Lecture 21: Magnetostatics, Continued

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For a current moving in a circle of radius a,

$$A_y \mapsto A_\varphi = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \mathrm{d}\varphi' \, \frac{\cos(\varphi')}{\sqrt{a^2 + r^2 - 2ar\hat{x} \cdot \hat{x}'}} \tag{1}$$

where $\hat{x} \cdot \hat{x} = \cos(\gamma) = \sin(\theta)\cos(\varphi')$.

We expand

$$\frac{1}{\sqrt{a^2 + r^2 - 2ar\cos(\gamma)}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos(\gamma)) \tag{2}$$

Recall (from Jackson) that

$$P_l(\hat{x}\cdot\hat{x}') = P_l^0(\cos(\theta))P_l^0(\cos(\theta)) + 2\sum_{m=1}^{\infty} \frac{(l-m)!}{(l+m)!} P_l^m(\cos(\theta))P_l^m(0)\cos(m[\varphi - \varphi']) \tag{3}$$

$$A_\varphi = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \mathrm{d}\varphi' \cos(\varphi') \left[\frac{r_<^l}{r_>^{l+1}} P_0(\cos(\theta))P_0(0) + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} 2 \sum_{m=0}^{l} (\dots) \right] \tag{4}$$

This removes all but the $m = 1$ term:

$$\int_0^{2\pi} \mathrm{d}\varphi' \cos(\varphi')\cos(m(\varphi - \varphi')) = \delta_{ml}\pi \tag{5}$$

$$A_\varphi = \frac{\mu_0 I a}{4\pi}(2\pi) \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \underbrace{P_l^1(\cos(\theta))P_l^1(0) \underbrace{\frac{(l-1)!}{(l+1)!}}_{\frac{1}{l(l+1)}}}_{\frac{(-1)^{s+1}(2s-1)!!}{2^{s+1}s!(2s+2)}} \tag{6}$$

where $l = 2s + 1$.

$$A_\varphi = -\frac{\mu_0 I a}{2} \sum_{s=0}^{\infty} \frac{r_<^{2s+1}}{r_>^{2s+2}} \left[\frac{(-1)^2(2s-1)!!}{2^{s+1}s!(2s+2)} P_{2s+1}^1(\cos(\theta)) \right] \tag{7}$$

Now we need to figure out what $\vec{B}$ is! We will once more rewrite this expression so that it matches what Jackson uses:

$$A_\varphi = -\frac{\mu_0 I a}{4} \sum_{s=0}^{\infty} \frac{r_<^{2s+1}}{r_>^{2s+2}} \frac{(-1)^s(2s+1)!!}{2^{s+1}s!(s+1)} P_l^1(\cos(\theta)) \tag{8}$$

$$\vec{B} = \boldsymbol{\nabla} \times \vec{A} \tag{9}$$

$$B_r = \frac{1}{r\sin(\theta)} \frac{\partial}{\partial\theta}(\sin(\theta)A_\varphi) \tag{10}$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) \tag{11}$$

(this is symmetric about φ so we don't need to calculate that component)

If you plug this vector potential into these formulas, you should use the trick that

$$\sin(\theta) = (1-x^2)^{1/2} \tag{12}$$

for small angles, which is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)^{1/2}P_l^1 = \frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)^{1/2}(-1)(1-x^2)^{1/2}P_l = \frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_l = l(l+1)P_l \tag{13}$$

Finally:

$$B_\theta = \frac{\mu_0 I a}{2r}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+1)!!}{2^s s!}\frac{r_<^{2s+1}}{r_>^{2s+2}}P_{2s+1}(\cos(\theta)) \tag{14}$$

and

$$B_r = -\frac{\mu_0 I a}{4}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+1)!!}{2^s(s+1)!}\begin{cases} -\frac{2s+2)}{(2s+1)}\frac{1}{a^2}\left(\frac{r}{a}\right)^{2s} & r<a \\ \frac{1}{r^3}\left(\frac{a}{r}\right)^{2s} & r>a \end{cases} P_{2s+1}^1(\cos(\theta)) \tag{15}$$

In retrospect, it might make more sense to write this in cylindrical coordinates. We call our current $\vec{J} = I_0\delta(\rho-a)\delta(z)\hat{\varphi}$.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' \tag{16}$$

$$\begin{aligned}\vec{A} = &\frac{\mu_0}{4\pi}\int I_0\delta(\rho'-a)\delta(z')[-\sin(\varphi')\hat{i}+\cos(\varphi')\hat{j}]\rho'\,\mathrm{d}\rho'\,\mathrm{d}\theta'\,\mathrm{d}z' \\ &\times\frac{4}{\pi}\int_0^\infty \cos(k(z-z'))\,\mathrm{d}k\left[I_0(k\rho_<)K_0(k\rho_>)+2\sum_{m=1}^{\infty}I_m(k\rho_<)K_m(k\rho_>)\cos(m(\varphi-\varphi'))\right]\end{aligned} \tag{17}$$