## Lecture 14: The Multipole Expansion

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Imagine we have a region of charge with density $\rho(\vec{x})$. The potential for this can be expanded as

$$\frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')d^3\sum_{l=1}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma) \tag{1}$$

where $\cos\gamma = \hat{x}\cdot\hat{x}'$. This can be written as

$$\frac{1}{4\pi\varepsilon_0}\overset{Q}{\cancel{\left[\int \rho(\vec{x}')d^3x'\right]}}\frac{1}{r}+\frac{1}{4\pi\varepsilon_0}\left[\int \rho(\vec{x}')d^3x'\hat{x}\cdot\hat{x}'\right]\frac{r'}{r^2}+\frac{1}{4\pi\varepsilon_0}\left[\int \rho(\vec{x}')d^3x'\left(\frac{3}{2}(\hat{x}\cdot\hat{x}')^2-\frac{1}{2}\right)\right]\frac{r'^2}{r^3}+\dots \tag{2}$$

This is the multipole expansion. We can further simplify the first term:

$$\frac{1}{4\pi\varepsilon_0}\frac{q}{r}+\frac{1}{4\pi\varepsilon_0}\frac{\left(\int \rho(\vec{x}')\vec{x}'d^3x'\right)\cdot\hat{x}}{r^2}+\frac{\rho(\vec{x}')\left[\frac{3}{2}(\vec{x}'\cdot\hat{x})^2-r'^2\hat{x}\cdot\hat{x}\right]}{4\pi\varepsilon_0 r^3} \tag{3}$$

This last term is the quadrupole moment:

$$\frac{1}{4\pi\varepsilon_0}\underbrace{\int d^3x'\rho(\vec{x}')\left[\frac{3}{2}x_i'x_j'-\frac{1}{2}r'^2\delta_{ij}\right]}_{Q_{ij}}\hat{x}_i\hat{x}_j\frac{1}{r^3} \tag{4}$$

Therefore, the potential can be written, in general as

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}+\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{x}}{r^2}+\frac{1}{4\pi\varepsilon_0}\frac{\sum_{i,j}Q_{ij}\hat{x}_i\hat{x}_j}{r^3}+\frac{1}{4\pi\varepsilon_0}\frac{\sum_{i,j,k}Q_{ijk}\hat{x}_i\hat{x}_j\hat{x}_k}{r^4}+\dots \tag{5}$$

**Remark.**

$$\int \rho(\vec{x}')d^3x'[\text{homogeneous polynomial of degree } l \text{ in } x_1', x_2', x_3'] \tag{6}$$

where the polynomial can be expanded in terms of $P_l(\hat{x}\cdot\hat{x}')$. This "Q" is traceless. Therefore there are $2l+1$ degrees of freedom per multipole moment.

Recall that

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m}\frac{4\pi}{2l+1}\left(\frac{r'^l}{r^{l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{7}$$

so

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')d^3x'\sum\frac{4\pi}{2l+1}\left(\frac{r'^l}{r^{l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{8}$$

which is equal to

$$\frac{4\pi}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\underbrace{\sum_{m=-l}^{l}}_{2l+1\text{ terms}}\frac{1}{2l+1}\frac{1}{r^{l+1}}\underbrace{\left[\int Y^*_{lm}(\theta',\phi')r'^{l}\rho(\vec{x}')d^3x'\right]}_{q^*_{lm}=q_{l,-m}(-1)^m}Y_{lm}(\theta,\phi) \tag{9}$$

We can construct $Y_{lm}$'s as homogeneous polynomials on $x-\imath y$, $x+\imath y$ and $z$.

$$q_{00}=\frac{1}{\sqrt{4\pi}}Q \tag{10}$$

$$q_{11}=-\sqrt{\frac{3}{8\pi}}\int(x'-\imath y')\rho(\vec{x}')d^3x'=-\sqrt{\frac{3}{8\pi}}(p_1-\imath p_2) \tag{11}$$

$$q_{10}=\sqrt{\frac{3}{4\pi}}\int z'\rho(\vec{x}')d^3x'=\sqrt{\frac{3}{4\pi}}p_3 \tag{12}$$

$$q_{22}=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\int(x'-\imath y')^2\rho(\vec{x}')d^3x'=\frac{1}{4}\sqrt{\frac{15}{2\pi}}(Q_{11}-2\imath Q_{12}-Q_{22}) \tag{13}$$

### 0.0.1 Dipole Case

Suppose we have $Q=0$ and a point dipole $\vec{p}$.

$$\Phi=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{x}}{r^2} \tag{14}$$

Assuming $r\neq 0$, can write the electric field from

$$\vec{E}=-\nabla\Phi=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}\cdot\hat{x})\hat{x}-\vec{p}}{r^3}\right] \tag{15}$$

$$\vec{E}(\vec{x})=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}\cdot\hat{n})\hat{x}-\vec{p}}{|\vec{x}-\vec{x}_0|^3}\right] \tag{16}$$

This is actually not correct. There are no point dipoles for electric fields. Atoms can have induced dipole moments, but there is no solution at $x_0$, the center of the dipole. To find this term, suppose we have a distribution $\rho(\vec{x}')$ which creates an electric field. Say we take a point $\vec{y}$ and a sphere around this point, and average the electric field in this region. Say $\vec{y}=\vec{0}$ for convenience.

$$\int_{\text{ball around }\vec{0}}\vec{E}d^3y=\frac{4\pi}{3}\vec{E}(\vec{0}) \tag{17}$$

If the ball contains the charge,

$$\int_{\text{ball around }\rho(\vec{x})}=-\frac{1}{3\varepsilon_0}\vec{p} \tag{18}$$

If you were to repeat this process for our point dipole, you would find that this integral evaluates to 0. It misses this $-\frac{1}{3_0}\vec{p}$ term. For an ideal point dipole,

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}\cdot\hat{n})\hat{x}-\vec{p}}{|\vec{x}-\vec{x}_0|^3}-\frac{4\pi}{3}\vec{p}\delta(\vec{x})\right] \tag{19}$$