Problem Chosen	2020 MCM/ICM Summary Sheet	Team Control Number
REPLACE ME WITH PROBELM		REPLACE ME WITH TEAM NUMBER

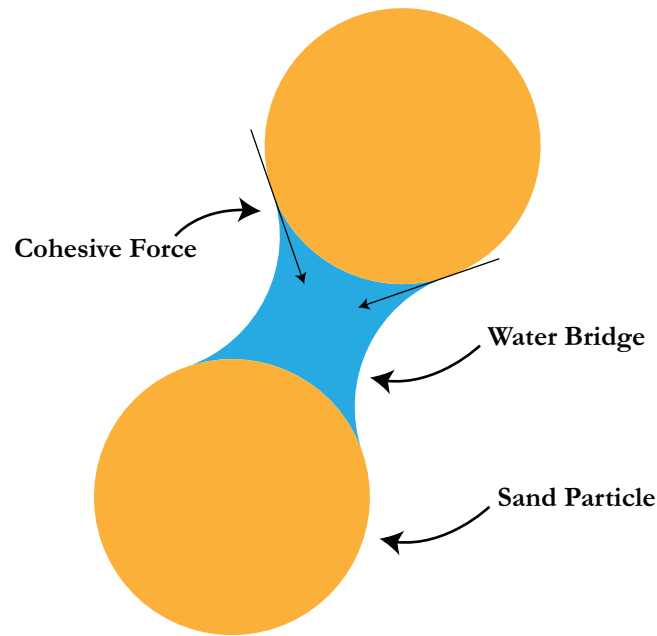# Kingdom Built on a Pile of Sand:Slow and Steady

Contents

Figure 1: Water Bridge Between Sand Particle

1 Introduction

1.1 Problem Background

Sunshine, clear blue sea and golden color sand always seem to leave people in a happy state of mind. And a beach is where these three are combined, drawing people all around towards it. Sand, the granular matter formed by constant brushing of flowing water, however, can react with water in a different way, despite the fact that people refer to it as non-stable or unreliable. On a beach, where the already formed granular sand and the rise and fall of sea wave lies together, a new buff can be added to our flowing friend, a wetted state.

Magically but not randomly, sand gets sticky when combined with water, due to the most obvious physical theorem: surface tension and atmosphere pressure. From previous people's work, we've know that this buff comes from the water bridge formed between sand particles, which can significantly cluster together during the increasing water-sand portion[1]–[3]. Kudrolli and Arshad has visualized the bridge between sand like Figure 1.

And that's the magic that glue our favorite sand castle together, which is one of the most entertainment for enthusiastic beach goers. However, being built near the water that melts mud and our wet sand, all sand castles have to face the fate that they'll get too wet to hold its own weight and the impact from sea waves. That's because the water bridge has another property of clustering together[3]. When you throw a pile of sand into water, they behave just as when they're completely dry, melting down like fluid. Thus beach castle builder might want to make their sand castle last longer than those build arbitrarily than others, which is also the purpose of this article.

1.2 Our Work

Normally people would sculpt their kingdom from a pile of tedious wet sand. To simplify the model and grasp main threads, we'll focus our research on this single, nondescriptive mound of wetted sand. In this article, we develop a method for determining the most long-lasting three dimensional shape for a tedious sand castle, which will be later referred to as *sand castle foundation*.

Our model addresses the problem of constructing a three dimensional structure by projection, which makes the construction of individual part of our model easy to implement. Firstly we determine the optimal *side view* of our *sand castle foundation* by analysing and extracting information from similar research on wet granular material[2]. Then we construct the best *top view* of the *sand castle foundation* by constructing a **Cellular Automata**.

Side Slope Determination   You've probably seen questions about the slope of a pile of dry sand in junior high school practice books. In our research, we address the problem about dry sand as a normal junior high student would. However, the analysis of wet sand requires some technique beyond low level physics. In our research, we establish this model using **Mohr-Coulomb Criterion**.

Top View Determination   Our model analyses the impact of sea wave flow from a top view, or a *slice* of the *sand castle foundation*. Then we can combine the sliced impaction result together with the *side slope* analysis to form a possible 3D object. Inside our model we construct a **Cellular Automata** since the shape of the sand slice edge isn't as obvious the side slope may be, that is, it's merely possible to construct continuous function on the shape.

Other Factors   It's worth noting that the interaction between sand and sea wave is far a lot more complex than direct force impact or sand-to-water mixture proportion change. Some other factors include: gravity of sea water that submerge the sand underneath, influx of weaker sea wave and the gravity of water inside the wetted sand. During the assembly of two views, we'll address these other factors that may influence a *sand castle foundation*'s life span.

2 Assumptions & Nomenclature

2.1 Assumptions

Several assumptions are made in order to simplify our model.

1) The weather condition is suitable for sand castle building, namely, there is no wind and...

2) The height of wave is the same, which we set as...

3) Our optimization goal is to acquire the largest top size while minimizing the lost of sand. Intuitively, if the side of the foundation is built as a gentle slope, the possibility of collapsing will be drastically reduced. However, with a fixed volume of sand, such a foundation will not allow any sophisticated carving.

4) Sand grains are regarded as tiny spheres. In our foundation, these grains are of the same size and are closely packed.

5) The top of the foundation is a flat platform whose surface is horizontal.

6) ...

2.2 Nomenclature

Table 1: States of Sand

Symbol	Meaning
τ	the shear stress on a plane
τ_f	the shear stress on the failure plane
μ	the internal friction coefficient
σ	the normal compressible stress on a plane
σ_f	the normal compressible stree on the failure plane
ρ	the density of sand
θ_c	the critical angle of a sandpile
D	the height from the top of the sand pile
P	the pressure[]
Γ	the surface tension of water
s_A	the adhesive stress
f_A	the adhesive force
V	the total amount of fluid present per particle contact
R	the radius of particle

3 Modeling Under Waves and Tides

The 3-dimensional shape constructing problem is divided into two sub-problems. We first establish the model using the **Mohr-Coulomb Criterion** to decide the shape of the slope. We then construct another model with a modified version of **Cellular Automata** to determine the best shape viewed from the top,...

3.1 Shape of the Slope: Mohr-Coulomb Criterion

We begin by determining the side shape of the sand castle foundation. Before Approaching the problem, we will briefly address the property of sand as a granular media.

In our assumptions, sand particles are considered as identical tiny spheres. If we zoom in to observe a pile of wet sand, there are the so-called liquid bridges formed between sand particles.

Various water contents produce different liquid bridge distributions, which will influence the properties of sand.

Table 2: States of Sand

Liquid Content	State	Description	left for future use
No	Dry	000	000
Small	Pendular	000	000
Middle	Funicular	000	000
Almost saturated	capillary	000	000
More	Slurry	000	000

When the wave gets into contact with the foundation, the surface area is in slurry state and there exists no cohesive interaction between the particles, which makes it very hard, if not impossible, to prevent sand loss. Nevertheless, collapses after the wave resides can cause more harm to the foundation, which can be avoided by alternating the shape.

For dry sand, the failure criterion is given in terms of the shear stress τ, the normal compressible stress σ and the internal friction μ as

$$\tau > \mu\sigma$$

This is simply the friction formula with different notations. For wet sand, we consider a sandpile with a normal adhesive stress s_A across every plane, in addition to the stress caused by weight. The equation(1) is then modified as

$$\tau > \mu(\sigma + s_A)$$

This is the so-called **Mohr-Coulomb criterion**. The stress resulting from the weight above the plane is shown in figure(2). Denote τ_f and μ_f as the shear stress and normal compressible stress at the failure plane, it is obvious from the schematic that they can be written as

$$\tau_f = \rho g D sin\theta_c \qquad and \qquad \sigma_f = \rho g D cos\theta_c$$

where θ_c is the critical angle, D is the height of the sandpile and ρ is the density of sand. Therefore, combine the equations above, to solve for θ_c is to solve the equation

$$\mu = tan\theta_c(D)\left(1 + \frac{s_A}{\rho g D cos\theta_c(D)}\right)$$

The only unknown factor is s_A, the adhesive stress across the plane. According to (Thomas C.H and Alex J.L -fix later)'s study, the value of s_A is determined by water content and there are three regimes as a function of the added-fluid volume. We now focus only on the state where the water content is close to saturation.

In this case, with water serves as lubricate, it makes sense to model sand particles as frictionless spheres. The pressure difference is then given by []

$$P = -\frac{\Gamma}{\sqrt{V/2\pi R}}$$

where Γ is the surface tension of the fluid, V is the total amount of fluid present per particle contact, and R the radius of the particle. The adhesive force is given by

$$f_A = 2\pi\Gamma R$$

Note that (with certain conditions like distance between grains remains constant)the term V which denotes the volume of liquid does not appear in equation(6).

Assume our foundation contains sand particles that are closely packed. Such structure means there is an average of 6 contacts per square.

3.2   Top View Shape

We used a modified version of **cellular automata** to simulate the impact of tides and waves. There are several factors that are taken into consideration:
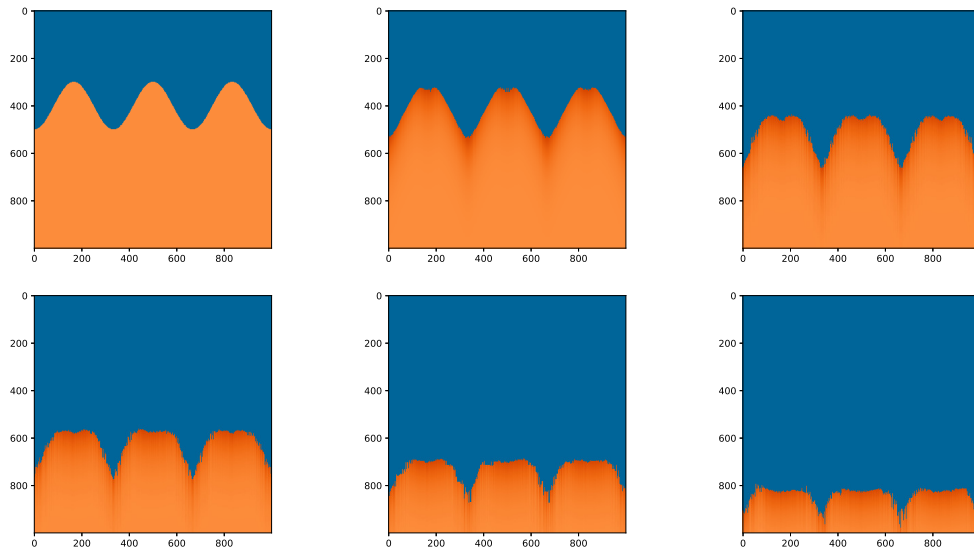
1) ...

2) ...

3) ...

Figure 2: A Windows Terminal.
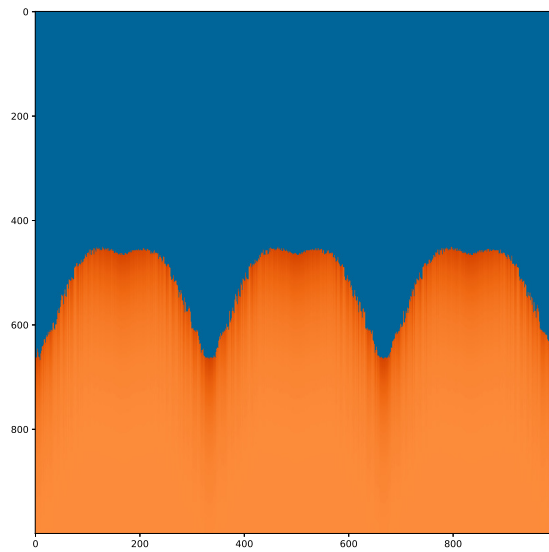
This is a test figure. Remember to delete me.

Figure 3: A Windows Terminal.

3.3   Calculate & Simulate Results

Briefly review the equation derived in section 3.1

$$tan\theta_c = \mu + \frac{\sqrt{2}\pi\mu\Gamma}{R\rho gD}sec[tan^{-1}(k)]$$

To solve for the best shape of the slope, the necessary information about sand is listed below

Table 3: States of Sand

Physical Quantities	Values	Units
Γ	72.8	mN/m
μ	0.55 0.60	-
ρ	1631	kg/m^3
R	0.05 2	mm
g	9.81	m/s^2

It is not sensible to embark on a trip to the beach on a stormy day, so we assume that the near-shore wave is gentle, with its height not exceeding 20cm, or $D \in [0, 20](unit : cm)$. We first assume that..

4 Modeling Under Rain

5 Determine the Best Sand-to-water Proportion

5.1 Assumptions & Nomenclature

ADD A CURVE FIGURE HERE, DRAWING!

As we mentioned above, the adhesive stress across the plane (s_A) is determined by water content. Consequently, we will find out how the cohesion in wet granular materials varies with different amount of liquid based on liquid bridge model to find an optimal sand-to-water mixture proportion.

There are five differing degrees of wetting in granular matter. As dry grains and slurry have no cohesion between particles, we just consider three other conditions. According to [2], we can see that the slope for low and large liquid content is much larger than that for intermediate liquid content. That is: the suction changes sharply as these three states transform each other. Consequently,we should keep the proportion of water to sand in steady for a long time, which means try to avoid transformation between three states.

we initially wish to create a three dimensional matrix to represent the situation. We still consider that the sandcastle C consists of cells mixed with sand and water. Then It's easy to Calculate each cell's sand-to-water mixture proportion from the amount of sand and water. ADD TWO CELLULAR FIGURES HERE, DRAWING!

5.2 Assumptions

1) waves and tides erode the sandcastle in a very short time, which means that the we can neglect Osmosis occurred during erosion.

2) After eroded, the liquid left on sandcastle has ample time to penetrate.

3) The evaporation of liquid is neglected.

5.3   Nomenclature

Table 4: Notation Table

symble	meaning
$c_{ijk}(t)$	the cell mixed with sand and water to form the sandcastle in time t
$s_{ijk}(t)$	the amount of sand in the cell in time t
$w_{ijk}(t)$	the amount of water in c_{ijk} in time t
$p_{ijk}(t)$	the proportion of sand-to-water mixture c_{ijk} in time t
$S(t)$	the amount of sand in the sandcastle int time t
$W(t)$	the amount of water in the sandcastle in time t
$P(t)$	the average proportion of sand-to-water mixture c_{ijk} in time t
$F(t)$	the erosion-Osmosis common effect of the sea water in time t

5.4   model

We define the expression when we just finish build the sandcastle.

$$P(t_0) = \frac{S(t_0)}{W(t_0)} = \frac{\sum_{i,j,k} s_{ijk}(t_0)}{\sum_{i,j,k} w_{ijk}(t_0)} = p_{ijk}(t_0)$$

To evaluate the existence of the sandcastle, we define that a cell is completely eroded by tides and waves as follow:

$$p_{ijk}(t) = \frac{s_{ijk}(t)}{w_{ijk}(t)} \le p_{erode}$$

And we define that the sandcastle is completely eroded as:

$$P(t) = \frac{\sum_{i,j,k} s_{ijk}(t_0)}{\sum_{i,j,k} w_{ijk}(t_0)} \le P_{erode}$$

where P_{erode} refers to proportion that Funicular transforms to capillary state. Then we define the flow equation during erosion as follow:

$$E = \begin{cases} F & \text{the cell is being eroded by sea} \\ 0 & \text{else} \end{cases}$$

As erosion occurs for a split second, we could just neglect the permeation to simplify our analysis. To obtain a probability model to evaluate the sand taken by waves, we need to acquire further understanding about the flow liquid. Here we use Navier-Stokes equations to describe the motion of viscous fluid substances.

$$x \, \rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) = -\nabla p + \nabla \cdot (\mu(\nabla v + (\nabla v)^T) - \frac{2}{3}\mu(\nabla v)I) + F\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} = 0 (continuity \ equations)$$

After eroded we should find the cell completely eroded by tides and waves. Then by reference to Darcy's law and Huygens–Fresnel principle, We consider water permeating the sandcastle as follow:

$$v = \frac{kA(p_b - p_a)}{\mu L} = \frac{kL(\Delta P)}{\mu}$$

where k is the hydraulic permeability of sea water, L is the length of one side of the cell, μ is the viscosity of sea water, Δ is the pressure differential between the two sides of the cell. As we assume that the sea water left on the surface of the sandcastle can totally penetrate. Then we could get for each cycle:

$$L(k, t, \mu, \rho, v, p) = \sum_{c_{ijk} \in C} \sum_{i-1}^{i+1} \sum_{j-1}^{j+1} \sum_{k-1}^{k+1} p_{mnp} \cdot E(m, n, p \neq i, j, k)(P(t) < P_{limit})(v \cdot \nabla v) + (\mu \cdot \nabla \mu) + (\rho \cdot \nabla \rho) = Con$$

Then We need to find out the longest time when the above constraints are met and the sandcastle exists.

6 Other Ways to Make Our Sand Castle Last Longer

6.1 Dig a Moat

Building a moat can help our sand castle foundation to survive the rising tide.

6.2 Make It Compact

There are two advantages of making a compact foundation.

More Contact Points

Fewer Cracks

6.3 Further From the Shore

According to our analysis in section 5, the optimal sand-to-water proportion is 1%. (This is not possible if the castle can be reached by wave...)

6.4   Finer Sand

Finer sand is a bonus because by constructing our sand castle foundation using it, the foundation has a greater critical angle. That is because the term R, which is a part of the denominator, decreases and (enlarge?not the proper word) the value of $tan\theta_c$.

6.5   Special Material

Glue   Sand sculptures seen in exhibitions or competitions have their surfaces covered with special glue. The glue is mainly used protect the sand sculpture from drying out or crumbling if the waves do not wash it away first, but it also keeps water out. In our model where the greatest enemy of the foundation is not collapses but erosion, such glue will help maintain the shape of the foundation.

Mixture   Concrete, a material that everyone is familiar with, has sand as its ingredient.

7   Sensitivity Analysis

7.1   Model Under Waves and Tides

Side View Shape   Here, insert the schematics I drew. They are far from perfect but they ARE a part of sensitivity analysis.

Top View Shape   DenDen dl has done a lot of parameter adjusting jobs...those are exactly what we need here.

7.2   Model Under Rain

Not now. Wait for tomorrow or something.

8   Strengths and Weaknesses

8.1   Strengths

Just like any other model, the one presented above has its Strengths and Weaknesses. Some major points are listed below.

   1) The use of Cellular Automata enables our model to predict erosion of any possible shape.

   2)

   3) ...

8.2   Weaknesses

1) Our model in section 3.1 does not consider the impact of the wave, which...

2) ...

3) ...

This is a test figure. Remember to delete me.
This is some example text[1].
I'm referring to footnote 1.

---

[1]Hello footnote

List of Figures

List of Tables

List of Listings

References

[1]   M. Pakpour, M. Habibi, P. Møller, and D. Bonn, "How to construct the perfect sandcastle,"
      Scientific reports, vol. 2, no. 1, pp. 1–3, 2012.

[2]   N. Mitarai and F. Nori, "Wet granular materials," *Advances in Physics*, vol. 55, no. 1-2,
      pp. 1–45, 2006.

[3]   A. Kudrolli, "Sticky sand," *Nature materials*, vol. 7, no. 3, pp. 174–175, 2008.