Problem Chosen

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2020 MCM/ICM Summary Sheet

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Summary

Kingdom Built on a Pile of Sand: Slow and Steady

Hello, dear reader of *Fun in the sun*! With summer drawing near, In this article, we will introduce little tricks about building the perfect sandcastle so that

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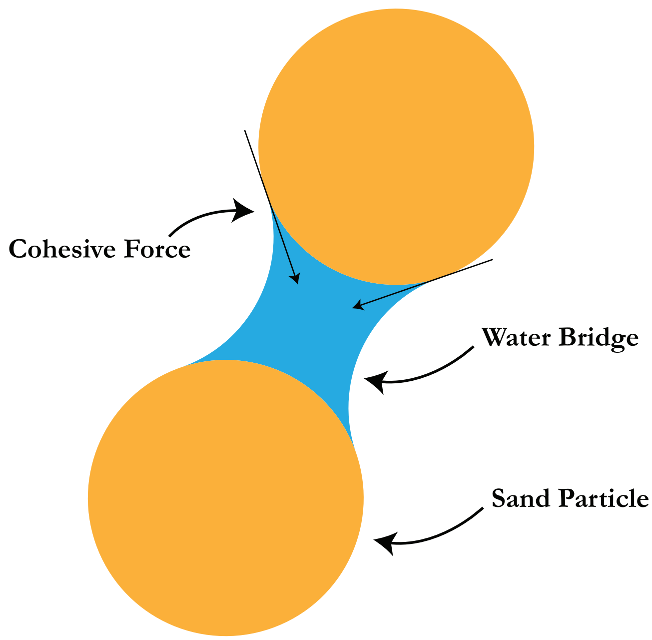


Figure 1: Water Bridge Between Sand Particle

1 Introduction

1.1 Problem Background

Sunshine, clear blue sea and golden color sand always seem to leave people in a happy state of mind. And a beach is where these three are combined, drawing people all around towards it. Sand, the granular matter formed by constant brushing of flowing water, however, can react with water in a different way, despite the fact that people refer to it as non-stable or unreliable. On a beach, where the already formed granular sand and the rise and fall of sea wave lies together, a new buff can be added to our flowing friend, a wetted state.

Magically but not randomly, sand gets sticky when combined with water, due to the most obvious physical theorem: surface tension and atmosphere pressure. From previous people's work, we've know that this buff comes from the water bridge formed between sand particles, which can significantly cluster together during the increasing water-sand portion[1]–[3]. Kudrolli and Arshad has visualized the bridge between sand like Figure 1.

And that's the magic that glue our favorite sand castle together, which is one of the most entertainment for enthusiastic beach goers. However, being built near the water that melts mud and our wet sand, all sand castles have to face the fate that they'll g et too wet to hold its own weight and the impact from sea waves. That's because the water bridge has another property of clustering together[3]. When you throw a pile of sand into water, they behave just as when they're completely dry, melting down like fluid. That is, sand castles lasts for only a period of time. Thus beach castle builder might want to make their sand castle last longer than those build arbitrarily than others, which is also the purpose of this article.

1.2 Our Work

Normally people would sculpt their kingdom from a pile of tedious wet sand. To simplify the model and grasp main threads, we'll focus our research on this single, nondescriptive mound of wetted sand. In this article, we develop a method for determining the most long-lasting three dimensional shape for a tedious sand castle, which will be later referred to as *sand castle foundation*.

Our model addresses the problem of constructing a three dimensional structure by projection, which makes the construction of individual part of our model easy to implement. Firstly we determine the optimal *side view* of our *sand castle foundation* by analysing and extracting information from similar research on wet granular material[2]. Then we construct the best *top view* of the *sand castle foundation* by constructing a **Cellular Automata**.

Side Slope Determination You've probably seen questions about the slope of a pile of dry sand in junior high school practice books. In our research, we address the problem about dry sand as a normal junior high student would. However, the analysis of wet sand requires some technique beyond low level physics. In our research, we establish this model using **Mohr-Coulomb Criterion**.

Top View Determination Our model analyses the impact of sea wave flow from a top view, or a *slice* of the *sand castle foundation*. Then we can combine the sliced impaction result together with the *side slope* analysis to form a possible 3D object. Inside our model we construct a **Cellular Automata** since the shape of the sand slice edge isn't as obvious the side slope may be, that is, it's merely possible to construct continuous function on the shape.

Other Factors It's worth noting that the interaction between sand and sea wave is far a lot more complex than direct force impact or sand-to-water mixture proportion change. Some other factors include: gravity of sea water that submerge the sand underneath, influx of weaker sea wave and the gravity of water inside the wetted sand. During the assembly of two views, we'll address these other factors that may influence a *sand castle foundation*'s life span.

2 Assumptions & Nomenclature

2.1 Assumptions

Several assumptions are made in order to simplify our model.

1) The weather condition is suitable for sand castle building, namely, there is no wind and...

2) The height of wave is the same, which we set as...

3) Our optimization goal is to acquire the largest top size while minimizing the lost of sand. Intuitively, if the side of the foundation is built as a gentle slope, the possibility of collapsing will be drastically reduced. However, with a fixed volume of sand, such a foundation will not allow any sophisticated carving.

4) Sand grains are regarded as tiny spheres. In our foundation, these grains are of the same size and are closely packed.

5) The top of the foundation is a flat platform whose surface is horizontal.

6) ...

2.2 Nomenclature

Table 1: States of Sand

Symbol	Meaning
τ	the shear stress on a plane
τ_f	the shear stress on the failure plane
μ	the internal friction coefficient
σ	the normal compressible stress on a plane
σ_f	the normal compressible stree on the failure plane
ρ	the density of sand
θ_c	the critical angle of a sandpile
D	the height from the top of the sand pile
P	the pressure[]
Γ	the surface tension of water
s_A	the adhesive stress
f_A	the adhesive force
V	the total amount of fluid present per particle contact
R	the radius of particle

3 Modeling Under Waves and Tides

The 3-dimensional shape constructing problem is divided into two sub-problems. We first establish the model using the **Mohr-Coulomb Criterion** to decide the shape of the slope. We then construct another model with a modified version of **Cellular Automata** to determine the best shape viewed from the top,...

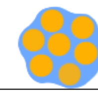
3.1 Shape of the Slope: Mohr-Coulomb Criterion

We begin by determining the side shape of the sand castle foundation. Before Approaching the problem, we will briefly address the property of sand as a granular media.

In our assumptions, sand particles are considered as identical tiny spheres. If we zoom in to observe a pile of wet sand, there are the so-called liquid bridges formed between sand particles.

Various water contents produce different liquid bridge distributions, which will influence the properties of sand.

Table 2: States of Sand

Liquid Content	State	Description	Schematic
No	Dry	No cohesive force exists	
Small	Pendular	Liquid bridges start to form	
Middle	Funicular	Liquid bridges and liquid-filled pores coexist	
Almost saturated	capillary	Almost all pores are filled with the liquid	
More	Slurry	No cohesive force exists	

When the wave gets into contact with the foundation, the surface area is in slurry state and there exists no cohesive interaction between the particles, which makes it very hard, if not impossible, to prevent sand loss in this process. Nevertheless, collapses after the wave resides can cause more harm to the foundation, which can be avoided by alternating the shape.

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For dry sand, the failure criterion is given in terms of the shear stress τ, the normal compressible stress σ and the internal friction μ as

$$\tau > \mu\sigma$$

This is simply the friction formula with different notations. For wet sand, we consider a sandpile with a normal adhesive stress s_A across every plane, in addition to the stress caused by weight. The equation(1) is then modified as

$$\tau > \mu(\sigma + s_A)$$

This is the so-called **Mohr-Coulumb criterion**. The stress resulting from the weight above the plane is shown in figure(2). Denote τ_f and μ_f as the shear stress and normal compressible stress at the failure plane, it is obvious from the schematic that they can be written as

$$\tau_f = \rho g D sin\theta_c \qquad and \qquad \sigma_f = \rho g D cos\theta_c$$

where θ_c is the critical angle, D is the height of the sandpile and ρ is the density of sand. Therefore, combine the equations above, to solve for θ_c is to solve the equation

$$\mu = tan\theta_c(D)\left(1 + \frac{s_A}{\rho g D cos\theta_c(D)}\right)$$

The only unknown factor is s_A, the adhesive stress across the plane. According to (Thomas C.H and Alex J.L -fix later)'s study, the value of s_A is determined by water content and there are three regimes as a function of the added-fluid volume. We now focus only on the state where the water content is close to saturation.

In this case, with water serves as lubricate, it makes sense to model sand particles as frictionless spheres. The pressure difference is then given by []

$$P = -\frac{\Gamma}{\sqrt{V/2\pi R}}$$

where Γ is the surface tension of the fluid, V is the total amount of fluid present per particle contact, and R the radius of the particle. According to the study of [fix later], the adhesive force in this state is given by

$$f_A = 2\pi\Gamma R$$

Note that (with certain conditions like distance between grains remains constant)the term V which denotes the volume of liquid per particle does not appear in equation(6).

We now focus on obtaining the value of s_A via f_A, whose value can be calculated with the equation above. Assume our foundation contains sand particles that are closely packed. In such a structure, we have $\phi_V = \sqrt(2)/6$, where ϕ_V stands for volume fraction which is defined as ratio of the volume of particles to the total volume. The average number of contacts per unit area will be $(3\phi_V/\pi R^2)$. With f_A representing cohesive force of a single liquid bridge, we can then write

$$s_A = \frac{f_A}{\sqrt{2}R^2}$$

Substitute the above result into Eq.(), we obtain the result

$$tan\theta_c = \mu + \frac{\sqrt{2}\pi\mu\Gamma}{R\rho g D} sec[tan^{-1}(\mu)]$$

3.2 Top View Shape

We used a modified version of **cellular automata** to simulate the impact of tides and waves. There are several factors that are taken into consideration:

3.3 Assumptions

1) We consider only the slice of the *sand castle foundation*, which is a two-dimensional plane. The sea wave is considered to come from above, which mean it impacts our *sand castle foundation* downwards.

2) Only a part of the *sand castle foundation* is taken into consideration since the chaos caused by flowing water would be too hard to simulation. So we'll only consider the impact angle's influence on both sand loss rate and water osmosis rate.

3.4 Nomenclature

We use `Python`, a script language to construct our **cellular automata**: *sand castle foundation* simulator, which provides ample 3rd party library to speed up development. The **cellular automata** is sealed in a class of python called `SandCastleSimulator2D`, which has the ability to:

1) Computing current sand surface slope, which is going to affect how sea wave is going to affect this small area of sand.

2) Reverse information about current simulation field, include the state of each individual sand cluster, the water-to-sand proportion of each cluster.

3) Simulate a small moment of wave impact, dropping sand and updating each cluster's *humidity*.

4) Comstructing a color map for this particular problem.

Table 3: Variable Nomenclature

Symblol	Meaning
`width`	The width of the simulation field along x axis. unit: sand cluster
`depth`	The depth of the simulation field along y axis. unit: sand cluster
`delta`	A value small enough for slope computation, determines the rate of sand property alteration per wave impact
`initial_edge`	The front edge of the *sand castle foundation*
`shear_rate`	How much the angle of impact influences the stay or leave of each cluster
`humidify_rate`	How much water can penetrate into sand through outer sand shell
`humidify_depth`	The reciprocal of water penetration decrease rate
`initial_humidity`	Initial humidity of each sand cluster
`osmotic_rate`	Similar as `humidify_rate`, affects sand cluster no matter the angle
`osmotic_depth`	Similar as `humidify_depth`, affects sand cluster no matter the angle
`slow`	Whether the model is running on high resolution mode
`osmosis`	Whether osmosis is considered (penetration no matter the angle)

Note that all values aren't exact corresponding to what there actual value in real world might be, since we've seldom needed to define unit in computation or trying to map variables to exact physical concept. Instead, during our simulation, we considerc more about development efficiency and the asymptotic complexity of the process so as to get more accurate results in a limited amount of time on a limited computer.

3.5 Implementation

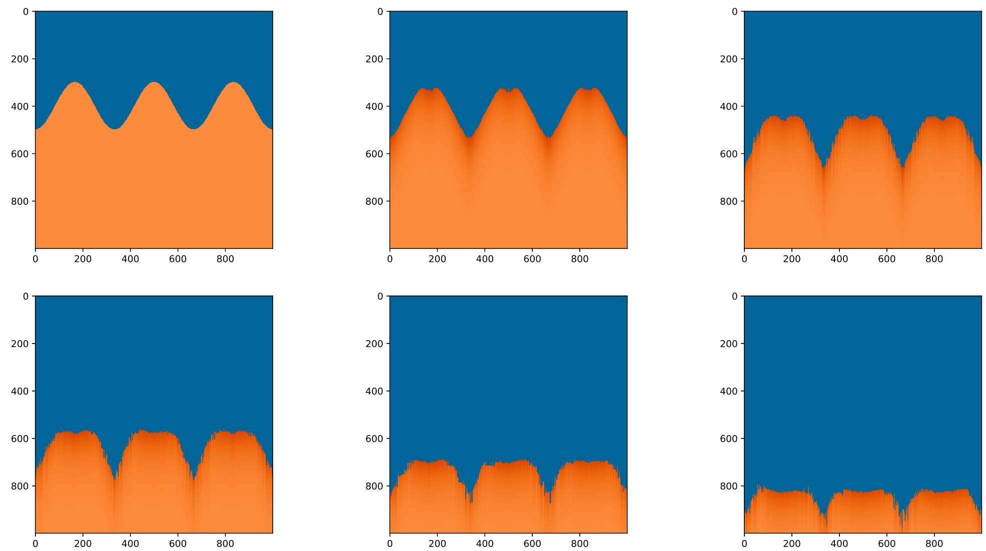

Figure 2: A Windows Terminal.

This is a test figure. Remember to delete me.

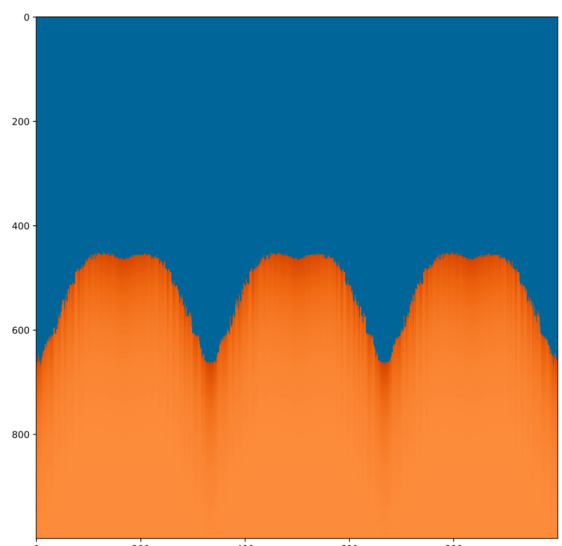

Figure 3: A Windows Terminal.

3.6 Calculate & Simulate Results

Briefly review the equation derived in section 3.1

$$tan\theta_c = \mu + \frac{\sqrt{2}\pi\mu\Gamma}{R\rho g D} sec[tan^{-1}(\mu)]$$

To solve for the best shape of the slope, the necessary information about sand is listed below

Table 4: States of Sand

Physical Quantities	Values	Units
Γ	72.8	mN/m
μ	0.55 0.60	-
ρ	1631	kg/m^3
R	0.05 2	mm
g	9.81	m/s^2

It is not sensible to embark on a trip to the beach on a stormy day, so we assume that the near-shore wave is gentle, with its height not exceeding 20cm, or $D \in [0, 20](unit : cm)$. We first assume that..

4 Modeling Under Rain

4.1 Model Adaptations

Rain differs from waves and sands in the following aspects: its continuity, and its quantity. So we modify our Cellular Automata model to adapt to the new conditions.

Adaptations

1) In our previous model, wave is modeled as discrete motions.

2) Rainwater can wet our model in a relatively short time.

3)

4.2 Shape Alteration

As a result, our model is not the most preferable option under the rain. Shape alterations have to be made if we want our sand castle foundation to last longer.

5 Determine the Best Sand-to-water Proportion

5.1 Assumptions & Nomenclature

ADD A CURVE FIGURE HERE, DRAWING!

As we mentioned above, the adhesive stress across the plane (s_A) is determined by water content. Consequently, we will find out how the cohesion in wet granular materials varies with different amount of liquid based on liquid bridge model to find an optimal sand-to-water mixture proportion.

There are five differing degrees of wetting in granular matter. As dry grains and slurry have no cohesion between particles, we just consider three other conditions. According to [2], we can see that the slope for low and large liquid content is much larger than that for intermediate liquid content. That is: the suction changes sharply as these three states transform each other. Consequently,we should keep the proportion of water to sand in steady for a long time, which means try to avoid transformation between three states.

we initially wish to create a three dimensional matrix to represent the situation. We still consider that the sandcastle C consists of cells mixed with sand and water. Then It's easy to Calculate each cell's sand-to-water mixture proportion from the amount of sand and water. ADD TWO CELLULAR FIGURES HERE, DRAWING!

5.2 Assumptions

1) waves and tides erode the sandcastle in a very short time, which means that the we can neglect Osmosis occurred during erosion.

2) After eroded, the liquid left on sandcastle has ample time to penetrate.

3) The evaporation of liquid is neglected.

5.3 Nomenclature

Table 5: Notation Table

symbol	meaning
$c_{ijk}(t)$	the cell mixed with sand and water to form the sand-castle in time t
$s_{ijk}(t)$	the amount of sand in the cell in time t
$w_{ijk}(t)$	the amount of water in c_{ijk} in time t
$p_{ijk}(t)$	the proportion of sand-to-water mixture c_{ijk} in time t
$S(t)$	the amount of sand in the sandcastle int time t
$W(t)$	the amount of water in the sandcastle in time t
$P(t)$	the average proportion of sand-to-water mixture c_{ijk} in time t
$F(t)$	the erosion-Osmosis common effect of the sea water in time t

5.4 model

We define the expression when we just finish build the sandcastle.

$$P(t_0) = \frac{S(t_0)}{W(t_0)} = \frac{\sum_{i,j,k} s_{ijk}(t_0)}{\sum_{i,j,k} w_{ijk}(t_0)} = p_{ijk}(t_0)$$

To evaluate the existence of the sandcastle, we define that a cell is completely eroded by tides and waves as follow:

$$p_{ijk}(t) = \frac{s_{ijk}(t)}{w_{ijk}(t)} \leq p_{erode}$$

And we define that the sandcastle is completely eroded as:

$$P(t) = \frac{\sum_{i,j,k} s_{ijk}(t_0)}{\sum_{i,j,k} w_{ijk}(t_0)} \leq P_{erode}$$

where P_{erode} refers to proportion that Funicular transforms to capillary state. Then we define the flow equation during erosion as follow:

$$E = \begin{cases} F & \text{the cell is being eroded by sea} \\ 0 & \text{else} \end{cases}$$

As erosion occurs for a split second, we could just neglect the permeation to simplify our analysis. To obtain a probability model to evaluate the sand taken by waves, we need to acquire further understanding about the flow liquid. Here we use Navier-Stokes equations to describe the motion of viscous fluid substances.

$$x\ \rho(\frac{\partial \upsilon}{\partial t}+\upsilon\cdot\nabla\upsilon) = -\nabla p+\nabla\cdot(\mu(\nabla\upsilon+(\nabla\upsilon)^T)-\frac{2}{3}\mu(\nabla\upsilon)I)+F\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\vec{v} = 0(continuity\ equations)$$

After eroded we should find the cell completely eroded by tides and waves. Then by reference to Darcy's law and Huygens–Fresnel principle, We consider water permeating the sandcastle as follow:

$$\upsilon = \frac{kA(p_b - p_a)}{\mu L} = \frac{kL(\Delta P)}{\mu}$$

where k is the hydraulic permeability of sea water, L is the length of one side of the cell, μ is the viscosity of sea water, Δ is the pressure differential between the two sides of the cell.As we assume that the sea water left on the surface of the sandcastle can totally penetrate. Then we could get for each cycle:

$$L(k,t,\mu,\rho,\upsilon,p) = \sum_{c_{ijk}\in C}\sum_{i-1}^{i+1}\sum_{j-1}^{j+1}\sum_{k-1}^{k+1} p_{mnp}\cdot E(m,n,p\neq i,j,k)(P(t)<P_{limit})$$
$$(\upsilon\cdot\nabla\upsilon)+(\mu\cdot\nabla\mu)+(\rho\cdot\nabla\rho) = Constant$$

Then We need to find out the longest time when the above constraints are met and the sandcastle exists.

6 Other Tricks to Make Our Sand Castle Last Longer

6.1 Better conditions to survive

As we acknowledge that the greatest threats to our sandcastle is the erosion of tides and wives. Consequently, a relatively safe place will be better if we want to protect it from erosion for a longer time. That means we can do something as follow:

Build a Moat Building a moat can help our sand castle foundation to survive the rising tide.

Further From the Shore Just stay in the "safety zone" and don't build your sandcastle too close to the shore will obriously reduce the damage caused by tides and waves.

Less Salty Water Choosing a beach near a less salty ocean may contribute to building a more concrete sand castle. This is because fresh water has a higher value of surface tension which appears in the numerator of the term $\frac{\sqrt{2}\pi\mu\Gamma}{R\rho g D}$. Thus Red Sea is not a perfect spot, but the Baltic Sea(poor English. better expression needed.)(footnote needed)

6.2 sharpen your design

make it compact In section 3.1, we assumed that sand particles are closely packed. In fact, there are some degree of internal erosion in the sandcastle along with the hydraulic pressures increase. The final value of the hydraulic conductivity also depends on the rate of erosion of the liner material.
According to study [4], The erodibility of a soil can be described in term of the rate of erosion. They find that Correlation between erodibility and Hydraulic Conductivity of Sand-Bentonite Mixtures can be expressed in the following form:

$$k = a(\varepsilon_e)^b$$

where a,b depends on the percentage of bentonite in the mixture (η).These studys indicate that the rate of erosion depends mainly on the porosity of the mixtures.Consequently,you had better make your works more compact to defend it from internal erosion.

Finer Sand Finer sand is a bonus because by constructing our sand castle foundation using finer sand, the foundation has a greater critical angle. That is because the radius of particles R which is a part of the denominator decreases and (enlarge?not the proper word) the value of $tan\theta_c$.

Special Material

Glue Sand sculptures seen in exhibitions or competitions have their surfaces covered with special glue. The glue is mainly used to protect the sand sculpture from drying out or crumbling if the waves do not wash it away first, but it can also keep water out. In our model where the greatest enemy of the foundation is not collapses but erosion, such glue will help maintain the shape of the foundation.

Mixture Concrete, a material that everyone is familiar with, has sand as its ingredient.

6.3 Less Salty Water

Choosing a beach near a less salty ocean may contribute to building a more concrete sand castle. This is because fresh water has a higher value of surface tension which appears in the numerator of the term $\frac{\sqrt{2}\pi\mu\Gamma}{R\rho g D}$. Thus Red Sea is not a perfect spot, but the Baltic Sea(poor English. better expression needed.)(footnote needed) Finer sand is a bonus because by constructing our sand castle foundation using it, the foundation has a greater critical angle. That is because the term R, which is a part of the denominator, decreases and (enlarge?not the proper word) the value of $tan\theta_c$.

6.4 Special Material

Glue Sand sculptures seen in exhibitions or competitions have their surfaces covered with special glue. The glue is mainly used protect the sand sculpture from drying out or crumbling if the waves do not wash it away first, but it also keeps water out. In our model where the greatest enemy of the foundation is not collapses but erosion, such glue will help maintain the shape of the foundation.

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7 Sensitivity Analysis

7.1 Model Under Waves and Tides

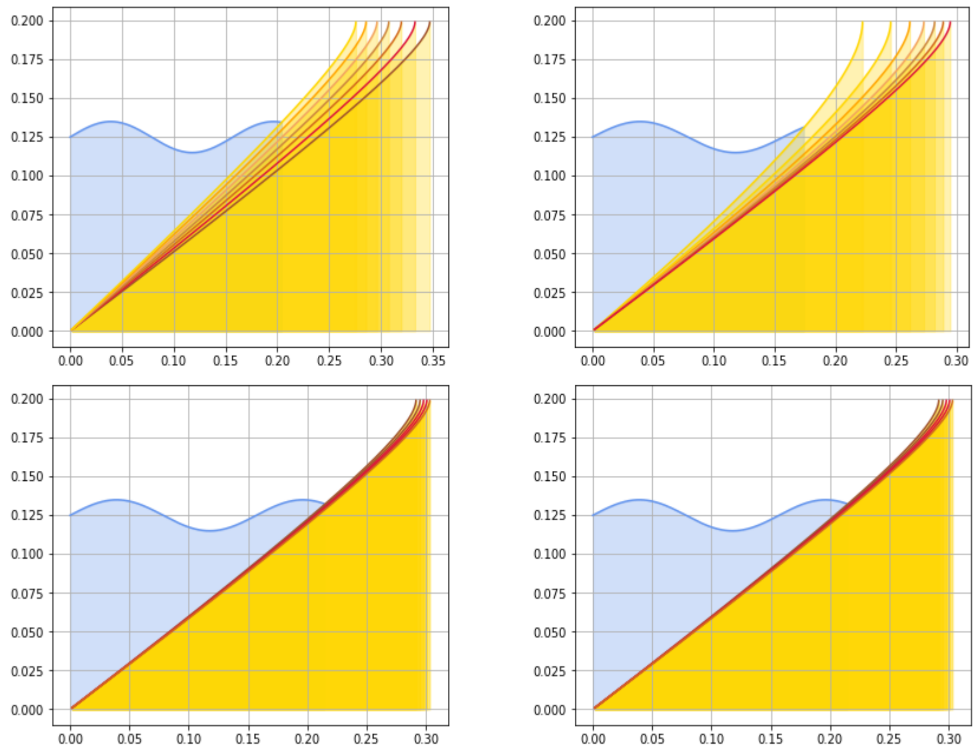
Side View Shape There are multiple factors that may affect our side view shape model. We consider four of them: the internal friction coefficient μ, radius of the sand particles R, density of sand ρ and total height of the sand castle foundation D.

Friction coefficient of sand varies by the amount of liquid in the system, the size of sand grains and other possible factors. Typically, this value ranges from 0.55-0.60. In figure[x.a], we test the effect of different friction coefficient ranging from 0.48 to 0.60. The curves with lighter colors, as illustrated in the figure, represent higher coefficients while those with darker colors represent lower coefficients.

Figure[x.b] shows how the side view shape is influenced by radius of the sand particles, which varies between 0.5 to 2.0 milimeters. Curves with lighter colors represent small particle radiuses and those with darker colors represent greater radiuses.

The effect of density is illstrated in Figure[x,c], which is not obvious compared with the former two factors.

Table 6



Top View Shape DenDen dl has done a lot of parameter adjusting jobs...those are exactly what we need here.

7.2 Model Under Rain

Not now. Wait for tomorrow or something.

8 Strengths and Weaknesses

8.1 Strengths

Just like any other model, the one presented above has its Strengths and Weaknesses. Some major points are listed below.

1) The use of Cellular Automata enables our model to predict erosion of any possible shape.

2)

3) ...

8.2 Weaknesses

1) Our model in section 3.1 does not consider the impact of the wave, which...

2) ...

3) ...

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This is some example text[1].
I'm referring to footnote 1.

[1]Hello footnote

List of Figures

List of Tables

List of Listings

References

[1] M. Pakpour, M. Habibi, P. Møller, and D. Bonn, "How to construct the perfect sandcastle," *Scientific reports*, vol. 2, no. 1, pp. 1–3, 2012.

[2] N. Mitarai and F. Nori, "Wet granular materials," *Advances in Physics*, vol. 55, no. 1-2, pp. 1–45, 2006.

[3] A. Kudrolli, "Sticky sand," *Nature materials*, vol. 7, no. 3, pp. 174–175, 2008.