$

The solution to eq (1) is

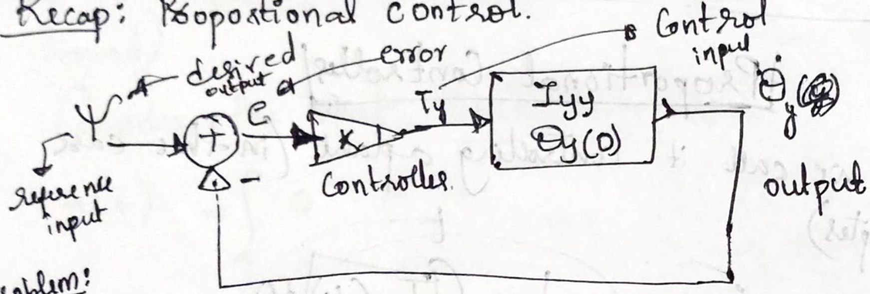
$$\boxed{\dot{\theta}_y(t) = \dot{\theta}_y(0)\, e^{-kt/I_{yy}}\, u(t)}$$

⇒ The angular velocity approaches to the desired angular velocity (zero) as t gets large if $k > 0$

Feedback Controll

09/08/2023

Recap: Propostional control.



Problem:

(1) given $\dot{\theta}_y(0)$ how to acheive

$$\dot{\theta}_y(t) = \Psi(t) = 0$$

Solution:
(1)

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) e^{-kt/I_{yy}} \cdot u(t)$$

⇒ as k increases, desired angular velocity is achieved same with large t.

$u(t)$: unit step

(2) Given $\dot{\theta}_y(0) = 0$ how to achieve

$$\Psi(t) = a u(t)$$

[we want to achieve a constant angular velocity from '0']

$$\dot{\theta}_y(t) = \frac{1}{I_{yy}} \int_0^t T_y(\tau)\, d\tau$$

$$= \frac{k}{I_{yy}} \int_0^t \left(\Psi(\tau) - \dot{\theta}_y(\tau)\right) d\tau$$

$$= \frac{k}{I_{yy}} \int_0^t a\, d\tau - \frac{k}{I_{yy}} \int_0^t \dot{\theta}_y(\tau)\, d\tau$$

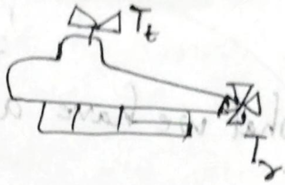
$$= \frac{k a t}{I_{yy}} - \frac{k}{I_{yy}} \int_0^t \dot{\theta}_y(\tau)\, d\tau$$

→ continuation

Solution : (2)
$$\boxed{\dot{\theta}_y(t) = a u(t)\left(1 - e^{-k t / I_{yy}}\right)}$$

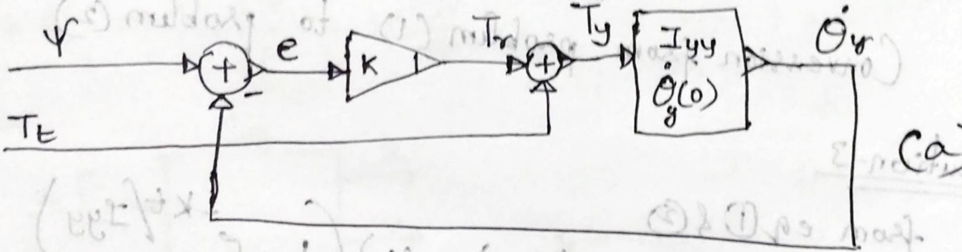
[using Laplace transformation]

2^{nd} term : traking error (exponential term - 2^{nd} term)



$T_y(t) = T_t(t) + T_r(t)$

Torque due to the ~~top~~ rotator.

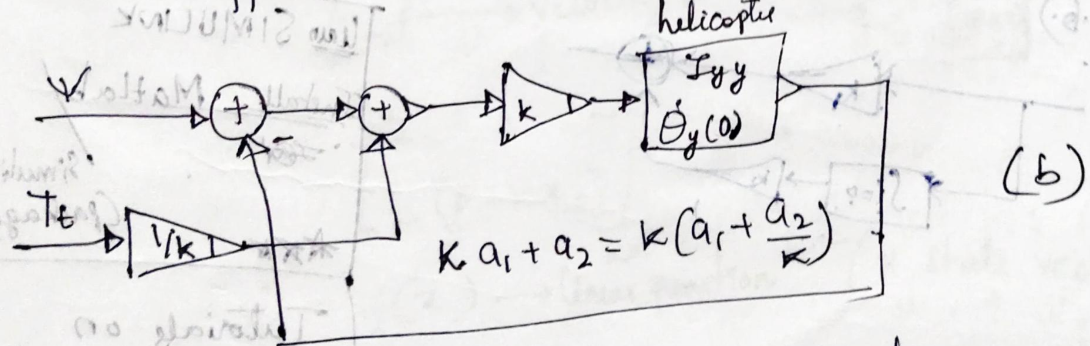


→ We can control only T_r (torque)

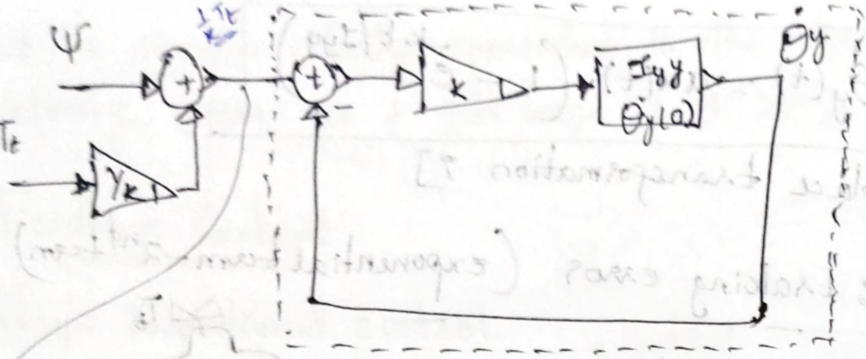
→ this problem is much more difficult to solve since we don't know T_t

⇒ So, Let's assume that $T_t = b u(t) \to ①$

suppose $\psi(t) = 0 \to ②$



(a) is transformed to (b) by ~~fac~~ considering
$$K a_1 + a_2 = K\left[a_1 + \frac{a_2}{K}\right]$$



(C)

what we have achieved from doing this transformations?

→ non zero angular velocity

Conversion from problem (1) to problem (2)

Solution-3

from eq ① & ②

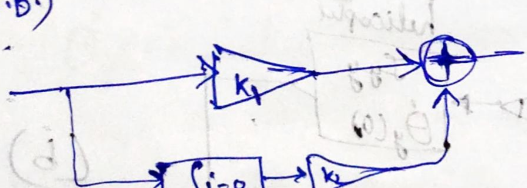
$$\dot{\theta}_y(t) = (b/k)\,u(t)\left(1 - e^{-kt/I_{yy}}\right)$$

[Similar to sol (2)]

$$\text{As } t \to \infty,\ \dot{\theta}_y(t) \to b/k$$

this is non zero

⇒ 7.b.)



Learn SIMULINK

Install Matlab / Simulink (package)

★★★

Tutorials on Simulink links in hompage

1st home exercise

7 questions (a & b)

MS Teams

Dynamical Systems (Rajeev Alur)

Continuous time models of systems using diff·

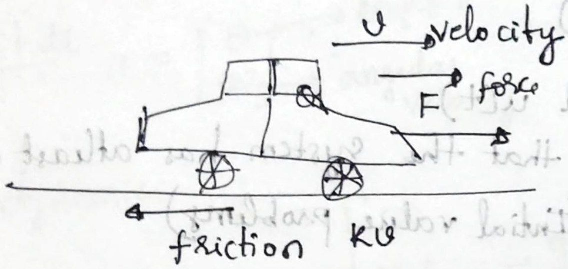
State space

change (country)

State:

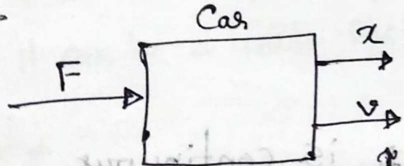
Ex: Car model.

1. State space representation



$$F - kv = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$$

Earlier

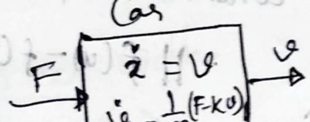


$$v(t) = v(0) + \int_0^t \cancel{\frac{F}{m}}\, d\tau \int_0^t \frac{(F-kv)}{m}\, d\tau$$

$$x(t) = x(0) + \int_0^t v(\tau)\, d\tau$$

state space

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} \rightarrow \text{position } x \\ \rightarrow \text{velocity } v \end{matrix}$$



$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{1}{m}(F - kx_2) \end{bmatrix}$$

$\dot{x} = f(x) \rightarrow$ not a linear function (x state vector)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F/m \end{bmatrix}$$

$\rightarrow$ if this is zero then it is linear function.

ψ — description

Ts. $\dot{\bar{x}} = f(x, u)$ — x: state, u: input (forgot in prev disc)

Output: $\psi = y = h(x, u)$

Q:- Given an input signal $u(t)$
When are we guarantee that the system has atleast or exactly one solution? (Intial value problems)

Intial Value problem:-

$\dot{x} = f(x)$ (System with no inputs called autonomous)

$x(0) = x_0$ ← constant.

(1) Atleast one solution if f is continious.
(2) Unique solution if f is lipschitz Continious

Def:- $f: \mathbb{R} \to \mathbb{R}$ is lipschitz continuous if there exists a constant c, such that ~~for~~ $\forall u, v \in \mathbb{R}$

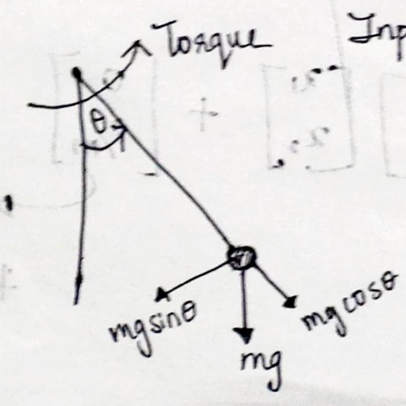
$$\| f(u) - f(v) \| \leq c \| u - v \|$$

Ex:- (1) ~~Any~~ linear function f is lip. continious

$\cancel{f(x) = A(x)}$ $f(x) = Ax + b$

(2) $f(x) = x^2, x^3, \ldots$ are not lip. continious.

Simple Pendulum



Input: external torque 'u'

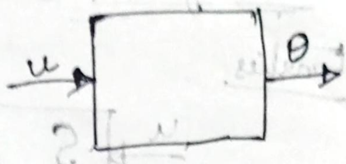
Convention

input variables: u
State variable: x
output variable: y

Dynamics!

$$u - mgl\sin\theta = ml^2\theta$$

(total torque)



State $x = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$ → θ: angle, ω: angular velocity.

dynamics $\dot{x} = \begin{bmatrix} \omega \\ u/ml^2 - \frac{g\sin\theta}{l} \end{bmatrix}$ not a linear system

if $u=0$ also it is not a linear system

⇒ if θ is small then $\sin\theta \approx \theta$ & $u=0$ then it can be a linear system.

Stability

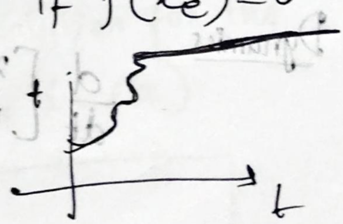
(1) BIBO stability (if input signal is bounded corresponding output signal is bounded)

(2) Lyapunov stability

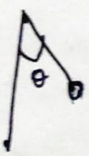
Applicable for closed systems. → no inputs

Assume:- System S with state vector x with dynamics lip. cont. and given by $\dot{x} = f(x)$

Def: A state x_0 is an equilibrium if $f(x_e) = 0$



Ex:-



$$\boxed{-\pi \le \theta < \pi}$$

$$d\theta = \omega.$$

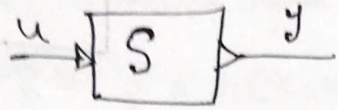
$$d\omega = -g\sin(\theta/l)$$

Equilibria (1) $\theta = 0, \ \omega = 0$

(2) $\theta = -\pi, \ \omega = 0.$

Linear Systems (State-Space View) 11/08/2023

Earlier



S is linear :

$S(a_1 u_1 + a_2 u_2) = a_1 S(u_1) + a_2 S(u_2)$

$\hookrightarrow$ super position principle

Linear State-space

$x : \mathbb{R} \to \mathbb{R}^n$ (state)

$u : \mathbb{R} \to \mathbb{R}^m$ (input)

$y : \mathbb{R} \to \mathbb{R}^k$ (output)

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}} \quad (1)$$

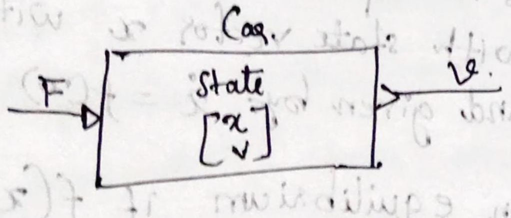
$x(t) \in \mathbb{R}^n$ $\quad y \in \mathbb{R}^p$

$A \in \mathbb{R}^{n \times n}$ $\quad C \in \mathbb{R}^{p \times n}$

$u(t) \in \mathbb{R}^m$ $\quad D \in \mathbb{R}^{p \times m}$

A, B, C, D representation of linear systems. $\quad B \in \mathbb{R}^{n \times m}$

Theorem :- If the ~~input~~ initial state $x(0) = 0$ then any system defined by (1) satisfies the superposition principle.



Dynamics

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ 1/m\,(F - kv) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k}{m} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{m} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -k/m \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

$$v = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

⟹ Superposition principle is satisfied only if initial state $\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Lyapunov Stability

* **Assume:** Closed continuous time system S.

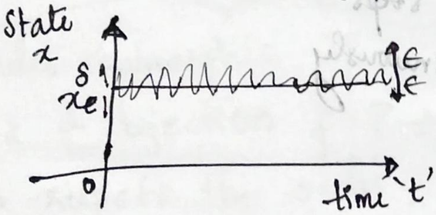
State $x(t) \in \mathbb{R}^n$

Dynamics $\dot{x} = f(x)$ → Lipc. cont.

($\dot{x}$ → time derivative)

* Equilibrium: $x_e \in \mathbb{R}^n$, such that $f(x_e) = 0$ → zero vector

(x_e → just a vector.)



Formally,

(1) The equilibrium x_0 is stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all states x_0 with $\| x_e - x_0 \| < \delta$, $\| x[x_0](t) - x_e \| < \epsilon$ for all $t \in \mathbb{R}_+$.

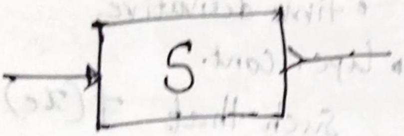
$x[x_0]$ → response signal, the unique solution for the initial value problem $x(0) = x_0$ & $\dot{x} = f(x)$

Alur's (notation)	Sis's (notation)
$S = \{x_1, x_2 \dots x_n\}$	$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ state
$I = \{u_1, \dots u_n\}$	$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$
$O = \{y_1, \dots y_n\}$	
$\frac{dS}{dt} = f(S, I)$	

(2) the equilibrium x_e is asymptotically stable if it is stable and there exists a $\delta > 0$ such that for all states x with $\|x_0 - x\| < \delta$, ~~then~~ $\lim_{t \to \infty} x[x_0](t)$ exists and equals x_e.

Discrete Dynamics ~~for~~ [Chap 3] 16/03/2023

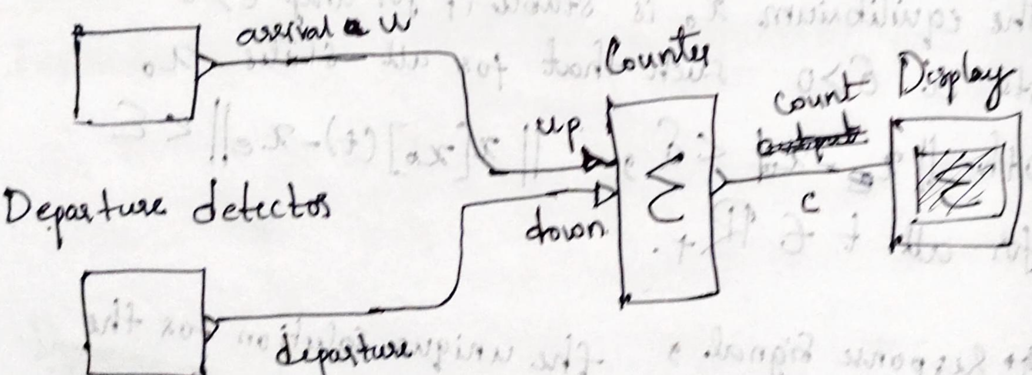
(assignment choose a & b vary k)



→ input & output signals are discrete
* discrete systems acts in discrete steps
* Continuous systems acts continuously

Parking Garage

Arrival detector (which generates arrival signal)



$u : \mathbb{R} \to \{\text{present, absent}\}$ → pure signals (either present or absent) and it doesn't have a value

$d : \mathbb{R} \to \{\text{present, absent}\}$

$c : \mathbb{R} \to \{\text{absent}\} \cup \mathbb{N}$

→ If 'u' and 'd' ~~present~~ at same time there is no change (in counter)

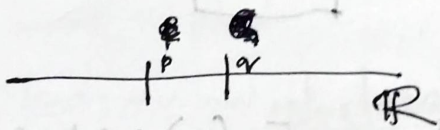
→ Inputs: u & d are discrete

Set $\mathbb{R}$ → uncountable.

$\mathbb{Z}, \mathbb{Q}, \mathbb{N}$ — Countable, enumerable. (Discrete)

(infinite Countable set)

<u>Discrete Signal</u>:- If it is present at the set of instants T then T is either finite or it is order isomorphic with $(\mathbb{N}, \leq)$



$\underbrace{(P, \leq_P), (Q, \leq_Q)}$

order isomorphic if there is a bijection $f : P \to Q$ which respects the order, i.e.,

$x <_P y \iff f(x) <_Q f(y)$

→ strictly less than (not reflexive) (irreflexive class)

$\mathbb{Q}$ & $\mathbb{N}$ are not order isomorphic.

$\mathbb{Q}$ $\mathbb{N}$

$x \to n$

$y \to n+1$

Ports $P = \{up, down\}$

Types of ports $V_{up} = V_{down} = \{present\}$

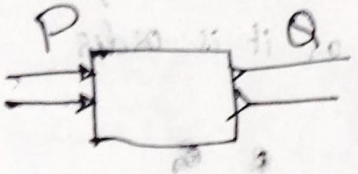
(Set of values up & down take)

For each input port $p \in P$ the set V_p denotes the values that every recieved when the input is present

Reaction :-

Behaviour of discrete systems ← sequence of steps (reactions)

Variation of the inputs :- Assignment of a value in V_p to each $p \in P$ (port)



$$S : (\mathbb{R}_+ \to Val(P)) \to (\mathbb{R}_+ \to Val(Q))$$

where, $Val(P)$: set of valuations of the ports in P

⇒ discrete systems are described using state machines usually FSM, but not always.

States $Q = \{0, 1, \dots, M\}$

Current count in parking model is state at any time t

FSM (Mealy machine)

(Q, q_0, I, O, δ)

Q :- finite set of states

q_0 :- initial state

I :- input alphabet (set of inputs)

O :- output alphabet (set of outputs)

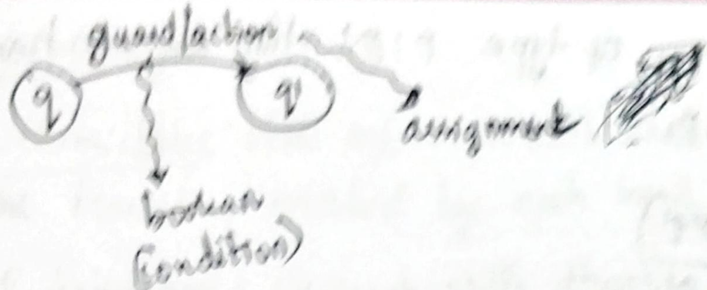
δ :- transition state

$$\delta : Q \times I \to Q \times O \quad \text{(deterministic)}$$

$$q \xrightarrow{i/o} q'$$

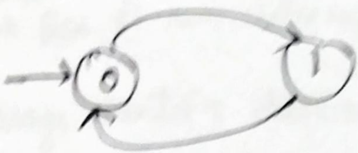
⇒ Always the systems (counter or car) we want it to be deterministic

⇒ For a given input signal, if we have a unique output signal then it is called deterministic.



Inputs : up, down : pure

Output: count : $\{0, \ldots, M\}$.



Discrete Dynamics : FSM

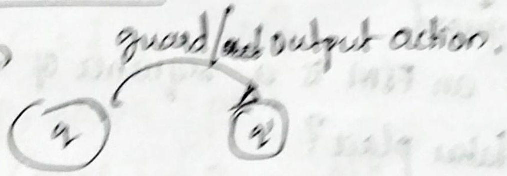
18/05/2023

Mealy m/c : (Q, q_0, I, O, δ)

states, initial state, input, output, transition funcn

$$\delta : Q \times I \to Q \times O$$

Convention



Garage counter

inputs : up, down : pure

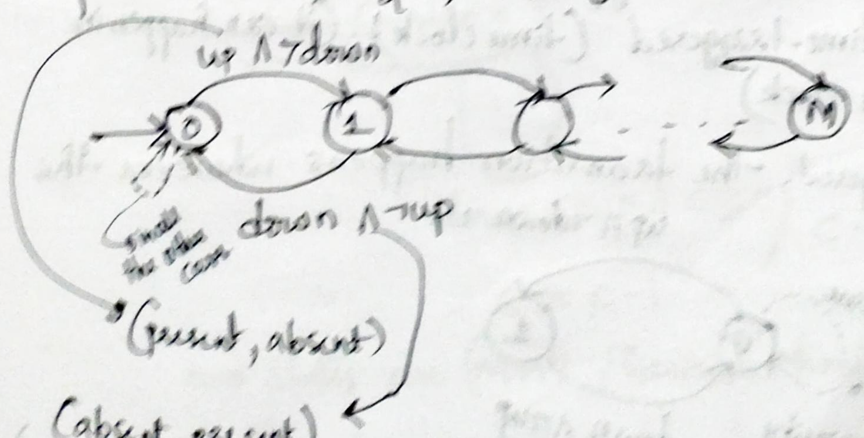
Output : Count : $\{0, \ldots, M\}$

P_1, P_2, P_3 : place

guard true

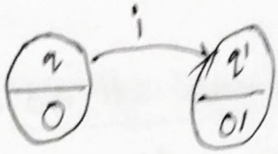
P_1

$P_1 \wedge P_2$



If in guard, P_1 is of type $p : \mathbb{N}$ then we can have guard as $(p > 5) \wedge p_1$

Moore :-

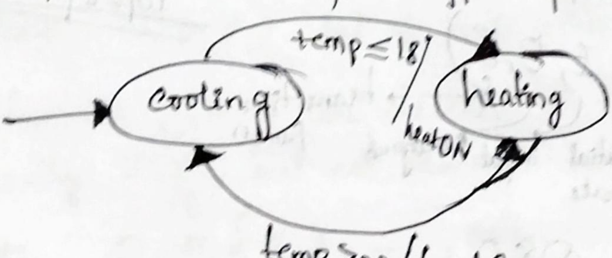


We choose mealy machines over moore because they are more compact.

Thermostat

input : temp : $\mathbb{R}$

Output : heatOn, heatOff : pure



FSM: the behaviour of an FSM is a sequence of reactions.

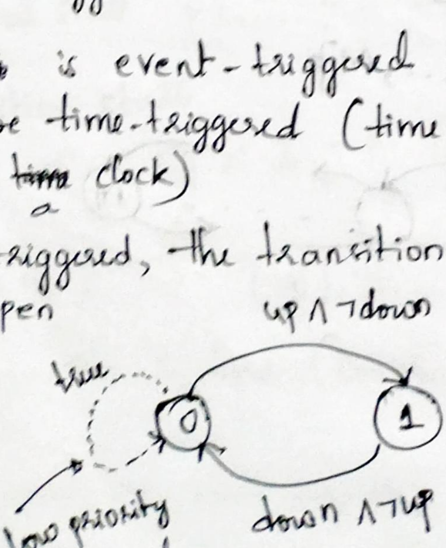
When does a reaction takes place?

— event-triggered

— time-triggered.

⇒ Thermostate is event-triggered (when temp ≤ 18), it can also be time-triggered (time clock). (it can happen in a tick of ~~time~~ a clock)

⇒ Even-triggered, the transition happens whenever the event happen



for default transitions (usually are not drawn)

Determinacy:-

Deterministic state m/c : in each state there is at most one transition enabled by each input value.

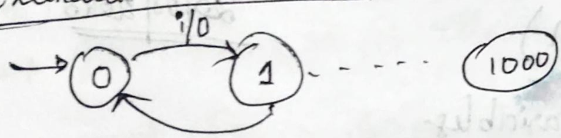
Receptiveness : for each state there is at least one transition possible on each input symbol.

⇒ ~~A~~ Son A non deterministic machine can also be receptive.

Garage counter : deterministic

Thermostate : deterministic

Extended state machines



{these many (1000) states is not convinient & time consuming}

⇒ We introduce variables.

for garage counter, we introduce variable 'c'

Variable : C : {0, ---, M}

Inputs : up, down : pure

Output : Count : {0, ..., M}

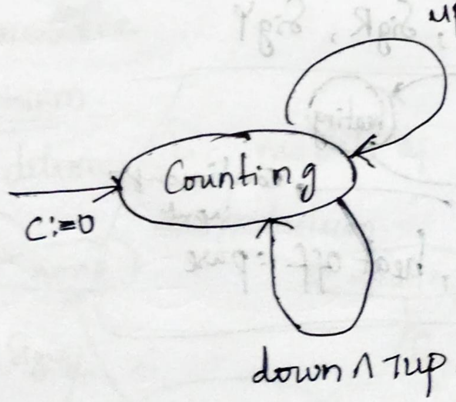
effects output signal

Output action

guard

$\underbrace{up \wedge \neg down \wedge c < M}_{guard}$ / $C+1$

$C := C+1$ → set action → effects internal variable



down ∧ ¬up ∧ C ≥ 0 / C − 1

$C := C-1$

⇒ States are (M+1) [Variable changes]

(Counting is not state, it is a location where (M+1) states occurs)

Variable Vs Signal

(up & down)

Variable retains value, but signal changes with instances.

Extended State Machine of Traffic light

Traffic light at a pedestrian crosswalk

- time triggered (once per second)
- locations : red, green, yellow, pending
- input : pedastrian : pure
- Output :- SigG, SigR, SigY.

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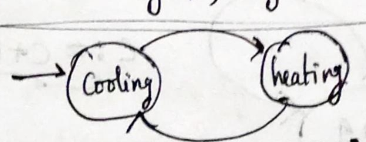
Extended state machines (ESM)

- augment FSMs with variables

Ex: ESM for traffic light at a pedastrian crosswalk.

* time-triggered : reacts once per second
* Reaction : variable count
* locations : red, green, yellow, pending
* pure input :- pedastrian (push the button)
* pure output : SigG, SigR, SigY

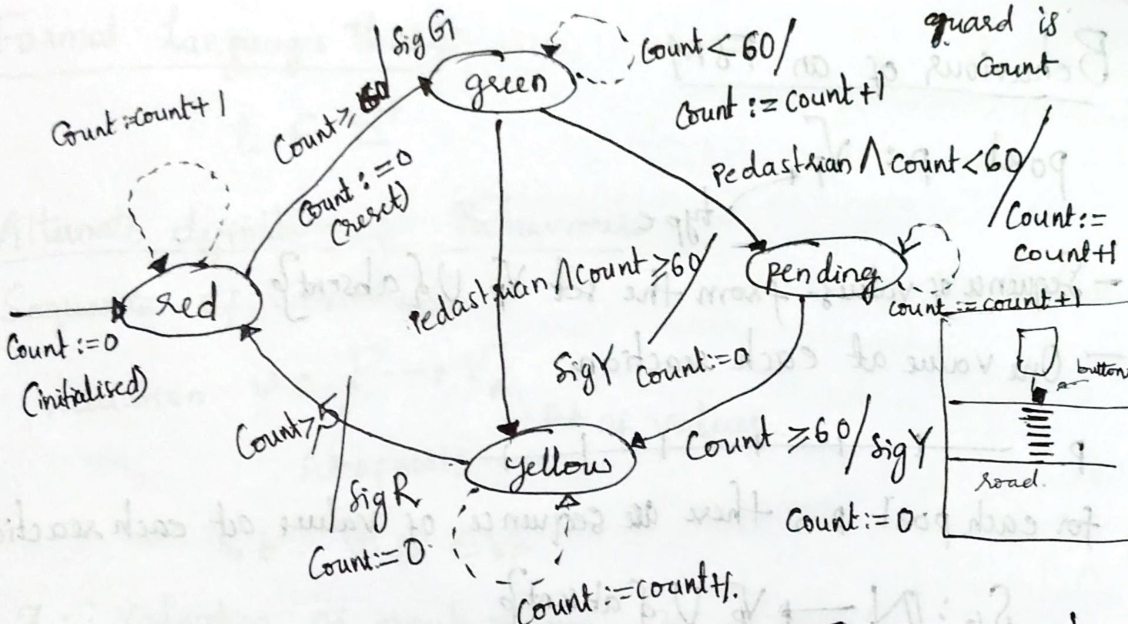
Thermostat



(read pdf) input :- temp : $\mathbb{R}$. continuous input

Something said

Output ⇒ heat on, heat off : pure



⇒ Pending location is ~~for~~ allow cars to move ~~fut~~ (for 60 secs)

⇒ green light is on for atleast 60 sec (Count ≥ 60 → 0)

var count : {0, ---, 60}, input : pedastrian; pure.

Output : SigR, SigG, SigY : pure.

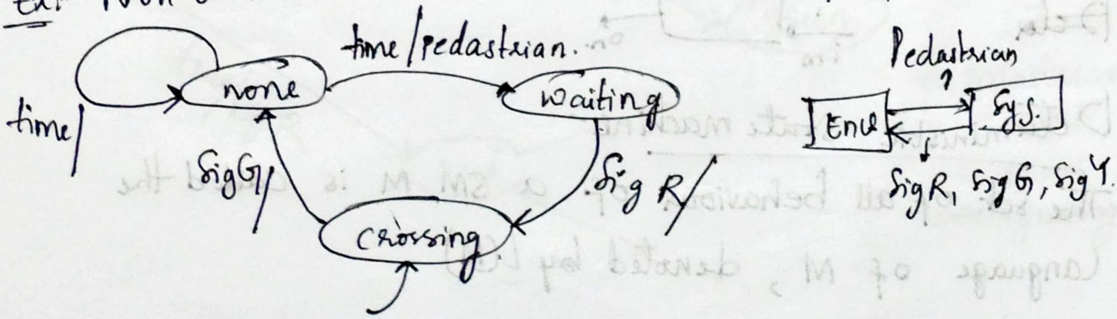
<u>No. of states</u>

$4 \times 61 = 244.$

Not all the 244 states are reachable.

<u>Non determinism</u>

Ex: Non deterministic model of arrival of pedastrians.



Behaviour of an FSM

port $p : V_p$ — type

- Sequence of values from the set $V_p \cup \{absent\}$
- One value at each reaction

for each port p, there are sequence of values at each reaction

$$S_p : \mathbb{N} \longrightarrow V_p \cup \{absent\}$$

Signal received or produced at port P

Behaviour of an ~~F~~M:

assignment of such a signal to each port s.t. ...

Behaviour of state machines

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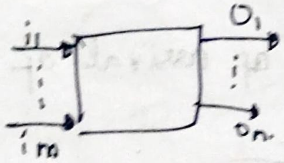
Behaviour : assignment of a signal to each port respecting its type s.t. the signal on any output sequence produced for the given input signals.

Signal $S_p : \mathbb{N} \rightarrow V_p \cup \{absent\}$

~~Deta~~



Deterministic state machine:

The set of all behaviour of a SM M is called the language of M, denoted by L(M)

Formal Languages Theory:-

$$L \subseteq \Sigma^*$$

Alternate defination of Behaviour:

Sequence of valuation

Valuation $v : P \rightarrow V_p$ — set of values.
P: set of ports

s.t. $v(p) \in V_p$

x_i : Valuation of input ~~values~~ ports at time i

y_i : Valuation of output ports at time i

$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots \Rightarrow$ Observation trace

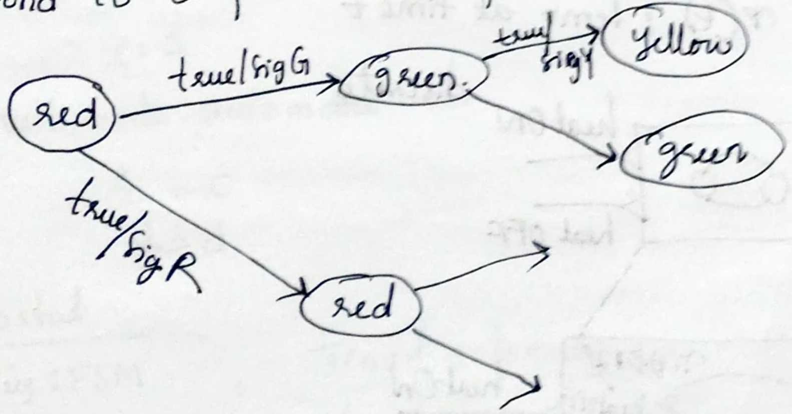
Execution trace : $s_0 \xrightarrow{x_0/y_0} s_1 \xrightarrow{x_1/y_1} s_2 \xrightarrow{x_2/y_2} s_3 \ldots\ldots$

⇒ We either use observation trace or execution trace.

Non deterministic ~~systems~~ SMs:-

Computation tree: represents all possible traces that correspond to a particular input sequence.

(ND machine)



⇒ guards are all True

⇒ pedastrian is input but we are ignoring.

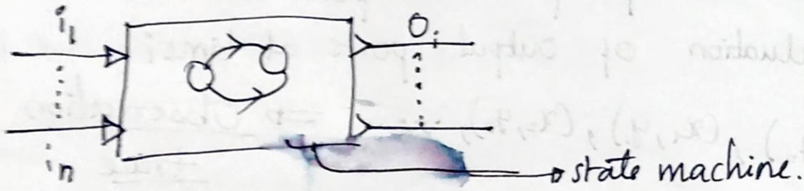
So far learnt topics:

(1) Continuous dynamics.
- Continuous signals.
- behaviour given by ODEs

(2) Discrete dynamics
- discrete signals.
- behaviour given by state machines

Next topic: (3) Hybrid systems: discrete + continuous dynamics.

Actor models for state machines:-



Thermostat

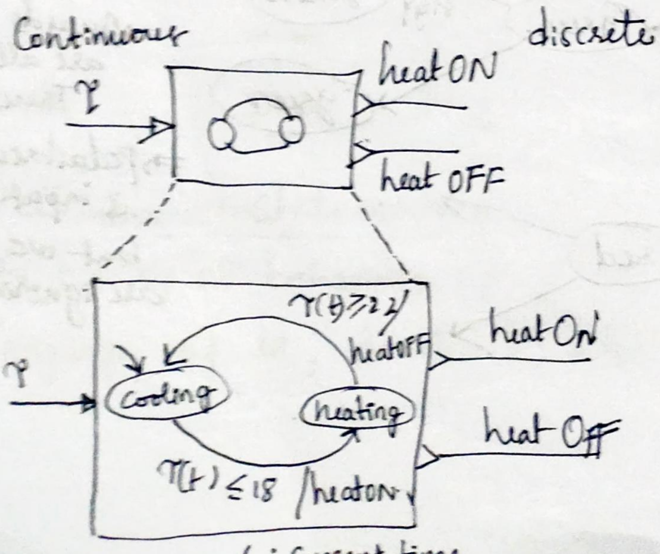
Earlier, input: temp : discrete (considered) signal

discrete signal input : temp : $\mathbb{N} \to \mathbb{R}$

But now, Cont. signal

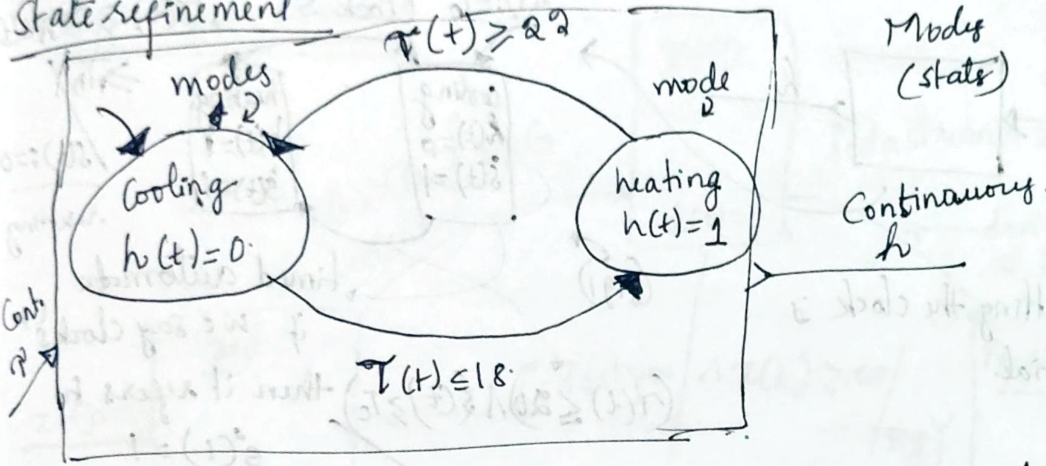
$\tau : \mathbb{R} \to \mathbb{R}$

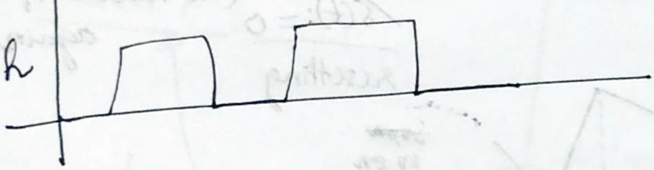
$\tau(t)$: temp at time t



t : Current time

State refinement





Continuous signal (it is defined at all non negative t)

Timed Automatics : Alur & Dill

Discrete dynamics (FSMs) + Clock variables

Clock variable : Continuous time variable (x), whose derivation is constant

$\dot{x}(t) = c$ $\quad \boxed{\dot{x}(t) = 1}$ $\quad x : \mathbb{R} \to \mathbb{R}$

⇒ We can set (or) reset its value to '0' (or) compare its value to any rational / integer / natural numbers.

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(1) timed automata

$\dot{x} = 1$

(2) multi-rate automata

$\dot{x} = c$

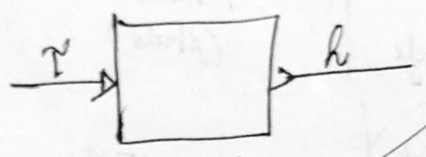
$\dot{y} = d$

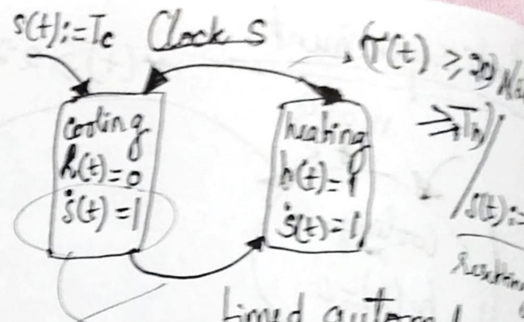
Thermostat

Earlier : FSM , Timed automata :

input $\tau : \mathbb{R} \to \mathbb{R}$

Output $h : \mathbb{R} \to \{0, 1\}$.





(fig1)

$\Rightarrow$ resetting the clock is essential.

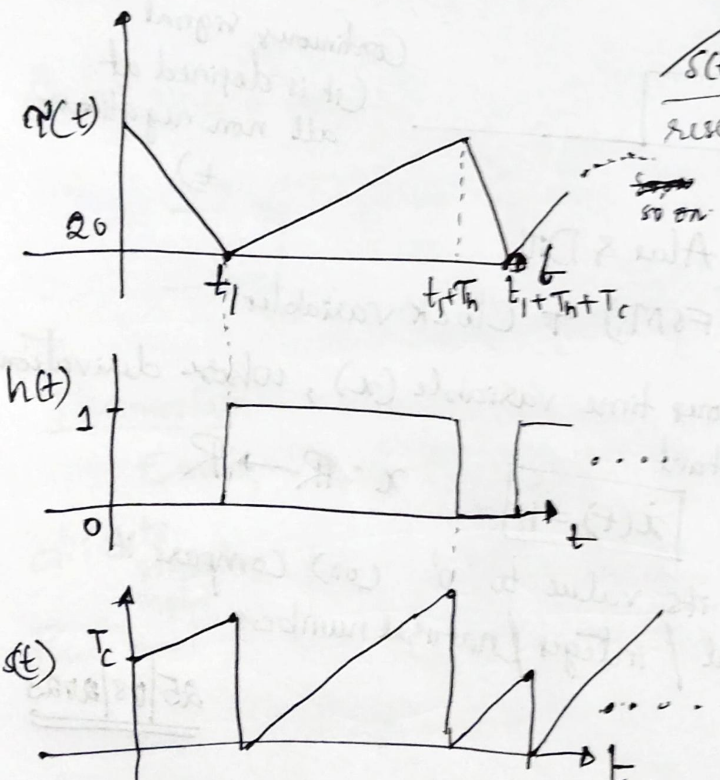
timed automata

If we say clock, then it refers to $\dot{s}(t) = 1$ (no need to specify again).

$(T(t) \leq 20) \wedge s(t) \geq T_c$

$s(t) := 0$

resetting



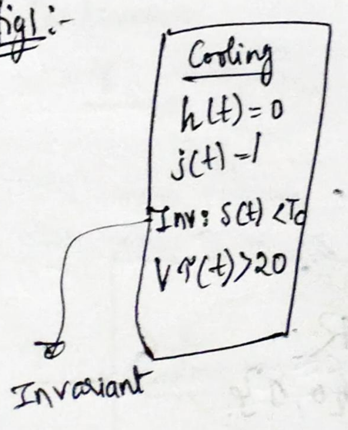
$:=$ assignment

$=$ equality

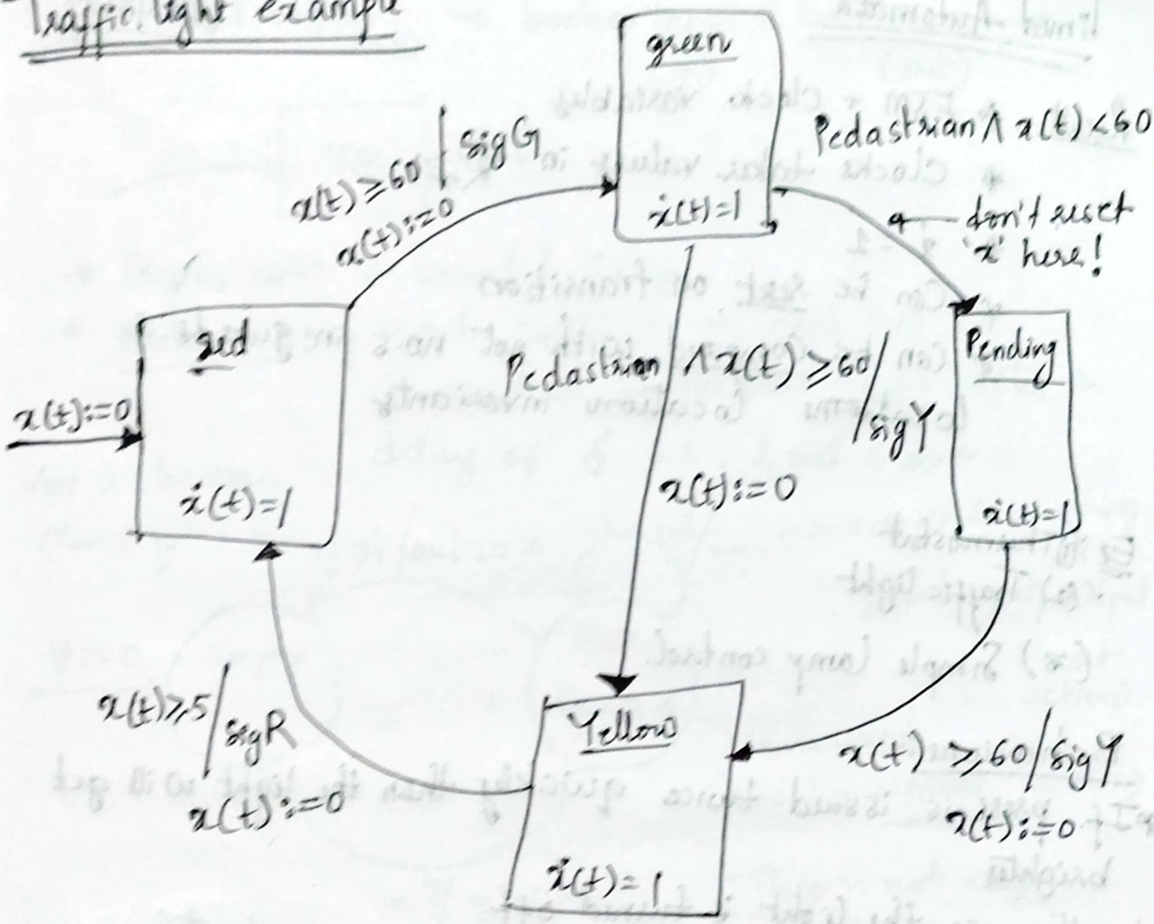
~~eg~~ edges semantics for transitions

— a transition is taken the moment it is true.

fig1:-



Traffic light example



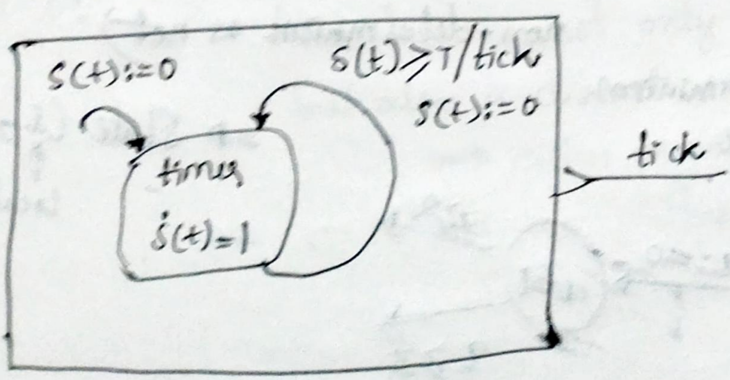
(Continuous variable)

clock : $x(t)$: $\mathbb{R}$

inputs : pedastrian : pure

outputs : sigR, sigG, sigY

→ Timed automata that generates a pure output tick every T time units.



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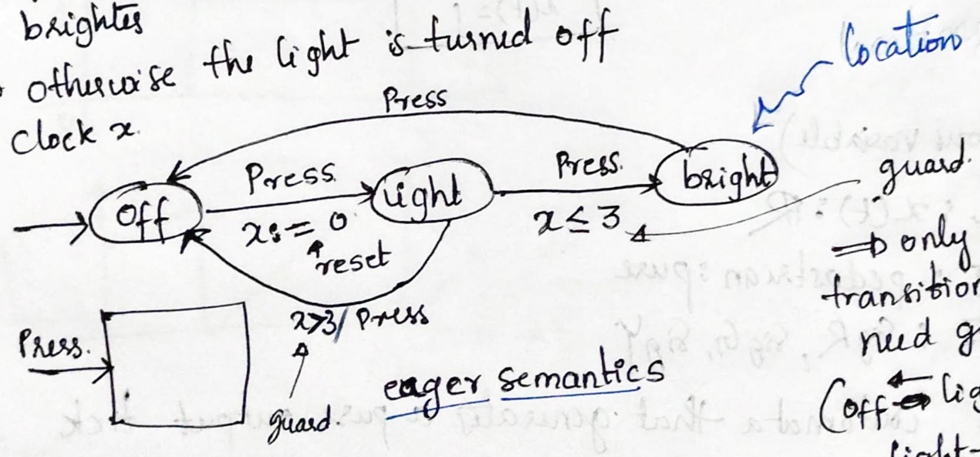
Timed "Automata"

Recall: * FSM + clock variables
* clocks takes values in $\mathbb{R}_{\geq 0}$
* $\dot{x} = 1$
* Can be reset on transition
* Can be compared with not no.s in guards & ~~location the~~ location invariants

Ex: (1) Thermostat
(2) Traffic light
(3) Simple lamp control.

Behaviour:-
* If press is issued twice quickly then the light will get brighter
* otherwise the light is turned off

clock x



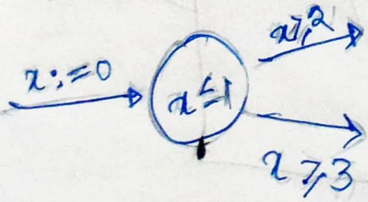
⇒ only two transitions need guards (off ← light & light → bright)

eager semantics

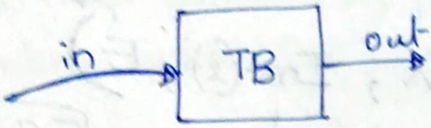
Uppaal: TA tool (Can check whether there is non-determinism or not)

(1) non determinism
(2) time lock

⇒ State (l, clock)
l = location, clock = time.



Ex:- Timed buffer (TB) → boolean input & boolean output. (in) (out)

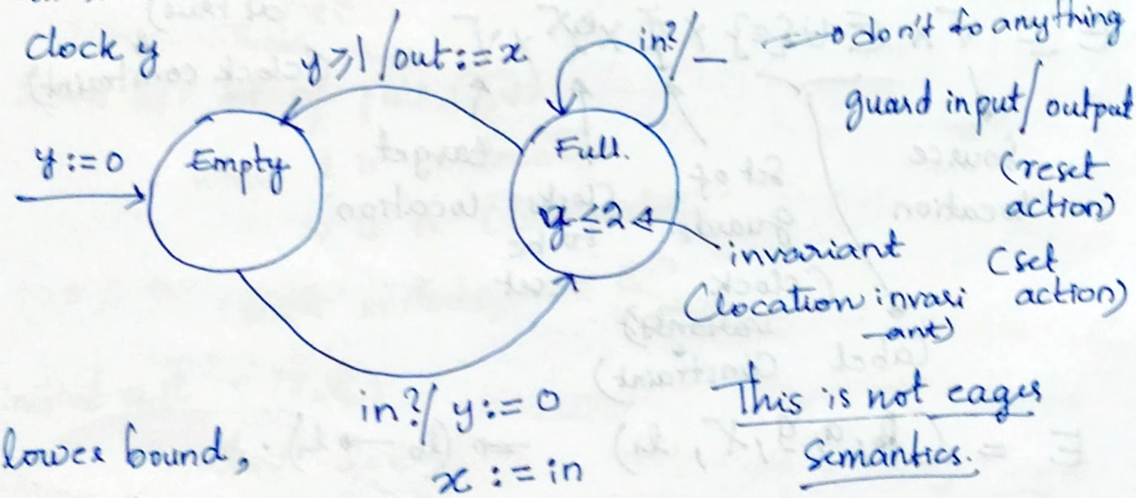


* Buffer with a bounded delay

* Behaviour: of input recieved on port 'in' is transmitted in the port 'out' after a delay of δ s.t. $l \leq \delta \leq u$

Var x : boolean

Clock y



This is not eager semantics.

lower bound, $l = 1$

upper bound, $u = 2$

→ If we use eager semantics, there is no way to say that delay δ is $l \leq \delta \leq u$

input in? → in? means 'in' is present 'in' is either '1' or '0'

Output o!

here 'in' is not only boolean but also a discrete signal

→ either present (or) absent

↳ either '0' or '1'

Timed Automata

Set of labels (Alphabet)

finite set of clocks

finite set of edges

Def: $A = (V, V_0, \Sigma, X, Inv(l), E)$

V: finite set of locations

$V_0 \subseteq V$ initial locations

$Inv(l)$: invariant for each location l (if not written it is taken as true) ⇓ (clock constraint)

~~$E \subseteq V \times \Sigma \cup \{\epsilon\} \times$ ... Source location~~

$$E \subseteq V \times \Sigma \cup \{\epsilon\} \times \Phi \times 2^X \times V$$

- V (first): Source location
- $\Sigma \cup \{\epsilon\}$: label
- Φ: Set of guards (clock ~~variants)~~ constraint)
- 2^X: Clocks to be reset
- V (last): target location

$E = (l_0, a, g, X, l_1) \rightarrow (l_0 \rightarrow l_1)$.

Clock constraint: Boolean combination of comparisons with integers

$$\Phi ::= x \sim c \mid \neg \Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi$$

$\sim c \; \{<, \leq, =, >, \geq\}$

$x \in X$ clock

Timed Automata: Semantics

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Given timed automata A, define an infinite state transition system $S(A)$ as follows.

Transition system $(Q, L, \rightarrow)$

$\rightarrow \subseteq Q \times L \times Q$

$(s, a, t) \in \rightarrow$

$\equiv$

$s \xrightarrow{a} t$

States Q: set of pair (l, v)

where l is a location

v is a clock vector.

with n clocks, $v \in \mathbb{R}^n_{\geq 0}$

initial state: (l_0, v_0)

where $l_0 \in V_0$

& $v_0(x) = 0$

for each clock x

Two types of transitions

(1) Time delay transition

$(l, v) \xrightarrow{d} (l, v+d)$ $d \in \mathbb{R}_{\geq 0}$ ↳ scalar

↑ vector

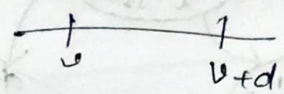
→ if both v and $v+d$ satisfy the ~~ive~~ invariant for location l

$Inv(l)$ [∵ then all intermediate pts also satisfy]

Clock constraints are of form:

$x \leq c \mid x < c \mid x = c$

$x > c \mid x \geq c$



if they are true at v & $v+d$ they also true at intermediate pts

(2) Switch Transition

$(l,v) \xrightarrow{a} (l',v)$

If there is an edge (l, a, g, X, l') — label; set of clocks to be reset; source location; guard; target location

v satisfies g &

$v' := v[X := 0]$

new clock vector; old c.v.

Ex:- Clocks:- x, y, z

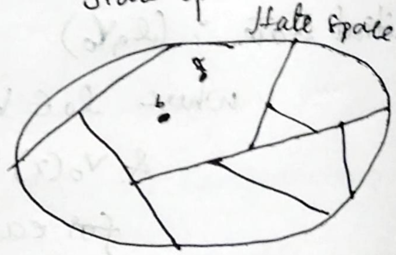
$(3,1,2)$ ($x\ y\ z$) $\Leftarrow$ ~~$(x>3) \wedge (y>2)$~~ $(x>2) \wedge (y \leq 2)$

Q:- Is an unsafe set of states reachable from the initial state?

e.g. (l, x) is reachable where $5 \leq x \leq 10$

location

State space



a → b in same is safe.

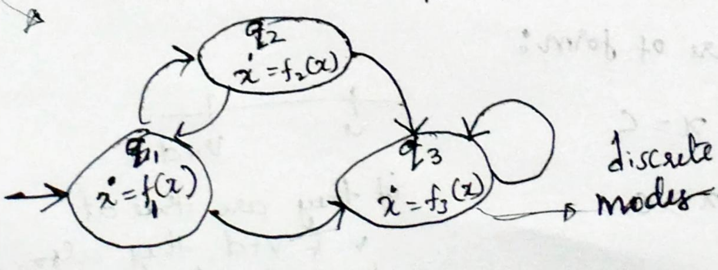
→ If we have FSM we can say if bad state is reachable or not.

Hybrid Systems :- discrete + Continuous dynamics.

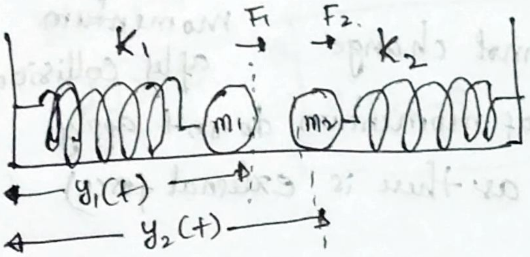
(1) Timed automata (simple not covers all)

(2) Hybrid systems (complex & cont. variable doesn't have simple behaviour)

(1) $\dot{x} = 1$ 2) $\dot{x} = f(x)$ vector



Ex (i) sticky masses attached to springs



* springs compressed/extended & released
* frictionless table.
* $k_1, k_2 \to$ spring constants

* P_1, P_2 : neutral positions of masses

APART

Force on left mass : $k_1(P_1 - y_1(x))$

Force on right mass : $k_2(P_2 - y_2(x))$

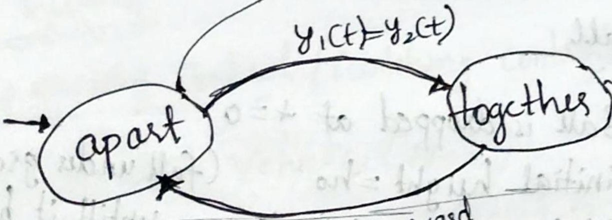
$$\ddot{y}_1(t) = k_1(P_1 - y_1(t))/m_1$$

$$\ddot{y}_2(t) = k_2(P_2 - y_2(t))/m_2$$

TOGETHER

$$y(t) = y_1(t) = y_2(t)$$

$$\ddot{y}(t) = \frac{k_1P_1 + k_2P_2 - (k_1+k_2)y(t)}{m_1 + m_2}$$



06/09/2023

$y_1(t) = y_2(t)$ → guard

→ set actions

$y := y_1(t)$

$\dot{y}(t) := \frac{1}{m_1+m_2}(m_1\dot{y}_1(t) + m_2\dot{y}_2(t))$

$y_1(0) = i_1$, $y_2(0) = i_2$, $\dot{y}_1(0) = 0$, $\dot{y}_2(0) = 0$

apart

$\ddot{y}_1(t) = k_1(P_1 - y_1(t))/m_1$

$\ddot{y}_2(t) = k_2(P_2 - y_2(t))/m_2$

together

$\ddot{y}(t) = \frac{k_1P_1 + k_2P_2 - (k_1+k_2)y(t)}{m_1+m_2}$

$y_1(t) = y(t)$

$y_2(t) = y(t)$

$(k_1 - k_2)y(t) + k_2P_2 - k_1P_1 > S$ → guard

$y_1(t) := y(t)$, $y_2(t) := y(t)$, $\dot{y}_1(t) := \dot{y}(t)$, $\dot{y}_2(t) := \dot{y}(t)$

Momentum before collision $\Rightarrow m_1\dot{y}_1(t) + m_2\dot{y}_2(t) = \underbrace{(m_1+m_2)\dot{y}(t)}_{\text{momentum after collision}}$

why? (Ans:- The momentum cannot change instantenously (Law of momentum doesn't apply here as there is external force)

$\Rightarrow$ S : stickiness of the two masses.
if $(F_2 - F_1)$ exceeds S then they come apart.

Clock constraints [TIMED AUTOMATA]

$$\Phi ::= x < c \mid x \leq c \mid x > c \mid x \geq c \mid \Phi \wedge \Phi$$

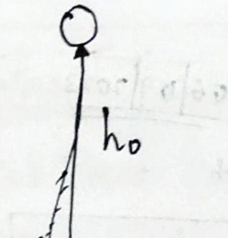
- no negation (so we can write $(x<c) \wedge (x \geq c)$)
- no disjunction (for equality $(x \leq c) \wedge (x \geq c)$)

Convex :- $X \subseteq \mathbb{R}^n$, $0 < \lambda < 1$

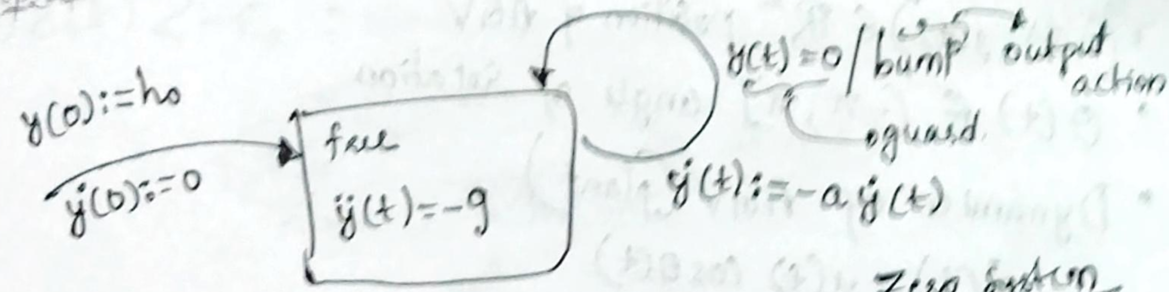
$$\forall x_1, x_2 : (x_1 \in X \wedge x_2 \in X) \Rightarrow (\lambda x_1 + (1-\lambda)x_2) \in X$$

---

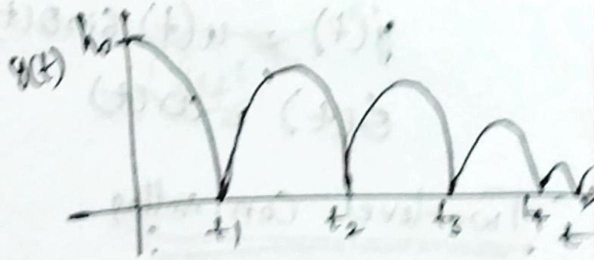
(2) Bounding Ball

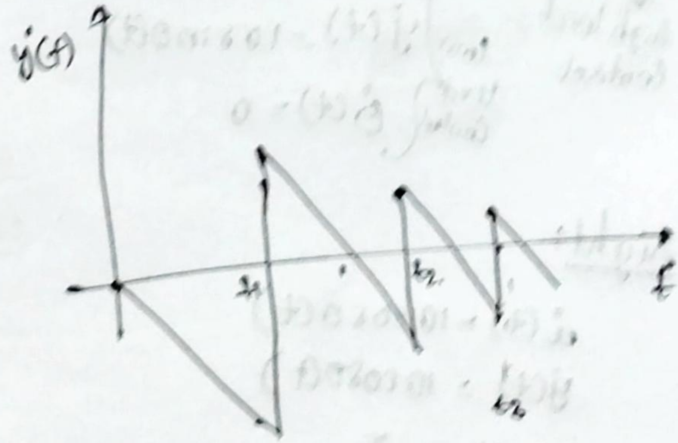


- Ball is dropped at $t=0$
- initial height $= h_0$ (fall under gravity untill it hits gnd)
- At time t_1 it hits the ground with velocity $\dot{y}(t_1) < 0$
- Collision is inelastic
- ball bounces back with velocity $-a\dot{y}(t_1)$ where $0 < a < 1$



Zero System

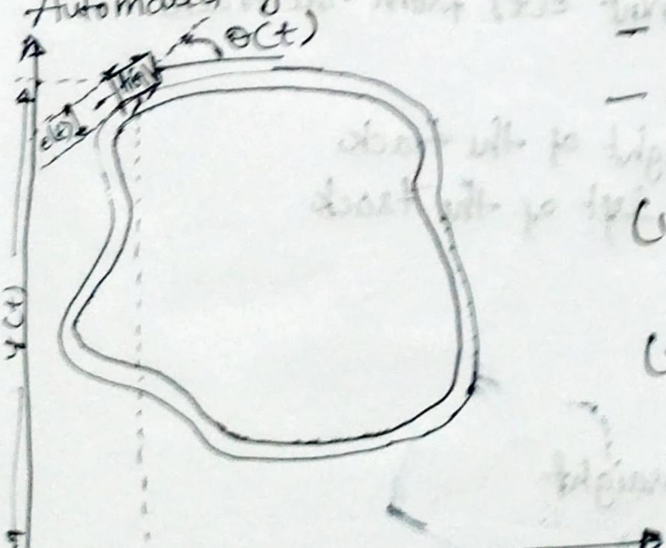




Hybrid Systems

Ex:- Supervisory Control (switching control) 08/09/2022

Automated guided vehicle (AGV)



- Closed track, painted
- AGV has two degrees of freedom
  - (1) Move forward with speed $0 \le u(t) \le 10$
  - (2) Rotate with angular speed $-\pi \le \omega(t) \le \pi$ rad/sec

• $(x(t), y(t)) \in \mathbb{R}^2$ position of AGV

• $\theta(t) \in [-\pi, \pi]$ angle of rotation

• Dynamics of AGV (plant)

$$\dot{x}(t) = u(t) \cos\theta(t)$$

$$\dot{y}(t) = u(t) \sin\theta(t)$$

$$\dot{\theta}(t) = \omega(t)$$

Two-level controller.

Straight: high level control

low level control

$$\begin{cases} \dot{x}(t) = 10 \cos\theta(t) \\ \dot{y}(t) = 10 \sin\theta(t) \\ \dot{\theta}(t) = 0 \end{cases}$$

Left:

$$\dot{x}(t) = 10 \cos\theta(t)$$

$$\dot{y}(t) = 10 \sin\theta(t)$$

$$\dot{\theta}(t) = \pi$$

Right:

$$\dot{x}(t) = 10 \cos\theta(t)$$

$$\dot{y}(t) = 10 \cos\theta(t)$$

$$\dot{\theta}(t) = -\pi$$

Stop:-

$$\dot{x}(t) = 0$$

$$\dot{y}(t) = 0$$

$$\dot{\theta}(t) = 0$$

* Sensors - photodiodes.
* track painted with light reflecting colour.
* Sensors estimates the displacement $e(t)$ from the track.

Convention

$e(t) < 0$: AGV is to the right of the track

$e(t) > 0$: AGV is to the left of the track

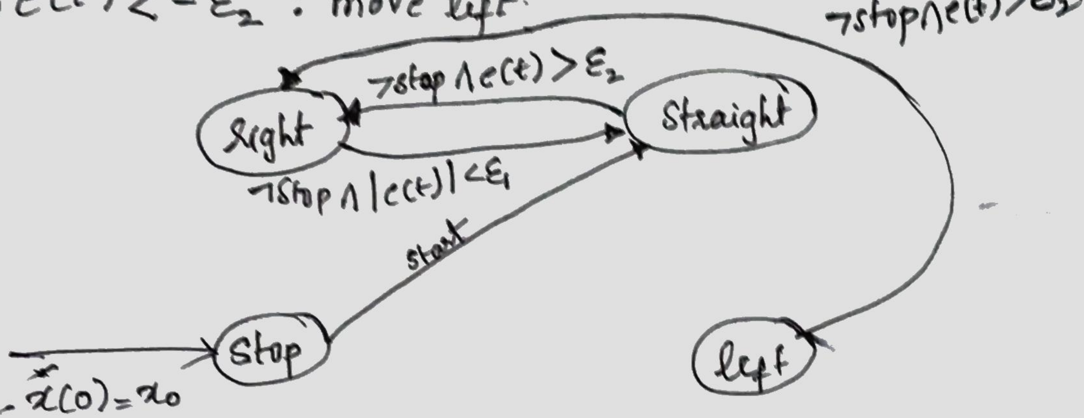
Two thresholds

$$\boxed{0 < \varepsilon_1 < \varepsilon_2}$$

(1) $|e(t)| < \varepsilon_1$: move straight

(2) $e(t) > \varepsilon_2$: move right

(3) $e(t) < -\varepsilon_2$: move left.



$x(0) = x_0$

$y(0) = y_0$

$\theta(0) = \theta_0$

positions

orientation.

$e(t) = f(x(t), y(t))$

e(t)