33-765 Homework 2

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4. Fun with the Transformation Theorem

Let X and Y be two real random variables, *independently* and *uniformly* chosen from the interval $[0, 1]$.

1. Define the new random variable $Z :=Y/X$. What is the probability density $\Pr_Z(z)$?

The transformation theorem for continuous random variables can be stated as

$$\Pr_F(f) = \int_{\mathbb{D}} \mathrm{d}x\, \delta(f - F(x)) \Pr_X(x)$$

In our case, $F(x, y) = x/y$. Since the variables are independent, we know that

$$\Pr_{X,Y}(x, y) = \Pr_X(x) \Pr_Y(y)$$

Now we must construct the transformation so that it picks out values of z weighted by the probabilities of getting x and y:

$$\Pr_Z(z) = \int_0^1 \int_0^1 \mathrm{d}x\, \mathrm{d}y\, \delta\left(z - \frac{y}{x}\right) \Pr_X(x) \Pr_Y(y)$$

I will now do a change of variables:

$$\beta = \frac{y}{x} \quad x\, \mathrm{d}\beta = \mathrm{d}y$$

$$\Pr_Z(z) = \int \mathrm{d}x\, |x| \int \mathrm{d}\beta \Pr_X(x) \Pr_y(\beta x) \delta(z - \beta) = \int_0^1 \mathrm{d}x\, |x| \Pr_Y(xz)$$

since $\Pr_X(x)$ is only nonzero for $0 \le x \le 1$. Finally, there are two cases. First, if $z > 1$, $0 \le x \le \frac{1}{z}$:

$$\int_0^{1/z} |x|\, \mathrm{d}x = \frac{1}{2z^2}$$

Second, if $z < 1$, x (which from the previous step is limited to be between 0 and 1) cannot make xz greater than 1 or less than 0, so the integral can be taken as

$$\int_0^1 |x|\, \mathrm{d}x = \frac{1}{2}$$

Therefore, the total function is

$$\Pr_Z(z) = \frac{\Theta(z - 1)}{2z^2} + \frac{\Theta(1 - z)}{2}$$

2. What is the probability that Z rounds to an even integer?

We want to sum over the probability distribution in intervals which round to even integers. Such intervals can be written

$$[2n - 1/2, 2n + 1/2) \quad n = 0^*, 1, 2, 3, \cdots$$

*In the range $0 < z < 1$, (we count rounding down to zero), the probability density will simply be $\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$.
For $z > 1$, we can do a definite integral:

$$\Pr(z \to \text{even}) = \frac{1}{4} + \sum_{n=1}^{\infty} \int_{2n-1/2}^{2n+1/2} \frac{1}{2z^2}\,\mathrm{d}z = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2}{16n^2 - 1} = \frac{1}{4} + \frac{4-\pi}{4} = \frac{5-\pi}{4}$$

Because we are considering 0 in this rounding, the section $z \in [0, 1]$ will dominate this summation, making the value strictly greater than $\frac{1}{4}$.

3. What is the probability that Z rounds down to an even integer?

Now the intervals of interest are

$$[2n, 2n + 1) \quad n = 0^*, 1, 2, 3, \cdots$$

*Again, for the zero case, the probability distribution will just be $\frac{1}{2}$.
For the rest of it, we have

$$\Pr(\text{floor}(z)\ \text{even}) = \frac{1}{2} + \sum_{n=1}^{\infty} \int_{2n}^{2n+1} \frac{1}{2z^2}\,\mathrm{d}z = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2(2n + 4n^2)} = 1 - \frac{\ln 2}{2}$$

Interestingly, this value is strictly greater than $\frac{1}{2}$, but this is again because of the uniformity of the $z \in [0, 1]$ case.

5. And Yet Another Application of the Transformation Theorem

Let X and Y be two independent real random variables which are both distributed according to a Gaussian with mean zero and variance one. Define the new random variable $Z = X/Y$. What is the probability density of Z?

Now the probability distributions are

$$\Pr_{X,Y}(\{x, y\}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\{x,y\}^2}{2}}$$

Using the mid-way result from the first problem, we have

$$\Pr_Z(z) = \int \mathrm{d}x\, |x| \Pr_Y(x) \Pr_X(xz)$$

I didn't quite work through all the math for this, but I think switching Y/X to X/Y only really switches the labeling of the distributions, which for this problem doesn't matter at all.
Next, we use the given Gaussian distributions:

$$\Pr_Z(z) = \frac{1}{2\pi} \int \mathrm{d}x\, |x| e^{-\frac{x^2}{2}} e^{-\frac{x^2 z^2}{2}} = \frac{1}{2\pi} \left[\frac{2}{1 + z^2} \right] = \frac{1}{\pi} \frac{1}{1 + z^2}$$

6. Poisson Distribution

Another (discrete!) distribution which one frequently encounters is the so-called "Poisson distribution". It is defined by

$$\Pr_{\mu}(n) = \frac{\mu^n}{n!}e^{-\mu} \quad n \in \mathbb{N}_0, \mu \in \mathbb{R}^+.$$

Show that $\Pr_\mu(n)$ is properly normalized and calculate its expectation value $\langle n \rangle$ and variance σ_n^2!

To normalize the discrete distribution, we just need to construct a sum over n:

$$1 = \sum_{n=0}^{\infty} \frac{\mu^n}{n!}e^{-\mu} = e^{-\mu}\sum_{n=0}^{\infty} \frac{\mu^n}{n!} = e^{-\mu}e^{\mu}$$

Next, we want to calculate the expectation values $\langle n \rangle$ and $\langle n^2 \rangle - \langle n \rangle^2$. For this, we need to calculate two infinite sums:

$$\langle n \rangle = \sum_{n=0}^{\infty} \frac{\mu^n}{n!}e^{-\mu}n$$

and

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} \frac{\mu^n}{n!}e^{-\mu}n^2$$

For the first sum, we can divide out the factor of n in the numerator:

$$\langle n \rangle = e^{-\mu}\sum_{n=0}^{\infty} \frac{\mu^n}{(n-1)!} = e^{-\mu}\mu\sum_{n=0}^{\infty} \frac{\mu^{n-1}}{(n-1)!} = e^{-\mu}\mu e^{\mu} = \mu$$

For the second sum, we have to reduce twice.

$$\langle n^2 \rangle = e^{-\mu}\sum_{n=0}^{\infty} \frac{\mu^n n}{(n-1)!} = e^{-\mu}\mu\sum_{n=0}^{\infty} \frac{\mu^{n-1} n}{(n-1)!} = e^{-\mu}\mu e^{\mu}(1+\mu) = \mu + \mu^2$$

so

$$\sigma^2 = \langle \mu^2 \rangle - \langle \mu \rangle^2 = \mu$$

7. More on the Poisson Distribution

In problem 5, we encountered the discrete Poisson distribution function $\Pr_\mu(n) = \frac{\mu^n}{n!}e^{-\mu}$. It is a good model to describe the number n of random events that independently occur in some interval of time, during which the *expected* number is μ.

1. It turns out that one way to think about the Poisson distribution is as follows: consider a Bernoulli process with N trials and success probability p, and imagine the limit in which $N \to \infty$, $p \to 0$, but $Np = \mu = \text{const}$. Show that in this limit the associated binomial distribution function $\Pr_{\text{bin}}(n; N, p)$ converges towards the Poisson distribution $\Pr_\mu(n)$!

   The binomial distribution function can be written as

   $$\Pr(x) = \binom{N}{n} p^x (1-p)^{N-n}$$

   Let's substitute $Np = \mu$ or $p = \frac{\mu}{N}$ and keep μ constant. Taking the limit as $N \to \infty$ will then effectively make $p \to 0$.

   $$\Pr(x) = \frac{N!}{n!(N-n)!}\left(\frac{\mu}{N}\right)^x \left(1 - \frac{\mu}{N}\right)^{N-n} = \frac{\mu^n}{n!}\left[\frac{N!}{(N-n)!}\frac{1}{N^n}\left(1 - \frac{\mu}{n}\right)^{N-n}\right]$$

Now we just need to show the stuff in the square brackets ends up looking like $e^{-\mu}$. First,

$$\frac{N!}{(N-n)!}\frac{1}{N^n} = \frac{N(N-1)(N-2)\cdots(N-n+1)(N-n)!}{(N-n)!N^n}$$
$$= \frac{N(N-1)(N-2)\cdots(N-n+1)}{N^n}$$

This part goes to 1 as $N \to \infty$ since the degrees of the numerator and denominator are both n (there are n N's multiplied in the numerator). The final term is

$$\left(1-\frac{\mu}{N}\right)^{N-n} = \left(1-\frac{\mu}{N}\right)^{N} \underbrace{\left(1-\frac{\mu}{N}\right)^{-n}}_{\to 1, N\to\infty}$$

The remaining term is

$$\left(1-\frac{\mu}{N}\right)^{N} = e^{-\mu}$$

2. Check this statement *numerically* by graphically comparing the distribution function $\mathrm{Pr}_{10}(n)$ with several Bernoulli distribution functions of increasingly large N and small p, such that $Np = 10$.

To complete this problem, I wrote the following Python code:

```
import numpy as np
from scipy.stats import binom, poisson
import matplotlib.pyplot as plt
fig, ax = plt.subplots(1, 1)

n_list = np.array([50, 60, 70, 100, 200, 400])
p_list = 10 / n_list
x = np.arange(poisson.ppf(0.01, 10), poisson.ppf(0.99, 10))
for i in range(len(n_list)):
    ax.plot(x, binom.pmf(x, n_list[i], p_list[i]),
            ms=8, label='Binomial: N = {}'.format(n_list[i]))
ax.plot(x, poisson.pmf(x, 10), ms=8, label='Poisson')
ax.legend(loc='best', frameon=False)
plt.show()
```

See Figure 0.1 for the resulting plot.

3. And finally: a neat application to the Poisson distribution. A support center receives calls from customers who need help with some product. The calls arrive randomly and independently of each other, but historical data shows that the center receives on average 10 calls per hour. Beyond a certain number of calls in any given hour, the support line is overwhelmed and the system collapses, so the center needs to make sure to employ enough operators to handle occasional rushes.

   (a) At least how many calls does the support center have to be able to handle within an hour so that the probability of being overwhelmed is less than 0.1%?

To find this value, we need to find where the cumulative distribution function of the Poisson distribution is equal to 0.999. The cumulative distribution function for the Poisson distribution is

$$CDF = \frac{\Gamma(\lfloor n+1 \rfloor, \mu)}{\lfloor n \rfloor !}$$

where $\Gamma(x, y)$ is the upper incomplete Γ function and the $\lfloor\rfloor$ symbols represent the floor function. From some guessing and checking, I've found this exceeds the tolerance at 21 calls per hour.

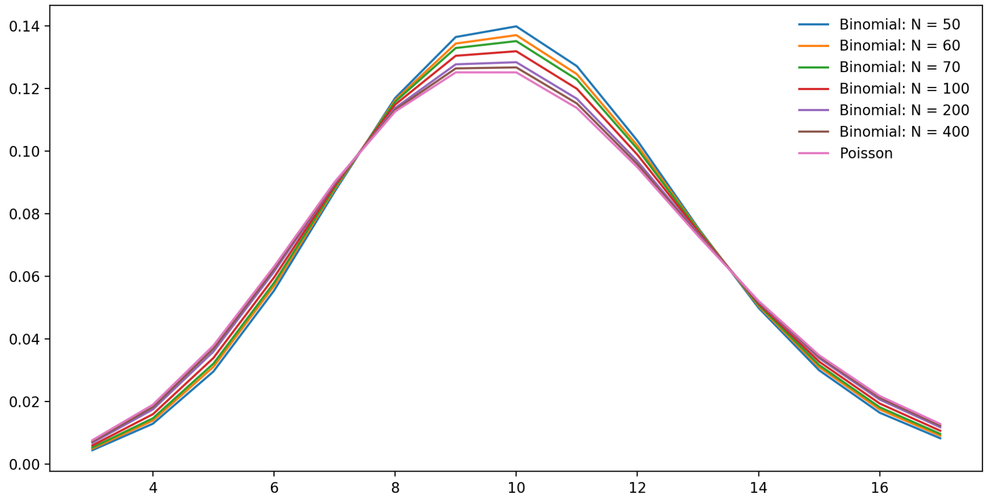

Figure 0.1: Binomial Distributions Approaching Poisson Distribution

(b) Repeat your calculation for three successively bigger call centers that receive on average 20, 50, and 100 calls per hour. Use your findings to argue why large call centers can be run more efficiently than small ones!

> The 20 call-per-hour center needs to be able to handle 35 calls, the 50 call-per-hour center needs to be able to handle 73 calls, and the 100 call-per-hour center needs to be able to handle 132 calls. This number does not scale linearly with the average amount of calls received by the center, so larger call centers need to hire less people proportional to the amount of calls they get to handle rushes.