HIGHER CYCLES IN THE (3X+1)-PROBLEM DISPROVED

1. Introduction

The (3x+1)-Problem consists of two rules building a Collatz-sequence:

$$n_{n+1} := \begin{cases} 3n_i + 1 & \text{if } n_i \text{is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{is even} \end{cases}$$

and a conjecture:

Conjecture 1. *Independent of the starting number $n_1 \in \mathbb{N}$ the Collatz-sequence ends into the cycle 4-2-1.*

Although the Problem is easy to understand, the conjecture is not proved or disproved yet. The Collatz-sequence could theoretically have two other alternative endings. It could end in a higher not yet known cycle or it could grow infinitely.

2. Generalization

The two operators of the Collatz-problem are not commutative. They could be made commutative by generalization . For this purpose the range of numbers was expanded from natural numbers to positive rational numbers with a finite amount of decimal places in the binary number system. That means The Collatz-operator C was transformed that it adds as many powers of 2 as the number consisted of. The division-operator D was not changed:

$x = y \cdot 2^n$, $x \in \mathbb{Q}$, $y \in \mathbb{N}$, $\frac{y}{2} \notin \mathbb{N}$, $n \in \mathbb{Z}$.

$C \equiv 3x + 2^n = 3y \cdot 2^n + 2^n = (3y + 1) \cdot 2^n$

$D \equiv \frac{x}{2} = y \cdot 2^{n-1}$

This generalization makes the two operators commutative: $C(D(x)) = D(C(x))$.

$C(D(x_0)) = C(D(y_0 \cdot 2^n)) = C(y_0 \cdot 2^{n-1}) = 3y_0 \cdot 2^{n-1} + 2^{n-1}$

$D(C(x_0)) = D(C(y_0 \cdot 2^n)) = D(3y_0 \cdot 2^n + 2^n) = 3y_0 \cdot 2^{n-1} + 2^{n-1}$

The Collatz-conjecture turns to:

Conjecture 2. *Independent of the starting number $x = y \cdot 2^n$, $x \in \mathbb{Q}$, $y \in \mathbb{N}$, $\frac{y}{2} \notin \mathbb{N}$, $n \in \mathbb{Z}$ the Collatz-sequence makes the $y = 1$.*

Because of the commutativity of the Operators and the reformulation of the Collatz-conjecture it is now unnecessary to apply the D-operator.

3. Transformation

In this approach the numbers were transformed in base 2 without executing the D-operator. Now there are simple rules to compute a new number applying the C-operator. This is makes it possible to make a small program for a Turing machine. A new digit is calculated by adding the old digit, the digit on the right of the old digit, the carry and 1, if it is the first 1 on the right. Each time the Turing machine applies the Collatz-operator on a number, the first 1 and the last 1 are moved to the left. This is the y- part of the number. The 2^n-part of the number consists of the lasting zeros.

It doesn't matter how big n is. Only the y-part is important.

Definition 3. Two numbers $a = y_a \cdot 2^n$ and $b = y_b \cdot 2^m$, $y_a,\ y_b \in \mathbb{N}$; $n,\ m \in \mathbb{Z}$ are Collatz-identical, if $y_a = y_b$. In formulas the sign $\overset{C}{=}$is used.

Example: numbers in base 2: $101 \overset{C}{=} 1010 \overset{C}{=} 101000 \overset{C}{=} 1,01$

Applying the same amount of Collatz-operators on two Collatz-identical numbers leads to two Collatz-idenical numbers

Therefore every C-operator moves the number to the left. The velocities of the first 1 of the number and the last 1 are different.

With v_1 the average velocity of the first 1 and v_2 the average velocity of the last 1 of a given number the three possible developments are:

$$\begin{cases} \text{If } v_1{<}v_2 & \text{the sequence converges, until a cycle is reached} \\ \text{If } v_1{=}v_2 & \text{a cycle is reached} \\ \text{If } v_1{>}v_2 & \text{the sequence diverges} \end{cases}$$

The third version of the Collatz-conjecture is:

Conjecture 4. *Starting from an arbitrary number $x = y \cdot 2^n$, $x \in \mathbb{Q}$, $y \in \mathbb{N}$, $\frac{y}{2} \notin \mathbb{N}$, $n \in \mathbb{Z}$ the speed of the first 1 and the last 1 will become 2 digits per applied C-operator .*

4. Analysis of the Collatz-velocity

4.1. **Investigation of cycles.** The velocity v_1 can be one or two digits per step if a single Collatz-step is executed. For infinite Collatz-steps

$$v_1 = \lim_{s\to\infty} \frac{\log_2(C^s(x) - \log_2(x+1)}{s} := \begin{cases} 2 & \text{if the sum of digits=1 (case 1)} \\ \log_2 3 & \text{if the sum of digits} \neq 1 \text{ (case 2)} \end{cases}$$

s is the number of the applied C-operators. One has to presume that there is a number that diverges to calculate $\lim\limits_{s\to\infty}(3x+1) = 3x$. The exact value of the limit in case 2 is less important than the limit is not element of $\mathbb{Q}$.

If there is a cycle, the numbers of the cycle have always to repeat after the same and finite counts of steps. Therefore in the case of a cycle $v_1 = v_2$ and v_1 and $v_2 \in \mathbb{Q}$. Only case 1 fits the requirement for a cycle. This is the known cycle. Any other cycle would have more than one digit and should have a $v_2 \notin \mathbb{Q}$. Therefore case 2 cant be a cycle and case 1 is the only cycle that exists. That is the prove that there is no other cycles but the 4-2-1-cycle.