<u>Draft papers to accompany the paper *Counterfactuals in Kantian causal networks* by M de Neeve</u>

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2 Preliminaries.

In this section we will introduce some fundamental notions that are to be used in what is to follow. We will just present the most important results, without aiming at a self-contained exposition; the reader is invited to explore the references when in need of a more systematic treatment of the concepts involved.

Definition 1. *Let $\mathcal{L}$ be a language of propositional logic over a finite set ATOM($\mathcal{L}$) of atomic sentences. A positive clause is a formula of the form $p_1, ..., p_n \to q$, where $p_1, ..., p_n, q \in ATOM(\mathcal{L})$. We refer to $p_1, ..., p_n$ as to the body of the clause, and to q as to the head of the clause. If the clause has the special form $\top \to q$, where $\top$ is the customary symbol for Verum (tautology), we refer to q as a fact. A positive program P is a finite set of positive clauses.*

Definition 2. *Let $\mathcal{L}$ be a language of propositional logic over a finite set ATOM($\mathcal{L}$) of atomic sentences. Let a literal be an atomic sentence or the negation of an atomic sentence. Thus, for all $p \in ATOM(\mathcal{L})$, both p and $\neg p$ are literals, and nothing else is a literal. A definite clause is a formula of the form $A_1, ..., A_n \to q$, where $A_1, ..., A_n$ are literals and $q \in ATOM(\mathcal{L})$. The nomenclature for definite clauses is analogous to the one for positive clauses exposed above. A definite program P is a finite set of definite clauses.*

The declarative semantics for logic programs is usually expressed in terms of *Herbrand Models*. Given a logic program P we will refer to the *Herbrand Universe* of P, HU_P, to the *Herbrand Base* of P, HB_P, to the three-valued immediate consequence operator of P, Φ_P, and to the usual two-valued immediate consequence operator T_P. We will also refer to the completion of a program P, $comp(P)$. Since for the moment we will consider only the propositional case (as in the above definitions), we notice that for a program P

formulated in the language $\mathcal{L}$: $HU_P = \{c\}$, where c is an arbitrary constant, and $HB_P = ATOM(\mathcal{L})$. We will therefore speak simply of *atoms* instead of *ground atoms*, and of *clauses* instead of *ground clauses*.

Definition 3. *Let P be a program. We define $||$ as a function: $|| : HB_P \to \mathbb{N}$ which assigns to every atom $p \in HB_P$ a natural number. We extend $||$ to comprise literals by positing: $|\neg p| = |p|$ for every atom $p \in HB_P$. We call $|p|$ (resp. $|\neg p|$) the level of p (resp. $\neg p$).*

We can now define what it means for a program to be acyclic.

Definition 4. *Let $\mathcal{L}$ be a language of propositional logic, P be a program, and $||$ be a level function from HB_P into $\mathbb{N}$. We say that P is acyclic with respect to $||$ iff, for every clause $A_1, ..., A_n \to q$ belonging to P, it holds that $|A_i| < |q|$ for every $1 \leq i \leq n$. We say that P is acyclic if there exists a level function $||$ with respect to which P is acyclic.*

We can now proceed to give the declarative semantics for positive and definite acyclic logic programs. We will give a three-valued semantics based on Kleene's strong three valued logic, which makes use of the three truth values t (True), f (False), u (Undefined). The semantics can be briefly explained as follows: negation turns True into False and viceversa, while leaving Undefined unchanged. A conjunction is true iff all of the conjuncts are true, false iff at least one conjunct is false, undefined otherwise. Disjunction is dual.

Definition 5. *A Herbrand Interpretation I is a pair (T,F) of disjoint sets of atoms. An atom p is true on I iff $p \in T$. An atom p is false on I iff $p \in F$. An atom p is undefined on I iff both $p \notin T$ and $p \notin F$ hold. We denote T with I^+ and F with I^-, so that $I = (I^+, I^-)$.*

Definition 6. *Given a logic program P and a clause $c \in P$, a Herbrand Interpretation M is a model for c iff it holds that: if $M \models_3 Body(c)$, then $M \models_3 Head(c)$, and if $M \not\models_3 Body(c)$, then $M \not\models Head(c)$. Given a logic program P, a Herbrand Interpretation M is a model for P iff M is a model for all the clauses $c \in P$.*

This three-valued semantics comprises, as a special case, a two-valued semantics in which the only possible truth-values are true or false. The two-valued semantics can be obtained from these definitions by letting no atom undefined, that is, for all atoms p and Interpretations I, either $p \in I^+$ or $p \in I^-$.

Definition 7. *Given a language $\mathcal{L}$ and a program P, the set of Herbrand Interpretations for P forms a complete partial order, such that $I \leq J$ iff $I^+ \subseteq J^+ \wedge I^- \subseteq J^-$.*

We can now define an operator Φ_P which takes as an input a Herbrand Interpretation for a program P and renders another Herbrand Interpretations as an output in the following way: $\Phi_P(I) = (T, F)$, where:

$$T = \{q \in HB_P \mid \text{there is a clause } A_1, ..., A_n \rightarrow q \text{ belonging to P } \mid I \models_3 A_1, ..., A_n\}$$
$$F = \{q \in HB_P \mid \text{there is a clause } A_1, ..., A_n \rightarrow q \text{ belonging to P } \mid I \not\models_3 A_1, ..., A_n\}$$

It is easy to see that $\phi_P(I) = (T, F)$ is such that $T \cap F = \emptyset$ and that the operator ϕ_P is monotone, in the sense that if $I \leq J$ then $\phi_P(I) \leq T_P(J)$. The following facts can also be proven:

Theorem 1. *Let $\mathcal{L}$ be a language of propositional logic and P be an acyclic program. Then:*

- *The operator ϕ_P has a least fixpoint.*
- *The least fixpoint LM_P of ϕ_P is the minimal Herbrand Model of P in the sense of the complete partial ordering as in the definition above.*
- *All the fixpoints of ϕ_p are models of the completion of P, comp(P), and every model of comp(P) is a fixpoint of ϕ_p.*

It is possible to compute the minimal model LM_P for a program P by starting with the empty Interpretation I and iteratively apply the operator ϕ_P, thus obtaining the sequence $M_0 \subseteq \Phi_P(M_0) \subseteq \Phi_P(\Phi_P(M_0)) \subseteq ... \subseteq \Phi_P^n(M_0)$. It is remarkable that the least fixpoint of ϕ_p is reached in finitely many steps ($n + 1$ steps if the program consists of n clauses). In the two-valued case, since no atom can be undefined, we can modify the definition of Φ_P by letting:

$$T = \{q \in HB_P \mid \text{there is a clause } A_1, ..., A_n \rightarrow q \text{ belonging to P } \mid I \models_3 A_1, ..., A_n\}$$

In which case we have back the usual definition of the two-valued immediate consequence operator T_P. It is then possible to compute the least-fixpoint of T_P by applying it iteratively to the Interpretation $I =< T, F >$ such that for all $p \in HB_P$, $p \in F$, i.e, the Interpretation that renders all atoms false. It is very important to notice that this is true only if the program P is acyclic.

It is well know that, given a general definite program P, the two-valued operator T_P does not always have a least fixpoint. For instance, it is easy to see that given the program $P = \{\neg p \rightarrow p\}$, T_P does not have a least fixpoint, since the value of p swings between true and false at each successive application of T_P. The fact that T_P does not have a least fixpoint for a general definite program P is due to the fact that the inclusion of negation in the body of the clause destroys its monotonicity; that is the reason why three-valued semantics was originally adopted, in order to be able to incrementally compute the least fixpoint for definite programs, i.e., programs with negation in the body of a clause. However, it can be shown that if the definite program P is acylic, that is, there is no recursion in the clauses of the program, then the two-valued operator T_P is monotone and therefore has a least fixpoint. In particular, if P is a definite acyclic program, then theorem 1, with T_P in place of Φ_P, holds. For acyclic programs, then, two-valued semantics is really a special case of the three-valued one, and T_P is the two-valued equivalent of Φ_P.

In what follows, apart from these notions about logic programs, we will assume familiarity with basic notions of graph theory, in particular with directed acyclic graphs, and with bayesian networks and bayesian probability theory. We will just give the following:

Definition 8. *Let $G =< V, E >$ be a directed acyclic graph. We say that G represents a causally sufficient causal structure C for a populations of units iff to each unit in C there corresponds a vertex in G, and there is a directed edge from A to B in G iff A is a direct cause of B. In this case, we will refer to G as to a "causal graph".*

We will now give the two axioms which connect a causal graph $G =< V, E >$ with probability distributions defined over V.

Axiom 1. *Let $G =< V, E >$ be a causal graph and P be a joint probability distribution over the vertices in V. We say that $< G, P >$ satisfies the* Markov Condition *iff for every vertex $W \in V$, $P(W|Parents(W), U) = P(Parents(W))$ for $U \subseteq V - (\{W\} \cup Descendents(W) \cup Parents(W))$; that is, W is conditionally independent of its non-descendendts given its parents.*

Axiom 2. *Let $G =< V, E >$ be a causal graph and P a joint probability distribution over the vertices in V. We say that $< G, P >$ satisfies the* Minimality condition *iff for every proper subgraph H of G with the same vertex V, it holds that $< H, P >$does not satisfy the* Markov condition.

It is easy to verify that if a causal network $< G, P >$ satifies the Markov condition, then its joint probability distribution can be factorised in the usual way:

$$P(X_1 = x_1, ..., X_n = x_n) = \prod_{v=1}^{n} P(X_v = x_v | X_j = x_j \text{ for all } X_j \in Parents(X_v))$$

Logic Programming and Causal networks: Two-Valued approach.

The purpose of this section is to explore the connection between logic programs and causal network, where the semantics for the programs is to be two-valued. We will first show how an acyclic logic program interpreted with the usual two-valued closed-world semantics, if properly translated into a causal network, enforces on the network the Minimality Condition. We will then show, viceversa, that if a causal network obeys Markov's Condition and the Minimality Condition, then it defines a logic program which obeys the two-valued closed-world semantics. This will be seen to establish a direct correspondence between causal networks and logic programs, and in particular between closed-word semantics and the Minimality Condition. We start by treating the case of positive programs, i.e., of programs in which negation cannot occur, and we move then to treat the case of definite programs. We start by giving a definition.

Definition 9 (Annotated graph)**.** *Let $\mathcal{L}$ be a language of propositional logic, P an acyclic positive program in which no atom p occurs more than once as the head of a clause. The* annotated graph *$G =< V, E >$ for P is obtained by letting $V = HB(P) = ATOM(\mathcal{L})$, and for every ordered couple $< A, B >$, where $A, B \in HB(P)$, there is a directed edge from A to B in G iff there is a clause $c \in P$ such that $A \in Body(c)$ and $B = head(c)$. We will call the program P as the program associated to the graph G.*

This definition intuitively captures the idea that the atoms in a positive program are to be thought as the vertices of the graph, i.e., as the random variables over which a probability distribution will be defined. Furthermore, in a logic program the elements in the body of a clause are interpreted as *direct causes* of the head of the clause, and therefore they are connected with directed edges. This naturally specifies a causal graph, which is easily seen to be acyclic, as follows.

Lemma 1. *Let $\mathcal{L}$, P, G as in the above definition. Then, G is acyclic.*

Proof. Since P is acyclic, for every clause $c \in P$ it holds that $|A_i| < |q|$ for all $A_i \in Body(c)$ with respect to some level function $||$. Suppose G were not acyclic. Then there would be a directed path from A to B and from B to A for some $A, B \in V$. Let $< A, X_1, ..., X_n, B >$ be the path from A to B (it must be finite, since the number of clauses in the program are finite and we are treating the propositional case). Let $< B, Y_1, ..., Y_n, A >$ be the path from B to A. From the construction of G and transitivity of $||$ we would then have that $|A| > |X_1| > ... > |X_n| > |B|$ and $|B| > |Y_1|, ..., > |Y_n| > |A|$, and therefore that $|A| > |A|$. Contradiction. Therefore G is acyclic. □

We are now able to associate, to an acyclic positive program P, an acyclic graph G whose verteces are the atoms of P. We would now have to define a joint probability distribution over G in order to obtain a causal network $< G, P >$. Since we are mainly interested in modelling causal reasoning, we are to make some assumptions that are justified by the fact that we are dealing with this specific domain. First, we will assume that literals (which are just atoms in the case of positive programs) identify elementary events which can either occur or not occur. In this interpretation, if the atom p represents the event "The Sun rises", we will postulate that it is clear what the event is and that it is clear whether the event happened or not. We then assume that probabilities range over the set $\{1, 0\}$, that is, either an event has probability 1 or it has probability 0. Then, if we assume that a program P is interpreted according to two-valued closed-world semantics, and we intepret each vertex of its annotated graph G as a random variable X with possible states $t = True$ and $f = False$, we can define a partial conditional probability distribution over $< G, V >$ as follows:

Definition 10 (Partial Probability Distribution)**.** *Let P be an acyclic positive program interpreted according to two-valued closed-word semantics, and $< G, V >$ be its annotated graph. We can define a partial conditional probability distribution $\mathcal{P}$ over V as follows. If $X \in V$, and $< Y_1, ..., Y_n >$ are the parents of X in V, then*

- $P(X = t | Y_1 = y_1, ..., Y_n = y_n) = 1$ *if* $y_1 = t, ..., y_n = t$
- $P(X = f | Y_1 = y_1, ..., Y_n = y_n) = 1$ *if* $y_i = f$*, where* $1 \leq i \leq n$

The purpose of this definition is to translate, into the language of probabilities, the usual closed-world semantics with which the program P is interpreted. Indeed, the two clauses above encode in the language of probability theory the behaviour of the operator T_P, in the sense that if P is a program in which the clause $p_1, ..., p_n \vdash q$ occurs, and I is a herbrand interpretation such that $I \models p_1, ..., p_n$, then we know that $T_P(I) \models q$. Analogously, If $parents(q) = \{p_1, ..., p_n\}$ have all value true, then $P(q = t) = 1$. In other words, a variable X is true with probability 1 if all of its parents (i.e., the body of the associated clause in the program) have the value true, otherwise it is false. We notice, however, that the notion of conditional probability employed here cannot be the usual one, that is, that which is due to Kolmogorov. In his treatment, indeed, conditional probability is defined from absolute probability as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Where A, B belong to the σ-algebra F of events. The shortcoming of this definition is that the conditional probability $P(A|B)$ is undefined when $P(B) = 0$; and this means that, since in our case we are dealing with events that can have probability zero, most of the times our conditional probability will be undefined. For instance, consider the conditional probability $P(X = t|Y_1 = t, Y_2 = t)$, when $P(Y_1 = t) = 1, P(Y_2 = t) = 0$. If we adopted Kolmogorov's definition, we would have:

$$P(X = t|Y_1 = t, Y_2 = t) = \frac{P(X = t \cap Y_1 = t \cap Y_2 = t)}{P(Y_1 = t \cap Y_2 = t)} = \frac{P(X = t \cap Y_1 = t \cap Y_2 = t)}{P(Y_1 = t|Y_2 = t)P(Y_2 = t)}$$

Since the last member of the equality is undefined (the denominator is zero), the conditional probability is undefined; but, according to our definition above, we would want it to be defined, namely, we would want $P(X = t|Y_1 = t, Y_2 = t) = 1$, despite the fact that $P(Y_2 = t) = 0$. In order to achieve this, we need to abandon Kolmogorov's probability and recur to Rényi's axiomatization of probability, which allows us to take conditional probability as primitive, and define in terms of conditional probability absolute probabilities. We briefly recall Rényi's axiomatization of probability, as follows:

Definition 11 (Rényi's axioms of probability)**.** *Let Ω be a non-empty set, $\mathcal{A}$ be an algebra on Ω, $\mathcal{B}$ be a non-empty subset of $\mathcal{A}$. The conditional probability function $\mathcal{P}$ is a function from the cartesian product between $\mathcal{A}$ and $\mathcal{B}$ to $\mathbb{R}$ such that:*

1 $P(A|B) \geq 0$

2 $P(B|B) = 1$

3 $P(A_1 \cup A_2 \cup ... \cup A_n|B) = \Sigma_{i=1}^{i=n} P(A_i|B)$, *where* $A_1, ..., A_n$ *are disjoint.*

4 $P(A_1 \cap A_2|B) = P(A_1|A_2 \cap B)P(A_2|B)$

Where the As belong to $\mathcal{A}$ and the Bs belong to $\mathcal{B}$.

Rényi's definition of probability assumes conditional probability as the primitive concept, that is, it takes as primitive a *conditional probability space* $< \Omega, \mathcal{A}, \mathcal{B}, P(A|B) >$. Absolute probabilities can then be defined from conditional probabilities as follows:

$$P(A) = P(A|\Omega)$$

We will therefore consider the partial conditional probability distribution $\mathcal{P}$ defined above as a partial specification of a conditional probability function in Rényi's sense. We can now show that if a positive program P is interpreted according to closed-word semantics, then extending the annotated graph of P to a network satisfying Markov's condition enforces on the network the minimality condition.

Lemma 2. *Let P be a positive program, $G =< V, E >$ its annotated graph, $\mathcal{P}$ the partial conditional distribution as in definition 10. Then if $< G, \mathcal{P} >$ is extended to a causal network S satisfying Markov's condition, S satisfies the Minimality Condition.*

Proof. Suppose $S =< G, \mathcal{P} >$ satisfies Markov's condition. Then for all random variables X such that $Parents(X) = \{Y_1, ..., Y_n\}$ and for all random variables W such that $W \notin (Parent(X) \cup Descentents(X))$, it is the case that $P(X = x|Y_1 = y_1, ..., Y_n = y_n, W = w) = P(X = x|Y_1 = y_1, ..., Y_n = y_n)$. Suppose now that S does not satisfy the Minimality Condition. Then there must be a random variable X such that $P(X = x|Y_1 = y_1, ..., Y_i = y_i, ..., Y_n = y_n) = P(X = x|Y_1 = y_1, ..., Y_{i-1} = y_{i-1}, Y_{i+1} = y_{i+1}, ..., Y_n = y_n)$, where $1 \leq i \leq n$; which means that the proper subgraph obtained from S by cutting the directed edge from Y_i to X still satisfies Markov's condition, i.e., X is conditionally independent of Y_i given $Y_1 = y_1, ..., Y_{i-1}, Y_{i+1}, ..., Y_n$. But then in turn this means that $P(X = t|Y_1 = t, ..., Y_i = t, ..., Y_n = t) = P(X = t|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t)$, which contradicts the partial definition of the probability distribution $\mathcal{P}$. Therefore S must satisfy the Minimality Condition. $\square$

Recall that the Causal Markov Condition and the Minimality condition are, in themselves, independent of each other. The Lemma then shows that if a positive program is interpreted according to its usual closed-worlds semantics, which determines the partial definition of the probability distribution $\mathcal{P}$, then any extension of its annotated graph to a causal network satisfying

Markov's Condition must satisfy the Minimality condition. We have then established a connection between a program P with its closed-world semantics and the Minimality Condition, in the sense that the former enforces the latter on the causal network. We will now show the other direction; that is, we will show that given a causal network which satisfies both Markov's Condition and the Minimality condtion, then this causal network enforces the usual closed-world semantics on the associated program P. We therefore need to modify the partial definition of the conditional probability distribution that has been given in definition 10, since there we assumed that the program P obeyed the closed-world semantics, while here we want to show that this semantics is actually enforced on the program by the fact that the annotated causal network satisfies the Markov condition and the Minimality Condition. We can therefore change definition 10 as follows:

Definition 12. *Let P be an acyclic positive program, and $< G, V >$ be its annotated graph. We can define a partial conditional probability distribution $\mathcal{P}$ over V as follows. If $X \in V$, and $< Y_1, ..., Y_n >$ are the parents of X in V, then*

- $P(X = t|Y_1 = y_1, ..., Y_n = y_n) = 1$ *if* $y_1 = t, ..., y_n = t$

The difference between this definition and definition 10 is that the latter already assumed closed-world semantics in the second clause, while the former just assumes the uncontroversial fact that if the parents are true, then X is true with probability 1. This new deinition will allow us to show how the Minimality Condition enforces us to interpret the program P with its usual closed-world semantics.

Lemma 3. *Let P be a positive program, $G =< V, E >$ its annotated graph, $\mathcal{P}$ the partial conditional probability distribution as in definition 12. Then for every extension of $< G, \mathcal{P} >$ to a causal network S satisfying the Causal Markov Condition and the Minimality Condition, S enforces on P a closed-world semantics.*

To prove this lemma, we are going to prove that if $< G, \mathcal{P} >$ is extended to a causal network that satisfies Markov's Condition and the Minimality Condition, then for any variable X the probability that this variables assumes the value True, given that one of its parents is false, is zero; which in turn means that the probability that X assumes the value False is one. We then derive the second clause of definition 10.

Proof. Suppose $S =< G, \mathcal{P} >$ satisfies the Causal Markov Condition and the Minimality Condition. Consider an arbitrary random variable X with

$Parents(X) = \{Y_1, ..., Y_n\}$. Suppose $P(X = t|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t) = 1$, where $1 \leq i \leq n$. Since we know that $P(X = t|Y_1 = t, ..., Y_i = t, ..., Y_n = t) = 1$, we conclude that $P(X = t|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t) = P(X = t|Y_1 = t, ..., Y_i = t, ..., Y_n = t)$. But since we know that $P(X = t|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t) + P(X = f|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t) = P(X = t|Y_1 = t, ..., Y_i = t, ..., Y_n = t) + P(X = f|Y_1 = t, ..., Y_i = t, ..., Y_n = t) = 1$, we can conclude that $P(X = f|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t) = P(X = f|Y_1 = t, ..., Y_i = t, ..., Y_n = t)$. But then this means that, in general, $P(X = x|Y_1 = y_1, ..., Y_{i-1} = y_{i-1}, Y_i = y_i, Y_{i+1} = y_{i+1}, ..., Y_n = y_n) = P(X = x|Y_1 = y_1, ..., Y_{i-1} = y_{i-1}, Y_{i+1} = y_{i+1}, ..., Y_n = y_n)$. We therefore conclude that X is conditionally independent of Y_i given $Y_1, ..., Y_{i-1}, Y_{i+1}, ..., Y_n$. But then the subgraph H obtained from S by cutting the directed edge between Y_i and X satifies the Causal Markov Condition, and therefore S does not satisfy the Minimality Condition. Contradiction. Therefore $P(X = t|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t) \neq 1$, which means that $P(X = t|Y_1 = t, ..., Y_{i-1} = t, Y_i = f, Y_{i+1} = t, ..., Y_n = t) = 0$. □

We have then the following theorem:

Theorem 2. *Let P be a positive program, and $G =< V, E >$ its annotated graph. If it is possible to define a conditional probability distribution $\mathcal{P}$ over G that satisfies the Causal Markov Condition, and such that $P(X = t|Y_1 = t, ..., Y_n = t) = 1$ for any node X and parents $Y_1, ..., Y_n$, then the network $< G, P >$ satisfies the Minimality condition iff P is interpreted according to closed world semantics.*

Proof. Direct consequence from lemma 2 and 3. □

What we have proven in the last theorem is a conditional result, namely, we have proven that if it is possible to define a conditional probability distribution over the annotated graph of a program P which satisfies Markov's Condition, then there is a direct correspondence between the Minimality Condition and the closed world semantics for P. We will now prove that such an extension always exists, that is, that it is always possible to define a conditional probability distribution over the annotated graph that satisfies Markov's Condition.

Theorem 3. *Let P be a positive program, and $G =< V, E >$ its annotated graph. Then there exists a conditional probability distribution $\mathcal{P}$ such that for any node X and parents $Y_1, ..., Y_n$, $P(X = t|Y_1 = t, ..., Y_n = t) = 1$*

and $\mathcal{P}$ satisfies the Causal Markov Condition, that is, $P(X|Y_1,...,Y_n,W) = P(X|Y_1,...,Y_n)$ *for* $W \subseteq (V \backslash Parents(X) \backslash Descendents(X))$.

In order to prove this theorem, it needs to be shown that for any three valued interpretation I_3:

(1) If $I_3 \models Parents(X), W$ then $\Phi_P^k(I_3) \models X$

(2) If $I_3 \models Parents(X)$ and $I_3 \not\models W$, then $\Phi_P^k(I_3) \models X$

Where Φ_P is the three-valued monotonic consequence operator for P, and $\Phi_P^k(I_3)$ is the least fixpoint obtained by k iterations on M_3. We then define $P(X = t|Parents(X) = t, W = w) = 1$ iff $\Phi_P^k(I_3) \models X$, and from (1) and (2) above it follows easily that for $W \in (V \backslash Parents(X) \backslash Descendents(X))$, $P(X|Parents(X), W = w) = P(X|Parents(X))$. Proving (1) and (2) is straightforward using the definition of the consequence operator (proof by contradiction). It is important, though, to define the set $\mathcal{B}$ of possible conditions for the conditional probability space in such a way that these are consistent with the iterated computation of the least fixed-point $\Phi_P^k(I_3)$. For instance, consider the positive program defined by the following two clauses: $Y \vdash X$, $Y \vdash W$. Clearly, $(Y = t, W = f)$ is not a consistent condition. Indeed, if an interpretation $I_3 = (I^+, I^-)$ of P is such that $Y \in I_3^+, W \in I_3^-$, then at the least fixed-point $F = \Phi_P^1(I_3)$ we have $F^+ = \{Y, W, X\}, F^- = \{W\}$, which is clearly a non well-defined model of P; therefore $P(X|Parents(X) = t, W = f)$ cannot be well-defined in this case, so $(Parents(X) = t, W = f)$ cannot be in $\mathcal{B}$. We then need to give a proper recursive definition for the admissible conditions that can be members of $\mathcal{B}$. Once a condition C is consistent according to this definition, then $P(X|C)$ is well-defined, and by (1) and (2) above we obtain the Markov Condition.

Definition 13 (Recursive consistent conditions)**.** *Let P be a program, $G =< V, E >$ its annotated graph, $X \in V$ a vertex of P. Then we define the set $\mathcal{B}_P$ of consistent conditions recursively as follows:*

1 If $W \in V$ is a root vertex such that $(\forall Y \in V)(Y \in Parents(X) \rightarrow W \notin Ancestors(Y))$, then $(Parents(X) = t, W = f) \in \mathcal{B}_P$

2 if $W \in V$ is a non-root vertex, then $(Parents(X) = t, W = f) \in \mathcal{B}_P$ iff $\forall Y(Y \in Parents(X) \rightarrow W \notin Ancestors(Y)) \wedge (\exists S \in V)(S \in Ancestors(W) \wedge \forall Y(Y \in parents(X) \rightarrow S \notin Descendents(Y)) \wedge (Parents(X) = t, S = f) \in \mathcal{B}_P$

3 If W is any vertex, then $Parents(X) = t, W = t) \in \mathcal{B}_P$

Lemma 4. *If $C = (Parents(X) = t, W = f)$ belongs to $\mathcal{B}_P$, then $\Phi_P^k(I_3)$ is well-defined, where $I_3^+ = \{Y : Y \in Parents(X)\}, I_3^- = \{W\}$.*

Proof Sketch. Suppose $(Parents(X) = t, W = f)$ is a condition of type 1. Then W is a root vertex and it is not an ancestor of any of the parents of X. Let I_3 be the interpretation defined as: $I_3^+ = \{Y : Y \in Parents(X)\}, I_3^- = \{W\}$. We can prove by induction on k that $\Phi_P^k(I_3)$ is well defined. Clearly, the base case, where $K = 1$, holds (since W cannot be an ancestor of any Y_i which is a parent of X). The inductive case is trivial. Suppose now $(Parents(X) = t, W = f)$ is a condition of type 2. We show that $\Phi_P^k(I_3)$ is well defined. We just show the base case, for $k = 1$. Let $F = \Phi_P^1(I_3)$. Suppose F is not well defined. Then it means that there is a vertex L such that $L \in F^+$ and $L \in F^-$. Furthermore, either $L \in Parents(X)$ or $L = W$. If $L \in Parents(X)$, then since $L \in F^-$ this means that $W \in Parents(L)$, but this contradicts the definition of the condition. Otherwise, if $L = W$, then since $L \in F^+$ this means that $Parents(W) = Parents(X)$. By the definition of the condition, there must be S such that $S \in Ancestors(W) \wedge \forall Y(Y \in Parents(X) \rightarrow S \notin Descendents(Y)$. Clearly, $S \notin Ancestors(Y)$ for any Y parents of X, for otherwise $Parents(X) = t, S = f$ would not be defined. But then, since S is an ancestor of W, there must be a finite path $R_1, ..., R_n$ from S to W. Now, R_n is a parent of W, and clearly $R_n \notin Parents(X)$ for what has been said above. Therefore $Parents(X) \neq Parents(W)$, contradiction. We therefore conclude that $\Phi_P^1(I_3)$ is well defined. The inductive case is trivial. □

The proof of the lemma concludes the proof of the theorem.

2.1 Definite programs

The results obtained in the previous section can be easily extended to programs with negation in the body of the clauses, due to the fact that we are considering acyclic programs, which, as we outlined above, have two-valued least fixpoints. First of all, we have to modify the definition of annotated graph as follows:

Definition 14. *Let $\mathcal{L}$ be a language of propositional logic, P an acyclic definite program in which no atom p occurs more than once as the head of a clause. The* annotated graph *$G =< V, E >$ for P is obtained by letting $V = HB(P) \cup N(P)$, where $N(P)$ denotes the set of negated literals that appear in P. For every ordered couple $< A, B >$, where $A, B \in HB(P) \cup N(P)$, there is a directed edge from A to B in G iff one of the following obtains:*

- *there is a clause $c \in P$ such that $A \in Body(c)$ and $B = head(c)$.*

- $A \in HB(P)$, $B \in N(P)$, *and* $B = \neg(A)$.

The difference between this definition and the previous one is that now, since negated atoms are allowed to appear in the body of a clause, we consider an atom p and its negation $\neg p$ as two different units in the graph, and, obviously, we draw a directed edge from the former to the latter: $< p, \neg p > \in E$. It is clear that the graph G obtained according to this definition is acyclic (it is sufficient to modify slightly the result of lemma 1 above, which we leave to the reader); G is then a DAG. We can now reformulate the definitions of the partial probability distribution which we gave in the case of positive programs, and this will allow us to prove the same results we obtained there for definite programs.

Definition 15. *Let P be an acyclic definite program interpreted according to two-valued closed-word semantics, and $< G, V >$ be its annotated graph. We can define a partial conditional probability distribution $\mathcal{P}$ over V as follows. If $X \in V$, and $< Y_1, ..., Y_n >$ are the parents of X in V, then*

- *if $X \notin N(P)$, then $P(X = t|Y_1 = y_1, ..., Y_n = y_n) = 1$ if $y_1 = 1, ..., y_n = t$;*
- *if $X \notin N(P)$, then $P(X = f|Y_1 = y_1, ..., Y_n = y_n) = 1$ if $y_i = f$, where $1 \leqslant i \leqslant n$.*
- *if $X \in N(P)$, then $P(X = t|Y_1)$ iff $Y_1 = f$*

Notice how the definition has changed to account for the presence of units representing negated atoms in the network. If a unit $X = \neg p$ belongs to $N(P)$, then according to the construction it will have only one parent, namely p, and it will be true with probability 1 if and only if p is false.

We have then the following equivalent of lemma 2:

Lemma 5. *Let P be a definite program, $G =< V, E >$ its annotated graph, $\mathcal{P}$ the partial conditional distribution as in definition 14. Then if $< G, \mathcal{P} >$ is extended to a causal network S satisfying Markov's condition, S satisfies the Minimality Condition.*

Proof. Analogous to the proof for lemma 2. □

The equivalent of definition 12 for definite programs is as follows:

Definition 16. *Let P be an acyclic definite program, and $< G, V >$ be its annotated graph. We can define a partial conditional probability distribution $\mathcal{P}$ over V as follows. If $X \in V$, and $< Y_1, ..., Y_n >$ are the parents of X in V, then*

- *if $X \notin N(P)$, then $P(X = t|Y_1 = y_1, ..., Y_n = y_n) = 1$ if $y_1 = t, ..., y_n = t$;*
- *if $X \in N(P)$, then $P(X = t|Y_1 = y_1) = 1$ iff $Y_1 = f$*

The sense of this definition should be clear. It is exactly equivalent to definition 12, with the difference that we include the clause for negation. This definition is justified in that it assumes only the uncontroversial facts of modus pones and the usual 2-valued semantics for negation. Nevrtheless, as before, this allows us to prove the following:

Lemma 6. *Let P be a definite program, $G =< V, E >$ its annotated graph, $\mathcal{P}$ the partial conditional probability distribution as in definition 12. Then for every extension of $< G, \mathcal{P} >$ to a causal network S satisfying the Causal Markov Condition and the Minimality Condition, S enforces on P the closed-world semantics.*

Proof. Analogous to the proof for the positive case, noticing that it has to be proven that:

$$P(X = f|Y_1 = y_1, ..., Y_n = y_n) = 1 \text{ if } y_i = f, \text{ where } 1 \leqslant i \leqslant n$$

For any $X \notin N(P)$. □

We obtain finally:

Theorem 4. *Let P be a definite program, and $G =< V, E >$ its annotated graph. If it is possible to define a conditional probability distribution $\mathcal{P}$ over G that satisfies the Causal Markov Condition, and such that $P(X = t|Y_1 = t, ..., Y_n = t) = 1$ for any node $X \notin N(P)$ and parents $Y_1, ..., Y_n$, then the network $< G, P >$ satisfies the Minimality condition iff P is interpreted according to closed world semantics.*

Proof. Follows directly from lemma 4 and 5 above □

Clauses with the same head

Up to now, we have restricted ourselves to programs in which no atom q was allowed to appear more than once as the head of a clause. This is not an essential restriction, and we will now show how it is possible to extend our results to programs in which this restriction has been waived. First of all, notice that if we have two clauses of the form $A_1 \wedge ... \wedge A_n \vdash q$

and $B_1 \wedge ... \wedge B_n \vdash q$, where the As and Bs denote literals, this essentially means the following: $(A_1 \wedge ... \wedge A_n) \vee (B_1 \wedge ... \wedge B_n) \vdash q$, where $\vee$ is the standard connective for disjunction. If we want to deal with this type of programs, we will then need to introduce an or-gate, which will have the usual interpretation that we assign to disjunction, and that will allow us to combine different rules with the same head into a unique representation. This is captured in the following construction.

Definition 17. *Let P be a program in which the same atom can occur as the head of more than one clause. Apply the following algorithm to obtain the associated normalized program P_n:*

1 *Let q an atom which in the program occurs as the head of more than one clause. Let $c_1, ..., c_n$ be these clauses. Each one of them will be of the form: $A_1, ..., A_n \vdash q$. For each clause c_i, replace it in the program with a clause of the form $A_1, ..., A_n \vdash q^i$, and add a clause of the form $q^i \vdash q$.*

2 *Repeat step 1 for every other atom q which occurs as the head of more than one clause.*

As it can be easily seen, the result of the application of this algorithm yields a program in which the clauses with the same head are limited to the form $q^i \vdash q$, for any q. This allows us to employ an or-gate in order to connect, in the graph, the different bodies $q^1, ..., q^n$ of these clauses to q. The result is at all analogous to having a clause of the form $(A_1, ..., A_n) \vee (B_1, ..., B_n) \vee ... \vdash q$ translated into the graph, but it is technically more pleasing, and uncontroversial.

Definition 18. *Let P be a program in which the same atom can occur as the head of more than one clause. We can obtain the annotated graph for P as follows:*

1 *Construct P_n as above expounded, and let the set of vertices V be $HB(P_n) \cup N(P_n)$, plus some OR-gates as in step 3.*

2 *Translate the clauses in P_n whose head does not appear more than once as outlined above in definition 14.*

3 *If q appears as the head of n clauses of the form $q^i \vdash q$, where $1 \leqslant i \leqslant n$, then connect each of the q^i in the graph to an OR-gate with a directed edge, and finally connect this OR-gate to q.*

The best way to explain this construction is by means of an illustration. Suppose that a logic program P has two clauses of the form $p \wedge p' \vdash q$ and $r \wedge r' \vdash q$. Then P_n will contain these four clauses: $p \wedge p' \vdash q'$, $r \wedge r' \vdash q''$, $q' \vdash q$, $q'' \vdash q$. Then the graph will have a directed edge from p and p' to q', and from q' to an OR-gate; analogously, there will be a directed edge from r and r' to q'' and from q'' to the OR-gate. Finally, the OR-gate is connected through a directed edge to q. We can specify the behaviour of the OR-gate by defining its conditional probability distribution as follows:

Definition 19 (OR-gate)**.** *Let P a program, $G =< V, E >$ its annotated graph, P_n its normalized version. We can modify the partial conditional probability distribution as in definition 15 (and, in an analogous way, as in definition 14) above by restricting the first two clauses to vertices $X \notin N(P_n)$ which are not OR-gates, and by adding the following:*

- *If X is an OR-gate, then $P(X = t|Y_1 = y_1, ..., Y_n = y_n) = 1$ iff $y_i = t$, where $1 \leqslant i \leqslant n$.*

Notice that if a program P does not have atoms that occur more than once as head, then $P = P_n$ and no OR-gate is introduced. Otherwise, P_n is computed and the graph is built according to the above definitions. Notice that definition 18 modifies the previous definitions of conditional probability distributions only in so far as OR-gates are concerned, for which the usual semantics is translated into the language of probabilities. It is a straightforward matter to check that all the results expounded above still hold in this construction.

THE LOGIC OF TIME AND THE CONTINUUM IN KANT'S CRITICAL PHILOSOPHY

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ABSTRACT. We aim to show that Kant's theory of time is consistent by providing axioms whose models validate all *synthetic a priori* principles for time proposed in the *Critique of Pure Reason*. In this paper the focus is on continuity properties, and on the use of the formalisation to shed some light on important Kantian concepts (for example the distinction between time as form of intuition and time as formal intuition), for which Kant's own explanations are all too brief. The outcome of the formalisation is a continuum that is neither classical nor intuitionistic, although it certainly resembles the latter more than the former. We explain the relation between this continuum and the category of causality. Our main tools are Alexandroff topologies and inverse systems.

CONTENTS

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1. INTRODUCTION: KANT'S THEORY OF TIME

The aim of the present work is that of providing a formalization of the theory of time which Kant developed during his critical period, in particular in the *Critique of Pure Reason*. This contribution belongs to the more general project, started in [2], of laying firm mathematical foundations for various aspects of Kant's Transcendental Philosophy. The advantages obtained by formalizing Kant's Transcendental Philosophy are substantial. Formalization can help clarify the structure of the intricate web of concepts which constitutes Kant's philosophy, and is thus exegetically advantageous, as it makes the structure of Kant's arguments explicit. The evaluation of the consistency of Kant's argumentation is thus simplified, and a more precise comparison of competing interpretations is made possible. This is readily seen in the case of Kant's theory of time, which, while certainly crucial for the general argument of the first *Critique*, is not obviously consistent.

From the perspective of formal ontology and of the philosophy of mathematics, one can interpret Kant's theory of time as an alternative view on the concept of the continuum. In particular, one can individuate two historically rival theories of the continuum. On the one hand, one might take the continuum as made up in some way by atomic, dimensionless entities (points). One obtains in this fashion the Cantorean continuum, which is the direct descendant of the medieval concept of *compositio ex punctis*. On the other hand, one might object to the constitution of extended entities from dimensionless points, and thus take the view of the continuum as a whole which "sticks together", which is given before its parts and can be infinitely divided, but is never exhausted. This was, for instance, the view of Aristotle. The tension between these two different views of the continuum is reflected also in Euclid's *Elements*, where a point is defined as "that which has no parts", but at the same time it is specified that "the extremity of a line are points". Kant's temporal continuum has notable similarities with the Aristotelian view of the continuum; in [], for instance, we find: "A point is like the now in time, i.e., not a part of it, but only the beginning, the end, or the division of time". Prima facie, it is thus inappropriate to model Kant's temporal continuum by means of the reals.

The formalization which we are going to develop will be based for the most part on the properties which are ascribed to the intuition of time in the *Transcendental Aesthetic* and the *Transcendental Analytic*, in the *Critique of Pure Reason*. Here, as well as in other texts of his critical period (some quoted here), Kant provides an analysis of the synthetic *a priori* principles valid of our representation of time. These principles are *synthetic* because they are not conceptual truths. The term '*a priori*' has different senses, of which the following is the most relevant for our present purpose: a principle is *a priori* if it is necessary and its necessity can be established without recourse to any particular sensory experience.'Necessary' is not to be interpreted as 'true in all possible worlds'. Logically speaking, necessary principles are preconditions for the possibility to make true judgements. As such they belong to what Kant calls 'transcendental logic', and our aim is to show that these principles constitute a *formal* logic of time, as part of the more project to present 'transcendental logic' as a logic in the full modern sense.

The synthetic *a priori* principles for time mix cognition and physics in intriguing ways, and we will comment on both aspects: cognition gets a separate section, and physical aspects are illustrated using passages from Barrow and Newton in our brief comments on Kant's principles for time.

1.1. **Synthetic a priori principles for time.**

1.1.1. Time is not an empirical concept that is somehow drawn from an experience. For simultaneity or succession would not themselves come into perception if the representation of time did not ground them a priori. Only under its presupposition can one represent that several things exist at one and the same time (simultaneously) or in different times (successively). (A30/B46)
Comment. One might think that the concept of time is acquired from natural timekeepers (in the 'starry heavens'). This leads to a relative concept of time, unless there is a way to compare the timekeepers; in other words, unless the temporal relations are *objective*. Even if time were a concept, that concept could not have emprical criteria for its applicability.
Newton: 'It is possible that there is no such thing as an equable motion by which time might be accurately measured. All motions can be accelerated or retarded, but the flowing of absolute time cannot be changed. The duration or perseverance in existence of things remains the same, whether the motions are swift or slow, or none at all; and therefore this duration ought to be distinguished from what are only sensible measures of it; and from which we infer it, by means of the astronomical equation.'

1.1.2. Time has only one dimension; different times are not simultaneous, but successive. (A31/B47)
Comment. To see that this is a synthetic judgement, one has to read 'times' as 'extended continua', not 'instants'. These 'times' can be compared to non-intersecting line segments, and simultaneity is a relation between such lines. Now consider the following passage from Isaac Barrow: 'Time, abstractly speaking, is the continuance of each thing in its own Being.' But since some things continue to exist longer than others, these times are durations with respect to the beings in question, and thus are relative measures. 'Time, absolutely, on the other hand, is a quantum; admitting (in its own way) the principal affections of quantity: equality, inequality and proportion.' Time is one-dimensional because this is the only way in which it can be absolute, or objective.

1.1.3. Time is no [...] general concept, but a pure form of sensible intuition. Different times are only parts of one and the same time. That representation, however, which can only be given through a single object, is an intuition. (A31-2/B47)
Comment. This is Kant's conclusion from 1.1.2. Concepts are by definition 'universal', meaning that they never pin down an object uniquely. This is because concepts (or at least basic concepts) arise by abstracting properties shared by different exemplars, which are compared in order to bring them under a concept. Since parts of time share all properties with time itself, and different times are parts of the one time, there is nothing to compare time with, hence time cannot be brought under a concept. The logical aspects of this situation are prominent in
Refl **4425** *Spatium est quantum, sed non compositum*. For space does not arise through the positing of its parts, but the parts are only possible through space; likewise with time. The parts may well be considered *abstrahendo a caeteris*, but cannot be conceived *removendo caetera*; they can therefore be distinguished, but not separated, and the *divisio non est realis, sed logica*.

Comment. This rules out constructions like that of the Cantor set, obtained by iterartively removing the middle third of intervals. The Cantor set is totally disconnected; by contrast, our axioms will have as a consequence that the continuum is strongly connected, for various senses of 'strong'.

1.1.4. The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation time must therefore be given as unlimited. But where the parts themselves [...] can be determinately represented only through limitation, there the entire representation cannot be given through concepts [...] but immediate intuition must ground them. (A32/B47-8)
Comment. It is tempting to gloss this by saying time is potentially infinite in the same sense as the intuitionistic conception of $\mathbb{N}$; but this cannot be right, since the successor operation adds to the integers already constructed, whereas we cannot add to time. Constructing a new natural number would correspond to create a proper part of the one objective time. As Kant puts it in *Refl.* **4756** 'All given times are parts of a larger time. Infinity.' However, this notion of infinity is partly topological: the 'given time' (obtained by 'limiting' time) is like a closed subinterval of an open interval.

1.1.5. The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. Space and time are *quanta continua* because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or a time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed. Magnitudes of this sort can also be called flowing, since the synthesis (of the productive imagination) in their generation is a progress in time, the continuity of which is customarily designated by the expression "flowing" ("elapsing"). All appearances whatsoever are accordingly continuous magnitudes [...] (A169-70/B211-2)
Comment. Continuity implies the possibility to construct arbitrarily fine subdivisions of time; but any concrete subdivision consists of a finite number of extended parts of time. To be able to iterate this construction, the parts of time must have the same continuity properties as time itself: 'Space therefore consists only of spaces, time of times.' Instants correspond to a kind of 'cut' in time, but time is not the set of cuts, unlike the unit (open) interval. Time exists as a whole prior to any subdivision – but how? Newton provides the ingredients for Kant's answer in *CPR*:
'All things endure precisely insofar as they remain the same at every time. The Duration of each thing flows, but its enduring substance does not flow, and does not change with respect to before and after, but always remains the same. Its actions, however, do change, but these are changed and manifested successively according to the will of that which endures...'

1.1.6. The three modi of time are *persistence*, *succession* and *simultaneity* [...] Only through that which persists does existence in different parts of the temporal series acquire a magnitude, which one calls duration. For in mere sequence alone existence is always disappearing and beginning, and never has the least magnitude. Without that which persists there is therefore no temporal relation. (A177/B219)
Comment. Kant will adopt Newton's linking of substance and persistence, but with a transcendental twist: without persistence no successiveness. Hence time cannot be modeled by a linear order only, one needs a timeless substrate as well. Given this, one can interpret successiveness as change of state, or *alteration* :

1.1.7. Here I add that the concept of alteration, and, with it, the concept of motion (as alteration of place), is only possible through and in the representation of time – that if this representation were not a priori (inner) intuition, then no concept, whatever it might be, could make comprehensible the possibility of an alteration, i.e., of a combination of contradictorily opposed predicates (e.g., a thing's being in a place and the not-being of the very same thing in the same place) in one and the same object. Only in time can both contradictorily opposed determinations in one thing be encountered, namely successively. Our concept of time therefore explains the possibility of as much synthetic a priori cognition as is presented by the general theory of motion, which is no less fruitful. (B48-9)

That is, now, the law of the continuity of all alteration, the ground of which is this: That neither time nor appearance in time consists of smallest parts, and that nevertheless in its alteration the state of thing passes through all these parts, as elements, to its second state. No difference of the real in appearance is the smallest, just as no difference in the magnitude of times is, and thus the new state of reality grows out of the first, in which it did not exist, through all the infinite degrees of reality, the differences between which are [arbitrarily small]. (A209/B254)
Comment. Here we have a necessary condition for the application of the category of causality; the latter says that the transition from state a to state b is given by a rule – here a function of time. The necessary condition expresses that the techniques of the differential and integral calculus can be applied – a small increment in the time variable leads to a small increment in the state variable.

1.1.8. *Refl.* **4756** Is there an empty time before the world and in the world, i.e., are two different states separated by a time that is not filled through a continuous series of alterations[?] The instant in time can be filled, but in such a way that no time-series is indicated.
All parts of time are in turn times. The instant. Continuity.
Comment. The instant is not a point, but a collection of pairwise overlapping events – for example, episodes retrieved from memory ('synthesis of reproduction in imagination', A101). Furthermore binding features together to create an appearance ('synthesis of apprehension in an intuition', A99) is a physical process of non-zero duration. Of particular interest here are appearances that limit parts of time, such as reading the position of the hands of a clock – such limits are never dimensionless.

1.1.9. We cannot think of a line without drawing it in thought, we cannot think of a circle without describing it, we cannot represent the three dimensions of space at all without placing three lines perpendicular to each other at the same point, and we cannot even represent time without, in drawing a straight line (which is to be the external figurative representation of time), attending merely to the action of the synthesis of the manifold through which we successively determine the inner sense, and thereby attending to the succession of this determination in inner sense. (B154).
Comment. The idea that time needs an outer represention that supports all temporal concepts (successiveness, duration ...) is a striking anticipation of the recent finding that duration estimates are much more reliable when the relevant events are projected on a timeline; see section 1.2.

As the aforementioned passages show, Kant's ontology of time cannot fully be captured by means of a set-theoretic construction of the continuum, by Dedekind cuts or equivalence classes of Cauchy sequences. The differences between the Kantian and the set-theoretic continuum, moreover, must become even more obvious when we consider how Kant glosses the properties of, e.g., infinity and continuity of time. Infinity (1.1.4) is explained by mereotopological means, and continuity (1.1.5) is defined as the absence of simple parts, along with the ontological primacy of extended parts over the points which bound them[1].

The Kantian notion of instant appears in 1.1.8. An instant being a boundary separating two parts of time does not mean that the instant is empty or atomic, but that the series of alteration of the substance within that instant is not specified. This hints at the fact that the Kantian instant can itself be "put under a magnifying glass", and can thus be refined to reveal further parts of

[1]It was customary before Dedekind to equate continuity and density, and to explain these two concepts interchangeably as "between any two points there is a third". Of course, this definition would not be acceptable for Kant, since points are not primitive entities in his ontology.

time. After all, since boundaries are parts of time, they must be extended (1.1.5); being extended, however, they must be infinitely divisible; thus any boundary can be split indefinitely. We shall see in the sequel that this aspect of Kant's notion of instant is nicely captured by the formalization by using a family of temporal structures which are linked by mappings representing divisibility.

Notice that these are not just sophistical subtleties, without consequences for the mathematical description of the temporal continuum. Kantian instants can only be given as boundaries ('limitations') separating two parts of time, hence time is logically prior to these boundaries; and since an interval of time is determined by two boundaries only, it may well be that the interval itself contains no further instants (as boundaries). Hence Kantian time may have far fewer points than the classical real continuum.

It is also important to highlight here the role which the categories, Kant's "pure concepts of the understanding", play in the constitution of time. Time, for example, is still seen as persisting through empty intervals, which is important as it allows Kant to talk of time elapsed, and hence of *duration* (1.1.6). Since the persistence of time and its measurability are seen as an effect of the category of substance, and its relation to the alteration of phaenomena is mediated by the category of causality (1.1.7), a full treatment of time would require formal analogues for the categories and for their role in the constitution of the intuition of time. Here our aims are more modest, although we outline a formal treatment of the category of causality.

1.2. **Kant and cognitive science.** There is an increasing awareness in cognitive science that Kant's treatment of space, time and causality is still a rich source of ideas; see [**?**] and [**?**] for recent illustrations. For example, in Kant's theory, the temporal continuum is the end-product of several distinct processes (time as form of intuition, the relational categories, the unity of apperception, time as non-observable, hence in need external (spatial) representation), which are not fully integrated in children (or even in adults). For instance, in children one observes the following [**?**]:

(1) There is no reliable correlation between causal and temporal order (i.e. children do not object to backwards causation) and children are at chance at inferring temporal order of hidden events from causal premises. In Kant's terminology, one would say that these children still lack the *category of causality*;
(2) The order of events is sometimes encoded in the child's mind, but generally not accessible to reasoning (e.g. children find it difficult to recite an event sequence in reverse order). In Kant's terminology, one would say that children have the *subjective* time, but not yet the *objective* time, where the italicised terms are to be taken in the Kantian sense, for which see section 2.1. One can also say that, at least w.r.t. time, these children lack the *capacity to judge* [**?**], which involves making temporal relationships explicit and reasoning about them.

But event order is just the simplest of temporal representations, and to handle more complex representations Kant's idea of the 'outer representation of time' mentioned in (1.1.9) has been invoked [**?**]. The authors point out that

> [M]ost everyday activities do not present with a neatly ordered row of events, contained and unfolding one at the time. Rather, most activities are temporally complex and involve asynchronous, partially overlapping events . . . Consider, for example, a theatre play in which the actors constitute the elements of the event (along with props). A theatre play reflects different temporal levels, including the real time of the performance,

> which commences at a specific hour and typically ends a couple of hours later. Furthermore, most events are composed of subelements, with individual temporal characteristics. In a play, the actors may enter and leave the scene simultaneously or separately, and they may appear for different periods of time.

and this provides the motivation for an experiment:

> In this study, children between 5 and 15 years and young adults observed a puppet show in which three puppets appeared on the scene during overlapping intervals of $30s$ to $90s$. At test, participants completed a conventional time estimation task and a timeline task in which they reconstructed the temporal pattern by drawing a timeline for each puppet. For all age groups, the timeline task produced more accurate duration judgments than the time estimation task. Preschoolers time estimation was at chance level, but their timeline performance was surprisingly good and age differences were eliminated in some task conditions.'

Space does not permit us to go deeper into the by now voluminous literare on temporal cognition, but we hope that our detailed presentation of Kant's theory of the components of time and how they interact, contributes to understanding the experimental results.

2. THE ORDER-THEORETIC AXIOMATIZATION FOR EVENTS

2.1. **'Subjective' and 'objective' time.** Since Kant does not tire of proclaiming that time is the form of sensibility and hence has no bearing upon things in themselves, one is easily led into thinking that time is subjective. There is an important sense in which this is not so, most easily explained using a spatial analogue: among spatial mental representations one may distinguish between those which are structured by a coordinate system centered upon the subject, and those where the coordinate system is centered upon an object in the visual field, which is for this reason represented as outside from the subject. One may call the first form of spatial representation 'subjective', and the second 'objective', even if one believes these are but different forms of mental representations. The same distinction applies to time; subjective time is centered upon the subject's *now*, with respect to which past, present and future are defined. As Kant puts it

> Our apprehension of the manifold of appearance is always successive, and is therefore always changing. We can therefore never determine from this alone whether this manifold, as object of experience, is simultaneous or successive . . .

The phrase 'our apprehension of the manifold' refers back to the *synthesis of apprehension in an intuition*: we cannot take in a sensory manifold at a glance, but must engage in a process of binding of features by 'running through [the sensory manifold] and holding together [the features found]' (A99). Linked to this first synthesis is the *synthesis of reproduction in imagination*; e.g. when admiring the beautiful facade of a large building the eyes must scan the surface and the and the sensory impressions obtained until *now*must be reproduced and bound to present impressions to generate a coherent and stable building-object.[2] But is clear that the movement of the eyes determines the order in which features are percceived, which is to say that this notion of succession is subject-centered. The use of the terms 'succession'/'successive' may lead one to think that subjective time is linearly ordered, but that is not so: even transitivity fails. Take any relation of succession, for instance $B(a, b)$:= 'a begins after b'. But the subjective content of this relation goes beyond the purely temporal. The synthesis of reproduction says that a reproduction

[2]Kant remarks (A99) that the same binding processes takes place when counting or constructing geometric figures and that "without it we could have a priori neither the representations of space nor of time".

of b must be synthesised with a. If B is transitive, this implies that reproduction of impressions arbitrarily distant in the past is considered to be possible.

The examples of reproduction supplied by Kant suggest that reproduction operates locally, but not globally, whence the failure of transitivity. The picture is rather that of 'islands in time': because of a developmental dissociation between the ability to remember past events and the ability to think of them as being arranged in a linear order, individual memories start off as unconnected

> There is no evidence that events are automatically coded by the times of their occurrence or that memory is temporally organized (Friedman 1993; 2004); many older events are difficult to discriminate by their ages (e.g., Friedman and Huttenlocher 1997) but are still presumably episodic memories; and it seems likely that we are poor at remembering the internal order of some episodic memories. [...] What appear to be genuine episodic memories are more like"islands in time" than memories one reaches by mentally traveling through some temporally organized representation. (Friedman 2007)

The quote strongly suggests that excluded middle in the form $B(a, b) \vee \neg B(a, b)$ will also fail, which removes yet another prerequisite for the linearity of time. It is possible to develop a formal theory of subjective time revolving around the idea of "islands in time" using maximal linearly ordered chains – but we shall not do so here, since Kant's main interest was the *internal* representation of time as centered upon *objects*. We shall call this 'objective time' for the sake of brevity, but it should be kept in mind that objective time is just as much a mental representation as subjective time.

Events receive a slightly different interpretation in the context of objective time. For Kant, the main question concerning time is one whose answer is announced in A177/B219:

> As regards their existence, appearances stand a priori under rules of the determination of their relation to each other in *one* time.

Formally, this means that there is a function that maps appearances to their position in time; we shall call this mapping the *tenure* function. Temporal positions are determined by, on the one hand, relations of simultaneity (or more broadly, overlap) and causal relations on the other hand. In both cases the relevaant relations can be expressed only if appearances are brought under concepts. If α is an appearance and C a concept, the *tenure* of (α, C) is a connected part of time during which C applies to α. The *tenure* of (α, C) will be called an *event*.Temporal relations such as $B(a, b)$ ('a begins after b) can now be expressed in terms of tenure events: let α be an appearance (which is held fixed for the purposes of this argument), a = *tenure*$((\alpha, C_0))$, b = *tenure*$((\alpha, C_1))$, and suppose the state of α changes from $\neg C_1$ to C_1 (Kant calls this an *alteration*). Since $\neg C_1 \wedge C_0, C_1 \wedge C_0$ are incompatible, by B48-9 (cf. 1.1.7 above), we must have that $C_1 \wedge C_0$ holds after $\neg C_1 \wedge C_0$ whence we obtain $B(a, b)$ if the tenure of $\neg C_1 \wedge C_0$ is non-empty; if it is empty $\neg B(a, b)$. Note that this form of excluded middle is still bound to the particular appearance α, w.r.t. which the tenure events are defined; it is not guaranteed that events derived from different appearances are comparable w.r.t. B. In principle we would have to index B with an appearance and the event structure associated to the appearance could be largely independent of other such event structures. In CPR a correspondence is established between these event structures – each centered upon a different object – via a notion of simultaneity (the 'Category of Community'). Because the notion of simultaneity is symmetric, no object constitutes a privileged temporal perspective, which is the strongest sense in which time can be objective. Below we adopt unrestricted excluded middle for temporal relations to achieve the same effect; that is, the tenure events occurrring in the relation need not tied to the same appearance.

2.2. **Relations between events.** The second and third Analogies argue that succession and simultaneity are the relevant relations between events. Above we introduced the relation B for 'begins after' but there are other such relations, for instance the relation P, representing 'total precedence: $P(a, b)$ iff the right boundary of a is to the left of the left boundary of b. P would satisfy axioms such as irreflexivity and transitivity, as suggested by Russell [8], Walker [10], Kamp [4] and Thomason [9].[3] That P cannot be the right primitive for our purposes was hinted at on page **??** , but it can also be shown by considering causality. A moment's reflection will show that this concept of precedence fits Hume, who conceived of causal chains as discrete, rather than Kant, who did not. Kant viewed effects in a causal chain as *alterations* – changes of state – and argued that such effects will generally be simultaneous with their causes, and moreover that *natura non facit saltus*: changes of state are continuous and themselves take time. This means that the left and right boundaries of an event (say the result of a state change), even though somewhat indefinite, will overlap with another event (the cause of that change). The appropriate representation of precedence is therefore the pair of predicates $B(e, d)$ for 'e begins after d (begins)' and $E(e, d)$ for 'e ends before d (ends)'. In addition we have a reflexive and symmetric predicate O for 'overlap', which in certain circumstances one can take to be transitive as well. In mereology, overlap would be interpreted as 'have a common part'; in a temporal context 'overlap' can have a wider meaning, for instance $O(a, b)$ iff part of a is simultaneous with part of b. Below, we give a representation theorem which shows that events can be taken to be parts of a single time.

In order to capture Kant's topological aspects of the temporal continuum, we shall also use, instead of the negations of (E, B, O), their respective antonyms (positive predicates) $(\breve{E}, \breve{B}, \breve{O})$. Thanks to the addition of the antonyms, we obtain two primitive reflexive and transitive predicates $\breve{E}, \breve{B}$ which will be used to treat the topological aspects of Kant's continuum, since reflexive and transitive relations (preorders) correspond naturally to topologies closed under arbitrary intersections (Alexandroff topologies).

Definition 1. *An* event structure *is a tuple $\mathcal{W} := (W; O, \breve{O}, E, \breve{E}, B, \breve{B})$, where W is a set of events.*[4]

For ease of exposition we introduce a notational convention

Definition 2. *Define the relation $\preceq$ by*

$$b \preceq a \Leftrightarrow \breve{B}(a, b) \wedge \breve{E}(a, b) \wedge O(a, b).$$

If $b \preceq a$ holds, we say that a covers *b.*

[3]We are however indebted to Thomason [9] for showing the power of (a modification of) Walker's axioms. The main difference between Thomason's methods and ours is the use of topology, made possible by a different choice of primitive predicates.

[4]An important difference with the literature referenced above is that the theory of these structures is developed without assuming from the outset finiteness of the domain; if finiteness is needed for a result, this will be indicated explicitly. Instead of assuming finiteness of the domain, we use the Alexandroff topology, which allows one to prove results for arbitrary domains that were previously obtained for finite domains only.

2.3. **A first order theory of objective time.** For event structures $\mathcal{W} := (W; O, \breve{O}, E, \breve{E}, B, \breve{B})$ we adopt the following axiom system GT_0[5]

(1) $E(a,b) \wedge \breve{E}(a,b) \to \bot$[6]
(2) $E(a,b) \vee \breve{E}(a,b)$[7]
(3) $B(a,b) \wedge \breve{B}(a,b) \to \bot$
(4) $B(a,b) \vee \breve{B}(a,b)$
(5) $\breve{E}(a,b) \vee \breve{E}(b,a)$[8]
(6) $\breve{B}(a,b) \vee \breve{B}(b,a)$
(7) $O(a,b) \wedge \breve{O}(a,b) \to \bot$
(8) $O(a,b) \vee \breve{O}(a,b)$
(9) $O(a,a)$
(10) $O(a,b) \to O(b,a)$
(11) $\exists c(a \preceq c \wedge b \preceq c)$
(12) $\breve{E}(a,b) \wedge \breve{E}(b,c) \to \breve{E}(a,c)$
(13) $\breve{B}(b,a) \wedge \breve{B}(c,b) \to \breve{B}(c,a)$
(14) $O(c,a) \wedge O(c,b) \wedge \breve{B}(a,b) \wedge \breve{E}(a,b) \to O(a,b)$
(15) $O(c,a) \wedge O(c,b) \wedge \breve{B}(a,b) \wedge \breve{B}(b,a) \to O(a,b)$
(16) $O(c,a) \wedge O(c,b) \wedge \breve{E}(a,b) \wedge \breve{E}(b,a) \to O(a,b)$
(17) (a) $O(c,a) \wedge O(c,b) \wedge E(a,b) \wedge \breve{E}(a,c) \to O(a,b)$
(b) $O(c,a) \wedge O(c,b) \wedge B(a,b) \wedge \breve{B}(a,c) \to O(a,b)$
(18) (a) $\breve{O}(c,a) \wedge \breve{O}(a,b) \wedge E(a,b) \wedge B(a,c) \to \breve{O}(c,b)$
(b) $\breve{O}(c,a) \wedge \breve{O}(a,b) \wedge B(a,b) \wedge E(a,c) \to \breve{O}(c,b)$
(19) (a) $O(c,a) \wedge O(c,b) \wedge \breve{E}(a,b) \wedge \breve{B}(a,c) \to O(a,b)$
(b) $O(c,a) \wedge O(c,b) \wedge \breve{B}(a,b) \wedge \breve{E}(a,c) \to O(a,b)$

2.4. **Comments on the axioms.**

2.4.1. *Time is a priori: top-down versus bottom-up.* One of the crucial properties which Kant ascribes to time is that of being an *a priori* representation (property 1.1.1 above), meaning that time is a representation which is not abstracted from experience (as might any empirical concept), but is a condition for experience to be possible in the first place. The central aim of this work is that of providing a formalization for Kant's temporal continuum on the basis of the axiom system above and of further, second order axioms which shall be introduced later. This raises the question, in what sense can an axiomatization of the temporal continuum be said to be *a priori*, in agreement with Kant'*dictum*?

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[5]'GT' stands for 'geometry of time'. a reference to Kant's insistence on the necessity of aan 'outer' (geometric) representation of time (B154). The '0' in GT_0 refers to the fact that the axioms are first order; when we construct continua from event structures we will need a higher order version of GT_0 comprising at least a comprehension axiom powerful enough to define topologies. Section 3.6 will briefly deal with the issue, how to incorporate higher order concepts in Kant's logic.

[6]E, $\breve{E}$ are antonyms.

[7]Excluded middle for $\breve{E}$; will be seen to be responsible for linearity of (objective) time.

[8]Follows from implicit anti-symmetry for E, plus axiom 2; can also be used to derive linearity of (objective) time. Implies $\breve{E}$ is reflexive.

This question is particularly important since the very idea of setting up axioms which events have to satisfy so that they can be combined to generate a linear order of instants representing time goes back to Russell [], and, one may argue, ultimately to Leibniz' relational view of time. In the Leibniz-Clarke correspondence, Leibniz maintains that time is the system of relations of precedence or succession which hold among empirical events, the "order of succession" of objects. This order is abstracted from the experience of empirical objects themselves, in a way which predates Frege's abstraction principles, and is thus *a posteriori*[9]. Kant, however, radically opposed the idea that time might be in any way abstracted from experience in this sense, as he claimed that time cannot be perceived, and that in any case it must already be there for temporal relations between events to be determined. Since our axiomatic approach seems *prima facie* similar to that developed by Russell [] and Thomason [] on ultimately Leibnizian grounds, one might wonder whether our approach is indeed in Kant's spirit, or even whether the distinction itself is amenable to formal treatment.

In this respect we note that there is indeed an essential difference between our approach and that of Russell-Leibniz. The latter is, in fact, a bottom-up approach: it starts from given events and their relations, which requires purely universal axioms for temporal relations, in this case $GT_0 - 11$; and these temporal relations are prior to time. The properties of the temporal relations allow one to define points as certain subsets of the set of events, and a linear order on these points, which represents time. On such a view, time is a higher-order concept, which is abstracted from first-order event structures; this seems to go against the Kantian doctrine that temporal relations and temporal parts already presuppose time itself, one of the arguments for both the a prioricity of time, and the undefinability of time. That this was not Kant's view can be seen, for instance, at A30/B46, where Kant is adamant that temporal relations have their intended interpretation (as relations between parts of time) only if time is already given (see p. 3).

By contrast our axiomatization proceeds in the top-down direction: given (a representation of) the whole of time they describe properties of its parts. As *Refl* **4425** (p. **??** makes clear, these parts are only virtual or potential, and cannot really be detached from the whole of time. Mathematically, this difference shows up in two aspects. The first is the strong connectivity properties of the Kantian continuum (theorem 7): whereas in a continuum like $[0, 1]$ each $x \in (0, 1)$ induces a decomposition of $[0, 1]$ into disconnected components, omitting an event from a Kantian continuum doesn't destroy connectedness. This is a consequence of axiom 11, which is very different in character from the other, universal, axioms. The latter deals with formal properties of events that are given *a posteriori* (e.g. the event initiated by turning on the heater, and terminated by switching it off), the former expresses the 'transcendental unity of apperception', the principle that governs all cognitive functions. The events posited by axiom 11 are *a priori* and do not depend upon sensibility, but they are instrumental in integrating the *a posteriori* events – what Kant called 'the comprehension of the manifold given in accordance with the form of sensibility in an intuitive representation.' The second aspect is that of our formal treatment of the potential infinite divisibility of time in Section 4.3, which relies on the notion of an inverse system of models of the axioms; the directedness condition on the index set of the inverse system is then interpreted as a higher order version of the synthesis of the unity of apperception, and is

[9]Space is defined in an analogous way as the "order of coexistence" of empirical objects; roughly speaking, humans form their concept of space by abstracting from empirical objects the "same place" relation, thereby defining space to be the class of all places. The Leibnizian relational view of space has also been subject of formal treatment; see, for instance, [].

also something that cannot be abstracted from experience, but which is necessary for experience. More on this topic in Section 4.3.1.

2.4.2. *Objective time and excluded middle.* Axioms 2, 4 and 8 express the law of excluded middle for the primitive predicates, in positive form. We argued above that these laws embody a form of simultaneity that makes truly objective time possible. If we want a clean separation between subjective and objective time, axioms 5, 6 have to go as well because they rely on axioms 2, 4. These axioms are independent of $GT_0 - 2, 4, 8, 5, 6$, as can be seen from the following structure

Lemma 1. *Consider an event structure $\mathcal{C}$ with domain $\{a, b, c\}$, with $O, \breve{E}, \breve{B}$ interpreted as reflexive, Oas symmetric, and satisfying in addition:*

$$O(c, a), O(c, b), \breve{E}(c, a), \breve{E}(c, b), \breve{B}(c, a), \breve{B}(c, b).$$

The pairs $(a, b), (b, a)$ are not contained in any of $O, \breve{O}, E, \breve{E}, B, \breve{B}$. Then

$$\mathcal{C} \models GT_0 - 2, 4, 8, 5, 6.$$

PROOF. Axioms 1, 3, 9 – 13 hold by construction. Axioms 14 – 18 hold because at least one conjunct in the antecedent is false. □

We add two observations that show the power of the excluded middle principles in the context of GT_0.

Lemma 2. *(i)* $\breve{E}(a, b) \wedge \breve{B}(a, b) \to O(a, b)$
(ii) $\breve{O}(a, b) \to B(a, b) \vee E(a, b)$*)*

PROOF. (i)Axiom 11 provides the covering event that is the prerequisite for applying axiom 14. (ii) Apply axioms 7, 2, 4. □

Lemma 3. $E(a, b) \to \breve{E}(b, a)$.

PROOF. Assume $E(a, b)$, then by axiom 1 $\neg\breve{E}(a, b)$, whence by axiom 5, $\breve{E}(b, a)$. □

2.4.3. *Axioms 12–18.* Axioms 12 and 13 express transitivity for the distinguished predicates $\breve{E}, \breve{B}$, which we know to be reflexive by axioms 5, 5. Transitivity is a strong principle in our context. An argument along the lines of the one given on p. 8 shows that $\breve{B}, \breve{E}$ are transitive w.r.t. a given object, but simultaneity (i.e. the category of community) must be invoked to obtain universal transitivity. The various transitivity properties are essential in capturing a defining characteristic of the Kantian continuum: that 'instants in time can be filled' (cf. 1.1.8 in section 1).

Axioms 14 – 18 express conditional transitivity for O and $\breve{O}$. One way to understand e.g. 18a is to introduce the abbreviation P for a transitive relation of 'total precedence' by $P(a, b) := \breve{O}(a, b) \wedge E(a, b)$; by applying lemma 2 to $\breve{O}(a, c) \wedge B(a, c)$ we obtain $\breve{O}(a, c) \wedge E(c, a)$ which is equivalent to $P(c, a)$, whence by the assumed transitivity of P, $P(c, b)$ and $\breve{O}(c, b)$. As explained above, there are Kantian reasons for not taking P to be primitive, hence we work with transitivity axiom 18a instead. We do have some properties of our chosen primitives that would follow trivially had we included P among the primitives; for instance the following property is a translation of $P(a, b) \wedge O(a, c) \to B(b, c)$ into our language

Lemma 4. $\breve{O}(a, b) \wedge E(a, b) \wedge \breve{B}(a, c) \to \breve{O}(a, c)$.

Together with axioms 1, 1 and 7, the principles of excluded middle also allow us to reformulate the order axioms in GT_0 in terms of the distinguished predicates $O, \breve{E}, \breve{B}$. For example

Lemma 5. *Given axioms 2, 4, 8, 1, 1 and 7:*
(i) axiom 17a is equivalent to

$$O(c,a) \wedge O(c,b) \wedge \breve{E}(a,c) \rightarrow \breve{E}(a,b) \vee O(a,b).$$

(ii) axiom 18a is equivalent to

$$O(c,b) \rightarrow O(c,a) \vee O(a,b) \vee \breve{E}(a,b) \vee \breve{B}(a,c)$$

Obviously the modified principles lack intuitive appeal, but they have the right preservation properties when we come to consider inverse systems of event structures and their limits.

2.5. **Finite model property.** The axioms collected in GT_0 are with one exception universal, as befits axioms concerned with the *form* of intuition. As a consequence, GT_0 is complete with respect to finite models. The proof uses the class of *geometric formulas*, which were argued to be the formal analogue of Kant's judgements in [2].

Definition 3. *A formula is* positive primitive *if it is constructed from atomic formulas using only* $\vee, \bigvee, \wedge, \exists, \bot$.

Definition 4. *A formula is* geometric *or* geometric implication *if it is of the form*

$$(\dagger) \forall \bar{x}(\theta(\bar{x}) \rightarrow \psi(\bar{x}))$$

where θ and ψ positive primitive.

It is easily seen that a geometric formula can be written as a conjunction of implications (†) where θ is a conjunction of atoms.

Theorem 1. *Let φ be a geometric formula in the signature of GT_0. Then $GT_0 \models \varphi$ iff φ holds on all finite models of GT_0.*

PROOF. Assume $GT_0 \not\models \varphi$, then for some countable structure $\mathcal{M}$,

$$\mathcal{M} \models GT_0 + \exists \bar{x}(\theta(\bar{x}) \wedge \neg\psi(\bar{x})).$$

Choose a tuple $\bar{a}\bar{c} \subseteq \mathcal{M}$ where $\bar{a}$ is an instantiation $\bar{a}$ of $\bar{x}$ such that $\mathcal{M} \models \theta(\bar{a}) \wedge \neg\psi(\bar{a})$, and $\bar{c}$ a set of of cover, then the generated submodel of $\mathcal{M}$ is the desired finite model. □

The formula expressing the existence of a universal cover, $\exists x \forall y(y \preceq x)$, is true on all finite models, but not on arbitrary models of GT_0, showing that theorem 1 cannot be extended beyond geometric formulas.

2.6. **Reversibility and apprehension.** Inspection of the axioms shows that they are invariant under the following simultaneous substitutions: $\breve{E}(a,b) \mapsto \breve{B}(a,b), \breve{B}(a,b) \mapsto \breve{E}(a,b), E(a,b) \mapsto B(a,b), B(a,b) \mapsto E(a,b)$. The net effect of these substitutions is to reverse the direction of time, in the following sense:

Theorem 2. *Let $\mathcal{W}$ be a model of GT_0, then $\mathcal{W}^{op}$ defined by the above substitutions is also a model of GT_0.*

We shall use this property of GT_0 to represent the reversibility of Kant's notion of "apprehension". In the Critique, an "apprehension" is a series of perceptions produced by the faculty of the imagination "running through and holding together" a given manifold of appearance. The apprehension of a manifold is guided by concepts and is directed towards the constitution of objects; for instance, in the perception of a house one apprehends its different parts (doors, windows, walls) and finally subsumes these perceptions under the concept "house". Here we are mainly concerned, however, with Kant's remark that if nothing other than the imagination is taken into account all apprehensions are reversible. This irreversibility is is stated in terms of a counterfactual: a series of observations of a house is reversible if it could have been obtained in reverse order, or, rather, if it could have been obtained in *any* order whatsoever; which is different from saying that both orders have been realised. This motivates the following definition:

Definition 5. *Let $\mathcal{W}$ be an event structure. An apprehension in $\mathcal{W}$ is a map $h : I \to W$ such that:*

$$x \leq y \textit{ implies } \breve{B}(h(x), h(y)) \wedge \breve{E}(h(y), h(x))) \textit{ for any } x, y \in I$$

We denote with $\mathcal{H}(\mathcal{W})$ the set of apprehensions in $\mathcal{W}$.

Having defined what an actual apprehension is, one can then define the notion of a possible apprehension:

Definition 6. *A possible apprehension in $\mathcal{W}$ is any map $c : I \to W$ which factorizes as a composition of maps $h_0, ..., h_n$, where $h_i \in \mathcal{H}(\mathcal{W}) \cup \mathcal{H}(\mathcal{W}^{op})$.*

One can easily check that for any finite event structure $\mathcal{W}$ and any possible apprehension $c = (a_0, ..., a_1)$, any permutation of c is a possible apprehension. Thus we model quite closely Kant's treatment of apprehensions as we find it, for instance, in A192/B237[10].

In apprehensions as defined above, the order of events is thus left, in Kantian terminology, *undetermined*. Consider for example events structures $\mathcal{W}_1, \mathcal{W}_2$ such that $\mathcal{W}_2$ is obtained from $\mathcal{W}_1$ by means of the simultaneous substitutions listed above. Suppose in $\mathcal{W}_1$ a precedes b, which can be represented as $\mathcal{W}_1 \models \breve{O}(a, b) \wedge B(b, a)$. In $\mathcal{W}_2$ we have $\breve{O}(a, b) \wedge E(b, a)$, which means that b comes before a. If $B(b, a)$ were also true on $\mathcal{W}_2$, we would have $O(a, b)$, a contradiction. This alright as long as these models are considered as part of what is produced by the imagination – then one model simply represents an imagined counterfactual situation. If we want to model objective time determination, i.e., an objective succession of states in the object itself, we must break the symmetry between $\mathcal{W}$ and $\mathcal{W}^{op}$; GT_0 must be changed so that models no longer come in symmetric pairs. This is not simply a matter of deleting some axioms:

Theorem 3. *The '(b) versions' of axioms 17, 18, 19 follow from the remaining axioms.*

We must therefore add axioms to GT_0 in order to break this simmetry. In particular, these axioms must yield a restricted subclass of possible apprehensions, so as to introduce irreversibility. Kant already suggests that this requires tying in time with the category of causality:

[10]An even better model of Kant's ideas would resort to tools from directed topology (see **??**), but we shall not do so here.

> One quickly sees that [...] appearance, in contradistinction to the representations of apprehension, can thereby only be represented as the object that is distinct from them if it stands under a rule that distinguishes it from every other apprehension, and makes one way of combining the manifold necessary [...] The **subjective sequence** of apprehension [...] proves nothing about the connection of the manifold in the object [...] this connection must therefore consist in the order of the manifold of appearance in accordance with which the apprehension of one thing (that which happens) follows that of the other (which precedes) **in accordance with a rule.**[11] (A191-B236/ A193-B238)

We shall investigate the role of causality for irreversibility in section 6.

3. Topologies supported by GT_0

Several properties of time listed by Kant concern its topology: time is infinitely divisible, time does not consist of instants, time is not 'mere sequence' but 'persistent', alterations are always gradual and continuous, and instants arise only as boundary points (which may themselves be extended).
This list makes clear that Kantian time cannot be represented by the real number line, with the topology generated by the open intervals. In fact, no single topology appears to be sufficient for deriving all features of the Kantian continuum, and ultimately we shall six topologies to account for these features.

3.1. **Topologies from $\breve{B}(a,b), \breve{E}(a,b)$.** The relations $\breve{B}(a,b), \breve{E}(a,b)$ are reflexive and transitive, and thus lend themselves to the following construction

Definition 7. *Let R be a reflexive and transitive relation on a set X. $G \subseteq X$ is R-*upwards closed *if $a \in G, R(a,b) \Rightarrow b \in G$. We omit reference to R when it is clear from the context. Arbitrary unions and intersections of upwards closed sets are again upwards closed. The upwards closed sets will be called* open*, and their complements, the downwards closed sets, will be called* closed*. The collection of open sets is called the* Alexandroff topology.

Definition 8. *An R-open set G is sub-basic if it is of the form $\{e \mid R(a,e)\}$..*

The relations $\breve{B}(a,b), \breve{E}(a,b)$ define Alexandroff topologies. When considering the space of events equipped with two different topologies we will use the term 'bitopological space'.

3.2. **Proximity.** The relations $O, \breve{O}$ also have topological meaning. If we define for sets of events A, B AOB iff $\exists a \in A \exists b \in B\ O(a,b)$ then O is a (weak) proximity relation. We shall also be interested in its dual $A\breve{O}B \Leftrightarrow \forall a \in A \forall b \in B\ \breve{O}(a,b)$, which one might call a remoteness relation. Although the relations $O, \breve{O}$ are symmetric, it is helpful to consider them as asymmetric operations on open sets, where the asymmetry derives from the existence of the two topologies $\breve{B}, \breve{E}$.

[11] The passage continues thus:

> [...] Only thereby can I be justified in saying of the appearance itself, and not merely of my apprehension, that a sequence is to be encountered in it, which is to say as much as that I cannot arrange the apprehension otherwise than in exactly this sequence.

Definition 9. *Let U be $\breve{E}$ open, and V $\breve{B}$ open.*
(i) $U\breve{O} = \{b \mid \forall a \in U\ \breve{O}(a,b)\}$
(ii) $\breve{O}V = \{a \mid \forall b \in V\ \breve{O}(a,b)\}$

Lemma 6. *(i) $U\breve{O}$ is $\breve{B}$ open. (ii) $\breve{O}V$ is $\breve{E}$ open.*

PROOF. Let $b \in U\breve{O}$ and $\breve{B}(b,c)$. Choose $a \in U$; to show $\breve{O}(a,c)$. We have $\breve{O}(a,b)$, which implies $E(b,a)$ or $B(b,a)$. The first possibility implies $b \in U$, quod non. But $B(b,a)$ together with $\breve{O}(a,b)$ and $\breve{B}(b,c)$ implies $\breve{O}(a,c)$. (ii) is proved similarly. □

3.3. **Topologies from the covering relation.** For the defined relation $\preceq$ (definition 2) we have

Lemma 7. *The relation $\preceq$ as defined in 2 is transitive and reflexive.*

PROOF. From axioms 14, 9, 6, 5, 13, 12. □

The covering relation $\preceq$ generates two topologies; it must be noted that sin both cases we are interested in the closed sets only. Initially we will use only the up-sets of $\preceq$ as closed sets, i.e. the sets C such that for any $a \in C$, the set of covers of a is a subset of C. Later we will use the down-sets as closed sets, i.e. the sets D such that for any $c \in D$, any $a \preceq c$; $a \in D$.

3.4. **Order topologies.** We shall see that a *part* of the structure of Kantian time is given by a complete separable linear order, i.e. a relation $<$ on a space X satisfying

(1) $<$ is transitive and antisymmetric
(2) $x = y \vee x < y \vee y < x$
(3) let $Z \subseteq X$ and $U := \{b \in X \mid \forall x \in Z(x \leq b\}$ non-empty, then U has a $\leq$-minimal element
(4) for a countable set $D \subseteq X$: $\forall x, y \in X(x < y \to \exists z \in D(x < z < y))$

If $(X, <)$ has no end-points, this topological space is order-isomorphic to the real numbers. In the theory to be presented the reals act as a kind of coordinatisation of the geometry of time, but their use does not extend beyond this; clearly the reals are useless as formal correlates of Kant's instants (cf. principle (1.1.8))

3.5. **Boundaries.** The topologies defined so far have a temporal meaning: e.g. the open sets of $\breve{E}$ represents the past, the up-sets of $\preceq$ the present and the open sets of $\breve{B}$ the future. Our ultimate aim is to show that the set of events can be given the structure of a one dimensional continuum, which may have some instants, all of which arise as boundaries. We argue as follows. Kant conceives of instants as limitations or boundaries. Now boundary is a topological concept, not order-theoretic. Informally, a temporal boundary in an event structure $\mathcal{W}$ determines a set of events *Past* in the past of that boundary, and likewise a set of events *Fut* which all lie in the future of the boundary. This does not mean that events in *Fut* are still to come; the division into *Past* and *Fut* is relative to the domain W, and all $e \in W$ may be situated in the real past. Temporal progression in the sense of *coming to be* will be represented not by a single event structure, but by a system of event structures linked by continuous maps. We will remind the reader of this observation later, to explain some otherwise puzzling results.

We have that $a \in$ *Past*, $\breve{E}(a,b) \Rightarrow b \in$ *Past* and $a \in$ *Fut*, $\breve{B}(a,b) \Rightarrow b \in$ *Fut*. Using the terminology introduced above, we say that *Past* is $\breve{E}$ open, while *Fut* is $\breve{B}$ open. Furthermore, under the given interpretation $a \in$ *Past* and $b \in$ *Fut* implies $\breve{O}(a,b)$, which in turn implies that *Past* and *Fut* are (set theoretically) disjoint. The complement of *Past* $\cup$ *Fut* can be viewed as a representation of the temporal boundary between *Past* and *Fut*, which we might as well call the present (*Pres*). These considerations are summarised in the following preliminary definition.

Definition 10. *Given an event structure* $\mathcal{W} := (W; O, \breve{O}, E, \breve{E}, B, \breve{B})$*, an* instant *in* $\mathcal{W}$ *is a triple* $(Past, Pres, Fut)$ *such that* $Past \cup Pres \cup Fut = W$*, Past is* $\breve{E}$*-open, and Fut is* $\breve{B}$*-open.*

Lemma 8. *Pres, the complement of* $Past \cup Fut$*, is* $\preceq$*-closed, in the sense that* $d \in Pres$ *and* $d \preceq c$ *implies* $c \in Pres$*.*

We now have to investigate whether the axioms GT_0 imply that the collection of instants $(Past, Pres, Fut)$ can be linearly ordered. Axioms 2, 4 – the law of excluded middle for $\breve{B}$, $\breve{E}$ – imply

Lemma 9. *For any two* $\breve{E}$*-open sets Past, Past': Past* $\subseteq$ *Past' or Past'* $\subseteq$ *Past and similarly for* $\breve{B}$*-open sets Fut, Fut'.*

Since pasts are linearly ordered, one may attempt to define a linear order $<$ on instants by

Definition 11. $(Past, Pres, Fut) < (Past', Pres', Fut')$ *if* $Past \subseteq Past'$*.*

This suggestion doesn't work for all instants as defined in definition 10. In fact, given *Past* only, *Fut* can be chosen independently subject to the constraint that *Past* and *Fut* are $\breve{O}$-remote, and it follows that the event structure is in a loose sense two-dimensional. This problem can be avoided if *Fut* is somehow completely determined by *Past*. As we shall see, this issue is connected to the nature of the boundary between *Past* and *Fut*, i.e. *Pres*. For instance, the present should not contain any pair of events a, b such that $\breve{O}(a, b)$; if there were such a pair (a, b), the present could be split into two parts, one containing a but not b, the other containing b but not a.[12] Moreover, if *Pres* is non-empty, one should not have that *Pres* and *Fut* are $\breve{O}$-remote; otherwise there would be no real distinction between *Pres* and *Fut*, given *Past*.

Definition 12. *A* boundary *is a triple* $(Past, Pres, Fut)$ *such that*
(i) $Past \cup Pres \cup Fut = W$
(ii) Past is $\breve{E}$*-open, and Fut is* $\breve{B}$*-open*
(iii) Past and Fut are $\breve{O}$*-remote*
(iv) for all $a, b \in Pres$*:* $O(a, b)$
(v) if Pres is non-empty, then Past and Pres are not $\breve{O}$*-remote, neither are Pres and Fut*
(vi) Past and Fut are maximal with respect to properties (i–v).

We obtain pairs $(Past, Fut)$ satisfying these specifications by a fixpoint construction, starting from a pair (U, V), U $\breve{E}$ open, V $\breve{B}$ open, both nonempty.

Theorem 4. *Let* U *range over* $\breve{E}$ *opens, and* V *over* $\breve{B}$ *opens.*
(i)(a) the mapping $L : U \mapsto \breve{O}(U\breve{O})$ *is monotone:* $U \subseteq U'$ *entals* $\breve{O}(U\breve{O}) \subseteq \breve{O}(U'\breve{O})$
(i)(b) the mapping $L : U \mapsto \breve{O}(U\breve{O})$ *is extensive:* $U \subseteq \breve{O}(U\breve{O})$ *(ii) the mapping* $R; V \mapsto (\breve{O}V)\breve{O}$ *is monotone and extensive*
(iii) for any U *the least fixpoint of* L *above* U *is* $\breve{O}(U\breve{O})$*; the least* $V' \supseteq V$ *with* $R(V') = V'$ *it is* $(\breve{O}V)\breve{O}$
(iv) if U *is a* L*-fixpoint, then* $V = U\breve{O}$ *is an* R*-fixpoint and we have* $\breve{O}V = U\breve{O}$*; conversely, if*

[12] This coherence requirement can also be expressed by demanding that *O* is transitive on *Pres*.

V is a R-fixpoint, then $U = \breve{O}V$ is an L-fixpoint. (v) the R- and L-fixpoints are preserved under arbitrary unions and intersections (vi) $\emptyset$ is a fixpoint of both operators.

PROOF. (i)(a) $U \subseteq U'$ entails $U'\breve{O} \subseteq U\breve{O}$, which in turn entails $\breve{O}(U\breve{O}) \subseteq \breve{O}(U'\breve{O})$
(i)(b) Lemma 6 implies $\breve{O}(U\breve{O})$ is $\breve{E}$ open. Choose $a \in U, b \in U\breve{O}$, then by definition $\breve{O}(a,b)$.
(ii) is proven similarly.
To prove (iii), observe that by (i) we have

$$\breve{O}(U\breve{O}) \subseteq \breve{O}(\breve{O}(U\breve{O})\breve{O})$$

and by (ii), setting $V = U\breve{O}$, we obtain the converse inclusion, whence

$$\breve{O}(U\breve{O}) = \breve{O}(\breve{O}(U\breve{O})\breve{O}).$$

(iv) $U = \breve{O}(U\breve{O})$ implies $(\breve{O}V)\breve{O} = (\breve{O}(U\breve{O}))\breve{O} = U\breve{O} = V$.
(v) By the properties of the Alexandroff topology, the set of $\breve{E}$ opens ($\breve{B}$ opens) forms a complete lattice, whence by the Tarski-Knaster theorem the set of fixpoints forms a complete lattice as well. □

Given a pair (U,V) we can construct two increasing sequences, where $\subseteq$ is interpreted coordinatewise:

$$(U,V) \subseteq (U,U\breve{O}) \subseteq (\breve{O}(U\breve{O}).U\breve{O}),$$

and

$$(U,V) \subseteq (\breve{O}V,V) \subseteq (\breve{O}V,(\breve{O}V)\breve{O}).$$

Since the $\breve{E}$ opens are linearly ordered by $\subseteq$, we may fuse the two sequences by ordering them linearly according to the first coordinate

$$(U,V) \leq (\breve{O}(U\breve{O}).V) \leq (\breve{O}V,V) \leq ((\breve{O}V),(\breve{O}V)\breve{O}),$$

which gives, for any $\breve{E}$-open U, least and greatest extensions that qualify as *Past*s.

Definition 13. *A pair (Past, Fut) is a fixpoint of the operation (L,R).*

Lemma 10. *Let a,b with $\breve{O}(a,b), E(a,b)$ be given, then there exist a pair (Past, Fut) with $a \in$ Past, $b \in$ Fut.*

PROOF. We construct a least fixpoint separating a and b. Let $U_a = \{c \mid \breve{E}(a,c)\}$. Since $\breve{O}(a,b)$, $b \in U_a\breve{O}$, $b \notin \breve{O}(U_a\breve{O})$ = Past. Since *Fut* is obtained as the least fixpoint above $U_a\breve{O}$, $b \in$ *Fut*. □

Theorem 5. *If (Past, Fut) is as in definition 13, then the corresponding triple (Past, Pres, Fut) is a boundary in the sense of definition 12.*

PROOF. We must show that for any $c,d \in$ *Pres*: $O(a,b)$. If on the contrary $\breve{O}(c,d)$, the preceding lemma shows that the pair (*Past*, *Fut*) wwould not be a fixpoint. The proof that *Past*, *Fut* are not $\breve{O}$-remote from *Pres* is similar. □

Corollary 1. *Every boundary (Past, Pres, Fut), where Past, Fut are non-empty, is obtainable by the procedure given in the proof of theorem 4.*

3.5.1. *Infinity.* Kant's notion of the infinity of time is expressed in procedural terms

> The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation time must there be given as unlimited. (A31-2/B47-8)

where we read 'limitations' as referring to the cognitive act of inserting a boundary. This insertion must not preclude further inserions, which is the case if in a boundary $(\mathit{Past}, \mathit{Pres}, \mathit{Fut})$, *Fut* is always nonempty; this will allow the insertion process described in section 4.3 to operate. Since Kant deemed an infinite past to be incomprehensible one might want to put bounds on inserting boundaries in *Past*.

3.6. **Higher-order logic and Kant.** Since we now consider not single events, but sets of events, such as, e.g. $\{b \mid \breve{E}(a,b)\}$, it is legitimate to ask whether such entities are admissible in a Kantian context. For Kant, infinite pluralities are acceptable if they can be generated by a constructive rule. Since there is an intimate connection between constructive rules and geometric formulas, we have to investigate whether the sets of interest – e.g. the least fixpoint of the operation $U \mapsto \breve{O}(U\breve{O})$ – are somehow definable geometrically. Using the principles of excluded middle, we may write $c \in \breve{O}(U\breve{O})$ as

$$\forall b(O(b,c) \to \exists a \in U\ O(a,b)).$$

Suppose we start with a $\breve{E}$ open U_0 which is given by a rule. Applying the operation, we obtain the $\breve{E}$ open U_1 defined by

$$U_1 = \{c \mid \forall b(O(b,c) \to \exists a \in U_1\ O(a,b))\},$$

from which it follows that U_1 is also given by a rule. Moreover U_1 is a fixpoint, hence the rule is determined in two steps.

Analogously one determines the rule for $\breve{B}$ open sets. By the complete symmetry of open and closed sets, one constructs geometric definitions of $\breve{E}$ closed and $\breve{B}$ closed sets, which then yield a rule to generate the present, even when infinite.

The only possible exception is the fixpoint W. However , in the presence of a universal cover c for W, one can show that $W = \breve{O}(\{c\}\breve{O})$, which generates W bottom-up. Hence so far the formalisation of time does not go beyond Kantian principles.

3.7. **The structure of the present.**

Lemma 11. *(i) In any boundary $(\mathit{Past}, \mathit{Pres}, \mathit{Fut})$ with Past and Fut non-empty, Pres is non-empty.*
(ii) If there exists a universal cover, i.e. c such that for all a, $a \preceq c$, then for all boundaries $(\mathit{Past}, \mathit{Pres}, \mathit{Fut})$ with Past and Fut non-empty, $c \in$ Pres.

PROOF. (i) Let $a \in \mathit{Past}$ and $b \in \mathit{Fut}$; by the covering axiom, one can choose c with $a, b \preceq c$. For the boundary given by corollary 10 we must have $c \in \mathit{Pres}$.
(ii) Suppose $c \notin \mathit{Pres}$, and assume $c \in \mathit{Past}$. Then $\mathit{Past}\breve{O} = \emptyset$, and hence $\mathit{Past} = \breve{O}(\mathit{Past}\breve{O})$ is the entire event structure, which is impossible since *Fut* is non-empty. □

The next corollary expresses that, given an event structure, for any boundary, a *Pres* does not contain a cut, i.e. a pair (a,b) with $\breve{O}(a,b)$.

Corollary 2. *Let $(\mathit{Past}, \mathit{Pres}, \mathit{Fut})$ be a boundary with Pres non-empty then any two events in Pres overlap.*

PROOF. If there were a, b with $\breve{O}(a,b), E(a,b)$ in *Pres*, $\{a\} \cup$ *Past*, $\{b\} \cup$ *Fut* could be extended to a pair of fixpoints, larger than *Past* and *Fut* respectively, a contradiction. □

While this result fully confirms Kant's remark cited as principle 1.1.8 above: 'The instant in time can be filled, but in such a way that no time-series is indicated,' nevertheless one can assign an orientation to a *Pres*:

Corollary 3. *Let Pres be non-empty. For any $c \in$ Pres there exist $a \in$ Past, $b \in$ Fut such that $O(a,c)$ and $O(b,c)$.*

PROOF. If for all $a \in$ *Past*, $\breve{O}(a,c)$ where $c \notin$ *Fut*, then *Past* would not be maximal. □

We have seen that defining instants as in definition 10 need not lead to a linear order. Definition 12 does better in this respect:

Theorem 6. *The set of boundaries of an event structure determines a linear order by means of the inclusion ordering on the Past's. The linear order is a complete lattice.*

PROOF. The inclusion order is linear by axiom 2. The second half follows from the Tarski-Knaster theorem (cf. theorem 4). □

Corollary 4. *The set of boundaries of an event structure is compact.*

PROOF. This follows from the fact that in a complete linear order, each closed interval is compact (Munkres Theorem 27.1), combined with lemma **??**. □

3.7.1. *Comparison with other theories of the present.* In the *Principles of psychology* [], William James discusses the notion of 'now' in the following terms:

> [T]he practically cognized present [i.e. the specious present] is no knife-edge, but a saddle-back, with a certain breadth of its own on which we sit perched, and from which we look in two directions of time. The unit of composition of our perception of time is a *duration*, with a bow and a stern, as it were–a rearward- and a forward-looking end. It is only as parts of this *duration-block* that the relation of *succession* of one end to the other is perceived. We do not first feel one end and then feel the other after it, and from the perception of the succession infer an interval of time between, but we seem to feel the interval of time as a whole, with its two ends embedded in it. The experience is from the outset a synthetic datum, not a simple one; and to sensible perception its elements are inseparable, although attention looking back may easily decompose the experience, and distinguish its beginning from its end. [3, p. 574-5]

The parallel between these remarks on the extended present and the above results, particularly Corollary 3, should be clear. It is also noteworthy that James' extended present, while given phenomenologically as a synthetic whole, can be further decomposed into parts whenever the attentional focus is directed back to it. This interpretation provides us with a cognitive justification for the continuous maps between event structures introduced in Definition 21 of Section 4.3, which we use to formalize the process of potential infinite divisibility of time. As we remarked above, the law of excluded middle forces not only the past, but also the future to be determined; events are in the future with respect to a temporal boundary, and not with respect to an agent. Thus, an event can be in the future of an instant despite it being already past. The perspectival *coming to be* of the future will be represented by an inverse system of event structures connected by maps, in which new maximal instants with respect to the order of Theorem 6 are introduced by adding "splittings" or "divisions" of events. Similarly, events can be "split" in order to introduce new instants which are not maximal with respect to the order of Theorem 6; we can interpret these new instants as arising from a process of analysis of past events by means of attention, along the lines outlined by James. The interested reader will be able to find a treatment of James' specious present, along with evidence of its relevance for contemporary neuroscience, in [**?**, **?**].

The Aristotelian theory of time bears marked similarities to Kant's. We do not have here the space to go into an in depth comparison of the two theories, but will only sketch some salient points, and compare our efforts with some recent attempts at formalizing the Aristotelian continuum [**?**, **?**].

The most fundamental understanding of continuity in Aristotle is that of a binary relation which holds between substances if these are such that they constitute a "unity":

> The continuous is a subdivision of the contiguous[13]: things are continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity. (227a10-15)

Thus, two substances are continuous if, when they are brought together, their respective boundaries "fuse" into each other and are ultimately absorbed into the whole; imagine, for instance, two bodies of water, which are continuous, versus two coins, which when brought together are merely contiguous.

A substance is then said to be continuous if any partition of the substance gives rise to two continuous substances by "actualizing" the boundary separating them, a boundary which is first only potential; contrary to merely contiguous subtances, which can be divided without the creation of anything new, the division of a continuous substance brings forth the "shared" boundary as a new entity. Thus a continuous magnitude enjoys a certain "viscosity" or "unity", which makes it impossible for it to arise by adding together indivisible points, since these cannot be continuous because they can be in contact only "as whole with whole" [14]. The temporal continuum cannot thus be made up from instants, or "nows". The "now" in time is but the division or the "link" of time, that which connects past and future. This basic description of the properties of "now" fits quite well with Kant's ideas and with the treatment above; the boundaries, as we have defined them, are indeed links between the past and the future. We depart from Aristotle, however, in taking boundaries to be extended, and therefore divisible; that this was likely Kant's own stance on the matter, in agreement with James, can be inferred by property 1.1.8 in Section 1.

Aristotle's notion of boundary gives his continuum a weak form of indecomposability: splitting any continuum into two parts will make actual the division between them, thus creating a new entity, the boundary itself. Hellman and Shapiro [**?**] note that both classical (Dedekind-Cantor) and intuitionistic theories of the continuum satisfy stronger forms of indecomposability. Moreover, they also note that their theory does not capture Aristotle's indecomposability, as they define points only as a superstructure in terms of cauchy sequences of nested regions defined by bisection (**??**, p.498). Our formalization does capture this aspect of Aristotle's theory more closely: splitting an event into two parts gives rise to a boundary separating these parts, albeit

[13] For Aristotle, two things are contiguous if their boundaries are in contact, i.e., when their extremities "are together" (226b21).

[14] See Aristotle's physics, , and the illuminating introduction in **??**.

an extended one. However, it will be seen in the next section that our topological approach also provides us with much stronger forms of indecomposability, indeed, much stronger than those satisfied by the classical continuum (see Section 4.1). Another important element of the Aristotelian view of the continuum is the rejection of actual infinities in favour of potential infinities; points come into existence only via the iterated process of division of parts of time, which are always finite in number[15]. Both formalizations of the Aristotelian continuum under consideration [**?**, **?**] have actual infinities, i.e., every model of the axioms will contain infinitely many regions. For instance, the axioms of [**?**] imply that the set of regions is atomless, and one can show that every interval is equal to the fusion of two nonoverlapping congruent parts. Hellman and Shapiro thus interpret Aristotle's "breaking in two" or "splitting" only as metaphorical; they note, however, that this poses a challenge in interpreting both Aristotle's notion of boundaries passing from potential to actual on breaking, and his views of potential infinity. We remarked above that our approach seems to capture reasonably well the former aspect; moreover, we provide in Section 4.3 a construction based on inverse systems of event structures which, in our opinion, provides a closer match to Aristotle's notion of potential divisibility. Note, however, that Hellman and Shapiro's axiomatization is more expressive than ours, as they include various geometric notions, such as congruence, allowing them to give a translation axiom and ultimately to prove the archimedean property for their "gunky" line. We can achieve a similar result only via the defined points, by embedding event structures into their linear order of boundaries.

The last point to which we turn our attention in this brief comparison is St. Augustine's argument against the extended present. We find it in book XI, chapter XV of [1]:

> If any fraction of time be conceived that cannot now be divided even into the most minute momentary point, this alone is what we may call time present. But this flies so rapidly from future to past that it cannot be extended by any delay. For if it is extended, it is then divided into past and future. But the present has no extension whatever.

The issue here seems to be that an extended present would have successive parts, some of which would be in the past, while others would be in the future; but then the present could be split into smaller parts, and it would be difficult to see how it would still be a present. This objection can be neutralized by noticing that it only holds if the domain of primitive objects are points, as in the classical continuum; then one could certainly split an extended present, represented by an interval, into a succession of two intervals, and conclude that the original interval was not a present after all. In such a setting, a present can only be an indivisible point. However in our formalization based on events it is the case that the present is extended, has a past-looking and a future-looking end (Corollary 3), but it does not have parts succeeding one another, as Corollary 2 shows. Thus it cannot be split within a given event structure. As we have remarked above, we shall see in section 4.3 that events can be split by means of maps between event structures. Adding "splittings" of an event in a boundary, effectively dividing it into two successive parts, will however just make the boundary "decay" into two new boundaries, one earlier than the other, which can be seen as arising due to the flow of time or the analysis of attention as explained above. The crucial intuition is that time boundaries are approximations of punctual presents, and therefore they can be refined, but are also indivisible within a given event structure.

[15]See book III, part VI of the physics.

Example 1. To show the distinctive character of the present as defined here, consider the structures:

(1) $W_- = \{a_1, \ldots, a_n\}$, and $i < j$ implies $E(a_i, a_j) \wedge \breve{O}(a_i, a_j)$. The boundaries are empty in this case, so events cannot be separated by boundaries. Furthermore, this structure is isomorphic to the segment $[1, n]$ with a subbase for the open sets given by $(j-1, j+1)$, which means the events are just points, and 'timeline' is disconnected.
(2) $W_+ = \{w, a_1, \ldots, a_n\}$, and $i < j$ implies $E(a_i, a_j) \wedge \breve{O}(a_i, a_j)$, with w acting as universal cover: for all i, $a_i \preceq w$. All boundaries in this structure are of the form $(\{a_1, \ldots, a_i\}, \{w\}, \{a_{i+1}, \ldots, a_n\})$, for $1 \leq i \leq n-1$.

Thus, the flow of time is not represented as a procession of *nows*, since are all identical. While discussing the category of substance (from which principle (1.1.6) cited above was taken), Kant argues that such an unchanging substrate is necessary to represent alteration.

Example 2. *Presents can become distinct if we take the synthesis of reproduction in imagination into account, which can be viewed as an operation that takes an event and puts it in a Pres. Elaborating example 26, reproduction in imagination may lead to boundaries of the form* $(\{a_1, \ldots, a_i\}, \{a_1, \ldots, a_i, w\}, \{a_{i+1}, \ldots, a_n\})$*, for* $1 \leq i \leq n-1$ *– a case of perfect recall.*

3.8. **Events as open intervals.** Theorem 6 raises the question, how the original event structure is related to the linear order obtained via boundaries. Ideally one should be able to map any event e to an open interval such that the topologically meaningful relations $O, \breve{B}, \breve{E}$ are preserved. The following definition owes its justification that two of our topologies are determined by the preorders $\breve{B}, \breve{E}$:

Definition 14. *A function* $f : \mathcal{W}_2 \longrightarrow \mathcal{W}_1$ *is* bi-continuous *if it preserves* $O, \breve{B}, \breve{E}$,[16] *hence also the covering relation* $\preceq$. *We will also refer to these maps as homomorphisms.*

The construction of a bi-continuous function F from events e to intervals defined by boundaries is based upon the following idea: w.r.t. the left endpoint of the interval, e is in the future; w.r.t. the right endpoint e is in the past. Let $V_e = \{b \mid \breve{B}(e, b)\}$, $U_e = \{A \mid \breve{E}(e, a)\}$. Then we construct two *Past*s as fixpoints
– for the left endpoint we get $\mathit{Past}^l_e = \breve{O}(\breve{O}V_e)\breve{O})$ (this set can be empty)
– for the right endpoint we obtain $\mathit{Past}^r_e = \breve{O}(U_e\breve{O})$.
Since $\breve{O}V_e \subseteq U_e$ and the fixpoint operator is monotone, we obtain $\mathit{Past}^l_e \subseteq \mathit{Past}^r_e$, and the *Past*'s have unique extension to boundaries, yielding the desired interval representation of e. One can show that the map from events to intervals is bi-continuous but we will not do so here, since we will later give a simpler but less general construction, that requires assumptions on the topology which will have been shown to hold at that stage.

4. TIME AS A CONTINUUM

Continuity occurs in various guises in the list of synthetic a priori principles for time:

[16]It is a general property of Alexandroff topologies that a mapping is continuous iff it is monotone w.r.t. the pre-orders defining the topologies; in our case this means that a mapping is bi-continuous if it is a homomorphism w.r.t. $\breve{B}, \breve{E}$. However, we need a stronger notion, which corresponds to what is known as 'proximally continuous' mappings on topological spaces with a proximity relation; since the relation O is a weak kind of proximity, we require preservation of O as well, without changing terminology.

(1) parts of times are themselves times
(2) time is infinitely divisible
(3) 'There is nothing simple in appearance, hence no immediate transition from one determinate state (not of its boundary) into another (*Refl.* **4756**)':.there are no leaps from one state to another
(4) time is not a 'mere series'
(5) duration is a continuous magnitude
(6) 'A hiatus, a cleft, is a lack of interconnection among appearances, where their transition is missing (*Refl.* **4756**)', hence there can be no clefts in time

One may group these aspects of continuity under four headings: connectedness, parthood, divisibility, metrics.

4.1. **Connectedness.** The textbook definition of 'space X is connected' is a kind of indecomposability condition: there are no disjoint non-empty open sets U, V such that $X = U \cup V$. As we shall see, Kant's notion of indecomposability is much stronger than this, but first we must define what it means for a space to be connected in a bi-topological setting. Since the $\breve{E}$ open sets are linearly ordered by inclusion, any event structure is trivially connected in the $\breve{E}$ topology (and likewise for the $\breve{B}$ topology). We therefore need both topologies:

Definition 15. *The event structure $\mathcal{W}$ is* asymmetrically connected *if there are no non-empty U, V such that U is $\breve{E}$-open, V is $\breve{B}$-open, $U \cap V = \emptyset$ and $U \cup V = W$.*

In particular, if the event structure $\mathcal{W}$ is asymmetrically connected, it cannot be written as *Past* $\cup$ *Fut*. As a consequence of the covering axiom we have

Lemma 12. *Event structures are asymmetrically connected.*

But the covering axiom implies still stronger forms of connectedness. We now focus attention on sets closed in both topologies. As we have seen, the *Pres* component of a boundary is such a set. It follows from lemma 12 that such sets are non-empty. But we also have

Lemma 13. *The intersection of any two $Pres_1$, $Pres_2$ is non-empty.*

PROOF. Choose $a \in Pres_1, b \in Pres_2$. If $a = b$, we are done. Otherwise, choose c with $a, b \preceq c$. By lemma 8, $c \in Pres_1 \cap Pres_2$. □

The following results show that there really is no way to detach a part of time from the temporal continuum. The first result reprises a theme broached in section **??**: that GT_0 describes time in top-down fashion. To formulate the result we need a definition

Definition 16. *An event e in an event structure $\mathcal{W}$ is* finite *if* $\exists a E(e, a) \wedge \exists b B(e, b)$.

new **Theorem 7.** *Let $\mathcal{W}$ be a finite model of GT_0; we know that this model is asymmetrically connected. Let $e \in \mathcal{W}$ be finite. Then $\mathcal{W} - \{e\}$ is a model of GT_0, hence asymmetrically connected.*

PROOF $\mathcal{W} - \{e\}$ is a substructure of $\mathcal{W}$, hence satisfies $GT_0 - 11$. Since $\mathcal{W}$ is a finite model, it has a universal cover, which is not finite in the sense of definition 16, and thus different from e. Hence $\mathcal{W} - \{e\}$ has a universal cover and therefore satisfies axiom 11. □

That is, one can remove an extended event without in any way affecting the unity of the Kantian continuum. This is characteristic of constructions where the whole (logically) precedes the parts. This result doesn't hold if axiom 11 is dropped, as one can see from

Example 3. *Consider $\mathbb{R}^+$ with events represented by intervals*

$$\{[n, n+2) \mid n \geq 0\}.$$

This set can be turned into a model of $GT_0 - 11$, and one can build a connected linear order of boundaries out of this event structure, since any two adjacent disjoint intervals are 'bridged by another interval. The first three events are $[0,2), [1,3), [2,4)$; if one removes $[1,3)$, the boundary between $[0,2)$ and $[2,4)$ is empty, and the structure becomes disconnected.

Definition 17. *A topological space is* ultraconnected *if any two non-empty closed sets have non-empty intersection.*[17]

In finite event structures ultraconnectedness takes on the following form

Lemma 14. *In finite event structures $\mathcal{W}$ there exists an event $w \in W$ such that for all Pres: $w \in$ Pres.*

While this notion of connectedness is obviously much stronger than connectedness of the real continuum, there's more to be had if we introduce 'parts of times'.

4.2. **Parthood.** 'Parts of times are themselves times': this we take to mean that a part of time satisfies GT_0. Furthermore, principle (**??**) above clearly states that time cannot be composed of parts of time; any such composition should fail to cover all of time. Since 'part' and 'cover' are more or less inversely related we adopt the following definition for 'part of time':

Definition 18. *An order ideal $\mathcal{I}$ w.r.t. $\preceq$ is a set of events satisfying*

(1) *$c \in \mathcal{I}$ and $a \preceq c$ implies $a \in \mathcal{I}$*
(2) *for any $a, b \in \mathcal{I}$ there exists $c \in \mathcal{I}$ such that $a, b \preceq c$*

We identify part of time *with an order ideal $\mathcal{I}$ w.r.t. $\preceq$*

The covering axiom yields

Lemma 15. *Every non-trivial down-set w.r.t. $\preceq$ is contained in a proper order ideal.*

The following observation corresponds to Kant's dictum that 'parts of time are times'

Lemma 16. *Let $\mathcal{I}$ be an order ideal, then the submodel generated by $\mathcal{I}$ satisfies GT_0.*

Example 4. *Consider again the event structure: $W = \{w, a_1, \ldots, a_n\}$, and $i < j$ implies $E(a_i, a_j) \wedge \breve{O}(a_i, a_j)$, with w acting as universal cover: for all i, $a_i \preceq w$. If the order ideal $\mathcal{I}$ contains two different a_i, a_j, it must contain w, whence $\mathcal{I} = W$. Hence the only order ideals are those of the form $\{a_j\}$. Note that the parts so defined are indeed separated by boundaries: between a_i and a_{i+1} one has the boundary (Past, Pres, Fut with Pres $= \{w\}$, Past $= \{a_1, a_i\}$, and Fut $= \{a_{i+1}, a_n\}$'*

Definition 19. *An ordered topological space X is* irreducible *if X cannot be written as the union of two proper order ideals.*

Lemma 17. *An event structure is irreducible.*

[17]In our bi-topological setting, this concept is non-trivial only for sets closed in both topologies, hence in the preceding definition 'closed' will be taken in this sense.

Proof. Suppose $W = A \cup B$, where A, B are proper closed subsets in the $\preceq$ topology; hence $W \neq A, W \neq B$. Pick $a \in A\mathcal{B}, b \in B\forall$. Choose c with $a, b \preceq c$. If $c \in A$, then $a, b \in A$, contradiction. Likewise for $c \in B$. Hence $W \neq A \cup B$. □

Theorem 8. *Consider an event structure with the topology generated by the $\preceq$ up-sets. Then any continuous function f with values in $\mathbb{R}$ is constant.*

Proofsketch. Assume the event structure is finite; generalising the argument to compact event structures is straightforward. The assumption implies that f takes values in a bounded interval, say $[0, 1]$. From the continuity of f it follows that $d \preceq e$ implies $f(d) \leq f(e)$. Suppose f takes at least two values, say 0 and 1. $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are closed, i.e. down-sets. We may assume these subsets are proper, whence it follows that the universal cover w is in neither of these sets. It follows that $0 < f(w) < 1$, and by continuity that the value of f on $f^{-1}(\{1\})$ must ≤ 1, a contradiction. □This argument can be recast as a proof of the invariance of substance. If e is an event, $f(e) \in [0, 1]$ is the amount of substance involved in e. We require f to be monotone w.r.t. $\preceq$, which entails continuity. It follows that f must be constant, i.e. in so far as substance is quantifiable it is invariant over time.

4.3. **Infinite divisibility.** Although Kant does not have a single notion of continuity, a recurring theme is that a continuum does not have smallest parts. For example, at any given stage we can have constructed only finitely many boundaries, but there is no fixed bound on how many boundaries can be constructed. By splitting events we can introduce new boundaries and make existing ones thinner:

Definition 20. *Let $mathcalW$ be an event structure and $c \in mathcalW$. A triple $< a, b, d >$* splits *c if $\breve{O}(a, b)$, $\breve{O}(a, d)$, $\breve{O}(d, b)$, $E(a, d)$, $E(d, b)$ and $\{a, b, d\} \preceq c$.*

Here is an example illustrating how we hope to achieve infinite divisibility. Suppose we have finite event structures $\mathcal{W}_1, \mathcal{W}_2$ such that the second differs from the first only in containing splittings of some events in $\mathcal{W}_1$. We thus take $\mathcal{W}_1 \subseteq \mathcal{W}_2$, but there is more to be said. We show that there is a surjective homomorphism $f : \mathcal{W}_2 \longrightarrow \mathcal{W}_1$. We set f equal to the identity on $\mathcal{W}_1$ considered as subset of $\mathcal{W}_2$; this makes f surjective. If $< a, b, d >$ splits $c \in W_1$, we may define f on these events by $f(a) = f(c) = c = f(b) = f(d)$.

Definition 21. *A homomorphism $f : \mathcal{W}_2 \longrightarrow \mathcal{W}_1$ is a* retraction *if f is surjective and equal to the identity on $\mathcal{W}_1$. In this case $\mathcal{W}_1$ is called a* retract*.*

Infinite divisibility will be represented by an inverse system of retracts; we briefly recall the relevant definitions.

Definition 22. *A directed set is a set T together with an ordering relation $\leq$ such that*

(1) *$\leq$ is a partial order, i.e. transitive, reflexive, anti-symmetric*
(2) *$\leq$ is directed, i.e. for any $s, t \in T$ there is $r \in T$ with $s, t \leq r$*

Definition 23. *Let T be a directed set. An inverse system of models indexed by T is a family of first order models $\{\mathcal{M}_s | s \in T\}$ together with a family of homomorphisms*

$$\mathcal{F} = \{h_{st} | s \geq t, h_{st} : \mathcal{M}_s \longrightarrow \mathcal{M}_t\}.$$

The mappings in $\mathcal{F}$ must satisfy the coherence requirement that if $s \geq t \geq r$, $h_{tr} \circ h_{st} = h_{sr}$.

In our setting – where models are event structures satisfying GT_0 – a homomorphism w.r.t. $\breve{E}, \breve{B}, O$ is bi-continuous in the Alexandroff topologies generated by $\breve{E}, \breve{B}$.

Definition 24. *Let $(T, \{\mathcal{M}_s \mid s \in T\}, \mathcal{F})$ be an inverse system. Let D_s be the domain of $\mathcal{M}_s$. Let $\mathcal{D} \subseteq \Pi_{s\in T} D_s$ the set of all ξ such that for $s \geq t$, $h_{st}(\xi(s)) = \xi(t)$. Define a model $\mathcal{M}$ with domain $\mathcal{D}$ by putting $\mathcal{M} \models R(\xi^1, \xi^2, \ldots)$ if for all $s \in T$, $\mathcal{M}_s \models R(\xi^1_s, \xi^2_s, \ldots)$. $\mathcal{M}$ is called the* inverse limit *of the given inverse system.*

Theorem 9. *Let $(T, \{\mathcal{W}_s \mid s \in T\}, \mathcal{F})$ be an inverse system of event structures where each domain W_s is compact in both Alexandroff topologies. Then the inverse limit is non-empty.*

Lemma 18. *Let $(T, \{\mathcal{M}_s \mid s \in T\}, \mathcal{F})$ be an inverse system of models with non-empty inverse limit $\mathcal{M}$. Then the projection π_s defined by $\xi \in \mathcal{M} \mapsto \pi_s(\xi) := \xi(s)$ is a homomorphism that in addition satisfies for $s \geq t$: $h_{st}(\pi_s(\xi)) = \pi_t(\xi)$.*

The next result says that with judiciously chosen axioms, the inverse limit inherits the properties of the models in the inverse system

Lemma 19. *Let $\varphi(x)$ be a geometric formula, $(T, \{\mathcal{M}_s \mid s \in T\}, \mathcal{F})$ an inverse system of event structures where each domain W_s is compact in both Alexandroff topologies. Let the limit $\mathcal{M}$ be non-empty, and suppose $\eta \in \mathcal{M}$. If for all $s \in T$, $\mathcal{M}_s \models \varphi(\eta_s)$, then $\mathcal{M} \models \varphi(\eta)$.*

Since GT_0 consists of geometric sentences, we get

Theorem 10. *Every inverse system of compact event structures $\mathcal{W}_s$ satisfying GT_0 has a unique inverse limit $\mathcal{W}$ which also satisfies GT_0.*

One way to force compactness on $\mathcal{W}_s$ is by means of the covering axiom plus a boundedness condition.

Definition 25. *Suppose there $a, b \in \mathcal{W}_s$ such that $\breve{O}(a, b) \wedge \breve{E}(b, a)$ and for all $x \in \mathcal{W}_s$, $\breve{B}(a, x) \wedge \breve{E}(b, x)$. Such $\mathcal{W}_s$ will be called* bounded.

Definition 26. *Let $\mathcal{W}_s$ be bounded. The covering axiom implies there exists w such that $\mathcal{W}_s \models \forall x(\breve{E}(w, x) \wedge \breve{B}(w, x))$. We call such a w a* universal cover *of $\mathcal{W}_s$.*

Lemma 20. *Suppose $\mathcal{W}_s$ has a universal cover. Then $\mathcal{W}_s$ is compact in both Alexandroff topologies.*

PROOF. Since an $\breve{E}$ or $\breve{B}$ open set containing the universal cover w must equal the whole space, every open cover of $\mathcal{W}_s$ has a finite subcover, whence it follows that $\mathcal{W}_s$ is compact. $\square$

Lemma 21. *Assume each $\mathcal{W}_s$ has a universal cover. Let $\omega \in \mathcal{W}$ be a thread such that each ω_s is a universal cover in $\mathcal{W}_s$. The formula defining universal cover is positive, whence by lemma 19ω is a universal cover in $\mathcal{W}$. Hence there exists a linear order (given by proper inclusion on the Pasts) of boundaries on $\mathcal{W}$ with non-empty Presents.*

4.3.1. *The unity of apperception.* The formalism of inverse systems and their limits was employed already in [2], where it was argued that it provides an accurate formalization of Kant's crucial notion of the *unity of apperception*, and of its role in constituting a unified experience. While we use inverse systems chiefly to formalize Kant's infinite potential divisibility of time, their interpretation as a formal correlate to the synthesis of the unity of apperception is still valid in a temporal setting; we now wish to comment on this aspect in slightly more detail.

In general terms, Kant holds that every appearance must be able to be accompanied by the "I think", which means that it must be thought of as *my* representation and must thus be ascribable to a stable "I". This "I" is the substratum to which everyone of my cognitive acts is attributed; without the possible ascription of appearances to the stable "I" these would be "nothing at all for me" (CPR, B132); in an extreme case, I would have a phenomenal experience which Kant describes as having "as multicoloured, as diverse a self as representations of which I am conscious" (CPR, B134), a sort of "scattered" consciousness. The necessity of a stable consciousness accompanying all my representations then translates into the necessity of a synthesis which unifies all these representations, by reagrding them as parts of one global representation. In other words, if the "I" must be able to accompany all my representations, then these representations must be mutually consistent, as they can be ascribed all to me without contradiction; if they are consistent, however, they can be "synthesized" into a unified representation.

Thus one can also read Kant's argument for the unity of apperception as a "transcendental argument"[18]. Our phenomenal experience exhibits stability and coherence since the same consciousness accompanies all my representations, so that they are all attributed to a stable self. This is a property of our phenomenal experience, constituting the first premise of the transcendental argument. We now attempt to reason to the conditions which make this property possible, and ask: how can the identity of the consciousness accompanying two distinct representations be thought at all? Kant's answer is that this is possible only if these two representations can be combined[19] into a more complex representation by a process of synthesis, so that the consciousness accompanying the complex representation ensures the identity of the "I's" accompanying its various parts. This is the synthesis of the unity of apperception, which gives essential unity to experience.

In the present setting one should think of the event structures M_s in an inverse system $(T, \{\mathcal{M}_s \mid s \in T\}, \mathcal{F})$ as representations of "local times", corresponding to specific experiential episodes, which are similar to the "islands of time" referred to above. The synthesis of the unity of apperception operates in this formalization at two levels. At the first order level we have Axiom 11, which, as we mentioned in Section 2.4.1, we interpret as an instance of the synthesis of the unity of apperception, which operates then already at a very early stage of cognitive processing. At this level, the unity of apperception ensures the unity of the events belonging to a single experiential episode, as we saw in Section 4.1. At the higher order level, the directedness of the index set T should also be interpreted as a formal correlate to the synthesis of the unity of apperception; any two local times are unified by this synthesis into a global time, thereby determining (in the sense in which Kant speaks of *time determination*) the temporal relations between their events. Indeed, given M_s, M_t, M_r in the inverse system with $s \leq r, t \leq r$, the comparative temporal position of two events $a \in D_s, b \in D_r$ is given by considering the inverse of the projection maps $f_{rs}^{-1}(a), f_{rt}^{-1}(b)$. In the general setting one does not assume surjectivity of the maps in $\mathcal{F}$; thus one of these inverse images might be empty, meaning that the event has been annhilated in the process of synthesis, as it happens when, for instance, the event in question is recognized as an illusion. In treating infinite divisibility below, however, we employ retraction maps, which are surjective. This restriction is justified by the fact that the primary focus of this paper is on the analysis of the *a priori* properties of time, and not of *a posteriori* phenomena such as temporal illusions. Note moreover that the directedness requirement on T

[18]For the notion of transcendental argument, see .

[19]Kant's technical term for this act is *combinatio*, or *Verbindung*.

ensures that there is a non-empty inverse limit of all the M_s. This means that all the local times must be mutually consistent and unifiable in a unique representation, which is what the synthesis of apperception is supposed to accomplish in Kant's framework.

Definition 27. *Infinite divisibility is represented by a countable inverse system of retracts (cf. definition 21)*

$$\ldots\ldots \mathcal{W}_s \longrightarrow_{f_{st}} \mathcal{W}_t \longrightarrow_{f_{tr}} \mathcal{W}_r \longrightarrow \ldots$$

such that for $s > t$ $\mathcal{W}_s$ contains all splittings of events in $\mathcal{W}_t$. We furthermore require that the $\mathcal{W}_s$ are at most countable and bounded.

That the inverse system consists of retracts is more or less forced upon us by the covering axiom, which implies that for any splitting $< a, b, d >$ of c there must be an event covering a, b, d; one might then just as well retain c. The covering axiom (and the requirement that the homomorphisms are retractions) will be seen to be essential in proving the 'unity of time'. Without the covering axiom, one could as inverse system representing infinite divisibility the ternary branching tree 3^ω, which gives rise to a totally disconnected inverse limit. It is possible to define a linear order on 3^ω which satisfies the universal axioms of GT_0, but this linear order will not be complete, and is in fact zero-dimensional, whereas time must be represented as one-dimensional.

Lemma 22. *The inverse system of definition 27 can be assumed to be an inverse sequence.*

PROOF By directedness one can choose a countable cofinal subsequence of the index set. The inverse limit determined by this subsequence is homeomorphic to the original inverse limit. □

Theorem 11. *Assume the event structures $\mathcal{W}_n$ are at most countable, compact, and represent infinite divisibility in the sense of definition 27. The inverse limit $\mathcal{W}$ satisfies GT_0, hence a linear order of boundaries can be constructed. The space of boundaries on $\mathcal{W}$ with the order topology is a separable complete connected compact second countable Hausdorff space.*

PROOF. Separability and the existence of a countable base for the topology follows from the countability of the $\mathcal{W}_n$. Completeness and compactness follow from theorem 6 and corollary 4; these imply connectedness (Munkres, theorem 24.1, p. 153) . To show that $\mathcal{W}$ has the Hausdorff separation property, let $(Past, Pres, Fut), (Past', Pres', Fut')$ be two boundaries with $Past \subset Past'$. Pick $c \in Past' - Past$ and let (a, b) split c. Then $Past \cup \{a\}$ generates a boundary strictly between the two given boundaries; this boundary can be used to define the required open sets. □

Definition 28. *We write ($\mathcal{B}(\mathcal{W}), \subset$) to indicate that the space under consideration is the set of boundaries on $\mathcal{W}$ with the linear order topology induced by strict inclusion on the Pasts.*

Since $(\mathcal{B}(\mathcal{W}), \subset)$ is second-countable and compact, (Engelking, theorem 4.2.8) implies

Corollary 5. *$(\mathcal{B}(\mathcal{W}), \subset)$ is metrisable.*

Theorem 12. *A compact connected separable linear order is homeomorphic to* $[0.1]$.

PROOF. See exercise 6.3.2(b) in Engelking, p. 373. □

Corollary 6. *$(\mathcal{B}(\mathcal{W}), \subset)$ is homeomorphic to $[0, 1]$. This holds for any $\mathcal{W}$ that can be represented as inverse limit event structures $\mathcal{W}_n$ are at most countable, compact, and represent infinite divisibility in the sense of definition 27.*

The topological structure guaranteed by infinite divisibility was set up so that we have enough points in an environment dominated by pointfree events. With this structure in place, we can now state

Theorem 13. *$\mathcal{W}$ can be embedded in ($\mathcal{B}(\mathcal{W}), \subset$).*

PROOF. Let e be an event in $\mathcal{W}$, then e will be represented as an open interval in ($\mathcal{B}(\mathcal{W}), \subset$). For the left endpoint we take the boundary whose past is the least fixpoint above $\bigcup\{\mathit{Past} \mid e \notin \mathit{Past}\}$, and for the right endpoint the boundary generated by $\bigcup\{\mathit{Fut} \mid e \notin \mathit{Fut}\}$. □

Corollary 7. *(of theorem 13). Any event e can be represented as an open interval in $[0, 1]$.*

PROOF. Compose the mapping of corollary 13 with the mapping given in theorem 6. □

4.3.2. *Similar yet different.* Corollary 6 may suggest that it has all been much ado about nothing; after endless diversions we're back at the unit interval. However, the existence of a homeomorphism falsely suggests that the homeomorphs are indistinguishable. To bring out the asymmetry we compare the homeomorphism $h : [0, 1] \longrightarrow (\mathcal{B}(\mathcal{W}'), \subset)$ with the lifted map $H : x \mapsto \{e \mid e \in \mathit{Pres}_{h(x)}\}$, where $x \in [0, 1]$ and $\mathit{Pres}_{h(x)}$ is the 'present' component in the boundary $h(x)$. Each $H(x)$ is closed with respect to the upwardly closed sets of intervals according to the covering relation $\preceq$. Clearly h has only information about the position of the boundary $h(x)$ in the ordering, whereas $H(x)$ also captures some of the internal structure of the boundary $h(x)$. The next theorem shows that the topology on $[0, 1]$ has no information whatsoever about the internal structure of the boundaries. First we introduce the relevant sense of continuity.

Definition 29. *A function H with domain $[0, 1]$ and range in the set of closed subsets of a space X is* lower hemi-continuous *(l.h.c.) if for closed $W \subset X$, the set*

$$\{x \mid H(x) \cap W \neq \emptyset\}$$

is closed in $[0, 1]$. H is continuous if it is l.h.c. and 'u.h.c.' – the precise meaning of which need not detain us.

Theorem 14. *If the map $H : x \mapsto \{e \mid e \in \mathit{Pres}_{h(x)}\}$ is continuous, it must be constant. In fact this holds as well if H has arbitrary closed sets as values.*

PROOF. This follows from ultraconnectedness – the fact that any two closed sets intersect – for since $H(x)$ is closed this implies that

$$\{x \mid H(x) \cap W \neq \emptyset\} = [0, 1].$$

□

4.4. **Duration.** The preceding material can be used to show that time is metrisable. This is of course a weak result, since the metric is not uniquely determined. Interestingly, Kant links the possibility of duration to the topological structure of time, e.g. in A177/B219:

> Only through that which persists does existence in different parts of the temporal series acquire a magnitude, which one calls duration. For in mere sequence alone existence is always disappearing and beginning, and never has the least magnitude. Without that which persists there is therefore no temporal relation.

On event structures, duration manifests itself in two ways: as duration of an event e, denoted $\delta(e)$ (the diameter of e) and the distance between (non-overlapping) events c and e, denoted $d(c, e)$. On the closed unit interval with metric ρ, both notions can be expressed using ρ.

Theorem 15. *Let ρ be a metric on $[0, 1]$. Let event c correspond to interval (s, t), and e to (u, v); suppose $t < u$. Then $\delta(e) := v - u$, $d(c, e) = d(e, c) := u - t$.*

5. Uniqueness of time: the hyperspace model

We next introduce a structure that corresponds to Kant's notion of time as forrmal inttuition, i.e. time as a unique object. This structure should be universal in the sense that all 'small' event structures should be embeddable into it. For ease of exposition we treat events as open intervals. The domain of the desired universal model must (at least) contain open intervals. Spaces of this type are generally known as *hyperspaces*. They may carry very different topologies; we first give a fairly standard definition, and will then motivate the changes we are forced to introduce.

Definition 30. *A* hyperspace *(or a space equipped with a hypertopology) is a topological space, which consists of the set $CL(X)$ of all closed subsets of another topological space X, equipped with a topology so that the* canonical map

$$x \mapsto \overline{\{x\}}$$

is a homeomorphism onto its image. As a consequence, a copy of the original space X lives inside hyperspace $CL(X)$.

We need not spell out this topology in detail, because our hyperspace must consist of open intervals $\subseteq [0, 1]$, whence the closed singleton $\{x\}$ is not in the space and the canonical map which sends x to $\{x\}$ cannot be defined. One can define the function from x to its closure in hyperspace, but it turns out this function cannot be continuous (theorem 14 translated to the present setting).

Definition 31. *Let Ω be the set of all open intervals on the unit interval $[0, 1]$. We define what is known as the 'hit or miss' topology $\mathcal{HYP}$ on Ω by means of a subbasis of open subsets of Ω as follows: (i) ('hit') given an open $G \subseteq [0, 1]$, define*

$$[G] = \{I \in \Omega \mid I \cap G \neq \emptyset\}.$$

and (ii) ('miss') given a closed set $F \subseteq [0, 1]$, define

$$[F^c] = \{I \in \Omega \mid I \cap F = \emptyset\}.$$

These conditions can be merged into a single condition defining a basis for the open sets: for open intervals $J_1, \ldots, J_n \subseteq [0, 1]$, let

$$[J_1, \ldots, J_n] := \{U \mid U \text{ open interval}, U \subseteq \bigcup_{i \leq n} J_i \ \wedge U \cap J_i \neq \emptyset \text{ for } i \leq n\}.$$

We shall refer to this topological space as the hyperspace on Ω.

The primitives of the axiomatic system GT_0 can be interpreted on Ω using the topology; for example, if event a is represented by interval (i, j), and b by (k, l), and we have $\breve{E}(a, b)$, i.e. '(i, j) does not end before (k, l)', then this is equivalent to

$$(k, l) \in \{I \in \Omega \mid I \subseteq [0, j)\},$$

and dually for $\breve{B}$. $\breve{O}(a, b)$ means that a, b are disjoint, and $(O(a, b)$ that a, b overlap; in other words, that a hits b. For fixed b, $\{a \mid O(a, b)\}$ is open, while $\{a \mid \breve{O}(a, b)\}$ is closed. In fact there exists a close relationship between our chosen primitives and the hyperspace topology:

Definition 32. *Let $\mathcal{W}$ be an event structure (possibly infinite). We define the relation $\sqsubseteq$ (called* hypercover*) by the condition: if $c_1, ..., c_n$ are such that $O(c_1, c_{i+1}$ and $\breve{E}(c_{i+1}, c_i)$ then*

$$a \sqsubseteq c_1, \ldots, c_n :\Longleftrightarrow \forall i O(a, c_i) \wedge \breve{B}(c_1, a) \wedge \breve{E}(c_n, a).$$

Theorem 16. *If $\mathcal{W} = \Omega$ with the canonical interpretation of the primitives $O, \breve{E}, \breve{B}$, then the basic open sets are of the form $\{a \in \Omega \mid a \sqsubseteq c_1, \ldots, c_n\}$.*

This result shows that a homomorphism $h : \Omega \longrightarrow \mathcal{M}$ preserving $O, \breve{E}, \breve{B}$ can be considered as a continuous map.

Theorem 17. *$(\Omega, \mathcal{HYP})$ is a model of GT_0.*

The remainder of this section will be devoted to showing that $(\Omega, \mathcal{HYP})$ is candidate for representing Kant's 'time as formal intuition'.

5.1. **Boundaries in hyperspace.** We have two ways to determine boundaries on hyperspace. The first is to use the general theorem 6: $(\mathcal{B}(\mathcal{W}), \subset)$ is homeomorphic to $[0, 1]$, for $\mathcal{W}$ as described in the theorem. This construction does not yield a canonical homeomorphism. The second construction uses the recipe of theorem 4. Given the interpretation of O, the present is generated by a unique point x in $[0, 1]$, and all (and only) open intervals I with $x \in I$ belong to that present.

Theorem 18. *In $(\Omega, \mathcal{HYP})$ a boundary $(Past, Pres, Fut)$ (corresponding to $x \in [0, 1]$) can be represented as follows ($_{hyp}$ indicates interpretation in $(\Omega, \mathcal{HYP})$):*
- $Past^x_{hyp} = [\ [0, x)\ ]$ (open)
- $Pres^x_{hyp} = \{I \in \mathcal{W} \mid x \in I\}$ (closed); not that $\{x\} \notin Pres^x_{hyp}$
- $Fut^x_{hyp} = [\ (x, 1]\ ]$ (open).

The result obtained in theorem 14 obviously applies in this setting: the preceding theorem does not mean the hyperspace – the intended model for time as 'formal intuition' – is indistinguishable from the unit interval.

5.2. **Finite quotients of hyperspace.** To prove metamathematical results about hyperspace, such as its universality, we must investigate the relation between finite event structures and hyperspace. Corollary 6 and related results tell us that we may represent a finite event structure $\mathcal{V}$ by means of two components: the first component is a finite set of points $x_1, \ldots, x_k$ together with intervals I_j that contain x, and possibly other points as well; these represent the (presents of) the boundaries. The second component consists of intervals that are disjoint from any $\{x_i\}$, and which must pairwise overlap so that no boundaries can be created between adjacent x_i, x_{i+1}.

The embedding proceeds as follows. The description just given shows that $\mathcal{V}$ is a subset of hyperspace, which we give the subspace topology derived from the hypercovering relation.

Definition 33. *Let the domain of $\mathcal{V}$ be $c_1, \ldots, c_n$; this set is a cover of $[0, 1]$. Without loss of generality we may assume the c_i are closed under taking unions. Choose an interval $I \in \Omega$ and*

$\{c_1^{'}, \ldots, c_k^{'}\} \subseteq \{c_1, \ldots, c_n\}$ *such that* $I \sqsubseteq c_1^{'}, \ldots, c_k^{'}$ *and* $c_1^{'} \cup \ldots \cup c_k^{'}$ *has the smallest diameter. Put*

$$h(I) = c_1^{'} \cup \ldots \cup c_k^{'}.$$

Lemma 23. $h : \Omega \longrightarrow \mathcal{V}$ *is continuous w.r.t. the hyperspace topology, surjective, as well as injective on its range, i.e.* h *is a retraction and a homomorphism w.r.t.* $O, \breve{E}, \breve{B}$.

Definition 34. *Let* $\mathcal{C} = \{c_1, \ldots, c_n\}$, $\mathcal{D} = \{d_1, \ldots, d_m\}$ *be coverings of* $[0, 1]$. $\mathcal{D}$ *is a* refinement *of* $\mathcal{C}$ *if every element of* $\mathcal{C}$ *can be written as a union of elements of* $\mathcal{D}$.

Lemma 24. *The refinement relation is a directed partial order* $\leq$.

Analogously to lemma 23 we then have

Lemma 25. *Let* s, t *be coverings such that* $s \geq t$. $h_{st} : \mathcal{V}_s \longrightarrow \mathcal{V}_t$ *is a retraction.* h *is a homomorphism w.r.t.* $O, \breve{E}, \breve{B}$.

Theorem 19. *Let* $(\{\mathcal{V}_s\}, \{h_{st}\}, \leq)$ *be the inverse system of all finite quotients of* $(\Omega, \mathcal{HYP})$ *, and let* $\mathcal{V}$ *be its inverse limit. Then there exists an injective homomorphism* $\iota : (\Omega, \mathcal{HYP}) \longrightarrow \mathcal{V}$.

PROOF. Define $\iota(I)$ to be the ξ such that each $\xi_s \supseteq I$ is constructed as in definition 33. $\square$

Theorem 20. *Let* φ *be a geometric sentence in the* $O, \breve{E}, \breve{B}$ *vocabulary. Then* $\mathcal{V} \models \varphi$ *iff* $(\Omega, \mathcal{HYP}) \models \varphi$.

This follows from the following three lemmas:

Lemma 26. Ω *is the direct limit of the* $\mathcal{V}_s$ *and if* φ *be a geometric sentence in the* $O, \breve{E}, \breve{B}$ *vocabulary, then* $(\Omega, \mathcal{HYP}) \models \varphi$ *implies, for all s,* $\mathcal{V}_s \models \varphi$.

Lemma 27. *Positive primitive formulas are preserved by the projections* $\pi_s : \mathcal{V} \longrightarrow \mathcal{V}_s$.

Lemma 28. *Let* $\theta(x)$ *be positive primitive, andd let* $\xi \in V$. *If for all s* $\mathcal{V}_s \models \theta(\xi_s)$*, then* $\mathcal{V} \models \theta(\xi)$.

The material in section 4 shows that finite event structures have a normal form – where events correspond to open intervals in the linear order of boundaries – which makes them isomorphic to some finite quotient. Hence the finite model property holds as well for these event structures. This gives

Theorem 21. *Let* φ *be a geometric sentence as above. Then* $(\Omega, \mathcal{HYP}) \models \varphi$ *iff for all finite event structures* $\mathcal{E}$, $\mathcal{E} \models \varphi$.

Corollary 8. *Let* φ *be a geometric sentence as above. Then* $(\Omega, \mathcal{HYP}) \models \varphi$ *iff* $GT_0 \models \varphi$.

We are now closing in on establishing the desired result that $(\Omega, \mathcal{HYP})$ is the representation time as a fully determined object which supports all 'concepts of time'. From [2, theorem ...] it follows that only geometric formulas have the property that what they express about finite event structures approximating time carries over to time as an object. Such sentences are called 'objectively valid' by Kant (B141-2). Formally we have

Definition 35. *A sentence* α *satisfying: for all inverse systems* $(\mathcal{M}_s)$*, if for all s,* $\mathcal{M}_s \models \alpha$ *implies* $\mathcal{V} \models \alpha$ *will be called* objectively valid.

For objectively valid sentences we have the following geometric interpolation theorem:

Theorem 22. *Let*
$alpha$ be an objectively valid sentence $GT_0 \models \alpha$.. Then there are geometric sentences $\theta_1, \ldots, \theta_n$ in the vocabulary $O, \breve{E}, \breve{B}$ such that $GT_0 \models \theta_1 \wedge \ldots \wedge \theta_n$ and $\theta_1 \wedge \ldots \wedge \theta_n \models \alpha$.

We may now strengthen corollary 8 to

Theorem 23. *Let α be an objectively valid sentence in the event language. Then $(\Omega, \mathcal{HYP}) \models \alpha$ iff $GT_0 \models \alpha$.*

PROOF. The right to left direction is trivial. For the other direction, assume $(\Omega, \mathcal{HYP}) \models \alpha$. Since $(\Omega, \mathcal{HYP}) \models GT_0$, and α is objectively valid, $(\Omega, \mathcal{HYP}) \models \theta_1 \wedge \ldots \wedge \theta_n$ for geometric θ_i. By corollary 8, $GT_0 \models \theta_1 \wedge \ldots \wedge \theta_n$, and it follows that $GT_0 \models \alpha$. □

Hence, there is essentially only one model for GT_0 plus the higher order 'infinite divisibility' condition, and unity of time follows.

5.3. **Form of intuition and formal intuition.** In the chapter of the Critique of Pure Reason entitled the *Transcendental Aesthetic* Kant describes time as a singular object of intuition, with considerations such as those expressed in properties (1.1.4,1.1.3) of section 1. The treatment of space and time in the Transcendental Aesthetic is however at least partially rewritten in a later chapter, the *Transcendental Deduction* of the B edition, where space and time are not merely taken as given (Kant says in the Aesthetic that they are "infinitely *given* magnitudes"), but they are instead seen as the results of a process of synthesis. This updated understanding of space and time is reflected in a famous footnote where Kant introduces the distinction between space and time as *forms* of intuition and as *formal* intuitions. The footnote reads as follows:

> Space, represented as object (as is really required in geometry), contains more than the mere form of intuition, namely the comprehension of the manifold given in accordance with the form of sensibility in an intuitive representation, so that the form of intuition merely gives the manifold, but the formal intuition gives unity of the representation. In the Aesthetic I ascribed this unity merely to sensibility, only in order to note that it precedes all concepts, though to be sure it presupposes a synthesis, which does not belong to the senses but through which all concepts of space and time first become possible. For since through it (as the understanding determines the sensibility) space or time are first given as intuitions, the unity of this a priori intuition belongs to space and time, and not to the concepts of the understanding (B161n).

This very brief explanation has been the subject of many a great interpretative effort, and even greater puzzlement, which is heightened by the fact that the term "formal intuition" does not occur very often in Kant's writings. We shall here try to shed some light on the problem by employing the formalism we have developed in the previous sections.

If we inspect the passage above, we find the following:

(1) although the distinction between *form of intuition* and *formal intuition* is initially formulated in terms of space, it is later applied as well to time
(2) time is represented as an *object*, which requires a *synthesis* of the manifold given by the form of intuition, thus bestowing *unity* to it
(3) through this unity *concepts of time* first become possible
(4) the unity of time is not given by the *concepts of the understanding*; however this unity is at the same time a consequence of how *the understanding determines the sensibility*

Even the reader unfamiliar with Kant's philosophy will recognize that the difficulty in interpreting this passage lies in understanding the purported meaning behind the obscure expressions which we have signalled above in italics. In particular, the crux of the matter lies in understanding exactly what kind of synthesis is mentioned in this passage. The opposition in the last point seems to support the following interpretation: this synthesis is not due to the categories (the "concepts of the understanding"), but it is the synthesis of the unity of apperception, the "effect of the understanding on sensibility" which had been mentioned a few pages earlier[20]. If this is correct, it then means that the dichotomy between time as form and as formal intuition is to be understood in terms of the synthesis of the unity of apperception. It is this synthesis which, operating on manifolds that are given first as temporal but in some sense "scattered", i.e., organized in merely contingent systems of temporal relations[21], unifies them into intuitive representations, and as a byproduct also first *produces* time as formal intuition ("for since through it [...] space or time are first given as intuitions"), i.e., as an "object"[22].

In the formalism we have proposed, the synthesis of the unity of apperception has two formal correlates: at the first-order level we have axiom 11, which ensures that any two events are unified as parts of an encompassing event, while at the second-order level we have the directedness condition on the inverse systems, which ensures that any two "local times" in the experiential history of an agent are part of one global time, the inverse limit. It is instructive to imagine how our experience would be like if these conditions were not in place. Renouncing the second-order unity of apperception would leave one with "islands of time", each accompanied by its own distinct consciousness. Renouncing the first-order unity of apperception would disrupt even the idea of "local times", as models of the axioms would now consist solely of empirical events without "transcendental events"; it would thus be possible to have a collection of models $\mathcal{M}_s$ of $Ax \setminus \{11\}$ each having in the domain only one empirical event. A being described by this collection of models would be "stuck in the present", the consciousness accompanying this fleeting event having no relation whatsoever to that accompanying the event just past or that about to come.

These remarks might suggest that a good analogue for Kant's formal intuition of time is just the inverse limit of an inverse system of models and retraction maps, where the models represent temporal experiences of a given agent. There are two problems with this, however.

The first problem is that it is unclear in what sense the formal intuition of time would be unique if every agent could have its own, totally unrelated to anybody else's.

The second, more substantial problem lies in the interpretation of those "concepts of time" mentioned at the third point above, which we have up to now ignored. It is clear that with this expression Kant could not have meant time as formal intuition, since the latter is of course not a concept. What concepts is Kant talking about here?

In order to find an answer to this question let us focus on the first sentence, "Space, represented as an object (as is really required in geometry)". The mention of geometry here is crucial: space as a formal intuition is a prerequisite for geometrical constructions, i.e., it is the space in which these constructions are carried out. Hence the concepts of space which are made

[20]This is, in particular, Beatrice Longuenesse's interpretation.

[21]Thus Kant at B67: "The time in which we place these representations [of outer sense], which itself precedes the consciousness of them in experience [...] already contains relations of succession, of simultaneity [...] now that which, as representation, can precede any act of thinking something is intuition and, if it contains nothing but relations, it is the form of intuition [...]".

[22]One must remember here that an intuition is what refers to an object immediately; thus it does make sense that if time is given as an intuition, it is also represented as an object.

possible by the synthesis of the unity of apperception (via space as formal intuition) must be what Kant terms "pure sensible concepts", e.g. the concept of triangle or of straight line, but also concepts which apply to geometrical objects, such as length, distance, spatial order, and so forth. Furthermore, that space is "represented as an object" also means that geometrical objects must be completely determined with respect to such "concepts of space" as distance or length; for instance, given two line segments in space, it must be completely determined whether one is longer than the other, or whether they intersect when they are prolonged. In Kantian terminology, this means that objects in space have to be "thoroughgoingly determined" with respect to their spatial properties; indeed, we maintain that for Kant it is this necessary thoroughgoing determination of objects in space with respect to spatial concepts which allows for synthetic *a priori* propositions, and such thoroughgoing determination is possible only because space is not abstracted from experience in the first place, hence it is not empirical[23]. This interpretation is also supported by a rare passage from Kant's notes in which "formal intuition" is mentioned:

> Synthetic propositions through concepts are *always a priori* and impossible; but through the construction of concepts (in sensible **formal intuition** in general) or the combination of universality with empirical synthesis in general they are not only possible, but also necessary. For experience is nothing other than synthetic [crossed out: cognition] connection of perceptions in one consciousness (as contained in it necessarily, hence universally) (*Refl.* 5928, our boldface.)

Where it clearly emerges that the possibility of synthetic *a priori* propositions about objects lies in the construction of such objects *a priori* in formal intuition, by means of which they are completely determined with respect to their spatio-temporal properties.

Mutatis mutandis, the "concepts of time" which are made possible by time as formal intuition must be those used in the synthetic a priori principles for time, in particular succession, simultaneity, duration, infinitude, continuous alteration, flowing magnitude, etc. Furthermore, events in time as formal intuition must be thoroughgoingly determined with respect to these concepts. If this is correct, it is then clear that the identification of time as formal intuition with the limit $\mathcal{W}$ of an inverse system of event structures and retractions is inadequate, since this limit does not, by itself, guarantee that the events in it are "thoroughgoingly determined" with respect to the concepts of time. For instance, if the linear order of boundaries $\mathcal{BW}$ were discrete, the concept of a determined duration of an event would lose its meaning, given that there could be no way to determine the duration of the stretch of time lying between two succesive boundaries. Thus events would also remain undetermined with respect to comparisons of length; similar considerations apply to the other concepts of time.

This discussion leads us to formulate the following *desiderata* for a mathematical analogue of time as formal intuition. First, it should support the possibility of intersubjective comparison of events and experiences. Second, it should be "thoroughoingly determined" in the formal sense that (i) it should provide a universal model for objectively valid judgments of temporal order of

[23]Thus Kant in the Opus Postumum: "Synthetic *a priori* propositions are only indirectly possible in philosophy, namely, in relation to objects of pure intuition in space and time, and to those objects' *existence* in space and time as their thoroughgoing determination [...] but the objects of sense are given in space and time only as things in appearance (*phaenomena*); that is, they are, by their form, not objects given purely and simply, but only subjectively, under the limitation of their principle [22:483]". In other words, geometrical objects in space are thoroughgoingly determined, hence synthetic *a priori* judgements about them are possible; while appearances are not "simple objects", i.e., pure, and therefore their thoroughgoing determination is only ideal.

events (see Section 5.2), and (ii) it should support full determination of events with respect to concepts of time, allowing for duration comparisons, full-fledged continuous functions, etc.

These considerations make it plain that the Hyperspace $\mathcal{HYP}$ model introduced in Section 5 is the mathematical structure we are looking for, according to the equation:

unity of apperception + infinite divisibility = formal intuition

Recall that according to property (1.1.3), time is an intuition and not a concept because "different times are only parts of one and the same time", and moreover "that representation [...] which can only be given through a single object, is an intuition". Indeed, $\mathcal{HYP}$ captures Kant's dictum, since every model of GT_0 can be embedded in $\mathcal{HYP}$; this also ensures the possibility of intersubjectivity, since via these embeddings the temporal relations between events belonging to experiential episodes of different agents can be determined (in Kant's speech, their "reciprocal time-determination" can be achieved). Most importantly, the universality of $\mathcal{HYP}$ with respect to objectively valid temporal judgments, along with the possibility of embedding any finite event structure as a subspace of the hyperspace, take care of the second of our *desiderata*, and provide us with a mathematical structure which embodies the "unicity" - in formal terms, the "universality" - of time as formal intuition.

It might be objected at this point that the embeddings of event structures into the hyperspace are by no means unique, and that therefore via its universality objectivity of time is only partially achieved; different embeddings correspond to quite different determinations of events with respect to, e.g., duration and intersubjective temporal order.

This objection is justified, however it must be put in its appropriate context. What we have shown in this paper is that Kant's theory of time can be given a consistent mathematical description, and that the principles listed at the beginning of this work do indeed enforce the existence of a *formal intuition* of time, completely determined with respect to judgements of temporal order, such that every appearance *can* and indeed *must* be there embedded and thus achieve full determinination with respect to the concepts of time which Kant mentions. We have thus given a proof of principle; the choice of a particular time-determination of appearances, i.e., of a particular embedding, goes beyond what is purely temporal, and demands also a treatment of space and of conceptual synthesis. Indeed, Kant writes at B155 (our emphasis):

> [...] time, although it is not itself an object of outer intuition at all, cannot be made representable to us except under the image of a line, insofar as we draw it, without which sort of presentation **we could not know the unity of its measure at all**, or likewise from the fact that **we must always derive the determination of the length of time or also of the positions in time for all inner perceptions from that which presents external things to us as alterable**; hence we must order the determinations of inner sense as appearances in time in just the same way as we order those of outer sense in space [...]

This passage perfectly resonates with recent findings in developmental psychology and psychophysics, which point out that the processing of temporal information is often mediated by spatial representation. In particular, an experiment by Carelli and Forman [**?**] reports that, for both adults and children, the accuracy in estimating durations of events greatly increases when these durations were expressed by situating the events on a time-line, rather than conveying the durations verbally using conventional numeric values. In Kantian terms, this can be glossed as evidence that the time-determination of an event with respect to the concept of "duration" first depends on its being represented outwardly as a line, as in the passage above. In terms of our formalism, this means that the specific embedding of an event structure $\mathcal{W}_s$ into the hyperspace $\mathcal{HYP}$ can be determined only by taking into account spatial or conceptual, i.e., non-temporal, information. Note, however, that this indeterminacy does not stem from the embedding of $\mathcal{W}_s$ into the space of boundaries on the limit $\mathcal{B}(\mathcal{W})$, since the latter is determined by construction; it stems instead from the fact that the homeomorphism between $\mathcal{B}(\mathcal{W})$ and $[0,1]$ does not determine the metric on $\mathcal{B}(\mathcal{W})$. Once the duration of events is determined via the outward representation of time, however, there exists one canonical embedding into $\mathcal{HYP}$, namely the identity (see Section 5.2), and the necessity of time-determination is then proven.

6. Alteration, causality and the continuum

We will now apply some of the techniques introduced above, to provide a formal analysis of Kant's views on the relation between causality and continuity, which figure in one of the proofs of the 'Second Analogy of Experience' (A189-211/B232-57). This synthetic a priori principle comes in two versions: in the first edition of the *Critique* we find

> (A) Everything that happens (begins to be) presupposes something which it follows according to a rule. (A169)

whereas the second edition is more specific about the nature of the rule, which must express a causal relation:

> (B) All alterations occur in accordance with the law of cause and effect.

The following passage (part of which was cited above as 1.1.7) contains in a nutshell all the ingredients necessry to explain the relation between continuity and causality, and we shall therefore proceed by numbering the steps in the argument, and formalise each of these in terms of event structures.

> (1) If a substance passes out of a state a into another state b, then the point in time of the latter is different from the point in time of the first state follows it.
> (2) Likewise the second state as a reality (in the appearance) is also distinguished from the first, in which it did not yet exist [...] The question therefore arises, how a thing passes from one state = a into another one = b.
> (3) Between two instants there is always a time, and (3a) between two states in those instances there is always a difference that has a magnitude (for all parts of appearances are magnitudes). (3b) Thus every transition from one state into another happens in a time that is contained between two instants, of which the former determines the state from which the thing proceeds and the second the state at which it arrives.
> (4) Both are therefore boundaries of the time of an alteration, consequently of the intermediate state between two states, and as such they belong to the whole alteration.
> (5) Now every alteration has a cause, which manifests its causality in the entire time during which the alteration proceeds. Thus this cause does not produce its alteration suddenly (all at once or in an instant), but rather in a time, so that as the time increases from the initial instant a to its completion in b, (5a) the magnitude of the reality $(b-a)$ is also generated through degrees that are contained between the first and the last.
> (6) All alteration is therefore possible only through a continuous action of causality [...]
> (7) That is, now, the law of the continuity of all alteration, the ground of which is this: That neither time nor appearance in time consists of smallest parts, and that nevertheless in its alteration the state of thing passes through all these parts, as elements, to its second state. No difference of the real in appearance is the smallest, just as no difference

> in the magnitude of times is, and thus the new state of reality grows out of the first, in which it did not exist, through all the infinite degrees of reality, the differences between which are all smaller than that between 0 and a. (A207-9/B253-4)

The link between the categories substance and causality on the one hand, and event structures on the other is the notion of *schema*. To explain Kant's notion of the 'schema of a category', we resort to a very anachronistic analogy: categories (such as substance and causality) act as theoretical terms which must be interpreted in the sensible world ('operationalised'). In fact these categories are interpret in time, which is not itself observable – hence the analogy is of limited value. The crucial concept here is 'interpretation' (in the logical sense): although causality is not definable using temporal terms, it must be interpretable in a theory of time.

The following passage introduces the contrasting schemata of substance and of causality:

> The schema of substance is the persistence of the real in time, i.e., the representation of the real as a substratum of empirical time determination in general, which therefore endures while everything else changes. (Time itself does not elapse, but the existence of that which is changeable elapses in it. To time, therefore, which is itself unchangeable and lasting, there corresponds in appearance that which is unchangeable in existence, i.e., substance, and in it alone can the succession and simultaneity of appearances be determined in regard to time.)
> The schema of the cause and of the causality of a thing in general is the real upon which, whenever it is posited, something else always follows. It therefore consists in the succession of the manifold insofar as it is subject to a rule.(A144/B183)

(1) Substance and change of state. Let $\mathcal{W}$ be an event structure. To interpret (1) in $\mathcal{W}$ we first apply the schema of substance, which suggests that a substance is interpreted by a universal cover w. The states of the substance are represented by events a, b satisfying $\breve{O}(a, b)$ and $B(b, a)$ (which just means that a is the initial state, b the final state).
(2) Transition. This sentence asks two questions, namely: how is it possible for state b to arise from state a?, and: how can state be real, whereas it wasn't real a while ago? The answer will be given in terms of the interplay between two forms of continuity: temporal continuity (3) and continuity in the appearances (3a), the latter treated in the 'Anticipations of perception' of which the B version reads

> In all appearances the real, which is an object of the sensation has intensive magnitude, i.e. a degree. (B207)

At this point we note that Kant draws a distinction between so-called "intensive" and "extensive" magnitudes, which correspond to two different changes of state. An intensive magnitude is a quantity that is (by and large) independent of the extent of a physical system; examples are colour, temperature, specific heat, and the strength of a gravitational field[24]. On the other hand we have alterations pertaining to the change in the extensive magnitude of the appearance. The most prominent example here is that of the motion of a point in space, which Kant understands as the continuous modification of the extended magnitude representing its trajectory[25]. This is in particular the "description" of a space mentioned in property 1.1.9 of Section 1, the core idea on which both Kant's theory of geometrical constructions and his attempt to develop an *a priori* theory of motion[26] rest, and which we have seen to be responsible for the determination of

[24]See Kant's anticipations of perception in the CPR, in particular B211.
[25]See the axioms of intuition, in particular A162/B203.
[26]See [6], [489].

events with respect to their duration (see Section 5.3). Given their structural similarity, we shall model the change of state with respect to an intensive magnitude and the change in the position of a point in its "describing" of a space in the same fashion.
(3, 3a) Magnitudes and times. We first illustrate the change in intensive magnitude alluded to in (a) by means of an example taken from the *Prolegomena*[**?**, §29] : 'If a body is illuminated by the sun for long enough, it becomes warm.' The logical form of this hypothetical judgement is something like

> If x is illuminated by y between time t and time s and $s - t > d$ and the temperature of x at t is v, then there exists a $w > 0$ such that the temperature of x at s is $v + w$ and $v + w > c$,

where d is the criterion value for 'long enough' and c a criterion value for 'warm'.[27] Intensive magnitudes call for a second hyperspace (together with its finite quotients); this is because Kant's argument hinges on the structural similarity between time and magnitudes. Let $(\{\mathcal{V}_s\}, \{h_{st}\}, \leq)$ be the temporal inverse system, and $(\{\mathcal{M}_s\}, \{g_{st}\}, \leq)$ the inverse system associated with the intensive magnitude.
(4, 5a) Boundaries. Choose a finite quotient $\mathcal{M}_r$ of the magnitude hyperspace, which has two boundaries corresponding to the values a and b. Its temporal counterpart must also have two boundaries; for ease of notation this structure will be taken to be $\mathcal{V}_r$. The existence of a causal rule implies there must be a function $f_r : \mathcal{V}_r \longrightarrow \mathcal{M}_r$ which is surjective and continuous w.r.t. the hyperspace topology.

Let then $e \in \mathcal{V}_r$, and let $g_r : \mathcal{M} \to \mathcal{M}_r$ be the retraction map from the magnitude inverse limit onto $\mathcal{M}_r$ defined in Definition 33. Then $g_r^{-1} \circ f_r(e)$ is a model of GT_0, which corresponds to all of the infinitely many "degrees of appearance" which, in Kant's terminology, are "traversed" during event e. In keeping with Kant's Aristotelianism (see Section 3.7.1), these infinitely many "degrees of reality" can be traversed during event e because they are "virtual", or, as Kant would put it, they are not "traversed in a series", but all at once - which is possible since they have not yet been specified by logical division (see property **??**). Indeed, Kant is adamant that infinite *series* cannot be intuited (property **??**).
(7) Limits. The finite quotients represent all possible ways of partitioning the temporal continuum (formally: the hyperspace). A function of time represents a *causal* relation if 'data points' that could have been observed but actually weren't, are also described by the relation. Formally this means that the causal relation should be defined for all possible refinements such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{V}_s & \xrightarrow{f_s} & \mathcal{M}_s \\ {\scriptstyle h_{st}}\downarrow & & \downarrow{\scriptstyle g_{st}} \\ \mathcal{V}_t & \xrightarrow[f_t]{} & \mathcal{M}_t \end{array}$$

This commutative diagram encapsulates the objective validity of the category; it implies that there is a unique surjective limit mapping $f : \mathcal{V} \longrightarrow \mathcal{M}$.

[27]Kant's examples of hypothetical judgements can all be brought in geometric form, such that $\vee$ does not occur positively.

REFERENCES

[1] Augustine, Saint and Augustinus, Aurelius and Outler, Albert Cook *Confessions and Enchiridion*. London, SCM Press. Volume 7. 1955.

[2] T. Achourioti and M. van Lambalgen. A formalisation of kant's transcendental logic. *Review of Symbolic Logic*, 4(2):254–289, 2011.

[3] W. James. *The principles of psychology*. Harvard University Press, 1983. Reprint of the 1890 edition.

[4] H. Kamp. Events, instants and temporal reference. In R. Baeuerle, U. Egli, and A. von Stechow, editors, *Semantics from different points of view*, pages 27–54. Springer Verlag, Berlin, 1979.

[5] I. Kant. *Critique of pure reason; translated from the German by Paul Guyer and Allen W. Wood*. The Cambridge edition of the works of Immanuel Kant. Cambridge University Press, Cambridge, 1998.

[6] I. Kant. *Metaphysical Foundations of Natural Science; translated from the German by Michael Friedman* A Companion to Kant, p. 236. Blackwell Publishing Ltd, 2006.

[7] I. Kant. *Notes and Fragments* The Cambridge edition of the works of Immanuel Kant. Cambridge University Press, 2005.

[8] Bertrand Russell. *Our Knowledge of the External World (Lecture IV)*. Allen and Unwin, London, 1914.

[9] S. K. Thomason. Free construction of time from events. *Journal of Philosophical Logic*, 18:43–67, 1989.

[10] A. G. Walker. Durées et instants. *Revue Scientifique*, 85:131–134, 1947.

[11] Longuenesse, Béatrice. Kant et le pouvoir de juger. Cambridge University Press, 1993.

[12] Michael, Friedman. Kant on geometry and spatial intuition. *Synthese*, 186:231-255, 2012.