Lecture 8: More on Y_{lm} Functions

September 23, 2019

N.B.

I will be using L for $\mathbb{L}$ from here onward.

If we don't have the full range of the spherical angles, we actually have to solve the original ∇^2 differential equations and can't use L and L^2 or the Y_{lm} functions.

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{\imath m\phi} \tag{1}$$

where

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l. \tag{2}$$

Orthogonality tells us:

$$\int Y_{lm}^*(\theta, \phi)Y_{l'm'}(\theta, \phi)d\Omega = \delta_{ll'}\delta_{mm'} \tag{3}$$

The spectral decomposition tells us:

$$\sum_{lm} Y_{lm}(\theta, \phi)Y_{lm}^*(\theta', \phi') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta') \tag{4}$$

since $\delta(f(x)) = \frac{1}{f'(x_0)}\delta(x - x_0)$.

N.B.

In EM, we write Y_{lm} such that $Y_{lm}(\theta, \phi) = (-1)^m Y_{l,-m}^*(\theta, \phi)$.

For general spherical solutions,

$\Phi = \sum g_{lm}(r)Y_{lm}(\theta, \phi)$ or

$$\frac{1}{r^2}\partial_r r^2 \partial_r g - \frac{l(l+1)}{r^2} = 0 \tag{5}$$

so $r^2\partial_r^2 g + 2r\partial_r g - l(l+1)g = 0$.

Suppose $g = r^\lambda$:

$$[\lambda(\lambda - 1) + 2\lambda - l(l+1)]r^\lambda = 0 \tag{6}$$

so $\lambda = l$ or $-(l+1)$

Therefore, the general solution in spherical systems (which use the periodicity in both ϕ and θ) is:

$$\Phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_l r^l + B_l r^{-(l+1)}]Y_{lm}(\theta, \phi). \tag{7}$$

0.0.1 Systems with ϕ-independence

If we have an axis of symmetry, set the z-axis as the axis of symmetry. Therefore solutions should be independent of the angle around the z-axis (ϕ).

This means $Y_{lm} \to Y_{l0}$ so $P_l^m \to P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l$, which are not normalized for "historical reasons". The differential equation then becomes:

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_l + l(l+1)P_l = 0, \; x \in [-1,1]. \tag{8}$$

Remark. There are other solutions $Q_l(x) = \frac{1}{2}P_l(x)\ln\left[\frac{1-x}{1+x}\right] + R_l(x)$ where R_l is a polynomial of degree $l-1$.

Additionally $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2l+1}{2}d_{ll'}$

Jackson notes some "easy" relations from Rodriguez's formula:

1. $\frac{d}{dx}P_{l+1} - \frac{d}{dx}P_{l-1} - (2l+1)P_l = 0$
2. $(l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0$
3. $P_{2k}(0) = \frac{(2k-1)!!}{2^k k!}(-1)^k$

It can be shown that:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \begin{cases} \sum \frac{1}{r^{l+1}} A_l P_l(\cos\gamma) & r > r' \\ \sum r^l B_l P_l(\cos\gamma) & r < r' \end{cases} \tag{9}$$

where γ is the angle between the vectors.

Equivalently,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{10}$$

where $r_<$ and $r_>$ correspond to the smaller and larger vector.