Electrodynamics

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Electrodynamics is the study of electrical force. Qualitatively, we observe that particles of the same type repel each other, while particles of different types attract each other.

Contents

1 Introduction 2
1.1 Maxwell equations 2
1.2 Electrodynamics to Special Relativity 2
1.3 Translations 2
1.4 Rotations 2
1.5 Galilean transformations 2
2 Electrostatics 3
2.1 Gauss's Law in Integral Form 3
2.2 Superposition Principle 3
2.3 Electric Field of a Point Charge (Coulomb's Law) 3
2.4 Continuous Charge Distributions and Dirac Delta 3
2.5 Scalar Potential 3
2.5.1 Poisson and Laplace Equations 3
2.5.2 Verifying the Potential Integral Solution 3
2.5.3 Physical Meaning of Potential: Work and Energy . 4
2.6 Potential Energy of a Charge Distribution 4
2.6.1 Discrete Charges 4
2.6.2 Continuous Distributions 4
2.6.3 Energy in Terms of Electric Field 4
2.6.4 Self-Energy Problem 4
2.7 Conductors in Electrostatics 4
2.7.1 Field and Charge Density at a Conductor Surface . 4
2.7.2 Electric Pressure on a Conductor 5
2.8 Boundary Value Problems in Electrostatics 5
2.8.1 Uniqueness Theorem 5
2.9 Green's Functions 5
2.9.1 Green's Identities 5
2.9.2 Solving Poisson's Equation using Green's Function 5
2.9.3 Dirichlet Green's Function (G_D) 6
2.9.4 Neumann Green's Function (G_N) 6
2.10 Method of Images 6
2.10.1 Example 1: Point Charge near a Grounded Conducting Plane 6
2.10.2 Example 2: Point Charge near a Grounded Conducting Sphere 6
2.11 General Method: Eigenfunction Expansion 7
2.11.1 Example: Infinite Space (No Boundaries) 7
2.11.2 Next Steps: Electrostatics in a Box 7
2.12 Electrostatics with Spherical Symmetry 7
2.12.1 Laplace in Spherical Coordinates 7
2.13 Other uses of spehrical harmonics 9
2.14 Dipole Moment 9
3 Magnetostatics 10
3.1 Current Density 10
3.2 Integral Form 10
3.2.1 Amperes Law 10
3.3 Vector Potential 10
3.3.1 Symmetry 10
3.4 Application: Magnetic Dipole 10
3.5 Relativity of $\vec{E}$ and $\vec{B}$ 11
3.6 Full Potentials 11
3.7 D'Alembert Operator 12
3.8 Gague Symmatry 12
4 Special Relativity 15
4.1 Relativistic distance 15
4.2 Poincare Group 15
4.3 Subgroups 15
4.3.1 Subgroup of the proper Lorentz transformations . 16
4.3.2 Subgroup of Rotations 16
4.4 Time Dilation 16
4.5 Relativistic Force 16
4.6 Relativistic Time 17
4.7 Momentum 17
4.8 Recap 18
4.9 Covariant 4-vectors 18
4.10 Vectors and Tensors 18
4.11 Tensors 19
4.12 Currents and Densities of Electric Charges 19
4.13 Energy-Momentum Tensor 20
4.14 Conservation of the Energy-Momentum Tensor 21
4.15 Electrodynamics as a relativistic Theory 21
4.15.1 Summary 22
4.16 Gauge Invariance 22
5 Radiation 24
5.1 Poynting vector 25
5.2 Scattering 26
5.2.1 Thompson scattering 26
5.2.2 Rayleigh Scattering 27
6 Macroscopic Maxwell Equations in a Medium 28
6.1 The Effective Theory 28
6.2 Atomic Approximation 28
6.3 Average atomic charge density 28
6.3.1 Metal 30
6.4 Reflection and Refraction 31
6.5 Fresnel Equations 31
7 QED 33
7.1 Classical propabilities 33
7.2 Quantum propabilities 33
7.3 Events 33
7.4 Feynman Rules 33
7.5 QED Thompson Scattering 34
7.6 QED Rayleigh Scattering 34
7.7 Glass Optics 34

1 Introduction

Electrodynamics is the study of electrical force. Qualitatively, we observe that particles of the same type repel each other, while particles of different types attract each other.

Qnauntitatively, we will observe a single charge surrounded by other charges and we will determine the force acting on the charge with:

$$\vec{F} = q\big(\vec{E} + \vec{v} \times \vec{B}\big)$$

$\vec{E}$ and $\vec{B}$ captiue the effects of all other charges in the system. We use the density of charges ρ and the current density $\vec{J}$ and determine the electric field $\vec{E}$ and the magnetic field $\vec{B}$.

$$\big(\rho, \vec{J}\big) \to \big(\vec{E}, \vec{B}\big) \to \vec{F}$$

1.1 Maxwell equations

Definition 1: Definition

We can then use the boundary conditions $\rho, \vec{J}$ to determine the electric field $\vec{E}$ and the magnetic field $\vec{B}$.

1.2 Electrodynamics to Special Relativity

We want to examine Maxwells equations in a different frame of reference.

- We have a translation:

$$\vec{r} \to \vec{r}' = \vec{r} + \vec{a}$$

- We have rotation:

$$\vec{r} \to \vec{r}' = R(\vec{r})$$

- We have galilean transformation:

$$\vec{r} \to \vec{r}' = \vec{r} + \vec{r}_0 + \vec{v}t$$
$$\vec{v} = \text{const}$$

We expect Maxwell Equations to be invariant under these transformations, but we observe that the equations are not invariant under galilean transformations.

1.3 Translations

If we move the origin from 0 to $0'$, we conserve the magnitude and direction of the electric field $\vec{E}$: $\vec{E}'(\vec{r}') = \vec{E}(\vec{r})$. So does everything else.

$$\vec{\nabla}' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$$

$$\Rightarrow \frac{\partial}{\partial x'} = \frac{\partial}{\partial (x+a)} = \frac{\partial}{\partial x} \Rightarrow \vec{\nabla}' = \vec{\nabla} \Rightarrow \nabla' \cdot \vec{E}'(\vec{r}') = \frac{\rho'(\vec{r})}{\varepsilon_0}$$

$$\Leftrightarrow \boldsymbol{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

1.4 Rotations

$$\vec{r} \to \vec{r}' = R(\theta, ...)\vec{r}$$

We write the rotation matrix in components:

$$r_i \to r_i' = \sum_{k=1}^{3} R_{ik} r_k$$

Definition 2: Definition

Rotations preserve lengths of vectors and angles between vectors. This is the defining property of a rotation. This implies that R is orthogonal:

$$R^T R = \mathbb{1}$$
$$\vec{r}'^2 = \vec{r}_i' \vec{r}_i' = (R_{ik} r_k)(R_{ie} r_e)$$
$$= R_{ik} R_{ie} r_k r_e$$
$$= R_{ki}^T R_{ie} r_k r_e$$
$$\Rightarrow r_i' r_i' = \left(R^T R\right)_{ke} r_k r_e$$
$$\left(R^T R\right)_{ke} = \delta_{ke} = \begin{cases} 1, k = e \\ 0, k \neq e \end{cases}$$
$$\Rightarrow (R^T R) = \mathbb{1}$$

An additional property of the rotation matrix is that its determinant is 1.

We have **inner product**:

$$\vec{A} \cdot \vec{B} \to \vec{A}' \cdot \vec{B}' = \vec{A} R^T R \vec{B} = \vec{A}\vec{B}$$

This confirms the invariance of the first and thrid maxwell equations.

We have **outer product**:

$$\big(\vec{A} \times \vec{B}\big) \to ... \to R\big(\vec{A} \times \vec{B}\big)$$

We see that the second and fourth maxwell equations are covariant under rotation.

1.5 Galilean transformations

We will see that the equations are not invariant under galilean transformations. We will arrive at the Lorentz transformation later, which is the correct transformation for a boost.

2 Electrostatics

Electrostatics deals with phenomena involving electric charges that are stationary or moving very slowly. Key assumptions include:

- **No electrical currents** (or negligible currents).
- **Steady density of charges**: $\rho(\vec{x}, t) = \rho(\vec{x})$.

The fundamental laws governing electrostatics are special cases of Maxwell's equations:

- **Gauss's Law**: Relates electric field to charge density.

$$\boldsymbol{\nabla} \cdot \left(\vec{E}\right) = \frac{\rho}{\varepsilon_0}$$

- **Faraday's Law (Static Case)**: Implies the electric field is conservative.

$$\boldsymbol{\nabla} \times \left(\vec{E}\right) = 0$$

2.1 Gauss's Law in Integral Form

Integrating Gauss's law over a volume V bounded by a surface S and applying the divergence theorem:

$$\int_V \mathrm{d}^3\vec{x}\, \boldsymbol{\nabla} \cdot \left(\vec{E}\right) = \int_V \mathrm{d}^3\vec{x}\, \frac{\rho}{\varepsilon_0}$$

$$\int_S \mathrm{d}\vec{S} \cdot \vec{E} = \int_V \mathrm{d}^3\vec{x}\, \frac{\rho}{\varepsilon_0} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

The electric flux through a closed surface S equals the total charge Q_{enc} enclosed within the volume V, divided by the permittivity of free space ε_0.

2.2 Superposition Principle

Since $\boldsymbol{\nabla} \cdot \left(\vec{E}\right)$ is linear in $\vec{E}$, if $\rho = \rho_1 + \rho_2$, and $\vec{E}_1, \vec{E}_2$ are solutions for ρ_1, ρ_2 respectively:

$$\boldsymbol{\nabla} \cdot \left(\vec{E}_1\right) = \frac{\rho_1}{\varepsilon_0} \text{ and } \boldsymbol{\nabla} \cdot \left(\vec{E}_2\right) = \frac{\rho_2}{\varepsilon_0}$$

Then the total field is $\vec{E} = \vec{E}_1 + \vec{E}_2$, satisfying $\boldsymbol{\nabla} \cdot \left(\vec{E}\right) = \frac{\rho_1+\rho_2}{\varepsilon_0}$. This allows us to find the field for complex distributions by summing the fieldd(S) of simpler components, like point charges.

2.3 Electric Field of a Point Charge (Coulomb's Law)

Consider a point charge q at the origin. Due to spherical symmetry, the electric field must be radial: $\vec{E} = E(r)\vec{r}$. Applying Gauss's law to a sphere of radius R:

$$\begin{aligned}\int_S \mathrm{d}\vec{S} \cdot \vec{E} &= \int_S R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi\, \vec{r} \cdot (E(R)\vec{r}) \\ &= E(R)(4\pi R^2) = \frac{q}{\varepsilon_0} \\ \Rightarrow E(R) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\end{aligned}$$

The electric field at position $\vec{x}$ due to a charge q at $\vec{y}$ is:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{y}|^2}\hat{r}_{\vec{x}-\vec{y}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{y}|^3}(\vec{x}-\vec{y})$$

This is **Coulomb's Law**.

For a system of N point charges q_i at positions $\vec{y}_i$, the total field at $\vec{x}$ is (by superposition):

$$\vec{E}(\vec{x}) = \sum_{i=1}^{N} \vec{E}_{i(\vec{x})} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\left|\vec{x}-\vec{y}_i\right|^3}(\vec{x}-\vec{y}_i)$$

Note: $\boldsymbol{\nabla} \times \left(\vec{E}\right) = 0$ because each term $\vec{E}_i$ is proportional to $\frac{\vec{x}-\vec{y}_i}{|\vec{x}-\vec{y}_i|^3}$, which is a gradient field (see below). Also, $\boldsymbol{\nabla} \cdot \left(\vec{E}_i\right) = 0$ for $\vec{x} \neq \vec{y}_i$.

2.4 Continuous Charge Distributions and Dirac Delta

Often, charges are described by a continuous density $\rho(\vec{x})$. The charge in an infinitesimal volume $\mathrm{d}V$ is $\mathrm{d}q = \rho(\vec{x})\,\mathrm{d}V$. The total field is obtained by integrating over the distribution:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{y}\, \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|^3}(\vec{x}-\vec{y})$$

A point charge q at $\vec{y}_0$ can be represented using the Dirac delta function:

$$\rho(\vec{y}) = q\delta^{(3)}(\vec{y}-\vec{y}_0)$$

where $\delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z)$ and $\int \mathrm{d}^3\vec{r}\,\delta^{(3)}(\vec{r}-\vec{r}_0)F(\vec{r}) = F(\vec{r}_0)$. A collection of point charges correspondd(S) to:

$$\rho(\vec{y}) = \sum_{i=1}^{N} q_i\delta^{(3)}(\vec{y}-\vec{y}_i)$$

2.5 Scalar Potential

Since $\boldsymbol{\nabla} \times \left(\vec{E}\right) = 0$ in electrostatics, the electric field can always be expressed as the gradient of a scalar potential Φ:

$$\vec{E} = -\boldsymbol{\nabla}\Phi = -\nabla\Phi$$

This is because $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Phi) = 0$ for any scalar function Φ.

Substituting this into Coulomb's law for a continuous distribution:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{y}\,\rho(\vec{y})\left(-\boldsymbol{\nabla}_{\vec{x}}\frac{1}{|\vec{x}-\vec{y}|}\right)$$

$$\text{since } \boldsymbol{\nabla}_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{y}|}\right) = -\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}$$

Assuming we can interchange gradient and integral:

$$\vec{E}(\vec{x}) = -\boldsymbol{\nabla}_{\vec{x}}\left[\frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\right]$$

Comparing with $\vec{E} = -\boldsymbol{\nabla}\Phi$, we identify the **scalar potential**:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

Working with the scalar Φ is often simpler than working directly with the vector field $\vec{E}$.

2.5.1 Poisson and Laplace Equations

Substituting $\vec{E} = -\boldsymbol{\nabla}\Phi$ into Gauss's law $\boldsymbol{\nabla} \cdot \left(\vec{E}\right) = \frac{\rho}{\varepsilon_0}$:

$$\begin{aligned}\boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla}\Phi) &= \frac{\rho}{\varepsilon_0} \\ \Rightarrow \nabla^2\Phi &= -\frac{\rho}{\varepsilon_0}\end{aligned}$$

This is the **Poisson equation**. It relates the potential to the charge density.

In regions where the charge density is zero ($\rho = 0$), the Poisson equation reduces to the **Laplace equation**:

$$\nabla^2\Phi = 0$$

2.5.2 Verifying the Potential Integral Solution

Does the integral form of Φ satisfy the Poisson equation?

$$\begin{aligned}\nabla^2_{\vec{x}}\Phi(\vec{x}) &= \nabla^2_{\vec{x}}\left[\frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\right] \\ &= \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{y}\,\rho(\vec{y})\nabla^2_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{y}|}\right)\end{aligned}$$

We know the result should be $-\frac{\rho(\vec{x})}{\varepsilon_0}$. This requires:

$$\nabla^2_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{y}|}\right) = -4\pi\delta^{(3)}(\vec{x}-\vec{y})$$

This identity is crucial. It shows that the Laplacian of the $\frac{1}{r}$ potential is zero everywhere except at the origin, where it represents a point source (the delta function).

2.5.3 Physical Meaning of Potential: Work and Energy

Consider the force on a test charge q in an electric field $\vec{E}$: $\vec{F} = q\vec{E} = -q\boldsymbol{\nabla}\Phi$. The work done **by an external agent** to move the charge q from point A to point B against the electric field is:

$$\begin{aligned} W_{A\to B} &= \int_A^B \vec{F}_{\text{ext}} \cdot \mathrm{d}\vec{l} = -\int_A^B \vec{F}\cdot \mathrm{d}\vec{l} \\ &= -\int_A^B \left(q\vec{E}\right)\cdot \mathrm{d}\vec{l} = q\int_A^B (\boldsymbol{\nabla}\Phi)\cdot \mathrm{d}\vec{l} \\ &= q[\Phi(\vec{x}_B) - \Phi(\vec{x}_A)] \end{aligned}$$

The work done dependd(S) only on the potential difference between the endpoints, not the path taken (confirming $\vec{E}$ is conservative). $q\Phi(\vec{x})$ represents the potential energy of the charge q at position $\vec{x}$.

2.6 Potential Energy of a Charge Distribution

What is the energy stored in assembling a configuration of charges?

2.6.1 Discrete Charges

Imagine bringing N charges q_i from infinity (where $\Phi = 0$) to their final positions $\vec{y}_i$.

- Bring q_1: $W_1 = 0$ (no field yet).
- Bring q_2: $W_2 = q_2(\Phi_1(\vec{y}_2) - \Phi_1(\infty)) = q_2\Phi_1(\vec{y}_2)$, where $\Phi_1(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{x}-\vec{y}_1|}$. $W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{y}_1-\vec{y}_2|}$.
- Bring q_3: $W_3 = q_3(\Phi_{1+2}(\vec{y}_3)) = q_3(\Phi_1(\vec{y}_3) + \Phi_2(\vec{y}_3))$ $W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_3}{|\vec{y}_1-\vec{y}_3|} + \frac{q_2q_3}{|\vec{y}_2-\vec{y}_3|}\right)$.
- Continue for all charges. The total work (potential energy W) is:

$$W = \sum_{n=2}^{N} W_n = \frac{1}{4\pi\varepsilon_0}\sum_{n=2}^{N}\sum_{i=1}^{n-1}\frac{q_iq_n}{|\vec{y}_n-\vec{y}_i|}$$

This can be written symmetrically by summing over all pairs (i,j) with $i \neq j$ and dividing by 2:

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{q_iq_j}{|\vec{y}_i-\vec{y}_j|}$$

Alternatively, recognize that $\sum_{j\neq i}\left(\frac{q_j}{4\pi\varepsilon_0|\vec{y}_i-\vec{y}_j|}\right)$ is the potential $\Phi_{\text{other}}(\vec{y}_i)$ at $\vec{y}_i$ due to all **other** charges.

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i\Phi_{\text{other}}(\vec{y}_i)$$

2.6.2 Continuous Distributions

Replace sums with integrals and q_i with $\rho(\vec{x})\,\mathrm{d}V$:

$$W = \frac{1}{8\pi\varepsilon_0}\int \mathrm{d}^3\vec{x}\int \mathrm{d}^3\vec{y}\,\frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

Recognizing the inner integral as related to the potential $\Phi(\vec{x})$ caused by $\rho(\vec{y})$:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

So, the energy becomes:

$$W = \frac{1}{2}\int \mathrm{d}^3\vec{x}\,\rho(\vec{x})\Phi(\vec{x})$$

2.6.3 Energy in Terms of Electric Field

Using $\rho = \varepsilon_0\boldsymbol{\nabla}\cdot\left(\vec{E}\right) = -\varepsilon_0\nabla^2\Phi$:

$$\begin{aligned} W &= \frac{1}{2}\int \mathrm{d}^3\vec{x}(-\varepsilon_0\nabla^2\Phi)\Phi(\vec{x}) \\ &= -\frac{\varepsilon_0}{2}\int \mathrm{d}^3\vec{x}\,\Phi\nabla^2\Phi \end{aligned}$$

Using the vector identity $\boldsymbol{\nabla}\cdot(\Phi\nabla\Phi) = (\nabla\Phi)\cdot(\nabla\Phi) + \Phi\nabla^2\Phi = |\nabla\Phi|^2 + \Phi\nabla^2\Phi$:

$$\begin{aligned} W &= -\frac{\varepsilon_0}{2}\int \mathrm{d}^3\vec{x}[\boldsymbol{\nabla}\cdot(\Phi\nabla\Phi) - |\nabla\Phi|^2] \\ &= -\frac{\varepsilon_0}{2}\int_S \mathrm{d}\vec{S}\cdot(\Phi\nabla\Phi) + \frac{\varepsilon_0}{2}\int_V \mathrm{d}^3\vec{x}\,|\nabla\Phi|^2 \end{aligned}$$

If the volume V extendd(S) to infinity where Φ and $\nabla\Phi$ vanish sufficiently fast, the surface integral is zero. Since $\vec{E} = -\nabla\Phi$, $|\nabla\Phi|^2 = \vec{E}^2$:

$$W = \frac{\varepsilon_0}{2}\int \mathrm{d}^3\vec{x}\,\vec{E}^2(\vec{x})$$

The energy can be thought of as stored in the electric field itself, with an energy density $\omega = \frac{\varepsilon_0}{2}\vec{E}^2$. Since $\vec{E}^2 \geq 0$, the electrostatic energy is always non-negative.

2.6.4 Self-Energy Problem

The continuous formula $W = \frac{1}{2}\int \rho\Phi\,\mathrm{d}V$ includes the interaction of infinitesimal charge elements $\rho(\vec{x})\,\mathrm{d}V$ with their own potential contribution, unlike the discrete sum $W = \frac{1}{2}\sum q_i\Phi_{\text{other}}(\vec{y}_i)$ where $i \neq j$. Consider the energy $W = \left(\frac{\varepsilon_0}{2}\right)\int \vec{E}^2\,\mathrm{d}V$ for a single point charge q. The field is $\vec{E} \propto \frac{q}{r^2}$, so $\vec{E}^2 \propto \frac{q^2}{r^4}$. The integral $\int\left(\frac{1}{r^4}\right)(r^2\,\mathrm{d}r\sin\theta d\theta d\varphi)$ diverges as $r \to 0$.

$$W \propto \int_0^R \frac{1}{r^4}r^2\,\mathrm{d}r = \int_0^R \frac{1}{r^2}\,\mathrm{d}r \to \infty$$

Classical electrodynamics predicts an infinite **self-energy** for a point charge. This divergence arises because the classical model allows charge to be concentrated at a single point.

In Quantum Electrodynamics (QED), self-energy calculations involve quantum corrections. While divergences still appear, they are handled through renormalization. The classical divergence can be seen as the $\hbar \to 0$ limit of the QED result, indicating it's a fundamental aspect of point-particle interactions in field theory, not just a mathematical artifact.

In practice, we often calculate **changes** in energy between configurations, where the (infinite) self-energy cancels out, leaving a finite, physically meaningful result.

2.7 Conductors in Electrostatics

Conductors contain charges free to move. In electrostatic equilibrium:

1. **Electric Field Inside is Zero**: $\vec{E}_{\text{inside}} = 0$. If not, charges would move until the field is cancelled.
2. **Potential Inside is Constant**: Since $\vec{E} = -\boldsymbol{\nabla}\Phi$, $\vec{E} = 0$ implies $\boldsymbol{\nabla}\Phi = 0$, so $\Phi = \text{const}$ inside and on the surface. The surface of a conductor is an equipotential.
3. **Net Charge Resides on the Surface**: From Gauss's law, $\boldsymbol{\nabla}\cdot\left(\vec{E}\right) = \frac{\rho}{\varepsilon_0}$. Since $\vec{E} = 0$ inside, $\rho = 0$ inside. Any net charge must be on the surface.
4. **Electric Field at the Surface is Normal**: $\vec{E}_{\text{surface}} \perp$ surface. If there were a parallel component $\vec{E}_{|}$, charges on the surface would move along the surface, violating equilibrium.

2.7.1 Field and Charge Density at a Conductor Surface

Apply Gauss's law to a small "pillbox" straddling the surface. The flux through the sides is negligible, and the flux through the bottom (inside) is zero ($\vec{E} = 0$). Only the top surface ΔS contributes:

$$\int \mathrm{d}\vec{S} \cdot \vec{E} = E_\perp \Delta S = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma \Delta S}{\varepsilon_0}$$

where σ is the surface charge density.

$$E_\perp = \frac{\sigma}{\varepsilon_0} \Rightarrow \vec{E}_{\text{surface}} = \left(\frac{\sigma}{\varepsilon_0}\right)\hat{n}$$

where $\hat{n}$ is the outward normal unit vector.

2.7.2 Electric Pressure on a Conductor

The electric field exerts an outward force on the surface charges. Consider the energy density just outside the surface: $\omega = \frac{\varepsilon_0}{2}\vec{E}^2 = \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{\sigma^2}{2\varepsilon_0}$. Imagine displacing a surface area ΔS outward by a small distance Δx. The work done by the electric field is $F\Delta x$. This work equals the change in field energy. The volume swept is $\Delta V = \Delta S \Delta x$.

$$\Delta W = \omega \Delta V = \left(\frac{\sigma^2}{2\varepsilon_0}\right)(\Delta S \Delta x)$$

Equating work done to energy change: $F\Delta x = \Delta W$.

$$F\Delta x = \left(\frac{\sigma^2}{2\varepsilon_0}\right)\Delta S \Delta x$$

$$\Rightarrow F = \left(\frac{\sigma^2}{2\varepsilon_0}\right)\Delta S$$

The pressure (force per unit area) is:

$$P = \frac{F}{\Delta} S = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon_0}{2}E^2$$

This outward pressure exists on the surface of any charged conductor.

2.8 Boundary Value Problems in Electrostatics

Often, we don't know the charge distribution everywhere but want to find the potential Φ (and thus $\vec{E}$) in a region V, given information on its boundary S. The governing equation is Poisson's: $\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$.

2.8.1 Uniqueness Theorem

Suppose Φ_1 and Φ_2 are two solutions to Poisson's equation in V for the same ρ. Let $\Phi_D = \Phi_1 - \Phi_2$. Then:

$$\nabla^2\Phi_D = \nabla^2\Phi_1 - \nabla^2\Phi_2 = \left(-\frac{\rho}{\varepsilon_0}\right) - \left(-\frac{\rho}{\varepsilon_0}\right) = 0$$

So Φ_D satisfies the Laplace equation in V. Consider the integral:

$$\int_V \mathrm{d}^3\vec{x}\, \boldsymbol{\nabla} \cdot (\Phi_D \boldsymbol{\nabla}\Phi_D) = \int_S \mathrm{d}\vec{S} \cdot (\Phi_D \boldsymbol{\nabla}\Phi_D)$$

Using the identity $\boldsymbol{\nabla} \cdot (f(A)) = (\boldsymbol{\nabla} f) \cdot (A) + f(\boldsymbol{\nabla} \cdot (A))$ with $f = \Phi_D$ and $(A) = \boldsymbol{\nabla}\Phi_D$:

$$\int_V \mathrm{d}^3\vec{x} \left[|\boldsymbol{\nabla}\Phi_D|^2 + \Phi_D(\nabla^2\Phi_D)\right] = \int_S \mathrm{d}\vec{S} \cdot (\Phi_D \boldsymbol{\nabla}\Phi_D)$$

Since $\nabla^2\Phi_D = 0$ in V:

$$\int_V \mathrm{d}^3\vec{x}\, |\boldsymbol{\nabla}\Phi_D|^2 = \int_S \mathrm{d}\vec{S} \cdot (\Phi_D \boldsymbol{\nabla}\Phi_D)$$

The term $\boldsymbol{\nabla}\Phi_D = \boldsymbol{\nabla}\Phi_1 - \boldsymbol{\nabla}\Phi_2 = -\left(\vec{E}_1 - \vec{E}_2\right)$. The integrand on the left is $\left|\vec{E}_1 - \vec{E}_2\right|^2 \geq 0$.

Case 1: Dirichlet Boundary Conditions If Φ is specified on the boundary S, then $\Phi_1 = \Phi_2$ on S, so $\Phi_D = 0$ on S. The surface integral becomes zero.

$$\int_V \mathrm{d}^3\vec{x} \left|\vec{E}_1 - \vec{E}_2\right|^2 = 0$$

Since the integrand is non-negative, it must be zero everywhere: $\left|\vec{E}_1 - \vec{E}_2\right|^2 = 0 \Rightarrow \vec{E}_1 = \vec{E}_2$ everywhere in V. If the fieldd(S) are the same, the potentials can differ only by a constant ($\Phi_1 = \Phi_2 + C$). Since $\Phi_1 = \Phi_2$ on the boundary, $C = 0$. **Conclusion**: The solution Φ to Poisson's equation is unique within V if Φ is specified on the boundary S.

Case 2: Neumann Boundary Conditions If the normal component of the electric field, $\vec{E} \cdot \hat{n} = -(\boldsymbol{\nabla}\Phi) \cdot \hat{n}$, is specified on the boundary S. Then $\boldsymbol{\nabla}\Phi_1 \cdot \hat{n} = \boldsymbol{\nabla}\Phi_2 \cdot \hat{n}$ on S, so $\boldsymbol{\nabla}\Phi_D \cdot \hat{n} = 0$ on S. The surface integral is $\int_S \Phi_D(\boldsymbol{\nabla}\Phi_D \cdot \hat{n})\,\mathrm{d}S = 0$. Again, $\int_V \left|\vec{E}_1 - \vec{E}_2\right|^2 \mathrm{d}S = 0$, implying $\vec{E}_1 = \vec{E}_2$ everywhere in V. **Conclusion**: The electric field $\vec{E}$ is unique within V if $\vec{E} \cdot \hat{n}$ is specified on the boundary S. The potential Φ is unique up to an additive constant.

These uniqueness theorems guarantee that if we find **a** solution satisfying the Poisson/Laplace equation and the given boundary conditions, it is **the** solution.

2.9 Green's Functions

Green's functions provide a general method for solving inhomogeneous differential equations like Poisson's equation with boundary conditions.

2.9.1 Green's Identities

Start with the divergence theorem for a vector field $\vec{F}$: $\int_V \boldsymbol{\nabla} \cdot \left(\vec{F}\right)\mathrm{d}V = \int_S \vec{F} \cdot \mathrm{d}\vec{S}$. Let $\vec{F} = \Psi\boldsymbol{\nabla}\Phi$. Then $\boldsymbol{\nabla} \cdot \left(\vec{F}\right) = \boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Phi + \Psi\nabla^2\Phi$.

$$\int_V (\boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Phi + \Psi\nabla^2\Phi)\,\mathrm{d}V$$

$$= \int_S (\Psi\boldsymbol{\nabla}\Phi) \cdot \mathrm{d}\vec{S}\,(\text{Green's First Identity})$$

Now swap Φ and Ψ:

$$\int_V (\boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Psi + \Phi\nabla^2\Psi)\,\mathrm{d}V = \int_S (\Phi\boldsymbol{\nabla}\Psi) \cdot \mathrm{d}\vec{S}$$

Subtracting the second from the first (and noting $\boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Psi = \boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Phi$):

$$\int_V (\Psi\nabla^2\Phi - \Phi\nabla^2\Psi)\,\mathrm{d}V$$

$$= \int_S (\Psi\boldsymbol{\nabla}\Phi - \Phi\boldsymbol{\nabla}\Psi) \cdot \mathrm{d}\vec{S}\,(\text{Green's Second Identity})$$

2.9.2 Solving Poisson's Equation using Green's Function

Let $\Phi(\vec{x})$ be the potential we want to find, satisfying $\nabla^2\Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\varepsilon_0}$. Let $\Psi(\vec{x})$ be a **Green's function** $G(\vec{x}, \vec{y})$ defined by:

$$\nabla^2_{\vec{x}} G(\vec{x}, \vec{y}) = -4\pi\delta^{(3)}(\vec{x} - \vec{y})$$

Here, $\vec{y}$ is a specific point where we want to evaluate the potential $\Phi(\vec{y})$, and the Laplacian is taken with respect to $\vec{x}$. The Green's function represents the potential at $\vec{x}$ due to a unit point charge at $\vec{y}$ (scaled by -4π).

Substitute $\Phi(\vec{x})$ and $\Psi(\vec{x}) = G(\vec{x}, \vec{y})$ into Green's Second Identity (integrating over $\vec{x}$ within volume V):

$$\int_V \left[G(\vec{x}, \vec{y})\left(-\frac{\rho(\vec{x})}{\varepsilon_0}\right) - \Phi(\vec{x})\left(-4\pi\delta^{(3)}(\vec{x} - \vec{y})\right)\right]\mathrm{d}^3\vec{x}$$

$$= \int_S [G(\vec{x}, \vec{y})\boldsymbol{\nabla}_{\vec{x}}\Phi(\vec{x}) - \Phi(\vec{x})\boldsymbol{\nabla}_{\vec{x}}G(\vec{x}, \vec{y})] \cdot \mathrm{d}\vec{S}_x$$

Evaluate the delta function integral: $\int_V \Phi(\vec{x})\big(4\pi\delta^{(3)}(\vec{x}-\vec{y})\big)\,\mathrm{d}^3\vec{x} = 4\pi\Phi(\vec{y})$ (assuming $\vec{y}$ is inside V).

$$-\frac{1}{\varepsilon_0}\int_V G(\vec{x},\vec{y})\rho(\vec{x})\,\mathrm{d}^3\vec{x} + 4\pi\Phi(\vec{y})$$
$$= \int_S [G(\vec{x},\vec{y})\boldsymbol{\nabla}_{\vec{x}}\Phi(\vec{x}) - \Phi(\vec{x})\boldsymbol{\nabla}_{\vec{x}}G(\vec{x},\vec{y})]\cdot\mathrm{d}\vec{S}_x$$

Rearranging to solve for $\Phi(\vec{y})$:

$$\Phi(\vec{y}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\vec{x},\vec{y})\rho(\vec{x})\,\mathrm{d}^3\vec{x}$$
$$+\frac{1}{4\pi}\int_S [\Phi(\vec{x})\boldsymbol{\nabla}_{\vec{x}}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\boldsymbol{\nabla}_{\vec{x}}\Phi(\vec{x})]\cdot\mathrm{d}\vec{S}_x$$

This formula gives the potential Φ at any point $\vec{y}$ inside V in terms of:

1. The charge density $\rho(\vec{x})$ inside the volume V.
2. The values of the potential $\Phi(\vec{x})$ and its normal derivative $\boldsymbol{\nabla}\Phi(\vec{x})\cdot\hat{n}$ on the boundary S.
3. The Green's function $G(\vec{x},\vec{y})$.

The simplest Green's function satisfying $\nabla^2 G = -4\pi\delta$ is $G_0(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|}$. However, this doesn't necessarily simplify the boundary integral.

The power of the method comes from choosing a specific $G(\vec{x},\vec{y})$ that satisfies certain boundary conditions itself, simplifying the surface integral. The general Green's function is:

$$G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} + F(\vec{x},\vec{y})$$

where $F(\vec{x},\vec{y})$ is any function satisfying Laplace's equation $\nabla^2_{\vec{x}}F(\vec{x},\vec{y}) = 0$ within V. We choose F to impose convenient conditions on G on the boundary S.

2.9.3 Dirichlet Green's Function (G_D)

Choose $F(\vec{x},\vec{y})$ such that $G_{D(\vec{x},\vec{y})} = 0$ for $\vec{x}$ on the boundary S. The surface integral term becomes:

$$\frac{1}{4\pi}\int_S \Big[\Phi(\vec{x})\boldsymbol{\nabla}_{\vec{x}}G_{D(\vec{x},\vec{y})} - 0\Big]\cdot\mathrm{d}\vec{S}_x$$
$$= \frac{1}{4\pi}\int_S \Phi(\vec{x})\big(\boldsymbol{\nabla}_{\vec{x}}G_{D(\vec{x},\vec{y})}\cdot\hat{n}\big)\,\mathrm{d}S_x$$

The solution is:

$$\Phi(\vec{y}) = \frac{1}{4\pi\varepsilon_0}\int_V G_{D(\vec{x},\vec{y})}\rho(\vec{x})\,\mathrm{d}^3\vec{x}$$
$$+\frac{1}{4\pi}\int_S \Phi(\vec{x})\left(\partial\frac{G_D}{\partial}n_x\right)\mathrm{d}S_x$$

This requires knowing the potential $\Phi(\vec{x})$ on the boundary (Dirichlet condition) and finding the specific Green's function G_D that vanishes on S.

2.9.4 Neumann Green's Function (G_N)

Choose $F(\vec{x},\vec{y})$ such that the normal derivative $\partial\frac{G_N}{\partial}n_x = \boldsymbol{\nabla}_{\vec{x}}G_{N(\vec{x},\vec{y})}\cdot\hat{n}$ is constant on the boundary S. Specifically, $\partial\frac{G_N}{\partial}n_x = -4\frac{\pi}{A}$, where A is the area of the surface S. (This choice ensures consistency, as $\int_S\left(\partial\frac{G_N}{\partial}n_x\right)\mathrm{d}S = \int_V \nabla^2 G_N\,\mathrm{d}V = -4\pi$). The surface integral term becomes:

$$\frac{1}{4\pi}\int_S\left[\Phi(\vec{x})\left(-4\frac{\pi}{A}\right) - G_{N(\vec{x},\vec{y})}\left(\partial\frac{\Phi}{\partial}n_x\right)\right]\mathrm{d}S_x$$
$$= -\frac{1}{A}\int_S \Phi(\vec{x})\,\mathrm{d}S_x - \frac{1}{4\pi}\int_S G_{N(\vec{x},\vec{y})}\left(\partial\frac{\Phi}{\partial}n_x\right)\mathrm{d}S_x$$

The first term is the average value of Φ on the boundary, $<\Phi>_S$. Since potential is defined up to a constant, this term can often be absorbed or set to zero.

$$\Phi(\vec{y}) = \frac{1}{4\pi\varepsilon_0}\int_V G_{N(\vec{x},\vec{y})}\rho(\vec{x})\,\mathrm{d}^3\vec{x}$$
$$-\frac{1}{4\pi}\int_S G_{N(\vec{x},\vec{y})}\left(\partial\frac{\Phi}{\partial}n_x\right)\mathrm{d}S_x + <\Phi>_S$$

This requires knowing the normal derivative of the potential $\partial\frac{\Phi}{\partial}n_x = -\vec{E}\cdot\hat{n}$ on the boundary (Neumann condition) and finding the specific Green's function G_N.

2.10 Method of Images

This is a powerful technique for finding the Green's function (usually G_D) for problems with simple boundary geometries (planes, spheres, cylinders) by introducing fictitious "image" charges outside the volume of interest. The potential within the volume is then calculated as the superposition of the real charges and these image charges. The image charges are chosen such that their combined potential satisfies the required boundary condition (e.g., $\Phi = 0$ or $\Phi = V_0$ on a conducting surface).

2.10.1 Example 1: Point Charge near a Grounded Conducting Plane

Consider a charge q at $\vec{y} = (d,0,0)$ with $d > 0$, and an infinite conducting plane at $x = 0$ held at potential $\Phi = 0$. We want the potential $\Phi(\vec{x})$ for $x > 0$. We need $G_{D(\vec{x},\vec{y})}$ such that $G_D = 0$ when $x_1 = 0$. Let $\vec{x} = (x_1,x_2,x_3)$ and $\vec{y} = (y_1,y_2,y_3)$. The mirror image position of $\vec{y}$ is $\vec{y}^* = (-y_1,y_2,y_3)$. Try the Green's function:

$$G_{D(\vec{x},\vec{y})} = \frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{|\vec{x}-\vec{y}^*|}$$

Check the boundary condition: If $\vec{x}$ is on the plane, $x_1 = 0$.

$$|\vec{x}-\vec{y}|^2 = (0-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2$$
$$= y_1^2 + (x_2-y_2)^2 + (x_3-y_3)^2$$
$$|\vec{x}-\vec{y}^*|^2 = (0-(-y_1))^2 + (x_2-y_2)^2 + (x_3-y_3)^2$$
$$= y_1^2 + (x_2-y_2)^2 + (x_3-y_3)^2$$

So $|\vec{x}-\vec{y}| = |\vec{x}-\vec{y}^*|$ when $x_1 = 0$. Therefore, $G_{D(\vec{x},\vec{y})} = 0$ for $\vec{x}$ on the plane. This G_D also satisfies $\nabla^2_{\vec{x}}G_D = -4\pi\delta(\vec{x}-\vec{y})$ for $x_1 > 0$ (since $\vec{y}^*$ is outside this region, $\nabla^2_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{y}^*|}\right) = 0$ for $x_1 > 0$).

The potential for $x > 0$ is given by the Dirichlet formula. If $\rho(\vec{x}') = q\delta(\vec{x}'-\vec{y})$ and $\Phi = 0$ on the boundary:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{x'>0} G_{D(\vec{x}',\vec{x})}q\delta(\vec{x}'-\vec{y})\,\mathrm{d}^3\vec{x}' + 0$$
$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}G_{D(\vec{y},\vec{x})}$$
$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{y}-\vec{x}|} - \frac{1}{|\vec{y}^*-\vec{x}|}\right]$$
$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{|\vec{x}-\vec{y}^*|}\right]$$

This is the potential due to the original charge q at $\vec{y}$ and an image charge $-q$ at $\vec{y}^*$.

2.10.2 Example 2: Point Charge near a Grounded Conducting Sphere

Consider a charge q at position $\vec{d}$ outside a grounded conducting sphere of radius R centered at the origin ($\left|\vec{d}\right| = d > R$). We want $\Phi(\vec{r})$ for $r > R$. We need $G_{D(\vec{x},\vec{r})}$ such that $G_D = 0$ when $|\vec{x}| = R$. The method of images suggests placing an image charge q' inside the sphere at position $\vec{d}'$ such that the potential from q and q' sums to zero on the sphere surface $r = R$. It can be shown that

this requires an image charge $q' = -q\frac{R}{d}$ placed at $\vec{d}' = \left(\frac{R^2}{d^2}\right)\vec{d}$. The potential outside the sphere is:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{r}-\vec{d}\right|} + \frac{q'}{\left|\vec{r}-\vec{d}'\right|}\right]$$

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left|\vec{r}-\vec{d}\right|} - \frac{\frac{R}{d}}{\left|\vec{r}-\left(\frac{R^2}{d^2}\right)\vec{d}\right|}\right]$$

This potential is zero for $|\vec{r}| = R$. Since $\Phi(\vec{r}) = \left(\frac{q}{4\pi\varepsilon_0}\right)G_{D(\vec{d},\vec{r})}$ for a single point charge q at $\vec{d}$ (and $\rho = 0$ elsewhere, $\Phi = 0$ on boundary), we can identify the Green's function:

$$G_{D(\vec{x},\vec{r})} = \frac{1}{|\vec{r}-\vec{x}|} - \frac{\frac{R}{|\vec{x}|}}{\left|\vec{r}-\left(\frac{R^2}{|\vec{x}|^2}\right)\vec{x}\right|}$$

This Green's function works for any charge distribution $\rho(\vec{x})$ outside the sphere ($|\vec{x}| > R$) when the sphere is grounded ($\Phi = 0$ on $r = R$).

2.11 General Method: Eigenfunction Expansion

A systematic way to construct Green's functions involves using a complete set of orthogonal eigenfunctions of the Laplace operator that satisfy the appropriate boundary conditions.

Consider the eigenvalue problem for the Laplacian in volume V:

$$\nabla^2\Psi_{n(\vec{x})} = -\lambda_n\Psi_{n(\vec{x})}$$

subject to boundary conditions on S. For Dirichlet conditions, we require $\Psi_{n(\vec{x})} = 0$ for $\vec{x}$ on S. The eigenfunctions Ψ_n form a complete orthonormal set:

- **Orthogonality**: $\int_V \Psi_m^*(\vec{x})\Psi_{n(\vec{x})}\,\mathrm{d}^3\vec{x} = \delta_{mn}$ (assuming normalization).
- **Completeness**: $\sum_n \Psi_n^*(\vec{y})\Psi_{n(\vec{x})} = \delta^{(3)}(\vec{x}-\vec{y})$.

We expand the Green's function $G(\vec{x},\vec{y})$ in terms of these eigenfunctions:

$$G(\vec{x},\vec{y}) = \sum_n c_{n(\vec{y})}\Psi_{n(\vec{x})}$$

Apply the defining equation $\nabla_{\vec{x}}^2 G(\vec{x},\vec{y}) = -4\pi\delta^{(3)}(\vec{x}-\vec{y})$:

$$\nabla_{\vec{x}}^2\sum_n c_{n(\vec{y})}\Psi_{n(\vec{x})} = \sum_n c_{n(\vec{y})}\nabla_{\vec{x}}^2\Psi_{n(\vec{x})}$$

$$= \sum_n c_{n(\vec{y})}\left(-\lambda_n\Psi_{n(\vec{x})}\right) = -4\pi\delta^{(3)}(\vec{x}-\vec{y})$$

Use the completeness relation to write the delta function:

$$\sum_n c_{n(\vec{y})}\left(-\lambda_n\Psi_{n(\vec{x})}\right) = -4\pi\sum_n \Psi_n^*(\vec{y})\Psi_{n(\vec{x})}$$

Comparing coefficients of $\Psi_{n(\vec{x})}$ on both sides:

$$-c_{n(\vec{y})}\lambda_n = -4\pi\Psi_n^*(\vec{y})$$

$$\Rightarrow c_{n(\vec{y})} = \left(4\frac{\pi}{\lambda_n}\right)\Psi_n^*(\vec{y})$$

Substituting back into the expansion for G:

$$G_{D(\vec{x},\vec{y})} = \sum_n\left(4\frac{\pi}{\lambda_n}\right)\Psi_n^*(\vec{y})\Psi_{n(\vec{x})}$$

This gives the Dirichlet Green's function once the eigenvalues λ_n and eigenfunctions Ψ_n satisfying $\Psi_n = 0$ on S are found.

2.11.1 Example: Infinite Space (No Boundaries)

The eigenfunctions are plane waves $\Psi_{\vec{k}(\vec{x})} = \left(\frac{1}{(2\pi)^{\frac{3}{2}}}\right)e^{i\vec{k}\cdot\vec{x}}$.

$$\nabla^2\Psi_{\vec{k}(\vec{x})} = -\left|\vec{k}\right|^2\Psi_{\vec{k}(\vec{x})} \Rightarrow \lambda_{\vec{k}} = \left|\vec{k}\right|^2$$

The sum becomes an integral over $\vec{k}$:

$$G(\vec{x},\vec{y}) = \int \mathrm{d}^3\vec{k}\left(4\frac{\pi}{\left|\vec{k}\right|^2}\right)\Psi_{\vec{k}}^*(\vec{y})\Psi_{\vec{k}(\vec{x})}$$

$$= \int \mathrm{d}^3\vec{k}\left(4\frac{\pi}{\left|\vec{k}\right|^2}\right)\left(\frac{1}{(2\pi)^3}\right)e^{-i\vec{k}\cdot\vec{y}}e^{i\vec{k}\cdot\vec{x}}$$

$$= 4\frac{\pi}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\,\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\left|\vec{k}\right|^2}$$

This integral evaluates to $\frac{1}{|\vec{x}-\vec{y}|}$, recovering the familiar free-space Green's function.

2.11.2 Next Steps: Electrostatics in a Box

The eigenfunction method can be applied to find the Green's function for a finite region, such as a rectangular box, by finding the appropriate eigenfunctions (e.g., sine functions satisfying zero boundary conditions on the walls).

$$\left(e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}}\right) \approx \sin(kx)$$

$$\Rightarrow \Psi_{lmn} = \sqrt{\frac{8}{abc}}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)$$

This satisfies the boundary conditions $\Psi = 0$ on the walls of the box.

$$\nabla^2\Psi_{lmn}(\vec{x}) = \lambda_{lmn}\Psi_{lmn}(\vec{x})$$

$$\lambda_{lmn} = -\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

The spectrum of eigenvalues is discrete $l, m, n \in \mathbb{Z}$. We can easily show, that the eigenfunctions are orthogonal, normalized, and complete.

$$G_{D(\vec{x},\vec{y})} = -4\pi\sum_{lmn=1}^{\infty}\left(\frac{1}{\lambda_{lmn}}\right)\Psi_{lmn}^*(\vec{x})\Psi_{lmn}(\vec{y})$$

This Green's function can be used to solve Poisson's equation for any charge distribution within the box, given the boundary conditions on the walls. Due to it's complexity, this is typically done numerically.

2.12 Electrostatics with Spherical Symmetry

For these problems, we can use spherical coordinates (r,θ,φ) and the Laplacian in spherical coordinates:

When we deal with dipole problems, we can estimate the potential at large distances from the dipole as a spherically symmetric problem.

We want to solve the Poisson equation:

$$\nabla^2\Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\varepsilon_0}$$

$$\Rightarrow G(\vec{r},\vec{r}')$$

$$\Rightarrow \nabla^2\Psi(\vec{r}) = \lambda\Psi(\vec{r})$$

2.12.1 Laplace in Spherical Coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} + \frac{\hat{A}}{r^2}$$

Where $\hat{A}$ is an operator that depends on the angular coordinates, which we will not need to solve because we are looking for spherically symmetric solutions.

$$\hat{A} = \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2}(\theta)\frac{\partial^2}{\partial\varphi^2}$$

$$\frac{\partial^2}{\partial\varphi^2}e_m(\varphi) = -m^2e_m(\varphi)$$

This is a harmonic oscillator equation.

$$e_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

We can verify, that this is an eigenfunction of the operator $\hat{A}$ with eigenvalue $-m^2$.

We still have to verify, that the boundary conditions are satisfied. The boundary conditions are:

$$\varphi \in [0, 2\pi]$$
$$e_{m(0)} = e_{m(2\pi)}$$
$$\Rightarrow m \in \mathbb{N}$$
$$1 = e^0 = e^{2\pi i m}$$

We can solve $\hat{A}$ now, by seperation:

$$\hat{A}(\theta,\varphi)Y(\theta,\varphi) = \rho Y(\theta,\varphi)$$
$$\Rightarrow Y(\theta,\varphi) = \sum_m c_m e_m(\varphi)$$
$$\hat{A}(\theta,\varphi)\left(c_{m(\theta)}e_{m(\varphi)}\right) = \rho\left(c_{m(\theta)}e_{m(\varphi)}\right)$$
$$\Rightarrow \left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2}(\theta)\frac{\partial^2}{\partial\varphi^2}\right]c_{m(\theta)}e_{m(\varphi)}$$
$$= \rho c_{m(\theta)}e_{m(\varphi)}$$

We see that e_m is an eigenfunction of $\frac{\partial^2}{\partial\varphi^2}$:

$$\Rightarrow \left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{-m^2}{\sin^2}(\theta)\right]c_{m(\theta)} = \rho c_{m(\theta)}$$
$$\rho = -l(l+1)$$
$$\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{-m^2}{\sin^2}(\theta)\right]P_l^m(\theta) = -l(l+1)P_l^m(\theta)$$

This is a difficult problem, but it is a one-dimensional problem.

$$m = 0$$
$$\Rightarrow \left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}\right]P_l^0(\theta) = -l(l+1)P_l^0(\theta)$$
$$P_l^0(\theta) = P_l(\theta) = P_l(\cos(\theta))$$

The $\cos(\theta)$ is a restriction, but if we can prove, that this solution is complete (with our conditions), we know that we dont need a more general solution.

$$P_l(\cos(\theta)) = \sum_n \beta_l^n(\cos(\theta))^n$$
$$P_l(\cos(\theta)) = \frac{1}{2^l}\frac{\mathrm{d}^l}{\mathrm{d}\cos(\theta)^l}(-\sin^2(\theta))^l$$
$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right]P_l(x) = 0$$

Now we need to solve it for arbitrary m.

$$\hat{A}_m(\theta,\varphi)Y_l^m(\theta,\varphi) = -l(l+1)Y(\theta,\varphi)$$
$$Y(\theta,\varphi) = \sum_m \Theta(\theta)\Phi(\varphi)$$

This factorized assumption relies on the fact, that the factorized solution is complete. We can show that this is true, by showing that the eigenfunctions are orthogonal and normalized.

$$x = \cos(\theta)$$
$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right]P_{l(x)} = 0$$
$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l$$
$$l = 0 : P_0(x) = 1$$
$$l = 1 : P_1(x) = x$$
$$\Rightarrow \left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + 2\right]x = 0$$
$$P_2(x) = \frac{1}{2 2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 = \frac{3}{2}x - \frac{1}{2}$$

We can see that the P_l are orthogonal and normalized.

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

Recap:

We wanted to solve the angular part of $\hat{A}(\theta,\varphi)Y(\theta,\varphi) = -(l+1)lY(\theta,\varphi)$.

$$Y_{lm} = e^{im\varphi}P_l^m(\cos(\theta))\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}$$

This is a complete solution on the space of periodic functions on the sphere with respect to θ. Completeness means:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta,\varphi)Y_{lm}(\theta',\varphi') = \delta(\varphi-\varphi')\delta(\cos)(\theta-\cos(\theta'))$$

Orthonormality means:

$$\int_{-1}^{1}\mathrm{d}\cos(\theta)\int_0^{2\pi}\mathrm{d}\varphi\, Y_{lm}^*(\theta,\varphi)Y_{l'm'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

These functions are called **spherical harmonics**.

$$\Psi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_{lm}\frac{r}{r}Y_{lm}(\theta,\varphi)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}R_{lm}\frac{r}{r} - \frac{l(l+1)}{r^3} = \lambda R_{lm}\frac{r}{r}$$

This can be solved with boundary conditions. For example:

- We have a conductor R_1 inside a circular building R_2 that is conductive:

  $$\Psi(r = R_1,\theta,\varphi) = \Psi(r = R_2,\theta,\varphi) = 0$$
  $$\Rightarrow \frac{R_{lm}}{r}\mid_{r=R_1=R_2} = 0$$

  In the case that $R_2 \to \infty$, we can model the conductor as a point.

  For $r \to 0$, we can solve the radial differential equation generally.

  $$R''(r) - \frac{l(l+1)}{r^2}R(r) = \lambda R(r)$$

  Take $R(r) \approx r^\alpha$ because, as $r \to 0$, all the higher powers vanish faster.

  $$\Rightarrow \alpha(\alpha-1)r^{\alpha-2} - \frac{l(l+1)}{r^2}r^{\alpha-2} = \lambda r^\alpha \approx 0$$
  $$\Rightarrow \alpha(\alpha-1) = l(l+1)$$
  $$\Rightarrow \alpha = -l \text{ or } \alpha = l+1$$

  The generic solution is:

  $$R_{lm}(r))\sum_{n=0}^{\infty}c_n r^{l+n} + \sum_{n=0}^{\infty}d_n r^{l+1+n}$$

This is a power series. To inspect it, we put it in the original equation. We have everything we need to calculate the coefficients.

2.13 Other uses of spehrical harmonics

If we have a local charge distribution and we want to calculate the potential at points that are very far away (imagine calculating the potential effects of a distant star on earth), we dont need to use Colombs law.

Sometimes we also want to calculate the charge distribution from a known potential. For this inverse problem, we can use spherical harmonics to calculate the charge distribution from the potential.

When the distance is much larger than the size of the charge distribution, we can expand:

$$\frac{1}{|\vec{r}-\vec{x}|} = \sum_n c_n P_{n(\theta)}$$

Where P_n is a legendre polynomial.

$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{(\vec{r}^2+\vec{x}^2-2|\vec{r}||\vec{x}|\cos(\theta))^{\frac{1}{2}}}$$
$$|\vec{x}| = \delta|\vec{r}|$$
$$\Rightarrow \frac{1}{\vec{x}-\vec{r}} = \frac{1}{|\vec{r}|}\frac{1}{\sqrt{1+\delta^2-2\delta\cos(\theta)}}$$
$$f(\delta,\theta) = \frac{1}{\sqrt{1+\delta^2-2\delta\cos(\theta)}}$$

f can be expanded

$$f(\delta,\theta) = \sum_n c_{n(\delta)} P_{n(\cos(\theta))}$$

- $f(-\delta,\theta) = \sum_n c_n(-\delta)P_{n(\cos\theta)} = \sum_n c_n(\delta)P_n(-\cos(\theta))$
$$= \sum_n c_n(\delta)(-1)^n P_n(\cos(\theta))$$
$$\Rightarrow c_n(-\delta) = (-1)^n c_n(\delta)$$
- $$f\left(\frac{1}{\delta},\theta\right) = \delta f(\delta,\theta)$$

For $\theta = 0 \Rightarrow P_n(1) = 1$:

$$\frac{1}{\sqrt{1+\delta^2-2\delta}} = \frac{1}{\sqrt{(1-\delta)^2}} = \frac{1}{1-\delta}$$
$$= \sum_n \delta^n$$

We can conclude:

$$\frac{1}{\sqrt{1+\delta^2-2\delta\cos(\theta)}} = \sum_n \delta^n P_n(\cos(\theta))$$
$$\Rightarrow \frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{|\vec{r}|}\sum_n \frac{|\vec{x}|^n}{(|\vec{r}|)^n} P_{n(\cos(\theta))}$$

Back to the original problem:

$$\Phi(\vec{r}) = \int \mathrm{d}^3\vec{x}\,\frac{\rho(\vec{x})}{|\vec{r}-\vec{x}|}\frac{1}{4\pi\varepsilon_0}$$
$$= \int \mathrm{d}^3\vec{x}\,\frac{\rho(\vec{x})}{4\pi\varepsilon_0}\sum_n P_n\left(\cos(\theta)\frac{|\vec{x}|^n}{(|\vec{r}|)^{n+1}}\right)$$

When we define $\theta = \theta(\theta_x,\varphi_x,\theta_0,\varphi_0)$, where x is the position of the charge distribution, we can calculate the potential at $\vec{r}$.

$$P_{n(\cos(\theta(\theta_x,\varphi_x,\theta_0,\varphi_0)))} = \sum_{l,m} Y_{lm}(\theta_0,\varphi_0)c_{lm}(\theta_x,\varphi_x)$$

The first factor corresponds to angular dependence in the observer, the second factor to the angular dependence of the charge distribution.

$$P_l(\cos(\theta(\theta_x,\varphi_x,\theta_0,\varphi_0)))$$
$$= \frac{4\pi}{1+2l}\sum_{m=-l}^{l} Y^*_{lm}(\theta_x,\varphi_x)Y_{lm}(\theta_0,\varphi_0)$$

Then

$$\Psi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{1+2l}\sum_{m=-l}^{l}\int \mathrm{d}^3\vec{x}\,\rho(\vec{x})\frac{|\vec{x}|^l}{(|\vec{r}|)^{l+1}}Y^*_{lm}(\theta_x,\varphi_x)Y_{lm}(\theta_0,\varphi_0)$$
$$= \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{1+2l}$$
$$\frac{Y_{lm}(\theta_0\varphi_0)}{|\vec{r}|^{l+1}}q_{lm}$$
$$q_{lm} = \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})|\vec{x}|^l Y_{lm}(\theta_x,\varphi_x)$$

This q_{lm} "knows" everything about the charge distribution. It is characteristic of the charge distribution.

q_{lm} is called the **multipole moment** of the charge distribution. It is a generalization of the dipole moment. The dipole moment is q_{10}, the quadrupole moment is q_{20}, and so on.

2.14 Dipole Moment

Suppose you have opposing charges with distance d between them. the charges are close to $\vec{r}$. The potential of the charges cancels with large $\vec{r}$. We can see, that the dipole moment is directionally dependent on the distance between the charges.

Compared to the Colomb dropoff of $\frac{1}{|\vec{r}|}$, the dipole moment behaves with $\frac{1}{|\vec{r}|^2}$, the quadrupole moment with $\frac{1}{|\vec{r}|^3}$, and so on.

Mutiple moments give us a deeper insight over the geometry of the charge distribution of distant systems.

3 Magnetostatics

Our electrostatic and magnetostatic equations are decoupled. For our case:

$$\boldsymbol{\nabla} \cdot \left(\vec{B}\right) = 0$$

$$\boldsymbol{\nabla} \times \left(\vec{B}\right) = \frac{\vec{J}}{\varepsilon_0 c^2}$$

This chapter includes:

- What is current density?
- Integral form of magnetostatics
- Vector potential $\vec{A}$
- Solution of magnetostatics
- Duality of $\vec{E}$ and $\vec{B}$

3.1 Current Density

The current density $\vec{J}$ is a vector field, that is a measure of the flow of electric charge per unit area. It is defined as:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \vec{J} \cdot \mathrm{d}\vec{s}$$

If we have a volume with a current flow:

$$\begin{aligned}
\frac{Q_0(t) - Q(t)}{\Delta t} &= \int \mathrm{d}^3\vec{x}\, \frac{\partial \rho}{\partial t} \\
&= -\int_{S(V)} \vec{J} \cdot \mathrm{d}\vec{s} \\
\Rightarrow \int_{V(S)} \mathrm{d}V\, \frac{\partial \rho}{\partial t} &= -\int_{V(S)} \mathrm{d}V\, \Delta \cdot \vec{J} \\
\Rightarrow \int_{V(S)} \left[\frac{\partial \rho}{\partial t} + \Delta \cdot \vec{J}\right] &= 0 \\
\Rightarrow \frac{\partial \rho}{\partial t} + \Delta \cdot \vec{J} &= 0
\end{aligned}$$

The is the continuity equation. It is already implied by the Maxwell equations.

We can write the current density as:

$$\vec{J} = \rho \vec{v}$$

3.2 Integral Form

$$c^2 \boldsymbol{\nabla} \times \left(\vec{B}\right) = \frac{\vec{J}}{\varepsilon_0}$$

$$\Rightarrow c^2 \int_{S(\Gamma)} \mathrm{d}\vec{s} \cdot \left(\boldsymbol{\nabla} \times \left(\vec{B}\right)\right) = \int_{S(\Gamma)} \mathrm{d}\vec{s} \cdot \frac{\vec{J}}{\varepsilon_0}$$

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

This is how many charges escape per time

3.2.1 Amperes Law

$$\Rightarrow \oint_{\Gamma} \vec{B} \cdot \mathrm{d}\vec{l} = \frac{I_{\text{Through } \Gamma}}{c^2 \varepsilon_0}$$

3.3 Vector Potential

$$\boldsymbol{\nabla} \times \left(\vec{B}\right) = 0$$

$$\Rightarrow \vec{B} = \boldsymbol{\nabla} \times \left(\vec{A}\right)$$

You can compute:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \left(\vec{A}\right)\right) &= \frac{\partial}{\partial x_i}\left(\boldsymbol{\nabla} \times \left(\vec{A}\right)\right)_i = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk} \frac{\partial A_k}{\partial x_j}\right) \\
&= \varepsilon_{ijk} \frac{\partial^2}{\partial x_i \partial x_j} A_k
\end{aligned}$$

This is a product of a symmetric tensor and an antisymmetric tensor, which is always zero.

We can construct the magnetic field completely through the vector potential.

$$\boldsymbol{\nabla} \times \left(\vec{B}\right) = \frac{\vec{J}}{\varepsilon_0 c^2} \Rightarrow \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \left(\vec{A}\right)\right) = \frac{\vec{J}}{\varepsilon_0 c^2}$$

$$\Rightarrow -\Delta^2 \vec{A} + \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \left(\vec{A}\right)\right) = \frac{\vec{J}}{\varepsilon_0 c^2}$$

We can also use:

$$\varepsilon_{ijk}\varepsilon_{klm} = \left|\begin{pmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{pmatrix}\right| = \delta_{il}\delta_{jm} - \delta_{im}\delta(jl)$$

The above solution if convenient, because it contains the laplacian. We can reuse much of electrostatics.

3.3.1 Symmetry

Many properties have to stay the same through different configurations.

$$\vec{A}(\vec{x}) \to \vec{A}'(\vec{x}') = \vec{A}(\vec{x}) + \boldsymbol{\nabla} \cdot (f(\vec{x}))$$

$$\vec{F} = q\vec{v} \times \vec{B} \text{ stays invariant}$$

$$\begin{aligned}
\vec{B}' &= \boldsymbol{\nabla} \times \left(\vec{A} + \Delta f\right) \\
&= \boldsymbol{\nabla} \times \left(\vec{A}\right) + \boldsymbol{\nabla} \times (\Delta f) = \vec{B}
\end{aligned}$$

We call this **Gauge Symmetry**. We can do this for any point in the full standard model.

$$\nabla^2\left(\vec{A}\right) - \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \vec{A}\right) = \frac{\vec{J}}{\varepsilon_0 c^2}$$

$$\nabla^2\left(\vec{A}'\right) - \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \vec{A}'\right) = \frac{\vec{J}}{\varepsilon_0 c^2}$$

$$\Rightarrow \vec{A}' = \vec{A} + \boldsymbol{\nabla} \cdot f$$

We need to construct the function f such that:

$$\vec{A}' = \vec{A} + \boldsymbol{\nabla} \cdot f$$

$$\boldsymbol{\nabla} \cdot \vec{A}' = 0 \Rightarrow \boldsymbol{\nabla} \cdot \vec{A} + \nabla^2 f = 0$$

$$\Rightarrow \nabla^2 f = -\boldsymbol{\nabla} \cdot \vec{A} = \text{Source}$$

We know how to solve the poisson equation. The fact that the above can happen means:

$$\nabla^2 \vec{A}' = \frac{\vec{J}}{\varepsilon_0 c^2}$$

Because of this symmetriy, for obtaining the magnetic field, we dont need to solve the regional equation $\nabla^2 \vec{A} - \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \vec{A}\right) = -\frac{\vec{J}}{\varepsilon_0 c^2}$, we can use the above equation instead.

We know the solution:

$$\vec{A}(\vec{x}) = \frac{1}{4\pi\varepsilon_0 c^2} \int \mathrm{d}^3\vec{y}\, \frac{\vec{J}(\vec{y})}{|\vec{x} - \vec{y}|}$$

3.4 Application: Magnetic Dipole

We have a current density in space $\vec{J}$ with currents at $\vec{x}$, and want to know $\vec{A}$ at $\vec{r}$.

We will use a 4-vector.

$$A(\vec{r}) = \int \mathrm{d}^3\vec{y}\, \frac{1}{4\pi\varepsilon_0 c^2}\vec{J}(\vec{y}) \cdot \left[\sum_{l=0}^{\infty} P_{l(\cos(\theta))}\frac{|\vec{x}|^l}{|\vec{r}|^{l+1}}\right]$$
$$= \int \mathrm{d}^3\vec{y}\, \frac{1}{4\pi\varepsilon_0 c^2}\vec{J}(\vec{y})\left[\frac{1}{|\vec{r}|} + \frac{\cos(\theta)|\vec{x}|}{|\vec{r}|^2} + ...\right]$$
$$= \frac{1}{4\pi\varepsilon_0 c^2}\left[\frac{1}{|\vec{r}|}\int \mathrm{d}^3\vec{x}\, \vec{J}(\vec{x}) + \frac{1}{|\vec{r}|^3}\int \mathrm{d}^3\vec{x}\, \vec{J}(\vec{x})(\vec{x}\cdot\vec{r}) + ...\right]$$

Using integration by parts, we can get:

$$A(\vec{r}) = \frac{1}{4\pi\varepsilon_0 c^2}\frac{\int \mathrm{d}^3\vec{x}\big(\vec{x}\times\vec{J}(\vec{x})\big)}{2}) \times \frac{\vec{r}}{r^3}$$

We discover the magnetic moment:

$$M = \frac{1}{2}\int \mathrm{d}^3\vec{x}\big(\vec{x}\times\vec{J}(\vec{x})\big)$$
$$\Rightarrow \vec{A}(\vec{r}) = \vec{\mu}\times\frac{\vec{r}}{r^3} \Rightarrow \vec{B} = (\vec{m}\cdot\Delta)\Big(\Delta\frac{1}{r}\Big)$$

3.5 Relativity of $\vec{E}$ and $\vec{B}$

We can use the duality of $\vec{E}$ and $\vec{B}$ to show that they are the same. We can use the Lorentz transformation to show that the electric field is a function of the magnetic field and vice versa.

$$F = qvB = \frac{1}{2\pi\varepsilon_0}\frac{qv}{c^2}\frac{I}{r}$$
$$= \frac{qS}{2\pi\varepsilon_0}\frac{\rho_-}{r}\frac{v^2}{c^2}$$

Lets change the reference frame. If we are in a reference frame, where the charge velocity is zero, the force vanishes.

This gets solved, because the charge density is dependent on the reference frame.

$$Nq = Q = \rho_{\text{rest}}\Delta V_{\text{rest}}$$

In the new frame:

$$Q = \rho_{\text{moving}}\Delta V_{\text{moving}}$$
$$\Rightarrow \rho_{\text{moving}} = \rho_{\text{rest}}\Delta\frac{V_{\text{rest}}}{\Delta}V_{\text{moving}}$$

But we are moving in one direction. The dimension in which we are moving is contracted.

$$\Rightarrow \rho_{\text{moving}} = \frac{\rho_{\text{rest}}}{\sqrt{1-\frac{v^2}{c^2}}}$$

This affects the electrons and positrons:

$$\rho' = \rho'_+ + \rho'_- = \frac{\rho_+}{\sqrt{1-\frac{v^2}{c^2}}} + \rho_-\sqrt{1-\frac{v^2}{c^2}}$$
$$\rho' = \frac{\rho_-}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^2}{c^2}$$
$$\Rightarrow F' = \frac{F}{\sqrt{1-\frac{v^2}{c^2}}}$$

Here F' is an electric force and F is a magnetic force. Thus we can see that the electric and magnetic forces are equivalent with a change in the reference frame.

3.6 Full Potentials

$$\boldsymbol{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\boldsymbol{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\boldsymbol{\nabla}\cdot\vec{B} = 0$$
$$\boldsymbol{\nabla}\times\vec{B} = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial\vec{E}}{\partial t}$$

We can derive:

$$\boldsymbol{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \boldsymbol{\nabla}\times\vec{A}$$
$$\boldsymbol{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}\big(\boldsymbol{\nabla}\times\vec{A}\big)$$
$$\Rightarrow \boldsymbol{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$
$$\Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\boldsymbol{\nabla}(\Phi)$$
$$\Rightarrow \vec{E} = -\boldsymbol{\nabla}(\Phi) - \frac{\partial\vec{A}}{\partial t}, \vec{B} = \boldsymbol{\nabla}\times\vec{A}$$

Maxwells equations imply charge conservation.

$$\varepsilon_0 c^2\boldsymbol{\nabla}\times\Big(\vec{B}\Big) = \vec{J} + \frac{\partial\vec{E}}{\partial t}\varepsilon_0$$
$$\Rightarrow \varepsilon_0 c^2\boldsymbol{\nabla}\cdot\Big(\boldsymbol{\nabla}\times\vec{B}\Big) = \boldsymbol{\nabla}\cdot\vec{J} + \frac{\partial\boldsymbol{\nabla}\cdot\vec{E}}{\partial t}\varepsilon_0$$
$$\Rightarrow 0 = \boldsymbol{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}\varepsilon_0 = 0$$

Thus, current is conserved. We integral over the entire universe:

$$\Rightarrow 0 = \int \mathrm{d}^3\vec{x}\,\boldsymbol{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\int \mathrm{d}^3\vec{x}\,\rho(\vec{x})$$
$$\Rightarrow \frac{\partial}{\partial t}Q_{\text{universe}} = 0$$

Thus, the charge in the universe is conserved.

$$\int_{\text{universe}} \mathrm{d}^3\vec{x}\,\boldsymbol{\nabla}\cdot\vec{J} = \int_{\text{edge}} \mathrm{d}\vec{s}\,\vec{J} = 0$$

We see, that no charge is leaving the unverse.

$$c^2\boldsymbol{\nabla}\times\vec{B} = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial\vec{E}}{\partial t}$$
$$\Rightarrow c^2\Big(\boldsymbol{\nabla}\times\Big(\boldsymbol{\nabla}\times\vec{A}\Big)\Big) = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right)$$
$$\Rightarrow c^{w(\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{A})-\nabla^2\vec{A})} = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right)$$
$$\Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A} = \frac{\vec{J}}{\varepsilon_0 c^2} - \boldsymbol{\nabla}\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \boldsymbol{\nabla}\cdot\vec{A}\right)$$

Similarly

$$\boldsymbol{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\text{and } \vec{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$
$$\Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = \frac{\rho}{\varepsilon_0} + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \boldsymbol{\nabla}\vec{A}\right)$$

We want the second terms for both cases to vanish.

3.7 D'Alembert Operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$
$$\Rightarrow \Box \vec{A} = \frac{\vec{J}}{\varepsilon_0 c^2}$$
$$\Rightarrow \Box \Phi = \frac{\rho}{\varepsilon_0}$$

3.8 Gague Symmatry

Change:

$$\Phi \to \Phi' = \Phi + \frac{\partial}{\partial t} f(\vec{x}, t)$$
$$\vec{A} \to \vec{A}' = \vec{A} - \boldsymbol{\nabla} f(\vec{x}, t)$$

With a substitution:

$$\Box \Phi' = \frac{\rho}{\varepsilon_0} + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \Phi'}{\partial t} + \boldsymbol{\nabla}\cdot\vec{A}'\right)$$
$$\Box \vec{A}' = \frac{\vec{J}}{\varepsilon_0 c^2} - \boldsymbol{\nabla}\left(\frac{1}{c^2}\frac{\partial \Phi'}{\partial t} + \boldsymbol{\nabla}\cdot\vec{A}'\right)$$

We can choose $f(\vec{x}, t)$ such that:

$$\frac{1}{c^2}\frac{\partial \Phi'}{\partial t} + \boldsymbol{\nabla}\cdot\vec{A}' = 0$$

Then we can work with the simpler equations:

$$\Box \Phi = \frac{\rho}{\varepsilon_0}$$
$$\Box \vec{A} = \frac{\vec{J}}{\varepsilon_0 c^2}$$
$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

We see, that the electromagnetic wave is a solution of electrical and magnetic fields, without any charge density.

$$\Box \Phi = \frac{\rho}{\varepsilon_0}$$
$$\Box \vec{A} = \frac{\vec{J}}{\varepsilon_0 c^2}$$

Given a solution of the wave equation $\Box \Phi_{\text{sol}} = \frac{\rho}{\varepsilon_0}$, then also $\Phi = \Phi_{\text{free}} + \Phi_{\text{Sol}}$, where $\Box \Phi_{\text{free}} = 0$ is a solution for empty space. Then:

$$\Box \Phi' = \Box \Phi_{\text{free}} + \Box \Phi_{\text{sol}}$$

We can see, that the solution is not unique. The uniqueness arises from boundary conditions.

If we want to solve the wave equation for the entire universe, we need to assume, that there are no waves outside the universe. We will use Green's function with boundary conditions that the fields vanish at infinity of space and time.

$$\Box G(\vec{x} - \vec{x}', t - t') = \delta^{(3)}(\vec{x} - \vec{x}')\delta(t - t')$$

If we find G, then we can find the solution:

$$\Phi(\vec{x}, t) = \int \mathrm{d}^3\vec{x}'\,\mathrm{d}t'\, G(\vec{x} - \vec{x}', t - t')\frac{\rho(\vec{x}', t')}{\varepsilon_0}$$
$$\Box \Phi(\vec{x}, t) = \int \mathrm{d}^3\vec{x}'\,\mathrm{d}t'\, G(\vec{x} - \vec{x}', t - t')\frac{\vec{J}(\vec{x}', t')}{\varepsilon_0}$$
$$= \int \mathrm{d}^3\vec{x}'\,\mathrm{d}t'\, \delta(\vec{x} - \vec{x}')\delta(t - t')\frac{\rho(\vec{x}', t')}{\varepsilon_0}$$
$$= \frac{\rho(\vec{x}, t)}{\varepsilon_0}$$

So this indeed works. The integration range in the full spacetime volume.

This means we also want to integrate all of time. The problem is, that we demant to know the charge density at all times to get the field at one time.

Let's try a different approach.

$$\Phi(\vec{x}, t) = \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3\vec{x}'\, G(\vec{x} - \vec{x}', t - t')\frac{\rho(\vec{x}', t')}{\varepsilon_0}$$
$$+ \int_{t}^{+\infty} \mathrm{d}t' \int \mathrm{d}^3\vec{x}'\, G(\vec{x} - \vec{x}', t - t')\frac{\rho(\vec{x}', t')}{\varepsilon_0}$$

The second term is unphysical, because it violates causality. We need the Green's function to vanish in the future. This can be done if

$$G(\vec{x} - \vec{x}', t - t') \propto \Theta(t - t')$$

This means that the Green's function cutts off for the future.

So we have to extend Maxwells theory, so that it incorporates causality explicitly. A theory which produces the causality respecting Green's function emergey by a limiting procedure.

We will modify the wave equation:

$$\frac{1}{C}\frac{\partial}{\partial t} = \lim_{\delta \to 0}\left[\frac{1}{c}\frac{\partial}{\partial t} + \delta\right]$$
$$\delta > 0$$
$$\Box \to \Box_\delta = \left(\frac{1}{c^2}\frac{\partial}{\partial t} + \delta\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \delta\right) - \nabla^2$$

At the end:

$$\lim_{\delta \to 0} \Box_\delta = \Box$$
$$\Box_\delta \vec{A}_{\delta(\vec{x},t)} = \frac{\vec{J}}{\varepsilon_0 c^2}$$
$$\to \vec{A}_{\delta(\vec{x},t)}$$
$$\to \lim_{\delta \to 0^+} \vec{A}_{\delta(\vec{x},t)} = \vec{A}(\vec{x}, t)$$

We introduce a 4-D fourier transform.

$$G(\vec{x} - \vec{x}', t - t') = \int \mathrm{d}E\,\mathrm{d}^3\vec{k}\,\frac{1}{(2\pi)^4}\tilde{G}\big(E, \vec{k}\big)e^{-iE(t-t') + i\vec{k}(\vec{x} - \vec{x}')}$$
$$\Delta t = t - t'$$
$$\Delta \vec{x} = \vec{x} - \vec{x}'$$
$$G(\delta\vec{x}, \Delta t) = \int \mathrm{d}E\,\mathrm{d}^3\vec{k}\,\frac{1}{(2\pi)^4}$$
$$\cdot\, e^{-i\left(c\Delta t E - \vec{k}\cdot\Delta\vec{x}\right)}\tilde{G}_{\delta\left(E,\vec{k}\right)}$$
$$\Box_\delta G(\delta\vec{x}, \Delta t) = \int \mathrm{d}E\,\mathrm{d}^3\vec{k}\,\frac{1}{(2\pi)^4}$$
$$\cdot\, e^{-i\left(c\Delta t E - \vec{k}\cdot\Delta\vec{x}\right)}\tilde{G}\big(E, \vec{k}\big)(-1)\Big((E + i\delta)^2 - \vec{k}^2\Big)$$
$$= \delta^{(3)}(\vec{x} - \vec{x}')\delta(t - t')$$
$$\text{with } 2\pi\delta(x) = \int \mathrm{d}\vec{k}\, e^{-ikx}$$
$$= \int \mathrm{d}E\,\mathrm{d}^3\vec{k}\,\frac{1}{(2\pi)^4}e^{-i\left(c\Delta t E - \vec{k}\cdot\Delta\vec{x}\right)}c$$

So, in Fourier space:

$$\tilde{G}\big(E, \vec{k}\big) = -\frac{c}{(E + i\delta)^2 - \vec{k}^2}$$
$$G(\Delta x, \Delta t) = \lim_{\delta \to 0}(-c)\int \mathrm{d}E\,\mathrm{d}^3\vec{k}\,\frac{1}{(2\pi)^4}\frac{e^{-i\left(c\Delta t E - \vec{k}\cdot\Delta\vec{x}\right)}}{(E + i\delta)^2 - \vec{k}^2}$$

This vanishes for $\Delta t < 0$. This implies that the Green's function is causal.

$$\mathrm{d}^3\vec{k} = \mathrm{d}k\, k^2\, \mathrm{d}\cos(\theta)\, \mathrm{d}\varphi$$

We can align $\Delta\vec{x} = \Delta x \hat{z}$. Then:

$$\vec{k}\cdot\Delta\vec{x} = k\Delta x\cos(\theta)$$

After integrating over angles, we get:

$$\begin{aligned} G(\Delta\vec{x},\Delta t) &= -c\int_0^\infty \mathrm{d}k\,\frac{|\vec{x}|}{|\vec{x}-\vec{x}'|}\left[e^{ik\Delta x}-e^{-ik\Delta x}\right] \\ &\quad\cdot\int_{-\infty}^{\infty}\mathrm{d}E\,\frac{e^{-iEc\Delta t}}{(E+i\delta)^2-k^2} \\ &= -c\int_0^\infty \mathrm{d}k\, e^{ik\Delta x}k\int_{-\infty}^{\infty}\mathrm{d}E\,\frac{e^{-iEc\Delta t}}{(E+i\delta)^2-k^2} \end{aligned}$$

The second integral demermines the causal behavior. We will use the residue theorem to compute this integral. We find poles at $(E+i\delta)^2 = k^2$. This gives us:

$$E = \pm k - i\delta$$

With $\delta\to 0$, we know that the poles approach the real axis from the bottom. This is important because limits are only unique for the same direction.

Assume $\Delta t < 0$ (violates causality). We close the contour in the upper half plane. We observe that the upper integrand vanishes at infinity and that the contour contains no poles. Thus, the integral vanishes.

The vanishing of the integrand means that the Green's function vanishes for $\Delta t < 0$. This is the causality condition.

For $\Delta t > 0$, we close the contour in the lower half plane. The integrand vanishes at infinity and the contour contains two poles.

$$\begin{aligned} &\int \mathrm{d}E\,\frac{1}{2\pi}k\frac{e^{-iEc(t-t')}}{(E+i\delta)^2-k^2} \\ &= -\Theta(t-t')i\pi\left[e^{-ikc\Delta t}-e^{ikc\Delta t}\right] \\ &G(\vec{x}-\vec{x}',t-t') \\ &= -\frac{ic\Theta(t-t')}{2(2\pi)^2|\vec{x}-\vec{x}'|}\int \mathrm{d}k\left[e^{-ik(c\Delta t-|\Delta\vec{x}|)}-e^{ik(c\Delta t+|\Delta\vec{x}|)}\right] \\ &= \frac{+Rc\Theta(t-t')}{4\pi|\vec{x}-\vec{x}'|}[\delta(c\Delta t-|\Delta\vec{x}|)-\delta(c\Delta t+|\Delta\vec{x}|)] \\ &\Rightarrow G(\vec{x}-\vec{x}',t-t') = \frac{1}{4\pi|\vec{x}-\vec{x}'|}\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right) \end{aligned}$$

Then for physical quantities:

$$\begin{aligned} \Phi(\vec{x},t) &= \int \mathrm{d}^3\vec{x}'\,\mathrm{d}t'\,G(\vec{x}-\vec{x}',t-t')\frac{\rho(\vec{x}',t')}{\varepsilon_0} \\ &= \frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3\vec{x}'\,\frac{\rho\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|} \end{aligned}$$

This is Colombs law in full, independent of the reference frame.

For $c\to\infty$, we get the static case and we get our normal Colombs law. We will also get the normal case, if the charge density is constant in time.

When the speed of light is finite, we have to take into account the time it takes for the field to propagate. This means that the field at a point in space is not only dependent on the charge density at that point, but also on the charge density at other points in space and past time.

$$\begin{aligned} t_{\text{retarded}} &= t - \frac{|\vec{x}-\vec{x}'|}{c} \\ G_{\text{retarded}} &= \frac{1}{4\pi|\vec{x}-\vec{x}'|}\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right) \end{aligned}$$

This is the time, that light needs to cover the distance between the observer and the charge density.

In analog, we can also use the **retarded Green's function** to compute the magnetic field. The derivation is of the same form. We get:

$$\vec{A}(\vec{x},t) = \frac{1}{4\pi\varepsilon_0 c^2}\int \mathrm{d}^3\vec{x}'\,\frac{\vec{J}\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}$$

We can write the potentials as a vector:

$$\begin{pmatrix}\Phi\\ \vec{A}\end{pmatrix} = \int \mathrm{d}^3\vec{x}'\,\frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{x}-\vec{x}'|}\begin{pmatrix}\frac{\rho\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{c}\\ \frac{\vec{J}\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{c}\end{pmatrix}$$

And use the identity:

$$\begin{aligned} &\frac{1}{|\vec{x}-\vec{x}'|}\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right) \\ &= \frac{2}{c}\delta\left((t-t')^2-\frac{|\vec{x}-\vec{x}'|^2}{c^2}\right)\Theta(t-t') \end{aligned}$$

This works on the principle of:

$$\begin{aligned} &2\delta(a^2-|b|^2)\Theta(a>0) \\ &= 2\delta(a-|b|)(a+|b|)\Theta(a>0) \\ &= 2\frac{\delta(a-|b|)}{2a}\Theta(a>0) \\ &= \frac{\delta(a-|b|)}{|b|} \end{aligned}$$

We also use:

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\delta\big((x-x_0)^2\big)f(x) = f(x_0)$$

We get:

$$\begin{aligned} &\Phi(\vec{x},t) \\ &= \int \mathrm{d}^3\vec{x}'\,\frac{1}{4\pi\varepsilon_0 c}\delta\left((t-t')^2-\frac{|\vec{x}-\vec{x}'|^2}{c^2}\right)\frac{\rho(\vec{x}',t')}{\varepsilon_0}\Theta(t-t') \end{aligned}$$

Our application is to calculate Φ for a charge q moving with a constant velocity $\vec{v}$. This will get us closer to understanding relativity.

Assume $t'=0, \vec{x}'=0$.

$$\begin{aligned} \vec{x}' &= \vec{v}t' \\ \rho(\vec{x}',t') &= q\delta^{(3)}(\vec{x}'-\vec{v}t') \\ &\Phi(\vec{x},t) \\ &= \frac{1}{2\pi\varepsilon_0 c}\int \mathrm{d}^3\vec{x}'\,\mathrm{d}t\,\delta\left((t-t')^2-\frac{|\vec{x}-\vec{x}'|^2}{c^2}\right)\Theta(t-t')\delta^{(3)}(\vec{x}'-\vec{v}t') \\ &= \frac{q}{2\pi\varepsilon_0 c}\int \mathrm{d}t'\,\Theta(t-t')\delta\left((t-t')^2-\frac{|\vec{x}-\vec{v}t'|^2}{c^2}\right) \end{aligned}$$

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$$

Lets look at the quantity:

$$(t-t')^2 - \frac{|\vec{x}-\vec{v}t'|^2}{c^2}, c=1$$

We decompose $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp, \vec{x}_\perp \cdot \vec{v} = 0$.

$$(t-t')^2 - \left|\vec{x}_\parallel - vt\right|^2 - |\vec{x}_\perp|^2$$
$$= t'^2(1-v^2) - 2t'\left(t - \vec{x}_\parallel v\right) + t^2 - \vec{x}_\parallel^2 - \vec{x}_\perp^2$$

We set

$$\tilde{t} = \frac{t - x_\parallel v}{\sqrt{1-v^2}}$$
$$\tilde{x_\parallel} = \frac{x_\parallel - vt}{\sqrt{1-v^2}}$$
$$\Rightarrow t = \frac{\tilde{t} + \tilde{x_\parallel} v}{\sqrt{1-v^2}}$$
$$\Rightarrow x_\parallel = \frac{\tilde{x_\parallel} + \tilde{t}v}{\sqrt{1-v^2}}$$
$$\Rightarrow t'^2(1-v^2) - 2t'\sqrt{1-v^2}\tilde{t} + \tilde{t}^2 - \tilde{x}^2$$
$$\tilde{x}^2 = \tilde{x_\parallel}^2 + \vec{x}_\perp^2$$

When we set:

$$t'\sqrt{1-v^2} = \rho$$
$$\Rightarrow \rho_\pm = \tilde{t} \pm |\tilde{x}|$$
$$\Rightarrow \Phi(x,t) = \frac{1}{4\pi\varepsilon_0} \frac{\frac{q}{\sqrt{1-\frac{v^2}{c^2}}}}{\sqrt{x_\perp^2 + \tilde{x_\parallel}^2}}$$
$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{\frac{\left(x_\parallel - vt\right)^2}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2} + x_\perp^2}}$$
$$\vec{A}(\vec{x},t) = \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{1-v^2}} \frac{\vec{v}}{c} \frac{q}{\sqrt{\frac{\left(x_\parallel - vt\right)^2}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2} + x_\perp^2}}$$
$$\Rightarrow \vec{A}(\vec{x},t) = \frac{\vec{v}}{c}\Phi(\vec{x},t)$$

For $v = 0$ we get the electrostatic case:

$$\begin{pmatrix} \Phi(\vec{x},t) \\ 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Phi\left(\tilde{\vec{x}},\tilde{t}\right) \\ \frac{\frac{\vec{v}}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\Phi\left(\tilde{\vec{x}},\tilde{t}\right) \end{pmatrix}$$

This is the Lorentz transformation. We can see that the electric and magnetic fields are equivalent in different reference frames. We also observe, that vector and scalar potentials transmorm similarly to time and space.

4 Special Relativity

We will once again work ourselfes up to the basics of special relativity. We remember Galilean transformations and Newtonian physics.

$$\vec{F} = m\ddot{\vec{r}}$$

Galinean transformations preserve time and space distances.

4.1 Relativistic distance

$$\Delta s^2 = c^2(t-t') - |\vec{x}-\vec{x}'|^2 \,\mathrm{d}^2 s = c^2\,\mathrm{d}^2 t - \mathrm{d}^2\vec{x}$$

$$= (c\,\mathrm{d}t \;\; \mathrm{d}x^1 \;\; \mathrm{d}x^2 \;\; \mathrm{d}x^3)\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

$$x^\mu = \begin{pmatrix} c\,\mathrm{d}t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

$$\mathrm{d}^2 s = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = c^2\,\mathrm{d}^2 t - \mathrm{d}^2\vec{x}$$

$\mathrm{d}^2 s$ is invariant under Lorentz transformations. This means its the same in all inertial frames. $g_{\mu\nu}$ is the metrix of the Minkowski space.

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + c^\mu$$

Λ^μ_ν is the Lorentz transformation matrix. c^μ is a constant vector. They are independent of x^μ.

In Einstein summation, we can write:

$$\mathrm{d}^2 s = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

This quantity is also called **Proper Time Squared** $\mathrm{d}^2 s = \mathrm{d}^2\tau$.

We can prove that the above equation:

$$\Rightarrow x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + \rho^\mu$$

which is a linear transformation.

$$\mathrm{d}^2 s = \mathrm{d}^2 s'$$
$$\Rightarrow g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = g_{\rho\sigma}\,\mathrm{d}x'^\rho\,\mathrm{d}x'^\sigma$$
$$= g_{\rho\sigma}\left(\frac{\partial x'^\rho}{\partial x^\mu}\,\mathrm{d}x^\mu\right)\left(\frac{\partial x'^\sigma}{\partial x^\nu}\,\mathrm{d}x^\nu\right)$$
$$= \left(g_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\right)\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$
$$\Rightarrow g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} = g_{\rho\sigma}$$

Lets differentiate:

$$\frac{\partial}{\partial x^\epsilon}\left[g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\right] = \frac{\partial}{\partial x^\epsilon}g_{\rho\sigma} = 0$$

After some comibinations and index changes, we get:

$$\Rightarrow 2\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\epsilon}g_{\mu\nu} = 0$$

This resolves to zero because g is just a constant matrix.

$$\mathrm{d}x^\mu \to \mathrm{d}x'^\nu = \frac{\partial x'^\nu}{\partial x^\mu}\,\mathrm{d}x^\mu$$

We expect that there is an inverse. Then:

$$\frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\rho} = \delta^{\nu\rho}$$
$$\Rightarrow 2\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\epsilon}\frac{\partial x^\epsilon}{\partial x'^\rho}g_{\mu\nu} = 0$$
$$\Rightarrow g_{\mu\nu}\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma} = 0$$

This also uses:

$$g_{\mu\nu}\,\mathrm{d}x'^\mu\,\mathrm{d}x'^\nu = g_{\mu\nu}\,\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$$
$$\Rightarrow g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\rho}\,\mathrm{d}x^\rho\frac{\partial x'^\nu}{\partial x^\sigma}\,\mathrm{d}x^\sigma = g_{\mu\nu}\,\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$$
$$\Rightarrow \left[g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} - g_{\rho\sigma}\right]\mathrm{d}x^\rho\,\mathrm{d}x^\sigma = 0$$
$$\Rightarrow g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} = g_{\rho\sigma}$$

We have proven, that if we go to a new frame, it is linear and constants Λ and ρ do not depend on coordinates.

4.2 Poincare Group

These transformations form a group called the **Poincare group** or **inhomogenious Lorentz group**.

A Group is a set with a product, which is associative, has an identity element and every element has an inverse. The product property is fulfilled by the composition of transformations.

$$x''^\mu = (\Lambda^\mu_{2\,\nu}x'^\nu + \rho^\mu_2)$$
$$= \left(\Lambda^\mu_{2\,\nu}\left(\Lambda^\nu_{1\,\rho}x^\rho + \rho^\nu_1\right) + \rho^\mu_2\right)$$
$$= \Lambda^\mu_{2\,\nu}\Lambda^\nu_{1\,\rho}x^\rho + \Lambda^\mu_{2\,\nu}\rho^\nu_1 + \rho^\mu_2$$
$$\Rightarrow \Lambda^\mu_{3\,\rho} = \Lambda^\mu_{2\,\nu}\Lambda^\nu_1$$
$$\Rightarrow \rho^\mu_3 = \Lambda^\mu_{2\,\nu}\rho^\nu_1 + \rho^\mu_2$$
$$\Rightarrow x''^\mu = \Lambda^\mu_{3\,\nu}x^\nu + \rho^\mu_3$$

Now the Lambdas satisfy:

$$g_{\mu\nu}\overline{\Lambda^\mu{}_\rho}\Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

If the above is satisfied by Λ_1 and Λ_2, then it is also satisfied by Λ_3.

$$g_{\mu\nu}\Lambda^\mu_{3\,\rho}\Lambda^\nu_{3\,\sigma} = g_{\rho\sigma}\left(\Lambda^\mu_{2\,\alpha}\Lambda^\nu_{1\,\rho}\right)\left(\Lambda^\mu_{2\,\beta}\Lambda^\beta_{1\,\sigma}\right)$$
$$= g_{\alpha\beta}\Lambda^\alpha_{1\,\rho}\Lambda^\beta_{1\,\sigma} = g_{\rho\sigma}$$

This proves the product rule.

The unity is:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu$$
$$\rho^\mu = 0$$
$$\Rightarrow g_{\mu\nu}\delta^\mu_\rho\delta^\nu_\sigma = g_{\rho\sigma}$$

The Poincare group is a group!

4.3 Subgroups

The Poincare group has a subgroup called the **Lorentz group**. This is the set of all transformations with $\rho^\mu = 0$ (this means no translation).

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Defining property of the Lambda matrix is:

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}$$
$$\det(\Lambda^T g\Lambda) = \det(g)$$
$$\Rightarrow \det(\Lambda) = \pm 1$$

The Lambdas $\det(\Lambda) = 1$ forms a subgroup called the **proper Lorentz group**.

$$g_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} = g_{00} = 1$$
$$\Rightarrow (\Lambda^{0}{}_{0})^2 - (\Lambda^{i}{}_{0})^2 = 1$$

We use the convention $\mu = 0, 1, 2, 3, i = 1, 2, 3$.

$$\Rightarrow \Lambda^{0}{}_{0} > 1 \text{ or } \Lambda^{0}{}_{0} < -1$$

The first case is called **time-like** and the second **space-like**. The first one preserves the time direction.

$$\mathrm{d}x'^{\mu} = \Lambda^{\mu}{}_{\nu}\,\mathrm{d}x^{\nu} = \Lambda^{\mu}{}_{0}\,\mathrm{d}x^{0}$$
$$\Rightarrow \mathrm{d}x'^{\mu} = \Lambda^{\mu}{}_{0} c\,\mathrm{d}t \Rightarrow \mathrm{d}t' = \Lambda^{0}{}_{0} c\,\mathrm{d}t$$

We do not want to reverse the time direction by chooseing other reference frames.

4.3.1 Subgroup of the proper Lorentz transformations

$$\det(\Lambda) = 1$$
$$\Lambda^{0}{}_{0} \geq 1$$
$$\Rightarrow (\Lambda^{0}{}_{0})^2 - (\Lambda^{i}{}_{0})^2 = g_{00} = 1$$
$$\Rightarrow (\Lambda^{0}{}_{0})^2 \geq 1$$

4.3.2 Subgroup of Rotations

$$\mathbb{R}^{T}_{3\times3} R_{3\times3} = 1_{3\times3}$$
$$\Lambda^{0}{}_{0} = \begin{pmatrix} 1 & & \\ 0 & 0 & R \end{pmatrix}$$

This is a subgroup.

We take two inertial frames O, O'.

O: A certain particle is at rest. O': The same particle is moving with velocity v for O'.

What is the connection between

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \rho^{\mu}$$
$$\vec{v} = v^{i} = \frac{\mathrm{d}x'^{i}}{\mathrm{d}t'} = c\frac{\mathrm{d}x'^{i}}{\mathrm{d}x'^{0}}$$
$$\mathrm{d}x^{\nu} = \{c\,\mathrm{d}t, 0, 0, 0\}$$
$$\Rightarrow \mathrm{d}x'^{\mu} = \Lambda^{\mu}{}_{0}\,\mathrm{d}x^{0}$$

For $\mu = 0$:

$$\mathrm{d}x'^{0} = \Lambda^{0}{}_{0}\,\mathrm{d}x^{0}$$

For $\mu = i$:

$$\mathrm{d}x'^{i} = \Lambda^{i}{}_{0}\,\mathrm{d}x^{0}$$
$$\Rightarrow \frac{v^{i}}{c} = \Lambda^{i}{}_{0}\frac{0}{\Lambda^{0}{}_{0}}$$
$$(\Lambda^{0}{}_{0})^2 - (\Lambda^{i}{}_{0})^2 = 1$$
$$\Rightarrow (\Lambda^{0}{}_{0})^2 \left[1 - \Lambda^{i}{}_{0}\frac{0}{\Lambda^{0}{}_{0}}\right] = 1$$
$$\Rightarrow (\Lambda^{0}{}_{0})^2 \left[1 - \frac{(v^{i})^2}{c^2}\right] = 1$$
$$\Rightarrow (\Lambda^{0}{}_{0})^2 = \frac{1}{1 - \frac{\vec{v}^2}{c^2}}$$
$$\Rightarrow \Lambda^{0}{}_{0} = \gamma = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}$$
$$\Lambda^{i}{}_{0} = \frac{v^{i}}{c}\Lambda^{0}{}_{0}$$
$$\Rightarrow \Lambda^{i}{}_{0} = \gamma\frac{v^{i}}{c}$$

In the simple case of O, O' maintaining parellel axes:

$$\Rightarrow \Lambda^{0}{}_{i} = \Lambda^{i}{}_{0} = \frac{v^{i}}{c}\gamma$$
$$\Lambda^{i}{}_{j} = \delta^{i}_{j} + \frac{v^{i}v^{j}}{\vec{v}^2}(\gamma - 1)$$
$$\Lambda(\vec{v})_{\vec{v}=0} = 1$$

4.4 Time Dilation

Consider a clock at rest in the frame O and moving with velocity $\vec{v}$ in the frame O'.

The proper-time intervall:

$$\mathrm{d}s = (c\,\mathrm{d}t \;\; 0 \;\; 0 \;\; 0)$$
$$\mathrm{d}s' = (c\,\mathrm{d}t' \;\; \mathrm{d}\vec{x})$$
$$\Rightarrow c^2\,\mathrm{d}t^2 - \mathrm{d}\vec{x}^2 = c^2\,\mathrm{d}t'^2 - \mathrm{d}\vec{x}'^2$$
$$\Rightarrow \mathrm{d}t' = \frac{\mathrm{d}t}{\sqrt{1 - \left|\frac{\mathrm{d}\vec{x}'}{\mathrm{d}t'}\right|^2}}$$
$$\Rightarrow \mathrm{d}t' = \gamma\,\mathrm{d}t$$

For an observer in the frame O', the clock is running slower by exactly:

$$\Delta t' = \gamma\Delta t = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}\Delta t$$

Since light has a frequency, it can function as a clock. The time dilation can be observed by looking at the frequency of light, which also corresponds to its color and energy.

For any moving object, observed from an observer O', the object has a radial component v_r off its velocity $\vec{v}$.

In frame O, where the light is at rest, the light has a frequency f and a wavelength λ.

In frame O', the light has a frequency f' and a wavelength λ'. ω_0 is the angular frequency of the light of the moving source.

$$c\Delta t_0 = c\Delta t' + v_r\Delta t'$$
$$\Rightarrow \Delta t_0 = \Delta t'\left(1 + \frac{v_r}{c}\right)$$
$$= \gamma\left(1 + \frac{v_r}{c}\right)\Delta t$$
$$\Rightarrow \omega_0 = \frac{2\pi}{\Delta t_0} = \omega\frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v_r}{c}}$$

The top of the fraction is the relativistic component, the bottom is the classical doppler component.

This allows us to determine how fast objects that emit light move towards or away from us.

4.5 Relativistic Force

We can also define a relativistic force. We have $\vec{F} = m\ddot{\vec{r}}$. We perform a Lorentz transformation to a different frame O', where $\vec{v} = 0$.

We introduce a more elegant way that compensates the fact that forces lead to acceleration.

$$f^{\mu} = mc^2\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2}$$
$$m = \text{const}$$

$\frac{\mathrm{d}}{\mathrm{d}\tau}$ signifies the derivative with respect to proper time $\mathrm{d}^2\tau = \mathrm{d}^2s$. This is the 4-vector for the force. Everything besides x^{μ} is invariant.

$$f'^{\mu} = mc^2 \frac{\mathrm{d}^2 \Lambda^{\mu}{}_{\nu} x^{\nu} + \rho^{\mu}}{\mathrm{d}\tau^2}$$
$$= \Lambda^{\mu}{}_{\nu} mc^2 \frac{\mathrm{d}^2 x^{\nu}}{\mathrm{d}\tau^2}$$
$$\Rightarrow f'^{\mu} = \Lambda^{\mu}{}_{\nu} f^{\nu}$$

At the rest frame $\mathrm{d}^2 x = 0 \Rightarrow \mathrm{d}^2 t = c^2\,\mathrm{d}^2 t$. So proper time becomes normal time.

$$f^{\mu}_{\text{rest}} = m \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}t^2}$$
$$\mu = 0 \Rightarrow f^0_{\text{rest}} = m \frac{\mathrm{d}^2 t}{\mathrm{d}t^2} = 0$$
$$f^i_{\text{rest}} = m \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}$$
$$\Rightarrow \vec{f}_{\text{rest}} = m \frac{\mathrm{d}^2 \vec{x}}{\mathrm{d}\tau^2} = \vec{F}_{\text{Newton}}$$
$$\Rightarrow f^{\mu}_{\text{rest}} = \begin{pmatrix} 0 \\ \vec{F}_{\text{Newton}} \end{pmatrix}$$

We transform to another frame O':

$$f'^{\mu} = \Lambda^{\mu}{}_{\nu} f^{\nu}_{\text{rest}}$$
$$f^0 = \Lambda^0{}_{\nu} f^{\nu}_{\text{rest}} = \Lambda^0{}_0 f^0_{\text{rest}} + \Lambda^0{}_i f^i_{\text{rest}}$$
$$= \Lambda^0{}_i f^i_{\text{rest}}$$
$$= v^i \frac{\gamma}{c} f^i_{\text{rest}}$$
$$= \gamma \frac{\vec{v} \cdot F_{\text{Newton}}}{c}$$
$$\Rightarrow f^0 = \gamma \frac{\vec{v} \cdot F_{\text{Newton}}}{c}$$
$$\Rightarrow f^i = \Lambda^i{}_{\mu} f^{\mu}_{\text{rest}}$$
$$= \Lambda^i{}_0 f^0 + \Lambda^i{}_j f^j_{\text{rest}}$$
$$= \Lambda^i{}_j f^j_{\text{rest}}$$
$$= \delta^i_j + \left[(\gamma - 1) \frac{v^i v^j}{v^2} \right] f^j_{\text{rest}}$$
$$= f^i_{\text{rest}} + (\gamma - 1) \frac{v^i}{v^2} \left(\vec{v} \cdot f^j_{\text{rest}} \right)$$
$$\Rightarrow \vec{f} = F_{\text{Newton}} + (\gamma - 1) \vec{v} \frac{\vec{v} \cdot F_{\text{Newton}}}{v^2}$$

When we write a vector with an arrow, we mean the spacial part $f^i = \vec{f}, f^{\mu} = \begin{pmatrix} f^0 & \vec{f} \end{pmatrix}^T$.

4.6 Relativistic Time

The Lorentz transformation can not always be inverted. This is because we can not invert $\tau(t)$ always. We cannot assume that we have four seperate equations. We take

$$\Omega = g_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}$$

This is identical in all reference frames.

$$f^{\mu} = m \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2}$$

If we want to solve this in 4 dimensions, we need to solve the time component. But in the rest frame, the time component is zero.

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = g_{\mu\nu} \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} + g_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}^2 x^{\nu}}{\mathrm{d}\tau^2}$$
$$= 2 g_{\mu\nu} \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}$$
$$= \frac{2}{mc^2} g_{\mu\nu} (mc^2\,\mathrm{d}^2 x^{\mu}\,\mathrm{d}^2 \tau) \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}$$
$$= \frac{2}{mc^2} g_{\mu\nu} f^{\mu} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}$$

Lets compute this in the rest frame:

$$f^{\mu}_{\text{rest}} = \begin{pmatrix} 0 & \vec{F}_{\text{Newton}} \end{pmatrix}$$
$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \frac{1}{c\gamma} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}t}$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$f^{\mu}_{\text{rest}} \frac{\mathrm{d}x^{\mu}_{\text{rest}}}{\mathrm{d}2} = 0$$
$$\Rightarrow g_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} = \text{const}$$

This is one more differential equation with

$$f^{\mu} = mc^2 \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2}$$

This allows us to claculate x^{μ} not only in relativistic time but also in normal time. This is not very practical, but it is possible.

The generalization of the Lorentz transformation is:

$$f^{\mu} = mc^2 \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2}$$

This transforms exactly like out coordinates.

$$f'^{\mu} = \Lambda^{\mu}{}_{\nu} f^{\nu}$$

4.7 Momentum

Classically, momentum is defined as:

$$p^i = m \frac{\mathrm{d}\vec{x}^i}{\mathrm{d}t}$$

In special relativity, it works the same way:

$$p^{\mu} = mc \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}$$

Where we derive after proper time τ.

$$p'^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}$$

Let's look at each component:

$$p^0 = mc \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = mc \frac{\mathrm{d}t}{\mathrm{d}\tau}$$
$$\text{recall } \mathrm{d}\tau = \sqrt{c^2\,\mathrm{d}^2 t} - \mathrm{d}^2 \vec{x}$$
$$= c\,\mathrm{d}t \sqrt{1 - \frac{\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}^2}{c^2}}$$
$$= c\,\mathrm{d}t \sqrt{1 - \frac{v^2}{c^2}}$$
$$\Rightarrow \frac{c\,\mathrm{d}t}{\gamma} = \mathrm{d}\tau$$
$$\Rightarrow p^0 = mc^2 \frac{\mathrm{d}t}{c \frac{\mathrm{d}t}{\gamma}}$$
$$\Rightarrow p^0 = mc\gamma$$

The other components are:

$$p^i = mc\frac{\mathrm{d}x^i}{\mathrm{d}\tau}$$
$$= mc\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\gamma}{c}$$
$$\Rightarrow p^i = m\gamma\frac{\mathrm{d}x^i}{\mathrm{d}t}$$
$$\Rightarrow \vec{p} = m\gamma\vec{v}$$

The momentum is a 4-vector.

$$p^0 = mc\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$
$$\approx mc\left[1+\frac{1}{2}\frac{v^2}{c^2}+o(v^4)\right] = mc+\frac{1}{2}cmv^2+...$$

We can see that the second term is the classical kinetic energy.

$$cp^0 = E_{\text{free particle}}$$

Lets solve this for v^2 and substitute it into $\vec{p}^2$. Then identify $E = cp^0$:

$$\Rightarrow E^2 = c^2\vec{p}^2 + m^2c^4$$

THis is the famous **energy-momentum relation**. This can be contrasted against the classical $E = \frac{\vec{p}^2}{2m}$ in $c \to \infty$.

All the qualtities we defined so far transform is the same way with ${\Lambda^\mu}_\nu$.

$$\mathrm{d}x'^\mu = {\Lambda^\mu}_\nu\,\mathrm{d}x^\nu$$
$$x'^\mu = {\Lambda^\mu}_\nu x^\nu + \rho^\mu$$
$$p^\mu = mc\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$
$$p'^\mu = {\Lambda^\mu}_\nu p^\nu$$
$$f'^\mu = {\Lambda^\mu}_\nu f^\nu$$

If an arbitraty object transforms with:

$$V^\mu \to V'^\mu = {\Lambda^\mu}_\nu V^\nu$$

(like a coordinate intervall), we call it a **contravariant 4-vector**.

4.8 Recap

As a recap, the most important property comes from the invariance of the spacetime interval:

$$g_{\mu\nu}{\Lambda^\mu}_\rho{\Lambda^\nu}_\sigma = g_{\rho\sigma}$$
$$\mathrm{d}^2s = \mathrm{d}^2s' = c^2\,\mathrm{d}^2t' - \mathrm{d}^2\vec{x}'$$

The Lorentz transformations are linear transformations of the coordinates of the spacetime and form a group. We call the Proper Lorentz transformations ($\det(\Lambda) = 1, {\Lambda^0}_0 \geq 1$) a "Physical" group:

4.9 Covariant 4-vectors

Some objects naturally want to transform differently. not with Λ, but with ${(\Lambda^{(-1)^\mu}}_\nu$. They want to transform in the reverse way. They are called **covariant 4-vectors**.

In relativity we have the following types of quantity:

- Covariant vectors
- contravariant vectors
- covariant tensors
- contravariant tensors
- and mixed tensors.

4.10 Vectors and Tensors

We have the metric:

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$$

We define the inverse with its defining property:

$$g^{\mu\nu} = \left(g^{-1}\right)_{\mu\nu}$$
$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$$

It happens, that the inverse of the metric is the metric itself:

$$g^{\mu\nu} = g_{\mu\nu}$$

Next we will look at the inverse Lorentz transformation. We expect:

$$\Lambda^{(-1)}(\vec{v}) = \Lambda(-\vec{v})$$

We define:

$$(-\Lambda)^\mu{}_\nu = \Lambda_{\mu\nu}$$

This is just a notation to denote the inverse. The inverse of an ovject can be noted by changing the upper to a lower index and vice versa.

$${\Lambda^\mu}_\rho{\Lambda_\mu}^\sigma = \delta^\sigma_\rho$$
$${\Lambda_\mu}^\nu = g_{\mu\rho}g^{\nu\sigma}{\Lambda^\rho}_\sigma$$
$${\Lambda_\mu}^a{\Lambda_\mu}^b = {\Lambda^\mu}_\sigma\left(g_{\mu\rho}g^{\beta\sigma}{\Lambda^\rho}_\sigma\right)$$
$$= g_{\beta\sigma}\left(g_{\mu\rho}{\Lambda^\mu}_\sigma{\Lambda^\rho}_\sigma\right)$$
$$= g^{\beta\sigma}g_{\alpha\sigma} = \delta^\beta_\alpha$$

From combining the result with the previous equations

$${\Lambda^0}_0(\vec{v}) = \gamma$$
$${\Lambda^i}_0 = {\Lambda^0}_i = \gamma\frac{v^i}{c}$$
$${\Lambda^i}_j = \delta^i_j + (\gamma-1)\frac{v^iv^j}{\vec{v}^2}$$

We obtain:

$${\Lambda_\mu}^\nu = {\left(\Lambda^{(-1)}\right)^\mu}_\nu$$
$$= g_{\mu\rho}g^{\nu\sigma}{\Lambda^\rho}_\sigma$$
$${\Lambda_0}^0 = g_{0\rho}g^{0\sigma}{\Lambda^\rho}_\sigma = g_{00}g^{00}{\Lambda^0}_0 = 1$$
$$\Rightarrow {\Lambda_0}^0 = {\Lambda^0}_0 = \gamma(\vec{v}) = \gamma(-\vec{v})$$

Lets look at a more complex component:

$${\Lambda_3}^0 = g_{3\rho}g^{0\sigma}{\Lambda^\rho}_\sigma$$
$$\Rightarrow {\Lambda_3}^0 = {\Lambda^3}_0$$
$$\Rightarrow {\Lambda_3}^0 = -{\Lambda^3}_0 = -\left(\gamma\frac{v^3}{c}\right)$$
$$\Rightarrow {\Lambda_3}^0(\vec{v}) = {\Lambda^0}_3(-\vec{v})$$
$${\Lambda_i}^0 = -{\Lambda^i}_0 = {\Lambda^i}_0(-\vec{v})$$
$${\Lambda_i}^j = {\Lambda^i}_j$$
$$= \delta^i{}_j + (\gamma(\vec{v})-1)\frac{v^iv^j}{v^2}$$
$$= \delta^i{}_j + (\gamma(-\vec{v})-1)\frac{-v^i+-v^j}{-v^2}$$
$$\Rightarrow {\Lambda_i}^j(\vec{v}) = {\Lambda^i}_j(-\vec{v})$$

We conclude:

$${\Lambda_\mu}^\nu = {\left(\Lambda^{(-1)}\right)^\mu}_\nu = g_{\mu\rho}g^{\nu\sigma}{\Lambda^\rho}_\sigma$$
$$= {\Lambda^\mu}_\nu(-\vec{v})$$

So the inverse Lorentz transformation is equivalent to boosting in the opposite velocity.

Let's see what objects transfrom with the inverse Lorentz transformation. We have the derivative:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial}{\partial x^\rho}$$
$$= \Lambda_{\mu\rho}\frac{\partial}{\partial x^\rho}$$

Let's justify this:

$$\delta^\mu_\nu = \frac{\partial x'^\mu}{\partial x'^\nu}$$
$$= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu}$$
$$\text{recall } (\mathrm{d}x'^\mu = \Lambda^\mu{}_\nu\,\mathrm{d}x^\rho)$$
$$= \Lambda_{\mu\rho}\frac{\partial x^\rho}{\partial x'^\nu}$$
$$\Rightarrow \Lambda_\mu{}^\alpha\Lambda^\mu{}_\rho\frac{\partial x^\rho}{\partial x'^\nu}$$
$$\Rightarrow \Lambda_\nu{}^\alpha = \delta^\alpha_\rho\frac{\partial x^\rho}{\partial x'^\nu}$$
$$\Rightarrow \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\nu} = \Lambda_\mu{}^\rho\frac{\partial}{\partial x^\rho}$$

The derivative likes to transform with the inverse Lorentz transformation.

This is the opposite of:

$$\mathrm{d}x^\mu \to \mathrm{d}x'^\mu = \Lambda^\mu{}_\nu\,\mathrm{d}x^\nu$$

Every object $\underline{U_\mu}$:

$$U_\mu \to U'_\mu = \Lambda_\mu{}^\nu U_\nu$$

is called **covariant vector**.

Covariant vectors have a contravariant dual. In special relativity the dual can be derived like this:

$$V_\mu = g_{\mu\nu}V^\nu$$

Proof:

$$V_\mu \to V'_\mu = g_{\mu\nu}V'^\nu$$
$$= g_{\mu\nu}\Lambda^\nu{}_\rho V^\rho$$
$$= g_{\mu\nu}\Lambda^\nu{}_\rho(g^{\rho\sigma}V_\sigma)$$
$$= \left(g^{\rho\sigma}g_{\mu\nu}\Lambda^\nu{}_\rho\right)V_\sigma$$
$$= \Lambda_{\mu\sigma}V_\sigma$$
$$\Rightarrow V_\mu \to V'_\mu = \Lambda_{\mu\sigma}V_\sigma$$

In summary: In special relativity we have coordinates that like to transform with the Lorentz transformation and objects like derivatives that like to transform with the inverse Lorentz transformation. For every covariant vector there is a dual contravariant vector and vice versa and the transformation is the inverse of the original transformation.

$$V_\mu = g_{\mu\nu}V^\nu$$
$$V^\mu = g^{\mu\nu}V_\nu$$

4.11 Tensors

Tensors hold more than one index

$$T^{\mu_1\mu_2} = T^{\mu_1\mu_2} = \Lambda^{\mu_1}{}_{\nu_1}\Lambda^{\mu_2}{}_{\nu_2}T^{\nu_1\nu_2}$$
$$T'_{\mu_1\mu_2} = \Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}T_{\nu_1\nu_2}$$
$$T'^{\mu_1\mu_2}{}_{\nu_1\nu_2} = T^{\alpha_1\alpha_2}{}_{\alpha_3\alpha_4}\Lambda^{\mu_1}{}_{\alpha_1}\Lambda^{\mu_2}{}_{\alpha_2}\Lambda_{\nu_1}{}^{\alpha_3}\Lambda_{\nu_2}{}^{\alpha_4}$$

This tells you how to transform a tensor with two contravariant indices and two covariant indices. The transformation is the same as for the coordinates.

Objects are invariant under Lorentz transformations. The action S is an example of an invariant object.

We have to define the **scalar product**:

$$A \cdot B = A_\mu B^\mu$$
$$= g_{\mu\nu}A^\mu B^\nu$$
$$= g^{\mu\nu}A_\mu B_\nu$$
$$A \cdot B \to A' \cdot B' = A'_\mu B'^\mu$$
$$= A_\mu\Lambda_\mu{}^\nu\Lambda^\mu{}_\rho B^\rho = A_\nu\delta^\nu_\rho B^\rho = A \cdot B$$

The Dalembert operator is invariant in every inertial frame:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$
$$\partial^\mu = g^{\mu\nu}\partial_\nu$$
$$= g^{\mu\nu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x_\mu}$$
$$\partial_\mu = \begin{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3}\end{pmatrix}$$
$$= \texttt{mat}\left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$
$$\partial^\mu = g^{\mu\nu}\partial_\nu = \begin{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} & -\vec{\nabla}\end{pmatrix}$$

For a contravariant differential we differentiate with respect to a covariant vector and vice versa. The derivative of a covariant vector is a contravariant vector and vice versa.

$$\partial^2 = \partial_\mu\partial^\mu$$
$$= \ldots = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \Box$$
$$\partial'^2 = \partial_\rho\Lambda_\mu{}^\rho\Lambda^\mu{}_\sigma\partial^\sigma$$
$$= \ldots = \partial^2$$

The Dalembert operator is invariant under Lorentz transformations. This means that the wave equation is invariant under Lorentz transformations.

4.12 Currents and Densities of Electric Charges

We will formulate a 4-vector, that combines the charge density and the current density and can be transformed with the Lorentz transformation.

$$\rho(\vec{x},t) = \sum_n q_n\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$
$$\vec{J}(\vec{x},t) = \sum_n q_n\frac{\mathrm{d}\vec{r}_n}{\mathrm{d}t}\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$
$$J^\mu(x^\mu) = \sum_n q_n\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\delta^{(3)}(x^\mu - \vec{r}_n(t))$$

Let's go through the components:

$$J^0(x^\mu) = \sum_n q_n\frac{\mathrm{d}(ct)}{\mathrm{d}t}\delta^{(3)}(x^\mu - \vec{r}_n(t))$$
$$= c\sum_n q_n\delta^{(3)}(x^\mu - \vec{r}_n(t))$$
$$\Rightarrow J^0 = c\rho$$
$$J^i(x^\mu) = \sum_n q_n\frac{\mathrm{d}x^i}{\mathrm{d}t}\delta^{(3)}(x^\mu - \vec{r}_n(t))$$
$$= \sum_n q_n\frac{\mathrm{d}r^i_n}{\mathrm{d}t}\delta^{(3)}(x^\mu 9 - \vec{r}_n(t))$$
$$\Rightarrow J^\mu = \begin{pmatrix}c\rho\\ \vec{J}\end{pmatrix}$$

We will show, that this transforms like a 4-vector:

$$J^\mu = \sum_n q_n \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\delta^{(3)}(\vec{x}^\mu - \vec{r}_n(t))$$
$$= \sum_n q_n \int \mathrm{d}t' \frac{\mathrm{d}x^\mu}{\mathrm{d}t'}\delta^{(3)}(\vec{x}^\mu - \vec{r}_n(t'))\delta(ct' - ct)c$$
$$x^\mu = (ct' \ \ \vec{x})$$
$$\Rightarrow J^\mu = c\sum_n q_n \int \mathrm{d}t' \frac{\mathrm{d}x^\mu}{\mathrm{d}t'}\delta^{(4)}(x^\mu - r_n^\mu(t'))$$
$$r^\mu(t') : r^i = r^i(t'), r^0 = ct'$$
$$\Rightarrow \delta^{(4)}(U^\mu) \to \delta^{(4)}(U'^\mu) = \delta^{(4)}(\Lambda^\mu{}_\nu U^\nu)$$
$$= \frac{\delta^{(4)}(U^\nu)}{\det(\Lambda) = 1} = \delta^{(4)}(U^\nu)$$
$$\text{also } c\,\mathrm{d}t = \mathrm{d}\tau\,\gamma$$
$$\Rightarrow c\frac{\mathrm{d}t'}{\mathrm{d}t'} = c\frac{\mathrm{d}\tau}{\mathrm{d}\tau} \Rightarrow J^\mu = c\sum_n q_n \int \mathrm{d}\tau \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\delta^{(4)}(x^\mu - r_n^\mu(\tau))$$
$$\Rightarrow J^\mu \to J'^\mu = \Lambda^\mu{}_\nu J^\nu$$

4.13 Energy-Momentum Tensor

Here we will formulate the analogue of the charge density and current density for energy and momentum. This is called the **Energy-Momentum Tensor**.

Assume particles with $P_n^\mu = m_n\gamma\frac{\mathrm{d}r_n^\mu}{\mathrm{d}t}$.

$$p_n^0 = m_n c\gamma$$
$$\vec{p}_n = m_n\gamma\vec{v}_n$$

Energy density:

$$\sum_n P_n^0\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$

Energy current density:

$$\sum_n P_n^0\frac{\mathrm{d}\vec{r}_n}{\mathrm{d}t}\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$

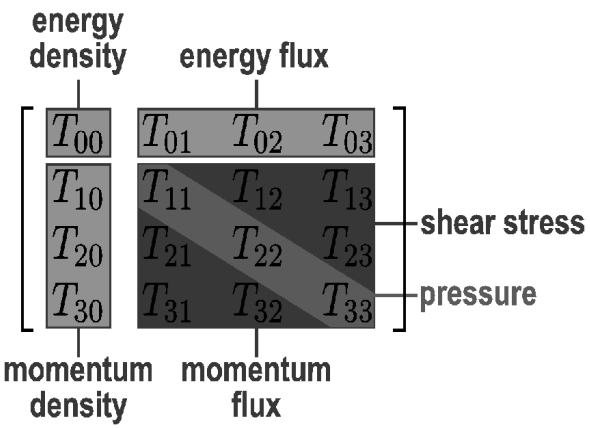
We can combine these two into a unified 4-vector:

$$\sum_n P_n^\mu\frac{\mathrm{d}r_n^\nu}{\mathrm{d}t}\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$

Now lets add the momentum for density and current density. For this we will generalize:

$$T^{\mu\nu} = \sum_n P_n^\mu\frac{\mathrm{d}r_n^\nu}{\mathrm{d}t}\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$

$\nu = \mu = 0$ gives us the energy density. $\mu = 1, \nu = 0$ we get the energy current density in the x^1 direction. $\mu = 1, \nu = 1$ gives us the momentum density in the x^1 direction. $\mu = 3, \nu = 2$ gives us the momentum current density in the x^3 direction (dont forget the factor c for the time component).



It can also be written as an integral over proper time:

$$T^{\mu\nu} = \sum_n \int \mathrm{d}\tau\, P_n^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta^{(4)}(x^\rho - r_n^\rho(\tau))$$

In this form, we see, that it is a contravariant tensor of rank 2.

$$T'^{\mu\nu} = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma T^{\rho\sigma}$$

The energy-momentum tensor is symmetric:

$$\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = mc\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$
$$\to \delta \to m_n c\frac{\mathrm{d}r_n^\nu}{\mathrm{d}\tau}\frac{1}{m_n c}$$
$$= P_n^\nu\frac{1}{m_n c}$$
$$= P_n^\nu\frac{1}{E_n}\gamma$$

Put this into the definition of the energy-momentum tensor:

$$T^{\mu\nu} = \sum_n P_n^\mu P_n^\nu\frac{1}{E_n}\gamma\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$
$$= \int \mathrm{d}\tau\, P_n^\mu P_n^\nu\frac{1}{E_n}\delta^{(4)}(x^\rho - r_n^\rho(\tau))$$
$$= T^{\nu\mu}$$

We can see, that the energy-momentum tensor is symmetric. This is a consequence of the fact, that the energy and momentum are conserved.

We can use densities and currents for other quantities, like energy-momentum, angular momentum, spin, temperature, etc. The energy-momentum tensor is the most important one.

We can prove, that the tensor is conserved with index notation. Lets look at the continuity equation:

$$\partial_\mu J^\mu = \partial_0 J^0 + \partial_i J^i$$
$$\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$$
$$\partial_i = \frac{\partial}{\partial x^i}$$
$$(\partial_i) = \vec{\nabla}$$
$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \vec{\nabla}\cdot\vec{J}$$
$$\Rightarrow \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = \partial_\mu J^\mu$$

Lets show, that this is zero:

$$\partial_i J^i = \frac{\partial}{\partial x^i}\left(\sum_n q_n \frac{\mathrm{d}\vec{r}_n^i}{\mathrm{d}t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))\right)$$
$$= \sum_n q_n \frac{\mathrm{d}\vec{r}_n^i}{\mathrm{d}t}\frac{\partial}{\partial x^i}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$\text{remember } \frac{\partial}{\partial x^i}\delta^{(3)}(\vec{x}-\vec{r}_n(t)) = -\frac{\partial}{\partial \vec{r}_n^i}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$\Rightarrow \partial_i J^i = -\sum_n q_n \frac{\mathrm{d}\vec{r}_n^i}{\mathrm{d}t}\frac{\partial}{\partial \vec{r}_n^i}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$= -\sum_n q_n \frac{\partial}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$= -\frac{\partial}{\partial t}\left(\sum_n q_n \delta^{(3)}(\vec{x}-\vec{r}_n(t))\right)$$
$$= -\frac{\partial}{\partial t}\rho(\vec{x},t)$$
$$\Rightarrow \partial_i J^i = -\frac{\partial}{\partial t}\rho(\vec{x},t)$$
$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$
$$\Rightarrow \partial_\mu J^\mu = 0$$

We will perform analogue calculations for the energy-momentum tensor:

4.14 Conservation of the Energy-Momentum Tensor

$$T^{\mu\nu} = \sum_n P_n^\mu \frac{\partial r_n^\nu}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$

What is $\partial_\nu T^{\mu\nu}$? We will see, that it is often zero.

$$\frac{\partial}{\partial x^i}T^{\mu i} = \frac{\partial}{\partial x^i}\left(\sum_n P_n^\mu \frac{\partial r_n^i}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))\right)$$
$$= \sum_n P_n^\mu \frac{\partial r_n^i}{\partial t}\frac{\partial}{\partial x^i}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$= \sum_n P_n^\mu \frac{\partial r_n^i}{\partial t}\left(-\frac{\partial}{\partial \vec{r}_n^i}\delta^{(3)}(\vec{x}-\vec{r}_n(t))\right)$$
$$= -\frac{\partial}{\partial t}\left(\sum_n P_n^\mu \delta^{(3)}(\vec{x}-\vec{r}_n(t))\right) + \sum_n \frac{\partial P_n^\mu}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$= -\frac{\partial}{\partial t}T^{\mu 0} + \sum_n \frac{\partial P_n^\mu}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$\Rightarrow \frac{\partial T^{\mu 0}}{\partial t} + \frac{\partial}{\partial x^i}T^{\mu i}$$
$$= \sum_n \frac{\partial P_n^\mu}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$\Rightarrow \partial_\nu T^{\mu\nu} = \sum_n \frac{\partial P_n^\mu}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$

For free particles, the momentum is conserved:

$$\frac{\partial P_n^\mu}{\partial t} = 0$$

Lets assume, that particles interact and change momenta only when they meet at collision points. Assume collisions conserve momentum. Let G be force density.

$$G^\mu = \sum_n \frac{\partial P_n^\mu}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$= \sum_{\text{collisions}} \delta(\vec{x}-\vec{x}_{\text{collisions}})\frac{\partial P_n^\mu}{\partial t}$$
$$= \sum_n \frac{\partial}{\partial t}P_n^\mu$$
$$= \sum_{\text{col}} \delta(\vec{x}-\vec{x}_{\text{col}})\frac{\partial}{\partial t}\left(\sum_{n\in\text{ col}} P_n^\mu\right)$$
$$= 0 \Rightarrow \partial_\nu T^{\mu\nu} = 0$$

In summary:

$$\partial_\nu T^{\mu\nu} = G^\mu = 0 \text{ often}$$
$$\partial_\nu T^{\mu\nu} = 0$$
$$\Rightarrow \frac{\partial}{\partial x^0}T^{\mu 0} + \frac{\partial}{\partial x^i}T^{\mu i} = 0$$
$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}\int \mathrm{d}^3\vec{x}\, T^{\mu 0} + \int \mathrm{d}^3\vec{x}\,\frac{\partial}{\partial x^i}T^{\mu i} = 0$$
$$\Rightarrow P^\mu = \int \mathrm{d}^3\vec{x}\, T^{\mu 0} = \text{const}$$
$$P^\mu = \int \mathrm{d}^3\vec{x}\sum_n P_n^\mu \frac{\partial r_n^0}{\partial t}\delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$= \int \mathrm{d}^3\vec{x}\sum_n P_n^\mu \delta^{(3)}(\vec{x}-\vec{r}_n(t))$$
$$= \sum_n P_n^\mu = \text{momentum of all particles}$$

This implies, that the momentum of all particles is conserved.

4.15 Electrodynamics as a relativistic Theory

$c = \varepsilon_0 = 1$.

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \rho$$
$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J}$$

Define a 4x4 matrix:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

In short:

$$F^{i0} = -E^i$$
$$F^{i,j} = -\varepsilon_{ijk}B^k$$

This matrix is called the **electromagnetic field strength tensor**. It is a contravariant tensor of rank 2.

$$B^i = -\frac{1}{2}\varepsilon^{ijk}F^{jk}$$

Maxwell's equations can be written in a more compact form:

$$\partial_\mu F^{\mu\nu} = J^\nu$$
$$\partial_\mu F^{\mu 0} = J^0 = \rho$$
$$\Rightarrow \partial_0 = F^{00} + \partial_i F^{i0} = \rho$$
$$\Rightarrow \partial_i F^{i0} = \rho$$
$$\Rightarrow \partial_i E^i = \rho \Rightarrow \left(\vec{\nabla}\cdot\vec{E} = \rho\right)$$
$$\partial_\mu F^{\mu i} = J^i$$
$$\Rightarrow \partial_0 F^{0i} + \partial_j F^{ji} = J^i$$
$$\Rightarrow -\frac{\partial E^i}{\partial t} + \partial_j\left(\varepsilon^{jik}B^k\right) = J^i$$
$$\Rightarrow -\frac{\partial E^i}{\partial t} + \varepsilon^{ijk}\partial_j B^k = J^i$$
$$\Rightarrow -\frac{\partial E^i}{\partial t} + \left(\vec{\nabla}\times\vec{B}\right)^i = J^i$$
$$\Rightarrow \vec{\nabla}\times\vec{B} + \frac{\partial\vec{E}}{\partial t} = \vec{J}$$

The other two equations are also satisfied. Lets look at the 4-vector potential:

$$A^\mu = \begin{pmatrix}\Phi & A^1 & A^2 & A^3\end{pmatrix}$$
$$\Rightarrow F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$\vec{E}0 = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$
$$E^i = -\partial_i\Phi - \partial_0 A^i$$
$$\Rightarrow F^{i0}\partial^i A^0 - \partial^0 A^i$$
$$\partial_\mu = \begin{pmatrix}\frac{\partial^2}{\partial t\partial\vec{\nabla}}\end{pmatrix}$$
$$\partial^\mu = \begin{pmatrix}\frac{\partial}{\partial t} & -\vec{\nabla}\end{pmatrix}$$
$$\Rightarrow F^{ij} = \partial^i A^j - \partial^j A^i$$
$$= -\frac{1}{2}\varepsilon^{ijk}B^k$$

Summary:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$\partial_\mu F^{\mu\nu} = J^\nu$$

In a new frame:

$$\partial'_\mu F'^{\mu\nu} = J'^\nu$$
$$\Rightarrow \partial_\rho \Lambda_\mu{}^\rho \Lambda^\mu{}_\sigma F^{\sigma\nu} = \Lambda^\nu{}_\kappa J^\kappa$$
$$\Lambda^\nu{}_\kappa \partial_\rho F^{\rho\kappa}$$
$$= \lambda^\nu{}_\kappa J^\kappa$$
$$\Rightarrow \partial_\rho F^{\rho\nu} = J^\nu$$

We can now define the electromagnetic 4-force:

$$f^\mu = qF^{\mu\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}\tau}$$

4.15.1 Summary

$$A^\mu = \begin{pmatrix}\Phi & \vec{A}\end{pmatrix} \text{ contravariant}$$
$$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

We also derived the Maxwell equations in a more compact form:

$$\partial_\mu F^{\mu\nu} = J^\nu$$

4.16 Gauge Invariance

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \mathrm{X}$$
$$F^{\mu\nu} \to F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$$
$$= \partial^\mu(A^\nu + \partial^\nu \mathrm{X}) - \partial^\nu(A^\mu + \partial^\mu \mathrm{X})$$
$$= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$
$$A'^\mu = A^\mu + \partial^\mu \mathrm{X}$$
$$\mu = 0 \Rightarrow \Phi' = \Phi + \frac{\partial \mathrm{X}}{\partial t}$$
$$\mu = i \Rightarrow A'^i = A^i + \partial^i \mathrm{X} = A^i - \partial_i \mathrm{X}$$
$$\Rightarrow \vec{A}' = \vec{A} - \vec{\nabla}\mathrm{X}$$

Lortenz force in a relativistic transformation:

$$f^\mu = F^{\mu\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}\tau}$$
$$\mu = i \Rightarrow f^i = qF^{i\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}\tau}$$
$$= q\left(F^{i0}\frac{\mathrm{d}x_0}{\mathrm{d}\tau} + F^{ij}\frac{\mathrm{d}x_j}{\mathrm{d}\tau}\right)$$
$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma}$$
$$= \gamma q\left(F^{i0}\frac{\mathrm{d}x_0}{\mathrm{d}t} - F^{ij}\frac{\mathrm{d}x^j}{\mathrm{d}t}\right)$$

The minus sign originates from bringing the index up.

$$= \gamma q\left(E^i - \left(-\varepsilon^{ijk}B^k\right)v^j\right)$$
$$= \gamma q\left(E^i + \left(\vec{v}\times\vec{B}\right)^i\right)$$

This is the Lorentz force caused by the electromagnetic field in a relativistic transformation.

$$\frac{\mathrm{d}x_\nu}{\mathrm{d}\tau} = g_{\nu\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

Lets look at what the energy-momentum tensor is for a system of particles. For a system of EM fields and particles, we have:

$$\partial_\nu T^{\mu\nu} = G^\mu$$
$$G^\mu = \sum_n \frac{\partial P_n^\mu}{\partial t}\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$
$$= \sum_n \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\partial P_n^\mu}{\partial \tau}\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$
$$= \sum_n \frac{\mathrm{d}\tau}{\mathrm{d}t} f_n^\mu \delta^{(3)}(\vec{x} - \vec{r}_n(t))$$
$$G^\mu = \sum_n \frac{\mathrm{d}\tau}{\mathrm{d}t} q_n F^{\mu\nu}\frac{\mathrm{d}x_\nu}{\mathrm{d}\tau}\delta^{(3)}(\vec{x} - \vec{r}_n(\tau))$$
$$= F^{\mu\nu}\sum_n q_n \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}x_\nu}{\mathrm{d}\tau}\delta^{(3)}(\vec{x} - \vec{r}_n(\tau))$$
$$= F^{\mu\nu}\sum_n q_n \frac{\mathrm{d}x_\nu}{\mathrm{d}t}\delta^{(3)}(\vec{x} - \vec{r}_n(t))$$
$$= F^{\mu\nu}J_\nu$$
$$\Rightarrow \partial_\nu T^{\mu\nu} = F^{\mu\nu}J_\nu$$

The right hand side is a total derivative. We have a solution:

$$T^{\mu\nu}_{\text{em}} = F^{\mu\rho}F^{\rho\nu} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

This is the energy-momentum tensor of the electromagnetic field. It is symmetric and traceless. It is also gauge invariant. The energy-momentum tensor of the electromagnetic field is conserved.

$$\partial_\nu T^{\mu\nu}_{\text{em}} = -\partial_\nu T^{\nu\mu}_{\text{em}}$$
$$\Rightarrow \partial_\nu(T^{\mu\nu} + T^{\mu\nu}_{\text{em}}) = 0$$

A continuity equation.

We can define:

$$\Theta^{\mu\nu} = T^{\mu\nu} + T^{\mu\nu}_{\text{em}}$$
$$= \sum_n \frac{P_n^\mu P_n^\nu}{E_n} \delta^{(3)}(\vec{x} - \vec{r}_n(t)) + F^{\mu\rho} F_\rho{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$
$$\partial_\nu \Theta^{\mu\nu} = 0$$

The first term is charges and the second term is the electromagnetic field. This is the total energy-momentum tensor of the system.

What is really conserved is the total energy-momentum tensor:

$$P^\mu = \int \mathrm{d}^3\vec{x}\, \Theta^{\mu 0}$$
$$= \sum_n P_n^\mu + \int \mathrm{d}^3\vec{x}\, F^{\mu 0}_{\text{em}}$$
$$= P^\mu_{\text{charges}} + P^\mu_{\text{em}}$$

We call P^μ_{em} the momentum of the electromagnetic field.

$$P^\mu_{\text{em}} = \int \mathrm{d}^3\vec{x}\, T^{\mu 0}_{\text{em}}$$

$$\mu = 0 \Rightarrow P^0_{\text{em}} = \int \mathrm{d}^3\vec{x}\, T^{00}_{\text{em}}$$

Where T^{00}_{em} is the energy density of the electromagnetic field.

$$= F^{0\rho} F_\rho{}^0 + \frac{1}{4} g^{00} F_{\rho\sigma} F^{\rho\sigma}$$
$$= \frac{\vec{E}^2 + \vec{B}^2}{2}$$

This is positive definite. The electromagnetic field has positive energy density.

$$P^i_{\text{em}} = \int \mathrm{d}^3\vec{x}\, T^{i0}_{\text{em}}$$
$$T^{i0}_{\text{em}} = S^i = \left(\vec{E} \times \vec{B}\right)^i$$

This vector is called the **Poynting vector**. It describes the energy flux of the electromagnetic field. The Poynting vector is a 3-vector. It's nothing else than the 3-momentum density of the electromagnetic field.

Lets go back to:

$$\partial_\nu T^{\mu\nu}_{\text{em}} = -F^{\mu\nu} J_\nu$$
$$\mu = 0 \Rightarrow \partial_\nu T^{0\nu}_{\text{em}} = -F^{0\nu} J_\nu$$
$$\Rightarrow \partial_0 T^{00}_{\text{em}} + \partial_i T^{0i}_{\text{em}}$$
$$= -F^{00} J_0 - F^{0i} J_i$$
$$\Rightarrow \frac{\partial}{\partial t}\omega + \vec{\nabla} \cdot S$$
$$\omega = T^{00}_{\text{em}} = \frac{\vec{E}^2 + \vec{B}^2}{2}$$
$$S^i = T^{0i}_{\text{em}} = \left(\vec{E} \times \vec{B}\right)^i$$

The electromagnetic field carries energy and momentum.

$$\omega = T^{00}_{\text{em}} = \frac{\vec{E}^2 + \vec{B}^2}{2}$$
$$\vec{S} = T^{0i}_{\text{em}} = \left(\vec{E} \times \vec{B}\right)^i$$

When added to the energy and momentum of the electrical charges, it is conserved.

5 Radiation

Radiation is a physical phenomenon due to accelerated or decelerated electric charges.

Consider a charge in a generic trajectory:

$$\vec{r}(t) \neq \vec{r}_0 + \vec{v}t$$

What is the resulting $\vec{E}$ an $\vec{B}$? We first compute:

$$a^\mu(\vec{x}, t) \Rightarrow F^\mu \Rightarrow \text{consequences}$$

We have the starting point:

$$A^\mu(x) = \int \mathrm{d}^4x'\, G_{\text{ret}}(x - x')J^\mu(x')$$

The 4-current density J^μ corresponds to an arbitrary trajectory of a single charge q:

$$\begin{gathered} J^\mu(x') = J^\mu(\vec{x}', t') \\ = \left(J^0(\vec{x}', t') \;\; \vec{J}(\vec{x}', t')\right) \\ \vec{x}'(t') = \vec{r}(t') \end{gathered}$$

Where $\vec{r}(t')$ is our trajectory.

$$= \texttt{mat}\left(q\delta(\vec{x}' - \vec{r}(t')), q\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\delta(\vec{x}', \vec{r}(t'))\right)$$

First term is the charge density, second term is the current density. We take $\vec{r}$ to be very generic.

$$\begin{gathered} \dot{\vec{r}} = \vec{v} \\ \ddot{\vec{r}} = \vec{a} \end{gathered}$$

For $\vec{a} \neq 0$ we have radiation, which is outward emission of energy, capable of reaching infinite distances. This is on contrast to electrostatic fields, which are static and decay with distance.

$$\begin{gathered} a^\mu \\ F^\mu = \partial^\mu A^\nu - \partial^\nu A^\mu \\ \vec{S} \text{ Poynting vector} = \vec{E} \times \vec{B} \\ \frac{\mathrm{d}P}{\mathrm{d}t} \text{ power of radiation} = \int \vec{S} \cdot \mathrm{d}A \end{gathered}$$

$$J^\mu(x) = q\int \mathrm{d}\tau\, v^\mu \delta(x^\rho - \vec{r}^\rho(\tau))$$

$$v^\mu = \frac{\mathrm{d}r^\mu}{\mathrm{d}\tau}$$

$$A^\mu = \int \mathrm{d}^4x'\, G_{\text{ret}}(x - x')J^\mu(x')$$

$$G_{\text{ret}}(x - x') = \frac{1}{2\pi}\delta\Big((x - x')^2\Big)\Theta(x^0 - x'^0)$$

$$\Rightarrow A^\mu(x) = \frac{q}{2\pi}\int \mathrm{d}\tau\, v^\mu(\mathrm{T})\delta((x - \vec{r}(\tau))^2)\Theta(x^0 - \vec{r}^0(\tau))$$

$$\begin{gathered} (x - \vec{r}(\tau))^2 = 0 \\ \Rightarrow x^0 - r^0(\tau) = \pm|\vec{x} - \vec{r}(\tau)| \\ \Rightarrow x^0 - r^0(\tau) > 0 \\ \Rightarrow x^0 - r^0(\tau) = |\vec{x} - \vec{r}(\tau)| \end{gathered}$$

This will give us one solution for τ, which we call τ_0:

$$\delta((x - \vec{r}(\tau))^2) = \frac{\delta(\tau - \tau_0)}{\frac{\mathrm{d}}{\mathrm{d}\tau}[(x - \vec{r}(\tau))^2]}\bigg|_{\tau=\tau_0}$$

$$\text{remember } \delta(f(x)) = \frac{\delta(x - x_0)}{\left|\frac{\partial f}{\partial x}\right|_{x=x_0}}$$

$$\begin{gathered} \Rightarrow \frac{\partial R^2}{\partial \tau} = \frac{\partial}{\partial \tau}\left[g_{\mu\nu}(R^\mu R^\nu)\right], R = x - r(\tau) \\ = g_{\mu\nu}\frac{\partial R^\mu}{\partial \tau}R^\nu + g_{\mu\nu}R^\mu\frac{\partial R^\nu}{\partial \tau} \\ = 2g_{\mu\nu}R^\nu\frac{\partial R^\mu}{\partial \tau} \\ = 2R \cdot \frac{\partial R}{\partial \tau} \\ = 2(x - r(\tau)) \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}(x - r(\tau)) \\ = -2(x - r(\tau)) \cdot \frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau} \\ = 2(x - r(\tau)) \cdot v(\tau) \end{gathered}$$

This works because:

$$\begin{gathered} \int \mathrm{d}x\, \delta(f(x)) \\ = \int \frac{\mathrm{d}f(x)}{\left|\frac{\partial f}{\partial x}\right|}\delta(f(x)) \\ = \int \mathrm{d}y\, \frac{\delta(y)}{\left|\frac{\partial f}{\partial x}\right|_{y=0}} \end{gathered}$$

All together:

$$\begin{gathered} A^\mu(x) = \frac{q}{4\pi}\int \mathrm{d}\tau\, \frac{\delta(\tau - \tau_0)v^\mu(\tau)}{(x - r(\tau))v(\tau)} \\ \Rightarrow A^\mu(x) = \frac{v^\mu(\tau_0)}{(x - r(\tau_0))v(\tau_0)}\frac{q}{4\pi} \end{gathered}$$

This is called the **Retarded potential**. τ_0 corresponds to the past. It was discovered by Lienard and Wiechert.

In the static limit, the carge is in its rest frame.

$$\begin{gathered} v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \begin{pmatrix}1 & \vec{0}\end{pmatrix} \\ (x - r(\tau)) \cdot v = (x^0 - r^0(\tau)) - (\vec{x}^i - \vec{r}^i(\tau))\vec{v} = x^0 - r^0(\tau) \\ = |\vec{x} - \vec{r}| \end{gathered}$$

Last part from the Green function constraint.

$$\Rightarrow A^\mu(x) = \frac{q}{4\pi}\frac{\begin{pmatrix}1 & \vec{0}\end{pmatrix}}{|\vec{x} - \vec{r}(\tau)|}$$

Thus we recovered electrostatics.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

We will skip the derivation of the electromagnetic field tensor. Define:

$$R^\mu = x^\mu - \vec{r}^\mu(\tau_0)$$

Greens function requires $R^2 = 0 \Rightarrow R^0 > 0$.

$$\begin{gathered} \Rightarrow R^0 = \left|\vec{R}\right| \\ R^\mu = \begin{pmatrix}R^0 & \vec{R}\end{pmatrix} \\ = \left|\vec{R}\right|\begin{pmatrix}1 & \frac{\vec{R}}{|\vec{R}|}\end{pmatrix} \end{gathered}$$

We call $\hat{n} = \frac{\vec{R}}{|\vec{R}|}$.

$$R^\mu = (x^\mu - r^\mu(\tau))$$
$$= \left|\vec{R}\right|(1 \ \hat{n})$$

We can write:

$$\vec{B} = \hat{n} \times \vec{E}$$
$$\vec{E} = \frac{q}{4\pi(1-\hat{n}\cdot\vec{v})^3}\left\{\frac{1-\vec{v}^2}{\left|\vec{R}\right|^2}(\hat{n}-\vec{v}) + \frac{1}{\left|\vec{R}\right|}\hat{n}\times\left[(\hat{n}-\vec{v})\times\dot{\vec{v}}\right]\right\}$$

When $v = 0$, we get electrostatics.

$$\vec{E} = \frac{q}{4\pi}\left(\frac{1}{\left|\vec{R}\right|^2}\right)\hat{n} + \frac{1}{\left|\vec{R}\right|}\hat{n}\times\left[\hat{n}\times\dot{\vec{v}}\right])$$

5.1 Poynting vector

$$\vec{S} = \vec{E}\times\vec{B} = \vec{E}\times\left(\hat{n}\times\vec{E}\right)$$
$$= \left|\vec{E}\right|^2\hat{n} - \left(\vec{E}\cdot\hat{n}\right)\vec{E}$$

We will substitute the electric field (at rest frame):

$$\vec{S} = \hat{n}\frac{q^2}{16\pi^2\left|\vec{R}\right|^2}\left|\hat{n}\times\left(\hat{n}\times\dot{\vec{v}}\right)\right|^2 + O\left(\frac{1}{R^3}\right)$$
$$= \ldots = \hat{n}\frac{q^2}{16\pi^2\left|\vec{R}\right|^2}\left|\dot{\vec{v}}\right|^2\sin^2(\theta) + O\left(\frac{1}{R^3}\right)$$

The poynting vector satisfies the continuity equation.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = P = \vec{S}\cdot\mathrm{d}A$$

We can use spherical coordinates:

$$\mathrm{d}\vec{A} = \hat{n}\cdot\left|\vec{R}\right|^2\mathrm{d}\Omega$$
$$\Rightarrow \mathrm{d}P = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2\sin^2(\theta)\,\mathrm{d}\Omega$$
$$\Rightarrow \mathrm{d}\vec{A} \propto R^2$$
$$\vec{S} \propto \frac{1}{R^2} + O\left(\frac{1}{R^3}\right)$$

We see, that the Poynting vector depends on acceleration. The power, that we see as radiation, so as energy transfer at really large distances, will be the strict infinite limit.

$$\frac{\mathrm{d}P_{\text{radiation}}}{\mathrm{d}\Omega} = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2\sin^2(\theta)$$

This is the radiation power, that we see at large distances. It is proportional to the square of the acceleration.

To find the total power, we integrate over the solid angle:

$$P_{\text{rad}} = \frac{q^2}{4\pi}\frac{2}{3}\left|\dot{\vec{v}}\right|^2$$

If our acceleration is at a right angle to our line of sight, the radiation is maximal. If it is parallel, the radiation is zero.

In general, radiation is generated by any accelerated charge. The radiation is proportional to the square of the acceleration, and it is emitted in all directions, with a maximum in the direction of the acceleration.

Lets look at the component

$$\frac{1}{R}\hat{n}\times\left[(\hat{n}-\vec{v})\times\dot{\vec{v}}\right]$$

This is the radiation field, which is perpendicular to the direction of the acceleration. It is responsible for the long range radiation (because $\frac{1}{R}$). The other component is the Coulomb field, which is responsible for the short range interaction.

There is also a connection between acceleration and crossectional radiation power:

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}\Omega} = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2\sin^2(\theta)\,\mathrm{d}\Omega$$

This is the radiation power per solid angle. It is proportional to the square of the acceleration. The total radiation power is:

$$P_{\text{rad}} = \frac{q^2}{4\pi}\frac{2}{3}\left|\dot{\vec{v}}\right|^2$$

We can use **the Larmor formula** to compute the radiation power of a charge in an arbitrary trajectory. The Larmor formula is:

$$P_{\text{rad}} = \frac{q^2}{4\pi}\frac{2}{3}\frac{\left|\dot{\vec{v}}\right|^2}{c}$$
$$= \frac{q^2}{4\pi\varepsilon_0 c}\frac{2}{3}(\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}^2\frac{1}{m^2}$$

The above is definitely valide in the rest frame. We can use the Lorentz transformation to get an arbitrary $\vec{v}$.

$$P_{\text{rad}} = \frac{\text{Energy}}{\text{Time}} = P'_{\text{rad}}$$

This is invariant under Lorentz transformations. If we integrate over the radiation power (solid angle), we get the total radiation power, which is invariant, thus the same for all observers.

In the rest frame, we have:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \left(0 \ \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right)$$
$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = 0^2 - \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}^2$$
$$= -\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}^2$$
$$\Rightarrow p_{\text{rad}} = \frac{q^2}{4\pi}\frac{2}{2m^2}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}$$

This is the radiation power in the rest frame. It is valid generally.

Explicitly in termos of $\vec{v}, \dot{\vec{v}}$:

$$\ldots = P_{\text{rad}} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^6\left[\left|\dot{\vec{v}}\right|^2 - \left|\vec{v}\times\dot{\vec{v}}\right|^2\right]$$

We see that acceleration i still needed at very high velocities, for the charge to produce radiation.

For relativistic velocities, effectively $\gamma^6 \to \infty$. The radiation power increases dramatically. This is particularly important in particle colliders, where charged particles are accelerated to near light speeds. The enormous energy loss due to this radiation (called synchrotron radiation) is a major design constraint, especially in circular colliders where particles constantly undergo centripetal acceleration. For electrons, which have small mass, this radiation limits the maximum achievable energies in circular accelerators.

Thus comes the question about the most efficient way to accelerate charged particles.

We can have circular accelerators, where the particles are accelerated in a circular path, or linear accelerators, where the particles are accelerated in a straight line.

For circular accelerators:

$$\vec{v} \perp \dot{\vec{v}} \Rightarrow \left|\vec{v} \times \dot{\vec{v}}\right| = |\vec{v}|\left|\dot{\vec{v}}\right|$$

$$P_{\text{rad}} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^6|\vec{v}|^2 - |\vec{v}|^2\left|\dot{\vec{v}}\right|^2$$

$$= \frac{q^2}{4\pi}\frac{2}{3}\gamma^4\left|\dot{\vec{v}}\right|^2$$

For linear accelerators, we have:

$$\vec{v} \parallel \dot{\vec{v}} \Rightarrow \left|\vec{v} \times \dot{\vec{v}}\right| = 0$$

$$P_{\text{rad}} = \frac{q^2}{4\pi}\frac{2}{3}\gamma^6\left|\dot{\vec{v}}\right|^2$$

This means, that circular accelerators are much more efficient.

So far we have studied the radiation of a single charge only in the $\vec{v} \to 0$ limit. We can extend this to arbitrary $\vec{v}$.

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}\Omega} = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2 \sin^2\frac{\theta}{(1 - v\cos\theta)^6}, c = 1$$

This only holds for v approaching c. For very relativistic velocities, we encounter quantum effects, which are not covered by this formula.

5.2 Scattering

We have a charged particle, some radiation hits it and it scatters. Thus it gains energy and momentum.

We have different types of scattering:

- q is free (Thompson scattering)
- q is bound (Rayeigh scattering)

The difference is in how the incident radiation accelerates the charge.

5.2.1 Thompson scattering

$$\vec{E} = \hat{e}ER_e e^{i(\omega t - \vec{k}\cdot\hat{r})}$$

$$= \hat{e}E\cos(\omega t)$$

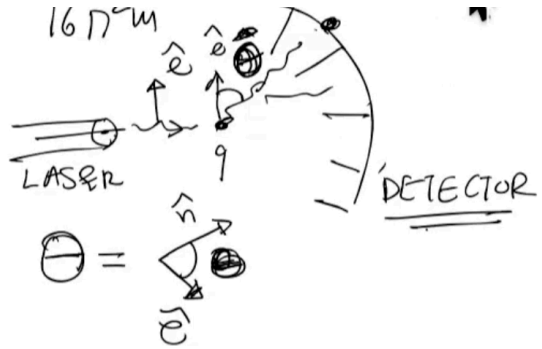
The force from the radiation on the still charge is:

$$\vec{F} = q\vec{E} = q\hat{e}E\cos(\omega t)$$

$$\Rightarrow \dot{\vec{v}} = \frac{\vec{F}}{m} = \frac{q}{m}\hat{e}E\cos(\omega t)$$

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}\Omega} = \frac{q^2}{16\pi^2}\left|\dot{\vec{v}}\right|^2\sin^2(\theta)$$

$$= \frac{q^4}{16\pi^2 m^2}\sin^2(\theta)E^2\cos^2(\omega t)$$



We can:

- Take a time average
- Convert it to a convenient angle, and then average over the vector of $\hat{e}$.

We have to average of all transverse directions, which is done by integrating over the solid angle.

- Measure cross-section

The time average is:

$$\frac{1}{T}\int_{0,T}\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}\Omega}\,\mathrm{d}t, T = 2\frac{\pi}{\omega}$$

$$=< \frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}\Omega} >= \frac{q^4}{32\pi m^2}E^2\sin^2(\theta)$$

This still depends on the electric field of the laser.

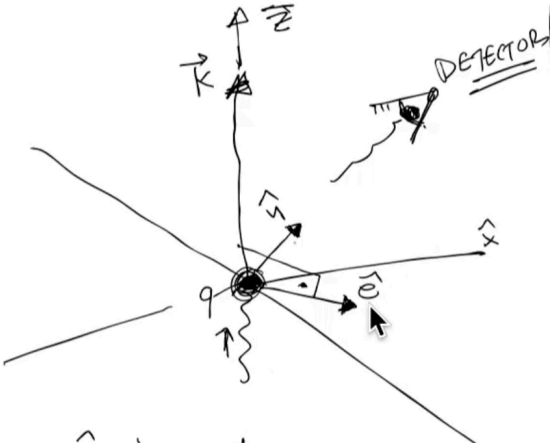
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} =< \frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}\Omega} >_{<} \text{Flux} >$$

Flux is energy per unit surface per unit time. The flux is:

$$< \text{Flux} >=< \left|\vec{S}\right| >= \frac{E^2}{2}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q^4}{16\pi m^2}\sin^2(\theta)$$

This is almost independent of the electric field, except for the angle.



$\hat{e}$ is on the x-y plane, because $\vec{k}$ is on the z-axis. There is another Ψ angle, which is random, because the laser is not polarized. We assume, that Ψ is uniformly distributed to cover the x-y plane. What is consistent is where the laser is pointing, which is $\hat{k}$.

$$\theta = f(\Theta, \Psi)$$

And then we can average over the Ψ angle.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q^4}{16\pi m^2}\frac{1 + \cos^2(\theta)}{2})$$

Keep in mind, that is assuming that the laser is unpolarized. Θ is the angle between the insident laser/radiation and the observer/detector.

This is the **differential cross-section** for Thompson scattering (EM and free charge q at rest). This is accurate in practice to approximately 1% (because we ignored quantum effects).

In this entire section, we have ignored what happens to the radiation after it hits the charge.

To consider that, let q be at rest $(m_q\ \vec{0})$. Let $P_\gamma = E_\gamma(1\ \hat{x})$. Let P_q^* be the final state of the charge after the radiation hits it.

$$P_q^* = \left(m_q + E_\gamma\ \hat{x}\right)$$

$$P_q^2 = \left(m_q + E_\gamma\right)^2 - E_\gamma^2\hat{x}^2$$

$$= m_q^2 + 2m_qE_\gamma \neq m_q^2$$

This means, that the charge has gained energy from the radiation. This violation can be explained by uncertainy principle with short time scales.

$$(2m_q E_\gamma)\Delta t \approx \hbar$$

Other weird things can happen during scattering, like an emission and reabsorbtion of a virtual photon, which is not detectable. This even contributes to the cross-section.

The corrections to the cross-section are usually small (on the order of 1%), but they can be significant in some cases.

Lets return to the base case of thompson scattering.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q^4}{16\pi m^2}(1+\cos^2(\theta))$$

We assume that this is independent of the frequency of the radiation. This was special to the case of the electron e^- being free and only valid classically.

The crossection takes its maxima on $\Theta = 0, \pi$. We except most of the radiation to be scattered in the forward and backward directions.

$$\sigma = \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{8\pi}{3} r_q^2$$

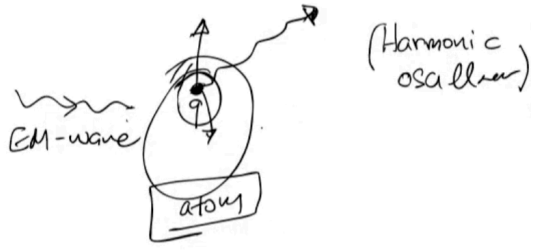
This is expained by the quantity called **Thompsons radius**:

$$r_q = \frac{q^2}{4\pi\varepsilon_0 mc^2}$$
$$\approx 2.8 * 10^{-15} m = 2.8 \text{ fm}$$

This is an effective radius of the electron if you assign it to a spherical shape.

5.2.2 Rayleigh Scattering

The electron is bound in an atom.



We can imagine the electron as a harmonic oscillator, which is driven by the incident radiation. We also have to account for friction. We model this as a damped harmonic oscillator:

$$\frac{qE\cos(\omega t)}{m} = \ddot{x} + \gamma\dot{x} + \omega_0^2 x$$
$$\vec{E} = E\hat{x}\cos(\omega t)$$

To solve this, we can use the ansatz:

$$\mathrm{Re}[q\left(E\frac{e^{i\omega t}}{m} = \ddot{x} + \gamma\dot{x} + \omega_0^2 x\right]$$
$$\Rightarrow x = x_0 e^{i\omega t}$$

When we differentiate, the exponential remains, and they will cancel out:

$$x = \left(\frac{qE\frac{e^{i\omega t}}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}\right)$$
$$\Rightarrow \vec{x} = \frac{\frac{q\vec{E}}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

γ is the damping coefficient, or friction coefficient. It is proportional to the frequency of the radiation, and it is inversely proportional to the mass of the electron.

$$\Rightarrow \dot{\vec{v}} = \frac{q\vec{E}(-\omega)^2}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$
$$\cos(\omega t) = \mathrm{Re}(e^{i\omega t})$$
$$= \mathrm{Re}(e^{-i\omega t})$$

For a free electron:

$$\omega_0 = 0, \gamma = 0$$
$$\Rightarrow \vec{d}(v)_{\text{Thompson}} = \frac{q\vec{E}}{m}$$
$$\Rightarrow \dot{\vec{v}}_{\text{Rayleigh}} = \vec{d}(v)_{\text{Thompson}} \frac{\omega^2}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

σ_{Rayleigh} is computed analogously to σ_{Thompson}:

$$\dot{\vec{v}}_T \to \vec{d}(v)_R = \vec{d}(v)_T \frac{f}{\omega}$$
$$\Rightarrow \omega_R = \omega_T \frac{\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2}$$

Thompson and Rayleigh scattering happen on the same magnitude, but rayleigh scattering is frequency dependent on ω_0, ω.

We use the principle $\hbar\omega_0 \approx$ energy of electron in the atom. The atom is not broken, so the electron is bound. $\hbar \ll \hbar\omega_0$. For $\hbar\omega \approx \hbar\omega_0$ we have ionization of the atom, which is not covered by this model.

We want to consider ω in the visible spectrum.

$$\Rightarrow \omega_R \approx \omega_T \frac{\omega^4}{\omega^4} < 1$$

It means at high frequencies, the Rayleigh scattering is higher.

If we consider full spectrum light radiating on bound electrions, we can see that the Rayleigh scattering is higher at high frequencies, and lower at low frequencies.

If we observe the light at a scattering direction different from the incident direction, we can observe more blue light than red light. This is called **Rayleigh scattering**.

If we observe the incident scattering, we see a reddish light, because the blue light is scattered more than the red light. This is why the sky is blue, and the sun is red at sunset.

6 Macroscopic Maxwell Equations in a Medium

We want to look at the Maxwell equations at large scales.

We have to find a way to simulate the interactions of the electromagnetic field without having to solve the microscopic equations of motion for each individual charge.

We are seeking an effective theory description. That means we will not use the full theory, but one that is valid at large scales. It relies on physical simplifications.

Macroscopically our full theory needs to be computed in the limit. This means replacing the full theory with the effective theory in the limit.

6.1 The Effective Theory

We use the fact that the atomic length is much smaller than our macroscopic length scales. Practically, if we take a metal, it will have some density of charges ρ and some current. The electrons are behaving as free parricles and can travel large distances.The distance covered by macroscopic currents are much larger than currents in an atom.

If we have a polarizen medium with atoms and electrons in each of them. If we shine radiations on these atoms without destroying them, the wavelength will be large for the incident beams. The beam will not interact with the individual atoms because it is much bigger than a single atom.

Thus we take maxwells theory $\partial_\mu F^{\mu\nu} = J^\nu$ and average over atomic scales. This will be our first examle of an effective theory.

6.2 Atomic Approximation

In an atom, the constituent charges are restricted to small distances but move very fast. Collective mostions of charges ar assumed to be slow and long ranged.

On our scales, we can average the position and velocity of charges.

$$g(\vec{x}) = \text{measured property of particle}$$
$$f(\vec{y}) = \text{propability/density of particle}$$
$$\int g(\vec{x}-\vec{y}) f(\vec{y})\, \mathrm{d}^3\vec{y} =< g(\vec{x}) >$$

Let $f(\vec{y})$ be smooth, positive definite and normalized $\int f(\vec{y}) = 1$.

We can average the current density J^μ.

$$J^\mu = J^\mu_{\text{free}} + J^\mu_{\text{atomic}}$$
$$\approx J^\mu_{\text{free}} + < J^\mu_{\text{atomic}} >$$

6.3 Average atomic charge density

$$\rho_{\text{atomic}} = \sum_{n\in \text{ atoms}} \rho_{(n)}$$
$$\rho_{(n)}(\vec{x}) = \sum_{j\in(n)} q_j \rho\big(\vec{x}-\vec{x}_y-\vec{x}_j\big)$$
$$< \rho_{\text{atomic}}(\vec{x}) >= \sum_n < \rho_{(n)}(\vec{x}) >$$
$$< \rho_n(\vec{x}) >= \int \mathrm{d}^3\vec{y} \left(\sum_j q_j \delta\big(\vec{x}-\vec{x}_y-\vec{x}_j-\vec{y}\big)\right) f(\vec{y})$$
$$< \rho_n(\vec{x}) >= \sum_j q_j f\big(\vec{x}-\vec{x}_y-\vec{x}_j\big)$$

We can use that $\big|\vec{x}-\vec{x}_y\big| \gg \vec{x}_j$. Thus we can taylor expand ind $\frac{|\vec{x}_j|}{|\vec{x}_y-\vec{x}_j|}$.

$$f\big(\vec{x}-\vec{x}_y-\vec{x}_j\big) \approx f\big(\vec{x}-\vec{x}_y\big) - \vec{x}_j\cdot\vec{\nabla} f\big(\vec{x}-\vec{x}_y\big) + o\big(\vec{x}_j^2\big)$$
$$\Rightarrow < \rho_n(\vec{x}) >= \sum_j q_j f\big(\vec{x}-\vec{x}_y\big) - \vec{x}_j\cdot\vec{\nabla} f\big(\vec{x}-\vec{x}_y\big) + o\big(\vec{x}_j^2\big)$$
$$\approx q_n f\big(\vec{x}-\vec{x}_y\big) - \left(\sum_j q_j\vec{x}_j\right)\cdot\vec{\nabla} f\big(\vec{x}-\vec{x}_y\big)$$
$$\approx q_n f\big(\vec{x}-\vec{x}_y\big) - \vec{\Phi}\cdot\vec{\nabla} f\big(\vec{x}-\vec{x}_y\big)$$

The first term is the average charge density of the atom, the second term is the dipole moment of the atom.

$$=< q_n\delta\big(\vec{x}-\vec{x}_y\big) > -\vec{\nabla}\cdot\Big(\vec{P}_n f\big(\vec{x}-\vec{x}_y\big)\Big)$$
$$=< q_n\delta\big(\vec{x}-\vec{x}_y\big) > -\vec{\nabla}\cdot < \vec{P}_n\delta\big(\vec{x}-\vec{x}_y\big) >$$

We need to sum over all the atoms:

$$< \rho_A(\vec{x}) >=< \sum_n q_n\delta\big(\vec{x}-\vec{x}_y\big) >$$
$$\vec{P} =< \sum_n \vec{P}_n\delta\big(\vec{x}-\vec{x}_y\big) >$$

Where $\vec{P}_n$ is the dipole moment of the atom.

$$\Rightarrow < \rho_A(\vec{x}) >= \rho_{A\text{ eff}} - \vec{\nabla}\cdot\vec{P}$$

$\vec{P}$ is also called the polarization vector of the medium.

$$\vec{p}_n = \sum_j q_j\vec{x}_j$$
$$\sum_j q_j f\big(\vec{x}-\vec{x}_y-\vec{x}_j\big) \approx g_j f\big(\vec{x}-\vec{x}_y\big) - \sum_j (\vec{x}_j q_j)\cdot\vec{\nabla} f\big(\vec{x}-\vec{x}_y\big)$$

Now we can do the same approximations for the current density $\vec{J}_A$.

We will use that $\big|\vec{x}_j\big| \ll \big|\vec{x}-\vec{x}_y\big|$ and

$$\vec{v}_j = \frac{\mathrm{d}\vec{x}_j}{\mathrm{d}t}$$
$$\big|\vec{v}_j\big| \ll \big|\vec{v}_y\big| < \vec{J}_n >= \sum_j q_j(\vec{v}_n+\vec{v}_j)\cdot f\big(\vec{x}-\vec{x}_y-\vec{x}_j\big)$$

We can use the same approximation as before:

$$f\big(\vec{x}-\vec{x}_y-\vec{x}_j\big) \approx f\big(\vec{x}-\vec{x}_y\big) - \vec{x}_j\cdot\vec{\nabla} f\big(\vec{x}-\vec{x}_y\big)$$
$$\Rightarrow < \vec{J}_n >= \sum_j q_j\vec{v}_j f\big(\vec{x}-\vec{x}_y\big)$$
$$+\sum_j q_j\vec{v}_y f\big(\vec{x},\vec{x}_y\big) - \sum_j q_j\vec{v}_j\Big(\vec{x}_j\cdot\vec{\nabla} f\big(\vec{x}-\vec{x}_y\big)\Big) + o$$

The first term ist dominant, the other two are of the same small order.

$$\sum_j q_j\vec{v}_j f\big(\vec{x}-\vec{x}_y\big) = \sum_j q_j\frac{\mathrm{d}\vec{x}_j}{\mathrm{d}t} f\big(\vec{x}-\vec{x}_y\big)$$
$$= \sum_j \frac{\mathrm{d}\vec{P}_j}{\mathrm{d}t} f\big(\vec{x}-\vec{x}_y\big)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \vec{P}_n f\big(\vec{x}-\vec{x}_y\big)\right) - \vec{P}_n\frac{\mathrm{d}}{\mathrm{d}t} f\big(\vec{x}-\vec{x}_y\big)$$
$$\Rightarrow \sum_j q_j\vec{v}_j f\big(\vec{x}-\vec{x}_y\big)$$
$$\approx \frac{\mathrm{d}}{\mathrm{d}t}\Big(\vec{P}_n f\big(\vec{x}-\vec{x}_y\big)\Big)$$

Further:

$$\sum_j q_j \vec{v}_j \big(\vec{x}_j \cdot \vec{\nabla} f(\vec{x} - \vec{x}_y)\big)$$

$$= \sum_j \Big\{ \frac{q_j}{2} \vec{v}_j \big(\vec{x}_j \cdot \vec{\nabla} f\big) - \frac{q_j}{2} \vec{x}_j \big(\vec{v}_j \cdot \vec{\nabla} f\big) \Big\} + \sum_j \frac{q_j}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big(\vec{x}_j \big(\vec{x}_j \cdot \vec{\nabla} f\big)\big)$$

We split of the term into a symmetric and an antisymmetric part.

The last term is a risky approximation, because it is not clear that the time derivative of a small quantity is also small. We will assume that it is small.

$$\Rightarrow \sum_j q_j \vec{v}_j \vec{x}_j \cdot \vec{\nabla} f(\vec{x} - \vec{x}_y)$$

$$\approx \sum_j \frac{q_j}{2} \vec{v}_j \big(\vec{x}_j \cdot \vec{\nabla} f(\vec{x} - \vec{x}_y)\big) - \vec{x}_j \big(\vec{v}_j \cdot \vec{\nabla} f(\vec{x} - \vec{x}_y)\big)$$

$$\vec{a} \times \big(\vec{b} \times \vec{c}\big) = \vec{a}\big(\vec{b} \cdot \vec{c}\big) - \vec{b}(\vec{a} \cdot \vec{c})$$

$$\Rightarrow = -\vec{\nabla} \times \big(\vec{m}_n f(\vec{x} - \vec{x}_y)\big)$$

$$\vec{m}_n = \sum_j \frac{q_j}{2} (\vec{x}_j \times \vec{v}_j)$$

Putting it all together we get:

$$< \vec{J}_A > = \sum_n < \vec{J}_n >$$

$$= \vec{J}_{\text{eff } A} \text{ pointlike with systems vel.}$$

$$+ \frac{\partial \vec{P}}{\partial t}$$

$$+ \vec{\nabla} \times \overrightarrow{M}$$

$$\overrightarrow{M} = \sum_n \big(\vec{m}_n f(\vec{x} - \vec{x}_y)\big)$$

The equation above describes the average angular momentum decomposition. Here, $< \vec{J}_A >$ represents the average total angular momentum of system A, which equals the sum of average angular momenta $< \vec{J}_n >$ of all constituent particles. This can be expressed as three components: (1) $\vec{J}_{\text{eff } A}$ - the effective angular momentum treating the system as a point-like object moving with the system's velocity; (2) $\frac{\partial \vec{P}}{\partial t}$ - the time derivative of linear momentum, accounting for changes in the system's overall motion; and (3) $\vec{\nabla} \times \overrightarrow{M}$ - the cross product between the gradient operator and magnetic moment $\overrightarrow{M}$. The magnetic moment itself is defined as the sum of individual moments $\vec{m}_n$ weighted by the distribution function $f(\vec{x} - \vec{x}_y)$.

Summary:

$$J^\mu = J^\mu_{\text{free}} + < J^\mu_{\text{atomic}} >$$

$$J^0 = J^0_{\text{eff}} \text{ atoms viewed as pointlike}$$

$$+ \vec{\nabla}\big(-\vec{P}\big) \text{ div of avg electric dipole moment}$$

$$\vec{J} = \vec{J}_{\text{eff}} \text{ everthing thats moving freely}$$

$$+ \frac{\partial \vec{P}}{\partial t} \text{ time derivative of avg electric dipole moment}$$

$$+ \vec{\nabla} \times \overrightarrow{M} \text{ magnetic moment of the system}$$

We can now write the macroscopic Maxwell equations in a medium:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \approx \frac{\rho_{\text{eff}} - \vec{\nabla} \cdot \vec{P}}{\varepsilon_0}$$

$$\Rightarrow \vec{\nabla} \cdot \big(\varepsilon_0 \vec{E} + \vec{P}\big) = \rho_{\text{eff}}$$

$$\vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\varepsilon_0 c^2} + \frac{\partial \vec{E}}{\partial t} \frac{1}{c^2}$$

$$\approx \frac{\vec{J}_{\text{eff}} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \overrightarrow{M}}{\varepsilon_0 c^2} + \frac{\partial \vec{E}}{\partial t} \frac{1}{c^2}$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{B} - \frac{\overrightarrow{M}}{\varepsilon_0 c^2} \right) = \frac{J_{\text{eff}}}{\varepsilon_0 c^2} + \frac{\partial \vec{E}}{\partial t} \frac{1}{\varepsilon_0 c^2} \big(\varepsilon_0 \vec{E} + \vec{P}\big)$$

$$\vec{H} = \vec{B} - \frac{\overrightarrow{M}}{\varepsilon_0 c^2}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_{\text{eff}}$$

$$\vec{\nabla} \times \vec{H} = \frac{J_{\text{eff}}}{\varepsilon_0 c^2} + \frac{\partial \vec{D}}{\partial t} \frac{1}{\varepsilon_0 c^2}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

All of this is an approximation.

Lets look at the speicla case of a material, that cannot get magnetized $\overrightarrow{M} = 0$ (e.g. a dielectric). $\vec{E}$ and $\vec{P}$ are correlated.

$$\vec{P} = 0 \text{ if } \vec{E} = 0$$

Ignoring nonlinear effects, we can write:

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

$$\chi = \text{electric susceptibility}$$

Maxwells equations simplify to:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0 (1 + \chi)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{J_{\text{eff}}}{\varepsilon_0 c^2} + \frac{\partial \vec{E}}{\partial t} \frac{1 + \chi}{c^2}$$

$$= \frac{J_{\text{eff}}}{\varepsilon_0 (1 + \chi) \left(\frac{c}{\sqrt{1+\chi}}\right)^2} + \frac{1}{\left(\frac{c}{\sqrt{1+\chi}}\right)^2} \frac{\partial \vec{E}}{\partial t}$$

We define:

$$c_m = \frac{c}{\sqrt{1 + \chi}}$$

$$\varepsilon = \varepsilon_0 (1 + \chi)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho_{\text{eff}}}{\varepsilon}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{J_{\text{eff}}}{\varepsilon c_m^2} + \frac{1}{c_m^2} \frac{\partial \vec{E}}{\partial t}$$

Where c_m is the speed of light in the medium and (it is complex because of losses), and ε is the permittivity of the medium.

To understand $\vec{P} = \chi \varepsilon \vec{E}$, we can look at the miscroscopic level.

$$\vec{E} = \vec{E}(\vec{x}, t) \text{ arbitrary}$$

In physics, some functions are plane waves. A supersposition of plane waves can represent any arbitrary function. Lets take a look at the plane wave:

$$\hat{E} = \vec{E} e^{i(\omega t - k\cdot\vec{x})}$$

For a single atom we approximate it as a dipole with the negative charge being somewhere outside and positive being in the center. If we subject the atom to the electromagnetic field, there will be some acceleration.

$$q\vec{E} = m\left(\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2 x\right)$$

$$\omega_0 = \text{resonance frequency of the atom}$$

We can be more precise, and refine the harmonic oscillator model to something nonlinear, but we will not do that here.

The solution for the microscopic model is:

$$\vec{x} = \frac{q\frac{\vec{E}}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

Same as in Rayleigh scattering. With this we can know die dipole moment of the atom:

$$\vec{p} = q\vec{x} = \frac{\frac{q^2}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}\vec{E}$$

$$\vec{p} = \text{dipole moment of the atom}$$

$$\Rightarrow < \vec{P} >= N\vec{x}$$

$$N = \text{density of charges}$$

$$\Rightarrow = \frac{\frac{Nq^w}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}\vec{E} = \vec{P}$$

$$\vec{P} = \text{polarization of the medium}$$

$$= \chi\varepsilon\vec{E}$$

$$\Rightarrow \chi(\omega) = \frac{N\frac{q^2}{m}}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

We can see that for a plane wave, the succeptibility is a complex function that is frequency dependent.

In Maxwell's equations, χ was used twice. To recalibrate the charges ε and to change the speed of light in the medium c_m. This implies that the speed of light in the medium and the permittivity of the medium are complex.

We consider an EM wave in a medium with:

$$\hat{J}_{\text{eff}} = \rho_{\text{eff}} = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t}\frac{1+\chi}{c_m^2}$$

If we combine them:

$$\left[\frac{1}{c_m^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}\right]\vec{E} = 0$$

$$\Rightarrow \hat{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

$$\Rightarrow k^2 = \frac{\omega^2}{c_m^2}$$

$$= \frac{\omega^2}{c_m^2}(1+\chi)$$

$$\Rightarrow v_{\text{phase}} = \frac{\omega}{|\vec{k}|} = \frac{c}{n}$$

$$n = \sqrt{1+\chi}$$

Lets look at n:

$$n = n_R - in_I$$

$$\vec{E} = \vec{E}_0 e^{i\omega\left[t - \frac{n_R}{c}\hat{k}\cdot\vec{x}\right]} e^{-\frac{\omega}{c}n_I\hat{k}\cdot\vec{x}}$$

$$\hat{k} \parallel \vec{x} \text{ if } n_I > 0$$

We found that:

$$\chi(\omega) = \frac{\frac{qN}{\varepsilon_0 m}}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

$$c_m = \frac{c}{\sqrt{1+\chi}}$$

$$\Rightarrow \text{Im}(\chi) \text{ supression of the wave in the medium}$$

6.3.1 Metal

We will now look at the special case of a metal:

$$\omega_0 = 0$$

$$\chi(\omega) = \frac{\frac{Nq^2}{\varepsilon_0 m}}{-\omega^2 + i\gamma\omega}$$

For a metal the γ is related to what we macroscopically call resistance or its inverse (conductivity).

γ is given untimately by Ohms law:

$$q\vec{E} = m\left(\ddot{\vec{x}} + \gamma\dot{\vec{x}}\right)$$

$$\vec{E} = \text{const}$$

$$q\hat{E} = m\left(\dot{\vec{v}} + \gamma\vec{v}\right)$$

$$\Rightarrow \vec{v} = \vec{v}_{\text{drift}} + \vec{v}_0 e^{-\gamma t}$$

v_0 vanishes at large times and:

$$\vec{v}_{\text{drift}} = q\frac{\vec{E}}{m\gamma}$$

This is the equilibrium drift velocity of the electrons in the metal when voltage acceleration and resistance are balanced.

$$\vec{J} = Nq\vec{v}_{\text{drift}} = \frac{nq^2}{m\gamma}\vec{E}$$

$$\Rightarrow \vec{J} = \sigma\vec{E}$$

$$\sigma = \text{conductivity} \Rightarrow \gamma = \frac{Nq^2}{m\sigma}$$

This relates the microscopic susceptibility to the macroscopic conductivity. Now we can relate χ to a macroscopic quantity.

We will look at the behavior in two limits:

- $\omega \ll \gamma$

Low frequency.

$$n = \text{index of refraction}$$
$$n^2 = 1 + \frac{\frac{Nq^2}{\varepsilon_0 m}}{-\omega^2 + i\omega\frac{Nq^2}{m\sigma}}$$
$$\Rightarrow n^2 \approx -i\frac{\sigma}{\varepsilon_0\omega}$$
$$\Rightarrow n^2\sqrt{\frac{\sigma}{\varepsilon_0\omega}}\frac{1-i}{\sqrt{2}}$$

We notice that the index of refraction is complex. The real part is the refractive index, the imaginary part is the absorption coefficient.

$$\begin{aligned}\vec{E} &= \vec{E}_0 e^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)} \\ &= \vec{E}_0 e^{i\omega\left(t - \frac{\vec{k}}{\omega}\cdot\vec{x}\right)} \\ &\quad {\scriptstyle (i\omega\left(t - \frac{|k|}{\omega}\times\cos\varphi\right)} \\ &\quad {\scriptstyle = \vec{E}_0 e^{i\omega(t - n\frac{x}{c}\cos\varphi)}} \\ &\quad {\scriptstyle = \vec{E}_0 e^{i\omega(t-(n_R \cdot i n_I)\frac{x}{c}\cos\varphi)}} \\ &\quad {\scriptstyle = \vec{E}_0 e^{i\omega(t - n_R\frac{x}{c}\cos\varphi)} e^{-n_I\omega\frac{x}{c}\cos\varphi}} \\ &\quad {\scriptstyle n_I = \sqrt{\frac{\sigma}{2\varepsilon_0\omega}}} \\ &\quad {\scriptstyle |\vec{E}| = |\vec{E}_0| e^{-\sqrt{\frac{\sigma\omega}{2\varepsilon_0}}x\cos\varphi}} \\ &\quad {\scriptstyle = |\vec{E}_0| e^{-\frac{x}{\delta}\cos\varphi}} \\ &= \vec{E}_0 e\end{aligned}$$

$$\left|\vec{E}\right| = \left|\vec{E}_0\right| e^{-\frac{x}{\delta}}$$
$$\delta = \sqrt{\frac{2\varepsilon_0}{\sigma}}\omega c$$

With a frequency of 10^4 MHz:

$$\delta \approx 6.7 * 10^{-7}\text{m}$$

- $\omega \gg \gamma$

High frequency limit.

$$n^2 = 1 + \frac{\frac{Nq^2}{\varepsilon_0 m}}{-\omega^2 + i\omega\frac{Nq^2}{m\sigma}}$$
$$\Rightarrow n^2 \approx 1 - \frac{\frac{Nq^2}{\varepsilon_0 m}}{\omega^2}$$
$$\Rightarrow \omega_p^2 = \frac{Nq^2}{\varepsilon_0 m}$$
$$\omega_p = \text{plasma frequency}$$
$$\Rightarrow n^2 \approx 1 - \frac{\omega_p^2}{\omega^2}$$

for $\omega_p < \omega \Rightarrow n^2 > 0 \Rightarrow n$ real. This means periodic wave propagation.

When $\omega > \omega_p \Rightarrow n^2 < 0 \Rightarrow n$ conplex. This means that the wave is damped and dies off after some length.

This means that metals become transparent at high frequencies. The plasma frequency is the frequency at which the metal becomes transparent.

6.4 Reflection and Refraction

We have materials with refraction indices n_1, n_2. An incident wave with angle φ_1 will be reflected and refracted at the interface between the two materials.

The wave satisfies Maxwell's equations at every point, especially at the interface.

Assume that n_1 is very different from n_2. Maxwell's equations in the boundary region:

$$\vec{\nabla}\cdot\vec{E} = -\frac{1}{\varepsilon_0}\vec{\nabla}\cdot\vec{P}$$
$$\Rightarrow \partial_x\vec{E}_x + \partial_y\vec{E}_y + \partial_z\vec{E}_z = -\frac{1}{\varepsilon_0}\left(\partial_x\vec{P}_x + \partial_y\vec{P}_y + \partial_z\vec{P}_z\right)$$

The interface is at $x = 0$. $\partial_x\vec{E}_x$ is large in comparison to the other terms. For close to the boudary:

$$\frac{\partial}{\partial x}E_x = -\frac{1}{\varepsilon_0}\frac{\partial}{\partial x}P_x$$
$$\Rightarrow \frac{E_{x_2} - E_{x_1}}{x_2 - x_1} = -\frac{1}{\varepsilon_0}\frac{P_{x_2} - P_{x_1}}{x_2 - x_1}$$
$$\Rightarrow E_{x_2}\varepsilon_0 + P_{x_2} = E_{x_1}\varepsilon_0 + P_{x_1}$$

We can repeat this for the other maxwell equations. We get:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\Rightarrow \frac{\partial E_z}{\partial x} \approx 0$$
$$\Rightarrow \frac{\partial E_y}{\partial x} \approx 0$$

As we go to the boundary we have a parallel and perpendicular component of the electric field. The parallel part is the same close to the boundary. Also the magnetic field is the same close to the boundary.

$$\vec{E}_{1_\parallel} = \vec{E}_{2_\parallel}$$
$$\vec{B}_1 = \vec{B}_2$$

The transverse components must fulfill the boundary conditions:

$$\left(\varepsilon_0\vec{E}_1 + \vec{P}_1\right)_\perp = \left(\varepsilon_0\vec{E}_2 + \vec{P}_2\right)_\perp$$

6.5 Fresnel Equations

Let's derive the Fresnel equations directly from the boundary conditions we established at the interface between two media. We'll use Maxwell's equations and the continuity conditions at the boundary.

Consider two media with different refractive indices n_1 and n_2, with the interface at $x = 0$. From our boundary conditions, we know:

$$\vec{E}_{1_\parallel} = \vec{E}_{2_\parallel}$$
$$\vec{B}_{1_\parallel} = \vec{B}_{2_\parallel}$$
$$\left(\varepsilon_0\vec{E}_1 + \vec{P}_1\right)_\perp = \left(\varepsilon_0\vec{E}_2 + \vec{P}_2\right)_\perp$$
$$\vec{B}_{1_\perp} = \vec{B}_{2_\perp}$$

For dielectric media where $\vec{P} = \varepsilon_0\chi\vec{E}$:

$$\varepsilon_0(1+\chi_1)\vec{E}_{1_\perp} = \varepsilon_0(1+\chi_2)\vec{E}_{2_\perp}$$
$$\Rightarrow \varepsilon_0\varepsilon_1\vec{E}_{1_\perp} = \varepsilon_0\varepsilon_2\vec{E}_{2_\perp}$$
$$\Rightarrow n_1^2\vec{E}_{1_\perp} = n_2^2\vec{E}_{2_\perp}$$

Let's consider a plane wave incident on the boundary:

$$\vec{E}_i = \vec{E}_{0_i} e^{i\left(\vec{k}_i\cdot\vec{x} - \omega t\right)}$$

For s-polarization (TE mode), where the electric field is perpendicular to the plane of incidence:

$$\vec{E}_{\text{inc}} + \vec{E}_{\text{refl}} = \vec{E}_{\text{trans}}$$

For the magnetic field, from Maxwell's equation $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, we get:

$$\vec{k}_{\text{inc}}\times\frac{\vec{E}_{\text{inc}}}{\omega} + \vec{k}_{\text{refl}}\times\frac{\vec{E}_{\text{refl}}}{\omega} = \vec{k}_{\text{trans}}\times\frac{\vec{E}_{\text{trans}}}{\omega}$$

Using $\left|\vec{k}\right| = n\frac{\omega}{c}$ and applying the boundary conditions:

$$n_1\left(\cos(\theta_1)\vec{E}_{\text{inc}} - \cos(\theta_1)\vec{E}_{\text{refl}}\right) = n_2\cos(\theta_2)\vec{E}_{\text{trans}}$$

Solving these equations simultaneously:

$$r_s = \frac{\vec{E}_{\text{refl}}}{\vec{E}_{\text{inc}}} = \frac{n_1 \cos(\theta_1) - n_2 \cos(\theta_2)}{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)}$$

$$t_s = \frac{\vec{E}_{\text{trans}}}{\vec{E}_{\text{inc}}} = \frac{2n_1 \cos(\theta_1)}{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)}$$

For p-polarization (TM mode), where the magnetic field is perpendicular to the plane of incidence:

$$\vec{E}_{\text{inc}} \cos(\theta_1) - \vec{E}_{\text{refl}} \cos(\theta_1) = \vec{E}_{\text{trans}} \cos(\theta_2)$$

Using the boundary condition for the magnetic field:

$$\left(\frac{n_1}{c}\right)\left(\vec{E}_{\text{inc}} + \vec{E}_{\text{refl}}\right) = \left(\frac{n_2}{c}\right)\vec{E}_{\text{trans}}$$

This yields:

$$r_p = \frac{\vec{E}_{\text{refl}}}{\vec{E}_{\text{inc}}} = \frac{n_2 \cos(\theta_1) - n_1 \cos(\theta_2)}{n_2 \cos(\theta_1) + n_1 \cos(\theta_2)}$$

$$t_p = \frac{\vec{E}_{\text{trans}}}{\vec{E}_{\text{inc}}} = \frac{2n_1 \cos(\theta_1)}{n_2 \cos(\theta_1) + n_1 \cos(\theta_2)}$$

These are the Fresnel equations derived directly from the electromagnetic boundary conditions. They describe how the amplitudes of reflected and transmitted waves relate to the incident wave at the interface between two media.

For completeness, Snell's law $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$ emerges from the requirement that the phase matching condition holds at the boundary, ensuring that the tangential components of the wave vectors are equal across the interface.

7 QED

Maxwell's equations are the classical limit of quantum electrodynamics (QED). In QED, we have to take into account the quantum nature of the electromagnetic field and the particles that interact with it.

In QED, everything is quantized in terms of propability amplitudes.

$$P = |M|^2$$
$$M = \text{propability amplitude}$$

We compute propabilities P for **Events**. Events are defined as a set of initial and final states.

7.1 Classical propabilities

convencionally, we know propabilities to work the following way:

$$P(A \to B) = |M(A - B)|^2$$
$$A \overset{C_1}{\to} B : P_1$$
$$A \overset{C_2}{\to} B : P_2$$
$$\Rightarrow P = P_1 + P_2$$

$$A \to C \to B$$
$$P(A \to B) \cdot P(C \to B) = P(A \to B)$$

7.2 Quantum propabilities

$$A \overset{C_1}{\to} B : M_1$$
$$A \overset{C_2}{\to} B : M_2$$
$$\Rightarrow M = M_1 + M_2$$
$$P = |M|^2 = P_1 + P_2 + 2\,\mathrm{Re}(M_1^* M_2)$$

Where $\mathrm{Re}(M_1^* M_2)$ is the interference term. This is the quantum mechanical way of computing propabilities.

$$A \to C \to B$$
$$M(A \to B) = M(A \to C) M(C \to B)$$

7.3 Events

An even can be that we have a photon at $t = 0$ and an electron and positron $t > 0$. This is a transition.

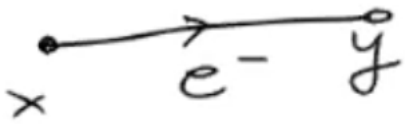
This can be complicated. We use the fact that all events can be composed of simpler events. Eventually we can deduce everything to a set of events that are simple enough to be computed.

Here we have 3 types of events:

1. Photon can move drom x to y, where x and y are points in space-time.

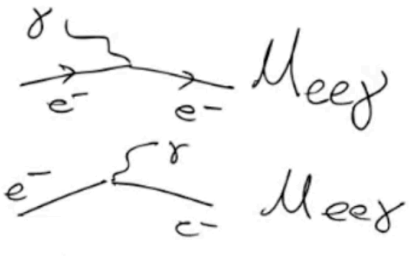
   For this we have an associated propability amplitude $M(x \to y)$.

2. Electron can move from x to y.



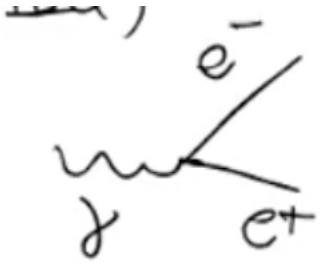
   For this we have an associated propability amplitude $M_{e(x \to y)}$.

3. At point x a photon is absorbed or emitted by an electron.



   For this we have an associated propability amplitude $M_{ee\gamma}$.

4. A photon can split into an electron and a positron.



   For this we have an associated propability amplitude $M_{\gamma \to ee}$.

Type 1 and 2 are the simplest and very similar.

7.4 Feynman Rules

A photon going from x to y is formulated as:

$$= \int \mathrm{d}^4 \vec{x}\, \frac{1}{(2\pi)^4} \left(\frac{e^{-ik\cdot(x-y)}}{k^2 + i\delta} \right) (-g^{\mu\nu}), \delta \to 0$$

This is a Green's function for the photon propagator:

$$\square \_\_ = -g^{\mu\nu} \delta^{(4)}(x - y)$$

k can be thought of as the momentum of the photon. If you integrate over all momenta, you get the full propability amplitude.

The most important contribution came from:

$$k^2 \approx 0 \to E_k^2 - \left|\vec{k}\right|^2 = 0$$
$$\Rightarrow E_k = \left|\vec{k}\right|$$

This means the photon is massless and travels at the speed of light. This is the rule of the photon propagator.

The $g^{\mu\nu}$ corresponds to the polarization of the photon. Most of the time we will use the same polarization for all photons, so it will equate to 1.

For the movement of an electron from x to y we have:

$$= \int \mathrm{d}^4 k\, \frac{1}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m_e^2 + i\delta} \cdot /(k + m1)$$
$$/(k) + m1 = \left(\gamma^\mu k_\mu + m_e 1\right)_{4\times 4}$$

The numberator is because $k^2 = m_e^2$ in special relativity.

The spinor is a mathematical object that describes the spin of the electron. It is a 4-component object that transforms under Lorentz transformations.

The final rule:

$$M_{ee\gamma} = -ie\gamma^\mu$$
$$e = \text{electric charge}$$
$$\gamma^\mu = \text{Spin of the electron}$$

If the particle is spinless:

$$M_{ee\gamma} = -ie$$

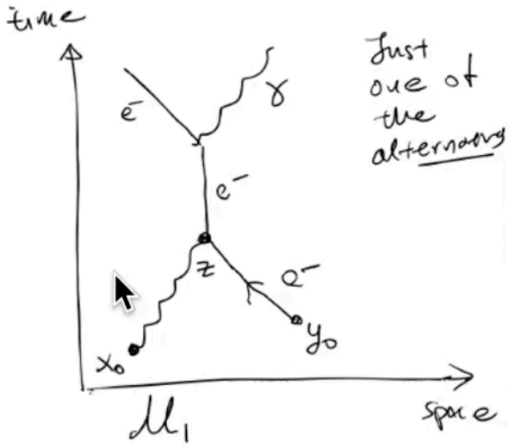
The **Fine Structure Constant** is the propability of an electron to emit or absorb a photon. It is a dimensionless constant that characterizes the strength of the electromagnetic interaction between charged particles. It is defined as:

$$\alpha = \frac{e^2}{4\pi} \leq 1\%$$

This means that the interaction between electrons and photons is very weak. This is why we can use the classical approximation of Maxwell's equations for macroscopic systems.

7.5 QED Thompson Scattering

An electron and photon meet in spacetime. The electron absorbs the photon and then emits it again. This is called Thompson scattering.



To compute the propability amplitude for this process, we can use the Feynman rules we established earlier.

$M_1 =$	
$M_\gamma(x_0 \to z)$	photon from x_0 to z
$\cdot\, M_e(y_0 \to z)$	electron from y_0 to z
$\cdot\, M_{ee\gamma}(z)$	photon absorbed by electron at z
$\cdot\, M_e(z \to z')$	electron from z to z'
$\cdot\, M_{ee\gamma}(z')$	photon emitted by electron at z'
$\cdot\, M_e(z' \to x_1)$	electron from z' to x_1
$\cdot\, M_\gamma(z' \to y_1)$	photon from z' to y_1

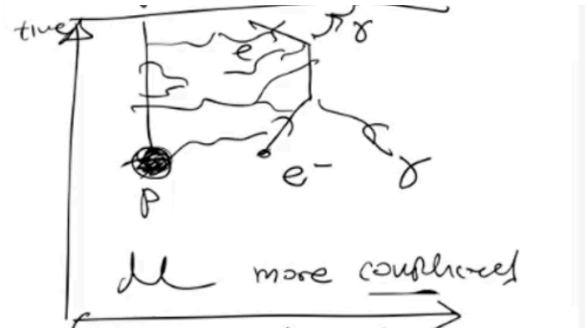
There are also other alternative processes that correspond to the same event. We can compute the propability amplitude for each of them and then sum them up.

We have to consider that most processes of an event are extremely complex and unlikely. We only consider the simplest processes that contribute significantly to the propability amplitude.

7.6 QED Rayleigh Scattering

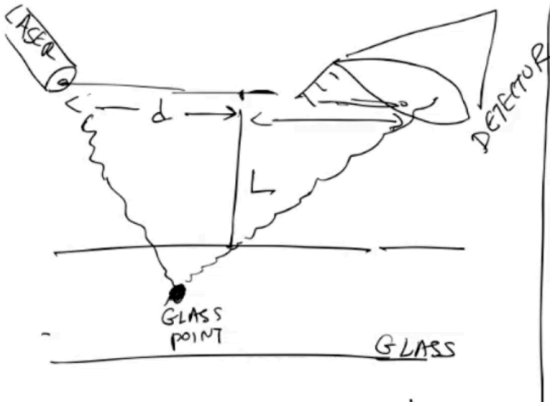
For Rayleigh scattering, we have to consider the interaction of a photon with an electron in the presence of an proton.

This makes the amplitude more complex and depends on the material.



7.7 Glass Optics

We produce a photon. In order to have reflection, the photon will arrive at the glass and interact with the electrons in the glass. A new photon will be emitted in the direction of the detector. We can see that this is a different photon.



We choose to compute only the simplest processes which contribute significantly to the propability amplitude.

$$\begin{aligned} &M(\gamma \text{ Laser} \to D) \\ &= M(\gamma @ \text{ Laser}) \cdot M_\gamma(L_{\to} X_{\text{Glass}}) \cdot M_R \cdot M_\gamma(X \to D) \\ &\approx M(\gamma @ \text{ Laser}) \cdot M_{\gamma \text{ classical}}(L \to X) M_R \cdot M_\gamma(X \to D) \\ &\approx M_R \cdot M(\gamma @ \text{ Laser}) \end{aligned}$$

We can assume that the photon travel in classical paths. To get a full answer, we have to intergrate over all posssible reflection points X in the glass.

$$\int \mathrm{d}^3X \, M_R \cdot M(\gamma @ \text{ Laser})$$

We assume the laser light is monochromatic. That also means that we can't know the exact time of emission of the photon (uncertainty principle).

We find that the amplitude to create a photon at the laser:

$$M(\gamma @ \text{ Laser}, t) \propto e^{i\omega t} \Rightarrow M(\gamma \text{ from Laser} \to D) = \int \mathrm{d}^3X \, e^{i\omega t} M_R$$

But the time before you detect it depends on the path length L. A photon that was created early, will have more possbility of paths compare to photons that were created later. This is why we have to integrate over all possible paths.