Lecture 38: The Ideal Fermi Gas
Friday, April 24, 2020

0.1 The Ideal Fermi Gas

For high temperatures, $f_+(\epsilon - \mu)$ is asymptotic to $e^{=\beta(\epsilon-\mu)}$. At low temperature, it is asymptotic to $1 - e^{\beta(\epsilon-\mu)}$. At $\epsilon = \mu$, it has a slope of $-\frac{1}{4}\beta$ and the distribution function equals 1/2. Now let's see how we can learn some facts about this distribution.

$$N = \sum_\alpha \langle n_\alpha \rangle = \sum_\alpha \frac{1}{e^{\beta(\epsilon_\alpha-\mu)}+1} = \int \mathrm{d}\epsilon\, \frac{D(\epsilon)}{e^{\beta(\epsilon=\mu)}+1}$$

$$U = \sum_\alpha \epsilon_\alpha \langle n_\alpha \rangle \sum_\alpha \frac{\epsilon_\alpha}{e^{\beta(\epsilon_\alpha-\mu)}+1} = \int \mathrm{d}\epsilon\, \frac{\epsilon D(\epsilon)}{e^{\beta(\epsilon=\mu)}+1}$$

We can additionally define the Fermi energy:

$$\epsilon_F \equiv \lim_{T\to 0} \mu(T, N)$$

Note that this is not the point where the distribution is 1/2, that point is μ. Instead, the Fermi energy lies halfway between the highest occupied state and the lowest unoccupied state.

Observe that

$$\frac{1}{1+x} + \frac{1}{1+\frac{1}{x}} = 1$$

This implies

$$\frac{1}{e^{\beta(\epsilon-\mu)}+1} = 1 - \frac{1}{e^{-\beta(\epsilon-\mu)}+1}$$

Let's imagine a two-state system with energies $E_1 = \epsilon$ and $E_0 = 0$. We can easily calculate

$$\begin{aligned} N &= \frac{1}{e^{\beta(E_0-\mu)}+1} + \frac{1}{e^{\beta(E_1-\mu)}+1} \\ &= \frac{1}{e^{-\beta\mu}+1} + \frac{1}{e^{\beta(\epsilon-\mu)}+1} \\ &= 1 - \frac{1}{e^{\beta\mu}+1} + \frac{1}{e^{\beta(\epsilon-\mu)+1}} \end{aligned}$$

Say we have one particle in the system. Then

$$\frac{1}{e^{\beta\mu}+1} = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

or

$$\beta\mu = \beta(\epsilon-\mu) \implies \mu = \frac{\epsilon}{2}$$

Now imagine a continuous spectrum of energies:

$$D(\epsilon) = (2s+1)\left(\frac{\sqrt{2\pi m}}{h}\right)^d \frac{V}{\Gamma\left(\frac{d}{2}\right)} \epsilon^{\frac{d}{2}-1}$$

We can calculate the number of particles by taking $T = 0$ and integrating up to the Fermi energy (let's also say these are spin-$\frac{1}{2}$ particles).

$$N = \int_0^{\epsilon_F} \mathrm{d}\epsilon\, 2(\cdots)\epsilon^{\frac{d}{2}-1} = 2\left(\underbrace{\frac{\sqrt{2\pi m\epsilon_F}}{h}}_{k_BT_F=\epsilon_F}\right)^d \frac{V}{\Gamma\left(\frac{d}{2}+1\right)}$$

In three dimensions,

$$N = 2\left(\frac{\sqrt{2\pi m\epsilon_F}}{h}\right)^3 \frac{V}{\frac{3}{4}\sqrt{\pi}} = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar}\right)^{3/2}\epsilon_F^{3/2}$$

where

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2\underbrace{n}_{N/V})^{2/3}$$

Also, at $T = 0$,

$$\begin{aligned} U &= \int_0^{\epsilon_F} \mathrm{d}\epsilon\, \epsilon D(\epsilon) \\ &= \int_0^{\epsilon_F} \mathrm{d}\epsilon\, 2\left(\frac{\sqrt{2\pi m}}{h}\right)^d \frac{V}{\Gamma\left(\frac{d}{2}\right)}\epsilon^{d/2} \\ &= 2\left(\frac{\sqrt{2\pi m}}{h}\right)^d \frac{V}{\Gamma\left(\frac{d}{2}\right)\left(\frac{d}{2}+1\right)}\epsilon^{\frac{d}{2}+1} \end{aligned}$$

Therefore

$$\frac{U}{N} = \frac{d}{d+2}\epsilon_F = \begin{cases} \frac{3}{5}\epsilon_F & d=3 \\ \frac{1}{2}\epsilon_F & d=2 \\ \frac{1}{3}\epsilon_F & d=1 \end{cases}$$

Also recall that we found $U = \frac{d}{2}PV$, so we can now calculate the pressure (again, at $T = 0$):

$$P = \frac{2}{d}\frac{U}{V} = 2\left(\frac{\sqrt{2\pi m}}{h}\right)^d \frac{\epsilon_F^{\frac{d}{2}+1}}{\Gamma\left(\frac{d}{2}\right)\left(\frac{d}{2}+1\right)}$$

We also worked out that $\epsilon_F \propto n^{2/d}$ so $P \propto \epsilon_F^{d/2+1} \propto n^{1+2/d} \propto V^{-\left(1+\frac{2}{d}\right)}$.This pressure is called the Fermi pressure, and what's amazing is that it's nonzero at $T = 0$, unlike the regular ideal gas. We can also calculate $\kappa_T = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_T$ or equivalently, the isothermal bulk modulus

$$\begin{aligned} K = -V\left.\frac{\partial P}{\partial V}\right|_T &= -V\left(-1-\frac{2}{d}\right)\frac{P}{V} = \left(1+\frac{2}{d}\right)P = \left(1+\frac{2}{d}\right)\frac{PV}{N}\frac{N}{V} \\ &= \left(1+\frac{2}{d}\right)\frac{\frac{2}{d}U}{N}\frac{N}{V} \\ &= \left(1+\frac{2}{d}\right)\frac{2}{d}\frac{d}{d+2}\epsilon_F\frac{N}{V} \\ &= \frac{2}{d}\epsilon_F\frac{N}{V} \end{aligned}$$

This says that the bulk modulus of an ideal Fermi gas at $T = 0$ is equal to $2/d$ times the Fermi energy divided by the volume per particle.

For small T, the thermodynamics is determined by $D(\epsilon)$ in the vicinity of ϵ_F. Generally, we want to calculate

$$I = \int \mathrm{d}\epsilon\, g(\epsilon) f_+(\epsilon - \mu)$$

The answer is the Sommerfeld Expansion, developed by Arnold Sommerfeld, and we will discuss this in the next lecture.