Lecture 6: Angular Momentum

Monday, January 27, 2020

Last time we were talking about representations of rotations, either the SO(3) or SU(2) groups. We decided to label our representations using a Casimir operator (for vector operators, we use J^2), and we chose our basis to diagonalize J_z. We then defined raising and lowering operators

$$J_\pm = J_x \pm \imath J_y$$

such that

$$J^2 \left|ab\right\rangle = a\hbar^2 \left|ab\right\rangle$$

$$J_z \left|ab\right\rangle = b\hbar \left|ab\right\rangle$$

and

$$J_\pm \left|ab\right\rangle \propto \left|a, \pm b\right\rangle$$

Now we want to determine the allowed values of b. Consider $J^2 - J_z^2 = J_x^2 + J_y^2$:

$$J^2 - J_z^2 = \frac{1}{2}\left[J_+ J_- + J_- J_+\right]$$

Recall that $J_\pm^\dagger = J_\mp$, so

$$J^2 - J_z^2 = \frac{1}{2}\left[J_+ J_+^\dagger + J_- J_-^\dagger\right]$$

Since $\left\langle\psi\middle|OO^\dagger\middle|\psi\right\rangle \geq 0$ (because $\left\|O\left|\psi\right\rangle\right\|^2 \geq 0$),

$$(J^2 - J_z^2) \geq 0 \implies (a - b^2) \geq 0 \implies |b| \leq |a|$$

Next, we will solve for $b_{\max}$ and $b_{\min}$:

$$J_- J_+ \left|b_{\max}\right\rangle = 0$$

since $J_+ \left|b_{\max}\right\rangle = J_- \left|b_{\min}\right\rangle = 0$.

$$J_- J_+ = J_x^2 + J_y^2 + \imath[J_x, J_y] = J_x^2 + J_y^2 - \hbar J_z$$

Therefore, we can rewrite this as

$$J_- J_+ = J^2 - J_z^2 - \hbar J_z$$

Let's now operate this on the $b_{\max}$ state:

$$0 = (J^2 - J_z^2 - \hbar J_z)\left|ab_{\max}\right\rangle = (\hbar^2)\left[a - b_{\max}^2 - b_{\max}\right]\left|ab_{\max}\right\rangle \implies a = b_{\max}(b_{\max} + 1)$$

We can do a similar calculation for $b_{\min}$ with $J_+ J_-$ to show that $a = b_{\min}(b_{\min} - 1)$. Finally, we can equate the a terms to show that

$$b_{\max}(b_{\max} + 1) = b_{\min}(b_{\min} - 1) \implies b_{\max} = -b_{\min}$$

The only way for this to be true is for $b_{\max} \in \frac{\mathbb{Z}}{2}$. Therefore, the number of states in a representation is $d = (2b_{\max} + 1)$. If $b_{\max}$ is a half-integer, this corresponds to representations of SU(2), whereas integer $b_{\max}$ give representations of SO(3). $d = 2$ are not "faithful" (one-to-one) representations of SO(3), but they are faithful representations of SU(2).

0.0.1 Matrix Representation

If we consider

$$\langle j'm'| J^2 |jm\rangle = \langle j'm'|jm\rangle \hbar^2 j(j+1) = \delta_{jj'}\delta_{mm'}\hbar^2 j(j+1)$$

so

$$J^2 = \mathbb{I} \cdot \hbar^2 j(j+1)$$

Next, consider

$$\langle j'm'| J_z |jm\rangle = \delta_{jj'}\delta_{mm'} m\hbar$$

so J_z is also diagonal:

$$J_z = \begin{bmatrix} m & & & & \\ & m-1 & & & \\ & & m-2 & & \\ & & & \ddots & \\ & & & & -m \end{bmatrix}$$

Finally, consider the ladder operators:

$$|J_\pm| \, |jm\rangle = c_\pm ||j, m \pm 1\rangle|^2$$

so

$$|c_\pm|^2 = \langle jm| J_\mp J_\pm |jm\rangle$$

For the c_+ case,

$$|c_+|^2 = \langle jm| \underbrace{J_x^2 + J_y^2}_{J^2 - J_z^2} - \hbar J_z |jm\rangle = \hbar^2 \left[j(j+1) - \underbrace{m(m+1)}_{m^2 - m} \right]$$

In general, we often write this constant with a phase:

$$|c_\pm|^2 = \hbar e^{\imath\varphi} \left[(j \mp m)(j \pm m + 1)\right]^{\frac{1}{2}}$$

so

$$\langle j'm'| J_\pm |jm\rangle = \hbar \delta_{jj'}\delta_{m',m+1} \left[(j \mp 1)(j \pm m + 1)\right]^{\frac{1}{2}}$$

0.0.2 Representations of Rotation Matrices

$$U(\hat{\mathbf{n}}, \theta) = e^{-\imath \hat{\mathbf{n}} \cdot \vec{\mathbf{J}} \theta}$$

We can write the general matrix elements as

$$\langle j'm'| e^{-\imath \hat{\mathbf{n}} \cdot \vec{\mathbf{J}} \theta} |jm\rangle = D^{(j)}_{mm'}(\hat{\mathbf{n}}, \theta)$$

These are known as the Wigner functions. The representations are labeled by j, so j' doesn't really matter here, it just specifies the dimensionality of the matrix.

0.0.3 Irreducible Representations

There are two types of representations, reducible and irreducible. An irreducible representation has no invariant subspaces. This means that there is no way to write it in block-diagonal form:

$$\begin{bmatrix} A_{n\times n} & & & \\ & B_{m\times m} & & \\ & & \ddots & \\ & & & Z_{l\times l} \end{bmatrix}$$