Lecture 26: Spinning Charged Sphere, continued

Monday, October 14, 2019

Example. From the last lecture, $\vec{J} = \sigma\omega a \sin(\theta)\delta(r-a)\hat{\varphi}$ and we are using $\vec{B} = -\boldsymbol{\nabla}\Phi_M$ with the boundary conditions $B_n^{(I)} - B_n^{(II)} = 0$ and $B_\theta^{(II)} - B_{\theta=\mu_0 K_\varphi}^{(I)} = \mu_0 \sigma a \omega \sin(\theta)$.

$$A_l = -\frac{l+1}{l}\frac{B_l}{a^{2l+1}}$$

This was from the first boundary condition. Next,

$$\sum_l \left[-\frac{1}{a}\frac{\partial}{\partial\theta}\frac{B_l}{a^{l+1}}P_l(\cos(\theta)) + \frac{1}{a}\frac{\partial}{\partial\theta}a^l A_l P_l(\cos(\theta))\right] = \mu_0\sigma\omega a\sin(\theta)$$

$$\sum_l \left[-\frac{B_l}{a^{l+1}}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}P_l(\cos(\theta)) + a^l A_l\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}P_l(\cos(\theta))\right] = \mu_0\sigma\omega a^2$$

$$\sum_l \left[+\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}(\cos(\theta))}P_l(\cos(\theta)) - a^l A_l\frac{\mathrm{d}}{\mathrm{d}(\cos(\theta))}P_l(\cos(\theta))\right] = \mu_0\sigma\omega a^2$$

$$\sum_l \left[\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}x}P_l + \frac{l+1}{l}\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}x}P_l\right] = \mu_0\sigma\omega a^2$$

$$\sum_l \left[\frac{2l+1}{l}\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}x}P_l\right] = \mu_0\sigma\omega a^2$$

There is no x dependence on the right side, and the only way to make that true on the left side is for $l = 1$. Therefore

$$B_1 = \frac{\mu_0\sigma\omega a^4}{3}$$

so

$$\Phi_M = \begin{cases} \frac{\mu_0\sigma\omega a^4}{3}\frac{\cos(\theta)}{r^2} & r > a \\ \frac{2\mu_0\sigma\omega a^4}{3} r\cos(\theta) & r < a \end{cases}$$

so

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{8\pi}{3}a^3\sigma\omega\right]\hat{z} \quad \text{if} \quad r < a$$

is constant inside the sphere. Outside the sphere the field, the field looks like a dipole field. $\diamond$

0.1 Materials with Magnetic Properties

In the macroscopic limit, we average out the microscopic distribution:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_{\text{free}(\vec{x}')}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x + \frac{\mu_0}{4\pi}\int\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,\mathrm{d}^3x$$

The second term here is basically

$$\vec{M}(\vec{x}')\times\boldsymbol{\nabla}\cdot\frac{1}{|\vec{x}-\vec{x}'|} \to -\boldsymbol{\nabla}\times\left[\vec{M}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\right]$$

Paramagnetic	Diamagnetic	Ferromagnetic
$-\vec{m}\cdot\vec{B} \iff k_B T$	Langevin Model	Heisenberg Model: $\mathbb{H} = -J\sum_{\langle ij\rangle}\vec{\delta}_i\vec{\delta}_j \to$ Ising Model in the Classical limit

Table 0.1.1: Three Different Types of Materials

so the integration gives us

$$-\int \boldsymbol{\nabla}\times\vec{M}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' + \int \frac{\boldsymbol{\nabla}\times\vec{M}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

so

$$\vec{A}_{\text{matter}} = \frac{\mu_0}{4\pi}\left\{\oint \frac{\hat{n}\times\vec{M}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}a' + \int \frac{\boldsymbol{\nabla}\times\vec{M}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'\right\}$$

From this we can find that the material description is reduced to

$$\vec{J}_{\text{matter}} = \boldsymbol{\nabla}\times\vec{M},$$

an effective medium current, and

$$\vec{K}_{\text{matter}} = \hat{n}\times\vec{M},$$

an effective surface current. From here, we modify Maxwell's equations to

$$\boldsymbol{\nabla}\times\vec{B} = \mu_0\vec{J}_{\text{free}} + \mu_0\boldsymbol{\nabla}\times\vec{M}$$

just like we did with the electric field (where $P_{\text{bound}} = -\boldsymbol{\nabla}\cdot\vec{P}$ and $\sigma_{\text{bound}} = \vec{P}\cdot\hat{n}$).