Lecture 3: Symmetries, Continued

Friday, January 17, 2020

Recall that we said there exist representations of groups which are quantum mechanically written as

$$U = e^{\imath \vec{\lambda} \cdot \vec{\mathbf{X}}}$$

where $\vec{\mathbf{X}}$ are called generators. The generators of continuous groups obey a Lie Algebra:

$$[X_i, X_j] = \imath f_{ijk} X_k$$

> **Definition 0.0.1** (Representation)**.** If we consider an abstract group space G and everything in that space is a group element, we know that if we pick out two elements g_1 and g_2 from that space and multiply them together, we will get an element $g_3 \in G$. This is a bilinear map because it takes two elements of one space and maps to a third element. This map happens to be a mapping $G \mapsto G$.
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> If we consider matrix representations, there is a mapping from the group elements to a matrix, and the product of those matrices must map to the representation of the third group element as above.

Suppose we have some operator $\hat{\mathbf{O}}$ acting on an eigenstate:

$$\hat{\mathbf{O}} |\psi\rangle = \lambda \psi$$

Suppose that G is a symmetry that leaves $\hat{\mathbf{O}}$ invariant.

$$\hat{\mathbf{O}} \to U \hat{\mathbf{O}} U^\dagger = \hat{\mathbf{O}}$$

Recall $U^\dagger U = 1$:

$$\hat{\mathbf{O}} U^\dagger U |\psi\rangle = \lambda |\psi\rangle$$

Multiply both sides by U:

$$(U \hat{\mathbf{O}} U^\dagger) U |\psi\rangle = \lambda U |\psi\rangle$$

However, since the symmetry leaves $\hat{\mathbf{O}}$ invariant, this is equivalent to

$$\hat{\mathbf{O}}(U |\psi\rangle) = \lambda (U |\psi\rangle)$$

so we find that $U |\psi\rangle$ is also an eigenvector. Essentially, we've found an additional solution by examining the symmetries of the system.

0.1 Conservation Laws

Symmetries imply conservation laws. Suppose we are given a Lagrangian:

$$L(x, \dot{x})$$

Suppose the Lagrangian is invariant under some group transformation $\vec{\mathbf{x}} \to \vec{\mathbf{x}}'$. There is an action

$$S = \int \mathrm{d}t \, L(x, \dot{x})$$

Minimizing this action gives us the equations of motion for the system:

$$x(t) \to x(t) + \delta x(t)$$

We are going to look for x's that minimize the action:

$$\begin{aligned}\delta S &= \int \left[\frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta \dot{x}}\delta \dot{x}\right] \mathrm{d}t \\ &= \int \left[\frac{\delta L}{\delta x}\delta x + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta L}{\delta \dot{x}}\delta x\right) - \delta x \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta x}\right] \\ &= \int_{t_i}^{t_f} \mathrm{d}t\, \delta x \left[\frac{\delta L}{\delta x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{x}}\right] + \underbrace{\frac{\delta L}{\delta \dot{x}}\delta x\Big|_{t_i}^{t_f}}_{0}\end{aligned}$$

Therefore, to minimize δS, we require

$$\frac{\delta L}{\delta x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{x}}$$

which are the Euler-Lagrange equations.

If we have a transformation that keeps the Lagrangian invariant, we can take a total derivative of the Lagrangian:

$$\delta L = \frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta \dot{x}}\delta \dot{x}$$

so

$$\int \left[\frac{\delta L}{\delta x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{x}}\right]\delta x + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta L}{\delta \dot{x}}\delta x\right] = 0$$

If we assume the Euler-Lagrange equations hold and we no longer take the end points to be fixed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta L}{\delta \dot{x}}\delta x\right] = 0 \qquad \text{(Noether's Theorem)}$$

Therefore, $\frac{\delta L}{\delta \dot{x}}\delta x$ is a constant along a classical trajectory.

Example. Suppose L is invariant under translations. Under translations, $\vec{\mathbf{x}} \to \vec{\mathbf{x}} + \vec{\epsilon}$ so $\delta\vec{\mathbf{x}} = \vec{\epsilon}$. Therefore, the corresponding conserved quantity is

$$\frac{\delta L}{\delta \dot{\vec{x}}}\vec{\epsilon}$$

If $\vec{\epsilon}$ does not change with time (fixed velocity), $\frac{\delta L}{\delta \dot{\vec{x}}} = \vec{\mathbf{p}}$ is conserved (momentum conservation). ◇

Example. Now consider a Lagrangian invariant rotations. $\delta L = 0$ and $\vec{\mathbf{x}} \to R\vec{\mathbf{x}}$. Recall we can represent a rotation by a unit vector and a magnitude:

$$R(\hat{\mathbf{n}}, \theta) = e^{\imath \vec{\mathbf{L}} \cdot \hat{\mathbf{n}}\theta}$$

Recall that $R^T R = 1$, so if we consider infinitesimal rotations, we find that

$$R^T R = 1 = (1 + \imath \vec{\mathbf{L}}^T \cdot \hat{\mathbf{n}}\theta)(1 + \imath \vec{\mathbf{L}} \cdot \hat{\mathbf{n}}\theta)$$

so

$$1 + \imath \vec{\mathbf{L}}^T \cdot \hat{\mathbf{n}}\theta + \vec{\mathbf{L}} \cdot \hat{\mathbf{n}}\theta + \mathcal{O}(\theta^2) = 1$$

so

$$\vec{\mathbf{L}}^T \cdot \hat{\mathbf{n}}\theta + \vec{\mathbf{L}} \cdot \hat{\mathbf{n}}\theta = 0$$

so

$$\vec{\mathbf{L}}^T = -\vec{\mathbf{L}}$$

so the generators are anti-symmetric.

$$\delta\vec{\mathbf{x}} = \vec{\mathbf{x}}' - \vec{\mathbf{x}} = e^{\imath \vec{\mathbf{L}} \cdot \hat{\mathbf{n}}\theta}\vec{\mathbf{x}} - \vec{\mathbf{x}} = (\vec{\mathbf{x}} + \imath \vec{\mathbf{L}} \cdot \hat{\mathbf{n}}\theta\vec{\mathbf{x}} - \vec{\mathbf{x}})$$

so

$$\delta\vec{\mathbf{x}} = \imath \vec{\mathbf{L}} \cdot \hat{\mathbf{n}}\theta\vec{\mathbf{x}}$$

Our conservation law is now

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta L}{\delta \dot{\vec{\mathbf{x}}}}\left(\imath \vec{\mathbf{L}}\cdot \hat{\mathbf{n}}\theta \vec{\mathbf{x}}\right)\right] = 0$$

There are three generators, and we will denote them using an upper index for now (a, b, c). The lower indices will be the matrix element.

$$(\vec{\mathbf{L}}) = (L)^a_{ij}$$

Recall the Lie algebra of the rotation group:

$$\left[L^a, L^b\right] = \imath\epsilon^{abc}L^c$$

The 3-by-3 representation of the L can be written

$$\imath L^a_{ij} = \epsilon^a_{ij}$$

Don't confuse this with the structure constants, although it is the same Levi-Civita tensor. This tells us that

$$\imath^2\epsilon^a_{ij}\epsilon^b_{jk} - \imath^2\epsilon^b_{ij}\epsilon^a_{jk} = \imath^2\epsilon^{abc}\epsilon^b_{ik}$$

The Levi-Civita Symbol

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

$$[\epsilon_{ija}\epsilon_{ijb} = \delta_{ab}A]\delta_{ab}$$

now

$$\epsilon_{ija}\epsilon_{ija} = A\delta_{aa}$$

or $6 = 3A$, or $A = 2$, so

$$\epsilon_{ija}\epsilon_{ijb} = 2\delta_{ab}$$

Finally,

$$\epsilon_{ija}\epsilon_{kla} = A\delta_{ik}\delta_{jl} + B\delta_{il}\delta_{jk} + C\delta_{ij}\delta_{kl}$$

If we interchange i and j, the right side must be antisymmetric. Therefore C is zero, since that term is symmetric in i and j. We can also conclude that $B = -A$ so

$$\epsilon_{ija}\epsilon_{kla} == A\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right]$$

Contract both sides with $\delta_{ik}\delta_{jl}$, and we find

$$6 = A\left[delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji}\right] = A[9-3] = 6$$

so $A = 1$:

$$\epsilon_{ija}\epsilon_{kla} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

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