Lecture 23: More about Multipole Expansions

Wed Oct 9 2019

0.1 Multipole Expansion for Vector Potential

Recall that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' \tag{1}$$

and

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + (\vec{x}'\cdot\nabla)\frac{1}{|\vec{x}|} + \frac{1}{2}(\vec{x}'\cdot\nabla)^2\frac{1}{|\vec{x}|} + \cdots \tag{2}$$

This is equal to

$$\frac{1}{|\vec{x}|} + x_i' x_i \frac{1}{|\vec{x}|^3} + \cdots \tag{3}$$

$$\partial_i(x_j J_i) = \delta_{ij}J_i + x_j\delta_i J_i = \delta_{ij}J_i = J_j \tag{4}$$

since $\boldsymbol{\nabla}\cdot\vec{J} = 0$ in the static case.

Therefore

$$\int \partial_i(x_j J_i)\,\mathrm{d}^3x = 0 = \int J_j\,\mathrm{d}^3x \tag{5}$$

Now we say that

$$\vec{A} = \frac{\mu_0}{4\pi}\overbrace{\int\frac{\vec{J}(\vec{x})}{|\vec{x}|}\,\mathrm{d}^3x}^{0} + \frac{\mu_0}{4\pi}\frac{1}{|x|^3}(\int \vec{J}(x')x_i'\,\mathrm{d}^3x')x_i + \cdots \tag{6}$$

Trick:

$$\int \partial_i(x_{j1}x_{j2}J_i) = \int (x_{j2}J_{j1} + x_{j1}J_{j2} + x_{j1}x_{j2}J_i = \int x_{(j1}J_{j2)} = 0 \tag{7}$$

Using this notation,

$$A_j = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\,J_j(x')x_i')(x_i\frac{1}{|\vec{x}|^3} + \cdots \tag{8}$$

This integrand is $J_j(x')x_i' = x'_{[i}J_{j]} + x'_{(i}J_{j)} = \frac{1}{2}(x_i'J_j - x_j'J_i) + \frac{1}{2}(x_i'J - j + x_j'J_i)$. We know the integral over the symmetrized part is zero from our trick, so

$$A_j = \frac{\mu_0}{4\pi}\frac{1}{2}\int \mathrm{d}^3x'\,[x_i'J_j - x_j'J_i]\frac{x_i}{|x|^3} + \cdots \tag{9}$$

A useful identity:

$$\varepsilon_{ijk}[\varepsilon_{klm}x_l'J_m] = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})x_l'J_m = x_i'J_j - x_j'J_i \tag{10}$$

Therefore

$$A_j = \frac{\mu_0}{4\pi}\frac{1}{2}[\int \mathrm{d}^3x'\,\vec{x}'\times\vec{J}(x')]\times\vec{x}\frac{1}{|x|^3} + \cdots \tag{11}$$

And we will call

$$\vec{m} = \frac{1}{2}\int \mathrm{d}^3x'\, \vec{x}' \times \vec{J}(x') \tag{12}$$

All together

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|x|^3} + \cdots \tag{13}$$

Luckily, when we compute $\vec{B} = \boldsymbol{\nabla}\times\vec{A}$, we find

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{x})\hat{x}}{|\vec{x}|^3} - \frac{\vec{m}}{|\vec{x}|^3}\right] \tag{14}$$

This expansion has a problem if we want to model point dipoles. If we take the average field over a ball, we can integrate in a ball which does not contain the dipole or in a ball that does contain it (similar to electric case):

$$\int_{\text{average over Ball(r)}} \vec{B}\,\mathrm{d}^3x = \int_{\text{ball}} \boldsymbol{\nabla}\times\vec{A}\,\mathrm{d}^3x = \oint_{S^2} (\hat{n}\times\vec{A})\,\mathrm{d}^2\Omega\, R^2 \tag{15}$$

This surface integral is

$$\oint_{S^2} (\hat{n}\times\vec{A})\,\mathrm{d}^2\Omega\, R^2 = \frac{\mu_0}{4\pi}\oint \hat{n}times \int \frac{J(x')\,\mathrm{d}^3x'}{|\vec{x}-\vec{x}'|} R^2\,\mathrm{d}\Omega = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\,\vec{J}(\vec{x}')\times\oint \frac{\hat{n}}{|\vec{x}-\vec{x}'|}R^2\,\mathrm{d}\Omega \tag{16}$$

where the surface integral here is equivalent to

$$\frac{4\pi}{3}\hat{x}'\frac{r_<}{r_>^2} \tag{17}$$

(we did this same derivation for the electric dipole)

Therefore,

$$\int_{\text{ball}} \vec{B}\,\mathrm{d}^3x = \begin{cases} \frac{\mu_0}{4\pi}\frac{4\pi}{3}\int \frac{J(x')\times R^3}{|x'|^2}\hat{x}' = \frac{4\pi}{3}R^3\vec{B}(0) & \text{outside} \\ \frac{\mu_0}{4\pi}\frac{4\pi}{3}\int J(x')\times\hat{x}'r'\,\mathrm{d}^3x' = \frac{\mu_0}{4\pi}\frac{8\pi}{3}\vec{m} & \text{inside} \end{cases} \tag{18}$$

In conclusion,

$$\int_{\text{average}} \vec{B}\,\mathrm{d}^3x = \begin{cases} \frac{4\pi R^3\vec{B}(0)}{3} & \text{outside} \\ \frac{2\mu_0}{3}\vec{m} & \end{cases} \tag{19}$$

This is following the notation Jackson, where some of the constants are absorbed into $\vec{m}$ to make it look similar to:

$$\int_{\text{average}} \vec{E}\,\mathrm{d}^3x = \begin{cases} \frac{4\pi R^3\vec{E}(0)}{3} & \text{outside} \\ -\frac{1}{3\varepsilon_0}\vec{p} & \text{inside} \end{cases} \tag{20}$$

This is all important for dealing with materials. If we had some structure with some microcurrents $\vec{j}$, we need to model the effects of these things. One way to do this is to take small regions and average them out over small volume elements.

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{x})\hat{x} - \vec{m}}{|x|^3}\right] + \frac{2\mu_0}{3}\vec{m}\delta(\vec{x}) \tag{21}$$

This last term has to be added to give us the correct average, just like in electric dipole.