Lecture 40: Wave Guides

Friday, November 15, 2019

Recall our discussion last lecture on perfect conductors with constant cross-section along the $\hat{\mathbf{z}}$-axis. By "perfect" we mean $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ and $\vec{\mathbf{B}} = \vec{\mathbf{0}}$ inside the material. In reality, even highly-conductive materials can have some fields breach the skin depth of the material, but we will ignore this for the present discussion. Recall that $\vec{\mathbf{E}}_{tangent}$ and $\vec{\mathbf{B}}_{\text{normal}}$ are both continuous at the boundaries of the conductor. With these boundary conditions, we can essentially say that

$$\vec{\mathbf{E}}_{\parallel}\Big|_{\text{surface}} = \vec{\mathbf{B}}_{n}\Big|_{\text{surface}} = \vec{\mathbf{0}}$$

If the conductor is straight along the $\hat{\mathbf{z}}$-axis, the propagation along this axis is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(x,y)e^{\pm \imath kz - \imath\omega t}$$

We will choose $+$, which represents waves going in the positive direction, so

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}(x,y)e^{\imath kz - \imath\omega t}$$

Inside the waveguide, $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{0}}$.

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} = +\imath\omega\vec{\mathbf{B}}$$

so

$$\begin{aligned} \partial_y E_z - \imath k E_y &= \imath\omega B_x \\ \imath k E_x - \partial_x E_z &= \imath\omega B_y \\ \partial_x E_y - \partial_y E_x &= \imath\omega B_z \end{aligned}$$

Similarly, $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = -\epsilon\mu\frac{\partial \vec{\mathbf{E}}}{\partial t}$:

$$\begin{aligned} \partial_y B_z - \imath k B_y &= -\imath\omega\epsilon\mu E_x \\ \imath k B_x - \partial_x B_z &= -\imath\omega\epsilon\mu E_y \\ \partial_x B_y - \partial_y B_x &= -\imath\omega\epsilon\mu E_z \end{aligned}$$

With six unknowns and six equations, we can probably solve this in terms of derivatives of the fields. If we solve this, we find (assuming $\omega^2\epsilon\mu \neq k^2$)

$$\begin{aligned} E_x &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_x E_z + \omega\partial_y B_z] \\ E_y &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_y E_z - \omega\partial_x B_z] \\ B_x &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_x B_z - \omega\epsilon\mu\partial\mu x E_z] \\ B_y &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_y B_z + \omega\epsilon\mu\partial_y E_z] \end{aligned}$$

If we find E_z and B_z, we get the other components. If $E_z = B_z = 0$ then this reduces to the case where $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \vec{\mathbf{0}}$ so $\vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\psi$ where the boundary conditions dictate that ψ is a constant, so there is no propagation.

If $E_z = 0$ we call these modes "TE" or "transverse-electric" modes, and if $B_z = 0$, we call these "TM" or "transverse-magnetic" modes.

By taking the curl of $\vec{\mathbf{E}}$ twice, we find that in general

$$(\nabla^2 + \omega^2\epsilon\mu)\begin{cases}\vec{\mathbf{E}} \\ \vec{\mathbf{B}}\end{cases} = \vec{\mathbf{0}}$$

However, with our boundary conditions applied, we can say

$$(\nabla_\perp^2 - k^2 + \omega^2\epsilon\mu)\begin{cases}\vec{\mathbf{E}} \\ \vec{\mathbf{B}}\end{cases} = \vec{\mathbf{0}}$$

where the perpendicular Laplacian refers to derivatives in only the x and y coordinates. Using the relations we found between the components, we can reduce our equations to

$$(\nabla_\perp^2 - k^2 + \omega^2\epsilon\mu)\begin{cases}\vec{\mathbf{E}}_z \\ \vec{\mathbf{B}}_z\end{cases} = \vec{\mathbf{0}}$$

From this, we see that $E_z\big|_{\text{surface}} = 0$ and $\vec{\mathbf{B}}_n\Big|_{\text{surface}} = \vec{\mathbf{0}}$.

Example. Let's look at a rectangular wave guide. We must impose boundary conditions on all four surfaces (not the ones parallel to the x/y-plane, just think of this as an infinite structure). Let's look for TE modes, where $E_z = 0$. We can write $B_z = X(x)Y(y)$ and set the boundaries at $x = 0, a$ and $y = 0, b$. Plugging in our definition of B_z,

$$\frac{X''}{X} + \frac{Y''}{Y} + (\omega^2\epsilon\mu - k^2) = 0$$

so we can say that $\frac{X''}{X} = -k_x^2$ and $\frac{Y''}{Y} = 0k_y^2$. Solving these, we find that

$$B_z = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)]$$

From our component relations, we have

$$B_x = \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_x B_z]$$

and

$$B_y = \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_y B_z]$$

By the boundary condition on the normal of B_z, we find that the derivatives in the equations above must be zero at the boundary, so we can show that $k_x a = n\pi$. Using B_y, we find that $k_y b = m\pi$:

$$B_z = A_{mn}\cos\Big(\frac{n\pi x}{a}\Big)\cos\Big(\frac{m\pi y}{b}\Big)e^{\imath kz - \imath\omega t}$$

Recall that we have to satisfy the condition $-k_x^2 - k_y^2 - k^2 + \omega^2\epsilon\mu$, or

$$k^2 = \omega^2\epsilon\mu - \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)$$

What this means is that there is a cutoff frequency below which no waves will propagate, if we choose m and n. If you check the velocities, you find that $v_p = \frac{\omega}{k} > \frac{1}{\sqrt{\epsilon\mu}}$ but $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} < \frac{1}{\sqrt{\epsilon\mu}}$, and in fact $v_p v_g = \frac{1}{\epsilon\mu}$.

Note that the condition that we cant have $B_z = E_z = 0$ implies that we can't propagate waves straight into the wave guide. We actually have to bounce around along the walls to maintain a propagating wave. $\diamond$

How can we generalize this? We can rewrite our previous equations as

$$\vec{\mathbf{E}}_\perp = \frac{\imath}{\mu\epsilon\omega^2 - k^2}[k\vec{\mathbf{\nabla}}_\perp E_z - \omega\hat{z} \times \vec{\mathbf{\nabla}}_\perp B_z]$$

$$\vec{\mathbf{B}}_\perp = \frac{\imath}{\mu\epsilon\omega^2 - k^2}[k\vec{\mathbf{\nabla}}_\perp B_z - \omega\hat{z} \times \vec{\mathbf{\nabla}}_\perp E_z]$$

We can see here that if we look only at TE or TM waves, we can reduce these further. For $E_z = 0$,

$$\vec{\mathbf{B}}_\perp = \frac{\imath k}{\mu\epsilon\omega^2 - k^2}\vec{\mathbf{\nabla}}_\perp B_z$$

and

$$\vec{\mathbf{E}}_\perp = \frac{-\imath\omega}{\mu\epsilon\omega^2 - k^2}\hat{z} \times \vec{\mathbf{\nabla}}_\perp B_z$$

and for $B_z - 0$,

$$\vec{\mathbf{E}}_\perp = \frac{\imath k}{\mu\epsilon\omega^2 - k^2}\vec{\mathbf{\nabla}}_\perp E_z$$

and

$$\vec{\mathbf{B}}_\perp = \frac{\imath\omega}{\mu\epsilon\omega^2 - k^2}\hat{z} \times \vec{\mathbf{\nabla}}_\perp E_z$$

We are looking for the solutions of

$$\left[\nabla_\perp^2 + (\omega^2\epsilon\mu - k^2)\right]\psi$$

for either $\psi\big|_S = 0$ or $\frac{\partial\psi}{\partial n}\big|_S = 0$, which we recognize as the Dirichlet and Neumann boundary conditions.