On the ℓ-adic Galois side of polylogarithm functional equations

Densuke Shiraishi (Tokyo Univ of Science)

A part of this talk is a joint work with

Hiroaki Nakamura (Osaka Univ)

23 th Sendai-Hiroshima

Outline

§1. Introduction

① Analogy between complex and ℓ-adic Galois polylogs

② Overview of main results

§2. ℓ-adic Galois polylogarithms

§3. Functional equations (Main results)

(i) Landen type

(ii) Spence-Kummer type.

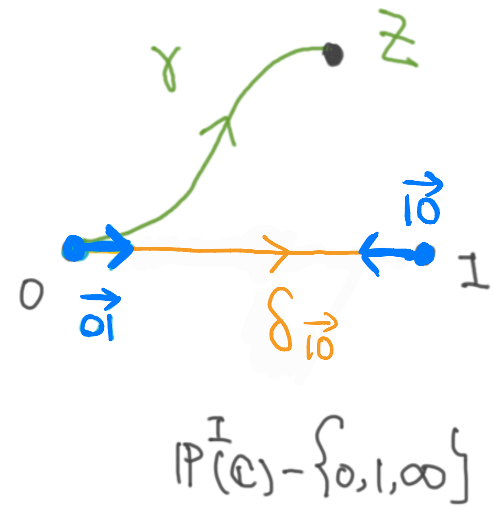
① classical polylogs and Galois polylogs

First, recall the classical (poly) logarithm :

Def For $k \in \mathbb{Z}_{\geq 1}$, $\gamma \in \pi_1^{top}(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}; \vec{01}, z)$,

- $$\mathrm{Li}_k(z;\gamma) := \begin{cases} \int_\gamma \mathrm{Li}_{k-1}(t;\gamma) \cdot \frac{1}{t}\,dt & (k \neq 1), \\ \int_\gamma \frac{1}{1-t}\,dt & (k=1). \end{cases}$$

- $$\log(z) \left(= \log(z;\gamma) \right) := \int_{\delta_{\vec{10}}^{-1} \cdot \gamma} \frac{1}{t}\,dt .$$



$$\boxed{\begin{aligned} \mathrm{Li}_k(z) : \pi_1^{top}(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}; \vec{01}, z) &\to \mathbb{C} \\ \gamma &\longmapsto \mathrm{Li}_k(z;\gamma) \end{aligned}}$$

Rem

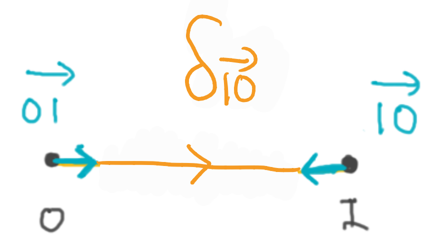
- $Li_2(z)$ is called the dilogarithm.
- $Li_3(z)$ is called the trilogarithm.

$\square$

Def

- $\zeta(k) := Li_k(\overrightarrow{10}\,;\,\delta_{\overrightarrow{10}})$

  : zeta value.



$\cap$

$\mathbb{P}^1(\mathbb{R}) - \{0, 1, \infty\}$

then.

$$\boxed{\zeta(k) \in \mathbb{R}}$$

$\square$

$\mathrm{Li}_k(z)$, $\zeta(k)$ were first studied by L. Euler in 1700s.

In a preprint published in 1999, Z. Wojtkowiak introduced the ℓ-adic Galois analog of $\mathrm{Li}_k(z)$ for any prime ℓ.

ℓ-adic Galois side $\longleftrightarrow$ classical (complex) side

$$\boxed{\mathrm{Li}_k^{\ell}(z;\gamma,\sigma) \in \mathbb{Q}_\ell}$$

where z : $\mathbb{Q}$-rational and $(\gamma,\sigma) \in \pi_1^{top}(\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\};\vec{01},z) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\boxed{\mathrm{Li}_k(z;\gamma) \in \mathbb{C}}$$

where z : $\mathbb{C}$-rational and $\gamma \in \pi_1^{top}(\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\};\vec{01},z)$

ℓ-adic Galois side	classical (complex) side
$\boxed{Li_1^{\ell}(z;\gamma,\sigma) = \rho_{1-z}(\sigma)}$ where $\rho_z : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell$: Kummer 1-cocycle along γ.	$\boxed{Li_1(z;\gamma) = -\log(1-z)}$ where $\log(z) \in \mathbb{C}$: the logarithm w.r.t. γ. $\log(z) = \int_{\delta_{\vec{10}}^{-1}\cdot\gamma} \frac{1}{t}\,dt$
$\boxed{\zeta_k^{\ell} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}_\ell}$ (defined by $\zeta_k^{\ell}(\sigma) := Li_k^{\ell}(\vec{10};\delta_{\vec{10}},\sigma)$) : the ℓ-adic Galois zeta value or called the ℓ-adic Soulé element e.g. $\boxed{\zeta_2^{\ell}(\sigma) = \frac{-1}{24}(\chi_\ell(\sigma)^2-1)}$ (χ_ℓ : ℓ-adic cyclotomic char)	$\boxed{\zeta(k) = Li_k(\vec{10};\delta_{\vec{10}})}$: the zeta value e.g. $\boxed{\zeta(2) = \frac{-1}{24}(2\pi i)^2}$ $= \frac{\pi^2}{6}$

The functional equations of ℓ-adic Galois dilogarithms studied by Nakamura-Wojtkowiak.

ℓ-adic Galois side	classical (complex) side	Underlying Geometry
$\boxed{\begin{aligned} & Li_2^{\ell}(z;\gamma,\sigma) + Li_2^{\ell}(1-z;\gamma',\sigma) \\ & + \rho_z(\sigma)\rho_{1-z}(\sigma) = \zeta_2^{\ell}(\sigma) \end{aligned}}$ (Nakamura-Wojtkowiak, 2012)	$\boxed{\begin{aligned} & Li_2(z;\gamma) + Li_2(1-z;\gamma') \\ & + \log(z)\log(1-z) = \zeta(2) \end{aligned}}$ (Euler, 1768)	$\mathbb{P}^1 - \{0,1,\infty\}$ $(\cong \mathcal{M}_{0,4})$
$\boxed{\begin{aligned} & Li_2^{\ell}\left(\tfrac{xy}{(1-x)(1-y)};\gamma_1,\sigma\right) - Li_2^{\ell}\left(\tfrac{x}{1-y};\gamma_2,\sigma\right) \\ & - Li_2^{\ell}\left(\tfrac{y}{1-x};\gamma_3,\sigma\right) + Li_2^{\ell}(x;\gamma_4,\sigma) \\ & + Li_2^{\ell}(y;\gamma_5,\sigma) = -\rho_{1-x}(\sigma)\rho_{1-y}(\sigma) \end{aligned}}$ (Nakamura-Wojtkowiak, 2012)	$\boxed{\begin{aligned} & Li_2\left(\tfrac{xy}{(1-x)(1-y)};\gamma_1\right) - Li_2\left(\tfrac{x}{1-y};\gamma_2\right) \\ & - Li_2\left(\tfrac{y}{1-x};\gamma_3\right) + Li_2(x;\gamma_4) \\ & + Li_2(y;\gamma_5) = -\log(1-x)\log(1-y) \end{aligned}}$ (Abel, 1827)	the moduli space $\mathcal{M}_{0,5}$ $(\dim = 2)$

In this talk, we discuss the functional equations of ℓ-adic Galois trilogarithms.

ℓ-adic Galois side	classical (complex) side	Underlying Geometry
?	$\mathrm{Li}_3(z;\gamma)+\mathrm{Li}_3(1-z;\gamma')+\mathrm{Li}_3\left(\frac{z}{z-1};\gamma''\right) = \zeta(2)\log(1-z;\gamma')-\frac{1}{2}\log(z;\gamma)\log(1-z;\gamma')^2+\frac{1}{6}\log(1-z;\gamma')^3$ (Landen, 1780)	$\mathbb{P}^1-\{0,1,\infty\}$
?	$\mathrm{Li}_3\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1\right)+\mathrm{Li}_3(xy;\gamma_2)+\mathrm{Li}_3\left(\frac{x}{y};\gamma_3\right) -2\mathrm{Li}_3\left(\frac{x(1-y)}{y(1-x)};\gamma_4\right)-2\mathrm{Li}_3\left(\frac{x(1-y)}{x-1};\gamma_5\right)-2\mathrm{Li}_3\left(\frac{1-y}{1-x};\gamma_6\right) -2\mathrm{Li}_3\left(\frac{y-1}{y(1-x)};\gamma_7\right)-2\mathrm{Li}_3(x;\gamma_8)-2\mathrm{Li}_3(y;\gamma_9)+2\zeta(3) =\log(y;\gamma_9)^2\log\left(\frac{1-y}{1-x};\gamma_6\right)-2\zeta(2)\log(y;\gamma_9)-\frac{1}{3}\log(y;\gamma_9)^3$ (Spence 1809, Kummer 1840)	?

② Overview of main thms

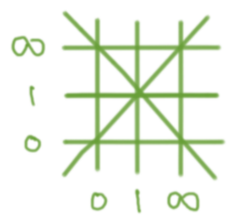
Underlying Geometry ⇝ Arithmetic

(i) Symmetry of $\mathbb{P}^1 - \{0,1,\infty\}$ ⇝ ℓ-adic Galois analog of Landen's equation

(ii) Symmetry of $V_{\text{non-fano}}$ ⇝ ℓ-adic Galois analog of Spence-Kummer's equation

$V_{\text{non-fano}} := M_{0,5} - \{1 = xy\}$

$\subset M_{0,5} := \mathrm{Conf}_2(\mathbb{P}^1 - \{0,1,\infty\})$



: non-fano arrangement

Note : It is noteworthy that both equations involves nontrivial ℓ-adic Galois extra terms in contrast to the classical (complex) versions.

§2 Recall the ℓ-adic Galois polylogarithm.

Let ℓ be a prime number. Suppose that z is $\mathbb{Q}$-rational.

We focus on the Galois action:

Galois group of $\mathbb{Q}$ $\qquad\qquad$ pro-ℓ étale path space

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \curvearrowright \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\};\ \overrightarrow{01},\ z\right)$$

$\uparrow$ comparison map

$$\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\};\ \overrightarrow{01},\ z\right)$$

<u>Def</u> For $\gamma \in \pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\};\overrightarrow{01},z)$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$f_\sigma^{z,\gamma} := \gamma \cdot \sigma(\gamma)^{-1}$$

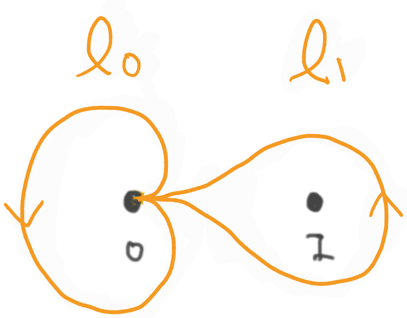
$$\in \underbrace{\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\},\ \overrightarrow{01}\right)}_{\text{pro-}\ell\text{ étale }\pi_1}. \qquad \square$$

To understand clearly the behavior of $f_\sigma^{z,\gamma}$,

we consider the ℓ-adic Magnus embedding

pro-ℓ étale π_1 non-commutative algebra.

$$\overline{\langle l_0, l_1 \rangle} \cong \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}, \vec{01}\right) \hookrightarrow \mathbb{Q}_\ell\langle\langle e_0, e_1 \rangle\rangle \quad : \text{multiplicative}$$



$$l_0 \longmapsto \exp(e_0) := \sum_{n=0}^{\infty} \frac{1}{n!} e_0^n$$

$$l_1 \longmapsto \exp(e_1)$$

$$f_\sigma^{z,\gamma} \longmapsto f_\sigma^{z,\gamma}(e_0, e_1)$$

étale loop   formal power series

Def $f_\sigma^{z,\gamma}(e_0,e_1) \in \mathbb{Q}_\ell\langle\langle e_0,e_1\rangle\rangle$ $\left(\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\right)$

is called the ℓ-adic Galois associator

associated to $\gamma \in \pi_1^{top}(\mathbb{P}^1(\mathbb{C})-\{0,1,\infty\}; \vec{01}, z)$ $\square$

Summary

$$
\begin{tikzcd}
G_{\mathbb{Q}} \times \pi_1^{top}(\vec{01},z) \arrow[r, "\text{Comparison map}"] & G_{\mathbb{Q}} \times \pi_1^{\ell\text{-ét}}(\vec{01},z) \arrow[r, "f"] & \pi_1^{\ell\text{-ét}}(\vec{01}) \arrow[r, hook, "\ell\text{-adic Magnus}"] & \mathbb{Q}_\ell\langle\langle e_0,e_1\rangle\rangle \\[-15pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-15pt]
(\sigma,\gamma) \arrow[r, mapsto] & (\sigma,\gamma) \arrow[r, mapsto] & f_\sigma^{z,\gamma} := \gamma\cdot\sigma(\gamma)^{-1} \arrow[r, mapsto] & f_\sigma^{z,\gamma}(e_0,e_1)
\end{tikzcd}
$$

Rem

ℓ-adic Galois side $\leftrightarrow$ classical (complex) side

$f_\sigma^{z,\gamma}(e_0,e_1) \in \mathbb{Q}_\ell\langle\langle e_0,e_1\rangle\rangle$

: ℓ-adic Galois associator wrt γ.

$\left(\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\right)$

$G_{\vec{01}}^{z,\gamma}(e_0,e_1) \in \mathbb{C}\langle\langle e_0,e_1\rangle\rangle$

: complex KZ associator wrt γ.

(that is a fundamental solution of the KZ-differential equation)

- $G_{\vec{01}}^{z,\gamma}(e_0,e_1) := \left(1+\sum_{i=1}^{\infty}\int_{\vec{01},\gamma}\underbrace{\omega\cdots\omega}_{i\text{ times}}\right)^{op}$ $\left(\omega := \frac{dz}{z}e_0+\frac{dz}{z-1}e_1,\ (e_0e_1)^{op}=e_1e_0\right)$
- $\frac{d}{dz}G=\left(\frac{e_0}{z}+\frac{e_1}{z-1}\right)G$: KZ-equation. (G is analytic in z with values in $\mathbb{C}\langle\langle e_0,e_1\rangle\rangle$)

$\square$

__Def__ For $k \in \mathbb{Z}_{\geq 1}$, $\gamma \in \pi_1^{top}\left(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}; \overrightarrow{01}, z\right)$,

and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

- $Li_k^{\ell}(z;\gamma,\sigma) := -1 \cdot \text{Coeff of } e_0^{k-1}e_1 \text{ in } \int_{\sigma}^{z,\gamma}(e_0,e_1)$

$$\boxed{Li_k^{\ell}(z) : \pi_1^{top}\left(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}; \overrightarrow{01}, z\right) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}_{\ell}}$$

: ℓ-adic Galois polylogarithm (Wojtkowiak's ℓ-adic iterated integral) $\square$

Def For $k \in \mathbb{Z}_{>1}$,

- $\zeta_k^\ell(\sigma) := Li_k^\ell(\vec{10}\,; \delta_{\vec{10}}, \sigma)$

$$\boxed{\zeta_k^\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}_\ell}$$

: ℓ-adic Galois zeta value. □

Rem

- $\zeta_k^\ell(\sigma) = \dfrac{1}{(1-\ell^{k-1})(k-1)!}\chi_k^\ell(\sigma)$: ℓ-adic Soulé character $\in \mathbb{Z}_\ell$

- $\zeta_{2k}^\ell(\sigma) = \dfrac{B_{2k}}{2(2k!)}\left(1-\chi_\ell(\sigma)^{2k}\right) \longleftrightarrow \zeta(2k) = \dfrac{B_{2k}}{2\cdot(2k)!}\cdot\left(-(2\pi i)^{2k}\right)$ □

  χ_ℓ : ℓ-adic cyc. char.

§ 3. Functional equations

First, we show the ℓ-adic Landen's equation.

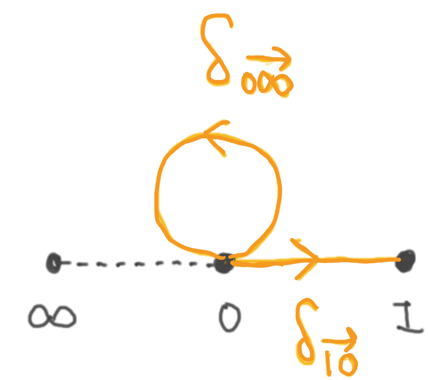
We focus on the symmetry of $\mathbb{P}^1 - \{0,1,\infty\}$:

$$\phi_{\overrightarrow{10}},\ \phi_{\overrightarrow{0\infty}} \in \mathrm{Aut}\left(\mathbb{P}^1 - \{0,1,\infty\}\right) \cong \mathfrak{S}_3 \qquad \boxed{\phi_{\overrightarrow{10}}(z) = 1-z\ ,\quad \phi_{\overrightarrow{0\infty}} = \frac{z}{z-1}}$$

For $\gamma \in \pi_1^{top}\left(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\} ; \overrightarrow{01}, z\right)$,

define

$$\boxed{\begin{aligned} \gamma' &:= \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}(\gamma) \ :\ \overrightarrow{01} \rightsquigarrow \overrightarrow{10} \rightsquigarrow 1-z \\ \gamma'' &:= \delta_{\overrightarrow{0\infty}} \cdot \phi_{\overrightarrow{0\infty}}(\gamma) \ :\ \overrightarrow{01} \rightsquigarrow \overrightarrow{0\infty} \rightsquigarrow \frac{z}{z-1} \end{aligned}}$$



the path system $(\gamma, \gamma', \gamma'') \rightsquigarrow \left(Li_3^{\ell}(z;\gamma,\sigma),\ Li_3^{\ell}(1-z;\gamma',\sigma),\ Li_3^{\ell}\left(\frac{z}{z-1};\gamma'',\sigma\right) \right)$

$\underline{\text{Thm}}$ (Nakamura - S. 2022 ; arXiv 2210.17182, Thm 1.1)

Fix a $\mathbb{Q}$-rational base point z of $\mathbb{P}^1 - \{0, 1, \infty\}$

and a path $\gamma \in \pi_1^{top}(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\} ; \vec{01}, z)$.

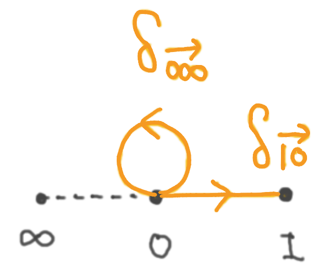
For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the following holds :

$$\boxed{\begin{aligned} &Li_3^{\ell}(z;\gamma,\sigma) + Li_3^{\ell}(1-z;\gamma',\sigma) + Li_3^{\ell}\left(\tfrac{z}{z-1};\gamma'',\sigma\right) \\ &= \zeta_3^{\ell}(\sigma) - \zeta_2^{\ell}(\sigma)\cdot \rho_{1-z}(\sigma) + \frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)^2 - \frac{1}{6}\rho_{1-z}(\sigma)^3 \\ &\quad - \frac{1}{2} Li_2^{\ell}(z;\gamma,\sigma) - \frac{1}{12}\rho_{1-z}(\sigma) - \frac{1}{4}\rho_{1-z}(\sigma)^2 \end{aligned}}$$

proof Compare the terms $e_0^2 e_1$ etc in

the algebraic relation (chain rule) of ℓ-adic Galois associators.

- $$f_\sigma^{\frac{z}{z-1},\gamma''}(e_0,e_1) = f_\sigma^{z,\gamma}(e_0,e_\infty)\cdot f_\sigma^{\overrightarrow{0\infty},\,\delta_{\overrightarrow{0\infty}}}(e_0,e_1)$$
- $$f_\sigma^{z,\gamma}(e_0,e_1) = f_\sigma^{1-z,\gamma'}(e_1,e_0)\cdot f_\sigma^{\overrightarrow{10},\,\delta_{\overrightarrow{10}}}(e_0,e_1)$$



where $e_\infty := \log\left(\exp(-e_1)\exp(-e_0)\right)$: the BCH sum

$$= -e_0 - e_1 \underbrace{+ \frac{1}{2}e_1e_0 - \frac{1}{2}e_0e_1 - \frac{1}{12}e_0^2e_1 + \cdots}_{\text{higher degree terms}}$$

$$\left(\log(e) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(e-1)}{n} \right)$$

$\blacksquare$

Rem By reinterpreting the above proof after replacing $f_\sigma^{z,\gamma}(e_0,e_1) \in \mathbb{Q}_\ell\langle\langle e_0,e_1\rangle\rangle$ by the KZ solution $G_0(z;\gamma)(e_0,e_1) \in \mathbb{C}\langle\langle e_0,e_1\rangle\rangle$, we obtain an algebraic proof of the Landen's equation of the classical trilogarithms $Li_3(z)$, $Li_3(1-z)$ and $Li_3\left(\frac{z}{z-1}\right)$.

ℓ-adic Galois side	classical (complex) side.
$f_\sigma^{\frac{z}{z-1}}(e_0,e_1) = f_\sigma^{z}(e_0,e_\infty)\cdot f_\sigma^{\overrightarrow{0\infty}}(e_0,e_1)$	$G_0\left(\frac{z}{z-1}\right)(e_0,e_1) = G_0(z)(e_0,e_\infty)\cdot \exp(\pi i e_0)$
$e_\infty = \log(\exp(-e_1)\exp(-e_0))$ $= -e_1 - e_0 + \underbrace{\text{higer degree terms}}$	$e_\infty = -e_1 - e_0$

↖ factor of ℓ-adic Galois extra terms !

□

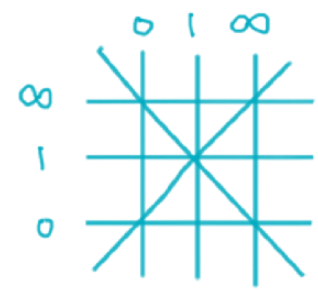
Next we show the ℓ-adic Spence-Kummer's equation.

We focus on the symmetry of $V_{\text{non-fano}} := \operatorname{Spec} \overline{\mathbb{Q}}\left[x, y, \frac{1}{xy(1-x)(1-y)(x-y)(1-xy)}\right]$

$$\left\langle (x,y) \mapsto \left(\frac{x(1-y)}{x-1}, \frac{y-1}{y(x-1)}\right) \right\rangle \circlearrowright V_{\text{non-fano}} \circlearrowleft \left\langle (x,y) \mapsto \left(x, \frac{1}{y}\right) \right\rangle$$

$$\cap \qquad\qquad\qquad\qquad\qquad\qquad \cap$$

$$\operatorname{Aut}(M_{0,5}) \qquad\qquad\qquad\qquad\qquad \operatorname{Aut}(M_{0,5})$$



$f_1 \downarrow \; f_2 \downarrow \; \cdots \downarrow f_9$

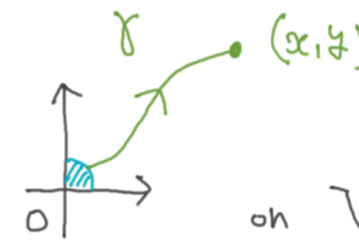
By this symmetry, we can consider:

$$\boxed{\{f_i\}_{i=1,\dots,9} : V_{\text{non-fano}} \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \mathbb{P}^1 - \{0,1,\infty\}}$$

$f_1(x,y) = \frac{x(1-y)^2}{y(1-x)^2}$, $f_2(x,y) = xy$, $f_3(x,y) = \frac{x}{y}$, $f_4(x,y) = \frac{x(1-y)}{y(1-x)}$, $f_5(x,y) = \frac{x(1-y)}{x-1}$,

$f_6(x,y) = \frac{1-y}{1-x}$, $f_7(x,y) = \frac{y-1}{y(1-x)}$, $f_8(x,y) = x$, $f_9(x,y) = y$

Let $\boxed{\vec{v} : \mathrm{Spec}\,\mathbb{Q}((t)) \to V_{\text{non-fano}}}$ be the $\mathbb{Q}$-rational tangential base point over the $\mathbb{Q}(t)$-rational pt (t^2, t).



on $V^{an}_{\text{non-fano}}$

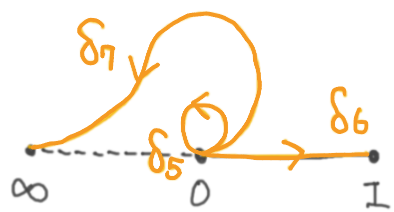
Given a $\mathbb{Q}$-rational base pt (x,y)

and $\gamma \in \pi_1^{top}(V^{an}_{\text{non-fano}}; \vec{v}, (x,y))$,

define the path system $\{\gamma_i\}_{i=1,\dots,9}$ by

$$\boxed{\gamma_i := \delta_i \cdot f_i(\gamma) : \vec{01} \rightsquigarrow f_i(\vec{v}) \rightsquigarrow f_i(x,y)}$$

- $\delta_i = 1$: trivial path $(i = 1,2,3,4,8,9)$
- $\left.\begin{array}{l}\delta_5 \\ \delta_6 \\ \delta_7\end{array}\right\}$: as in



Summary

$$\pi_1(V_{\text{non-fano}}) \to \pi_1(\mathbb{P}^1 - \{0,1,\infty\})$$

$$\gamma \rightsquigarrow \{\gamma_i\}_{i=1,\dots,9}$$

$$\rightsquigarrow$$

$$\{Li_3^{\ell}(f_i(x,y); \gamma_i, \sigma)\}_{i=1,\dots,9}$$

Thm (S. 2023; arXiv 2307.09414, Theorem 0.2)

For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the following holds:

$$\boxed{\begin{aligned}
&Li_3^{\ell}\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1,\sigma\right)+Li_3^{\ell}(xy;\gamma_2,\sigma)+Li_3^{\ell}\left(\frac{x}{y};\gamma_3,\sigma\right)\\
&-2Li_3^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right)-2Li_3^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right)-2Li_3^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right)\\
&-2Li_3^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right)-2Li_3^{\ell}(x;\gamma_8,\sigma)-2Li_3^{\ell}(y;\gamma_9,\sigma)+2\zeta_3^{\ell}(\sigma)\\
&=-\rho_{y,\gamma_9}(\sigma)^2\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma)+2\zeta_2^{\ell}(\sigma)\cdot\rho_{y,\gamma_9}(\sigma)+\frac{1}{3}\rho_{y,\gamma_9}(\sigma)^3\\
&-Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right)-Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right)+\frac{1}{2}\rho_{\frac{1-xy}{1-x},\gamma_5'}(\sigma)-\frac{1}{3}\rho_{y,\gamma_9}(\sigma)
\end{aligned}}$$

proof We use the algebraic relations arising from $\{\gamma_i\}_{i=1,\cdots,9}$:

$$\boxed{f_\sigma^{f_i(x,y),\gamma_i} = \left(\delta_i \cdot f_i\left(f_\sigma^{(x,y),\gamma}\right) \cdot \delta_i^{-1}\right) \cdot f_\sigma^{f_i(\vec{v}),\delta_i}} \quad \text{in } \mathbb{Q}_\ell\langle\langle e_0, e_1\rangle\rangle$$

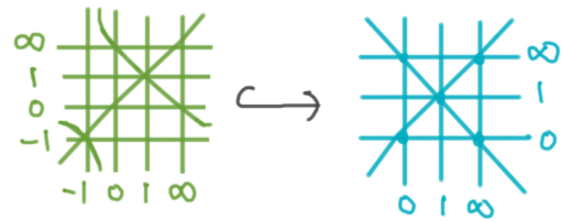
In order to compare coefficients on both sides,

we need to capture the ℓ-adic Galois associator explicitly :

$$\boxed{f_\sigma^{(x,y),\gamma} \in \pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}, \vec{v}\right)}$$

We focus on the key diagram :

$$\begin{array}{ccc} (x,y) & V_{B_3} & \xhookrightarrow{\text{open immersion}} \quad V_{\text{non-fano}} \\ \Big\downarrow & \text{finite étale Galois}\ \Big\downarrow \ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} & \\ \left(\left(\frac{1-x}{1+x}\right)^2, \left(\frac{1-y}{1+y}\right)^2\right) & M_{0,5} & \end{array}$$



$$V_{B_3} := \operatorname{Spec} \mathbb{Q}\left[x, y, \frac{1}{xy(1-x^2)(1-y^2)(x-y)(1-xy)}\right]$$

: the complement to the Coxeter arrangement of type B_3

By this diagram, we capture $\pi_1^{\ell\text{-ét}}(V_{\text{non-fano}})$ as an explicit subquotient of the well known fundamental group :

$$\pi_1^{\ell\text{-ét}}(M_{0,5}) \cong \widehat{P_4/\text{center}}^{\,\text{pro-}\ell} \qquad (P_4 : \text{the pure braid group})$$

Then. $\pi_1^{\ell\text{-ét}}\left(V_{\text{non-fano}}, \vec{v}\right) = \overline{\left\langle \begin{matrix} B_1, B_2, B_3 \\ B_4, B_5, B_6 \end{matrix} \;\middle|\; \text{Some relations} \right\rangle}$

where the topological generators $B_1, \cdots, B_6$ are described in terms of pure braids.

By a technical computation, we obtain

$$\boxed{\begin{aligned} \log\left(\mathfrak{f}_\sigma^{(x,y),\gamma}\right) = {} & \rho_x(\sigma) X_1 + \rho_y(\sigma) X_2 + \left(\rho_{\frac{x-y}{x}}(\sigma) + \rho_y(\sigma)\right) X_3 \\ & + \rho_{1-x}(\sigma) X_4 + \rho_{1-y}(\sigma) X_5 + \rho_{1-xy}(\sigma) X_6 \\ & + \cdots \text{ higher degree terms} \end{aligned}}$$

in the ℓ-adic Lie algebra of $\pi_1^{\ell\text{-ét}}$ over $\mathbb{Q}_\ell$ (where $X_i := \log(B_i)$).

the algebraic relation $\star$ of ℓ-adic Galois associators

$+$

the "tensor criterion" for functional equations of polylogarithms

due to Zagier, Nakamura-Wojtkowiak

$$\boxed{\begin{array}{l} \displaystyle\sum_{i=1}^{9} k_i \left(f_i \otimes f_i \wedge (f_i - 1)\right) = 0 \\ \quad \text{in } \mathcal{O}^\times/\overline{\mathbb{Q}}^\times \otimes \mathcal{O}^\times/\overline{\mathbb{Q}}^\times \wedge \mathcal{O}^\times/\overline{\mathbb{Q}}^\times \\ \text{where } V_{\text{non-fano}} = \mathrm{Spec}(\mathcal{O}) \end{array} \implies \sum_{i=1}^{9} k_i \cdot Li_3^{\ell}(f_i(x,y)) = \underbrace{\text{lower degree terms}}_{\text{the exact computation here requires } \star}}$$

$\rightsquigarrow$ we obtain ℓ-adic Spence-Kummer's equation. ∎

Rem By reinterpreting the above proof after replacing the ℓ-adic Galois associator "$f_\sigma^{f_i(\gamma)}$" $\in \mathbb{Q}_\ell\langle\langle e_0, e_1\rangle\rangle$ by

the Chen's formal power series "$\Delta_{f_i(\gamma)}$" $:= 1 + \sum_{k=1}^{\infty} \int_{f_i(\gamma)} \underbrace{\omega \cdots \omega}_{k \text{ times}} \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$

(where $\omega = \frac{dz}{z} e_0 + \frac{dz}{z-1} e_1$)

we obtain an algebraic proof of the Spence-Kummer's equation of the classical trilogarithm.

For $i = 1,2,3,4,5,6,7$, e_∞ appears in $\begin{cases} f_\sigma^{f_i(\gamma)} & (\ell\text{-adic case}) \\ \Delta_{f_i(\gamma)} & (\text{complex case}) \end{cases}$

$$e_\infty = \begin{cases} -e_1 - e_0 + \text{higer degree terms}, & (\ell\text{-adic case}) \\ -e_1 - e_0 & (\text{complex case}) \end{cases}$$

factor of ℓ-adic Galois extra terms !

$\square$

Some specializations of ℓ-adic Spence-Kummer's equation :

[S.2023]

Spence-Kummer

V non-fano

has the rich symmetry !!

$\text{Spence-Kummer} \xrightarrow{x=y} \text{distribution}$

[N-W 2020]

distribution

$\mathbb{P}^1 - \{0, \pm 1, \infty\}$

$\text{Spence-Kummer} \xrightarrow{x \to 0} \text{Landen}$

[N-S. 2022]

Landen

$\mathbb{P}^1 - \{0, 1, \infty\}$

$\text{Spence-Kummer} \xrightarrow{x = YZ,\ y = -Y/Z} \text{Newman}$

In progress

Newman

$\{X+Y+Z = XYZ\} \setminus \{\text{divisors}\}$

Thank you for your attention.