Lecture 22: Ring of Current in Cylindrical Coordinates

Mon Oct 7 2019

0.1 Ring of Current in Cylindrical Coordinates

From last time

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4}{\pi}\int_0^\infty \mathrm{d}k \cos(k(z-z'))\left[\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty} I_m(k\rho_<)K_m(k\rho_>)\cos(m(\varphi-\varphi'))\right] \tag{1}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' \tag{2}$$

$$= \frac{\mu_0}{4\pi}\int I_0\delta(\phi'-a)\delta(z')[-\sin(\varphi')\hat{\imath}+\cos(\varphi')\hat{\jmath}] \tag{3}$$

$$\times\frac{1}{|\vec{x}-\vec{x}'|}\rho'\,\mathrm{d}\rho'\,\mathrm{d}z'\,\mathrm{d}\varphi' \tag{4}$$

where $\hat{\varphi} = [-\sin(\varphi)\hat{\imath}+\cos(\varphi)\hat{\jmath}$. We can choose $\varphi = 0$ since we believe the system is symmetric about φ. By doing this, we can reduce the equation to

$$\vec{A}(\vec{x}) = \frac{\mu_0 I_0 a}{\pi}\int_0^\infty \mathrm{d}k\cos(kz)I_1(k\rho_<)K_1(k\rho_>)\hat{\jmath} \tag{5}$$

We can then use the previous formulation to write down the elements of the $\vec{B}$ field using the curl:

$$B_\rho = \frac{1}{\rho}\partial_z A_\varphi \tag{6}$$

and

$$B_z = \frac{1}{\rho}\partial_\rho(\rho A_\varphi) \tag{7}$$

or if we rewrite the potential with $\hat{\jmath} = \hat{\varphi}$

$$B_\rho = \frac{1}{\rho}\frac{\mu_0 I_0 a}{\pi}\int_0^\infty \mathrm{d}k\,(-k)\sin(kz)I_1(k\rho_<)K_1(k\rho_>) \tag{8}$$

and

$$B_z = \frac{\mu_0 I_0 a}{\pi}\int_0^\infty \mathrm{d}k\cos(kz)\begin{cases}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho I_1(ka)K_1(k\rho)\right] & \rho > a\\ \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho I_1(k\rho)K_1(ka)\right] & \rho < a\end{cases} \tag{9}$$