Lecture 24: Force Acting on a Localized Current Distribution

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From the previous lecture, we found that

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J}(x)\,\mathrm{d}^3x$$

We assume that

$$\vec{F} = q\vec{v} \times \vec{B}$$

or

$$\vec{F} = \int \vec{J} \times \vec{B}\,\mathrm{d}^3x$$

We will Taylor expand this in components (to deal with the cross product):

$$\begin{aligned} F_i &= \int \epsilon_{ijk} J_j \left[B_k(0) + x_l\partial_l\Big|_0 B_k + \frac{1}{2}(x_l\partial_l)^2\Big|_0 B_k + \cdots \right] \mathrm{d}x \\ &= \epsilon_{ijk} B_k(0) \cancelto{0}{\int J_j\,\mathrm{d}^3x} + \int \mathrm{d}^3x\, \epsilon_{ijk} J_j x_l \partial_l\Big|_0 B_k + \dots \end{aligned}$$

Recall the way we split up index notation in last lecture:

$$\int x_l J_j\,\mathrm{d}^3x = \int x_{[l}J_{j]} + \cancelto{0}{x_{(l}J_{j)}}\,\mathrm{d}^3x = \frac{1}{2}\int \underbrace{x_l J_j - x_j J_l}_{\epsilon_{ljm}(\vec{x}\times\vec{J})_m}\,\mathrm{d}^3x$$

so

$$\begin{aligned} F_i &= \epsilon_{ijk}\epsilon_{ljm}\frac{1}{2}\int (\vec{x}\times\vec{J})_m\,\mathrm{d}^3x\,\partial_l\Big|_0 B_k \\ &= (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) m_m \partial_l\Big|_0 B_k = m_k\partial_i\Big|_0 B_k - \underbrace{\cancelto{0}{m_i\partial_k\Big|_0 B_k}}_{\nabla\cdot B = 0} \end{aligned}$$

We can then say that

$$\vec{F} \approx \nabla\Big|_0 (\vec{m}\cdot\vec{B})$$

Recall that since $\nabla \times \vec{B} = 0$ (we suppose this magnetic field is external), $\partial_i B_k - \partial_k B_i = 0$, so

$$m_k\partial_i B_k = m_k \partial_k B_i$$

so

$$\vec{F} \approx (\vec{m}\cdot\vec{\nabla})\Big|_0 \vec{B}$$

What is the torque on this system?

Notation

Jackson uses "n", but we will use $\mathcal{T}$

$$\mathcal{T} = \int \vec{x} \times (\vec{J} \times \vec{B}) \, \mathrm{d}^3 x$$

Again, let's look at the elements:

$$\begin{aligned}
\mathcal{T}_i &= \int (J_i(x_k B_k) - B_i(x_k J_k)) \, \mathrm{d}^3 x \\
&= \int J_i(x_k B_k \, \mathrm{d}^3 x - \cancelto{0}{\int B_i x_k J_k \, \mathrm{d}^3 x}
\end{aligned}$$

because we can expand B_i as

$$B_i(0) + \vec{x} \cdot \nabla \bigg|_0 \vec{B} + \cdots$$

and

$$B_i(0) \int x_k J_k \, \mathrm{d}^3 x = 0$$

because these are symmetrized indices.

We can expand the other side as

$$\begin{aligned}
\mathcal{T}_i &= \int J_i x_k [B_k(0) + (\vec{x} \cdot \nabla) \bigg|_0 \vec{B} + \ldots] \, \mathrm{d}^3 x \\
&= \int \frac{1}{2}(x_k J_i - x_i J_k) B_k(0) \, \mathrm{d}^3 x + \ldots \\
&= \epsilon_{kil} m_l B_k(0) \\
&= \epsilon_{ilk} m_l B_k(0) \\
&= \vec{m} \times \vec{B}(0)
\end{aligned}$$

Remark

$$\begin{aligned}
\boldsymbol{\nabla} \times \vec{B} &= \mu_0 \vec{J} \\
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{B} &= \mu_0 \boldsymbol{\nabla} \times J \\
\boldsymbol{\nabla} \cancelto{0}{\boldsymbol{\nabla} \cdot \vec{B}} - \nabla^2 \vec{B} &= \mu_0 \boldsymbol{\nabla} \times \vec{J}
\end{aligned}$$

so

$$\nabla^2 \vec{B} = -\mu_0 \boldsymbol{\nabla} \times \vec{J}$$

From quantum mechanics, (circularly polarized light, for example), we know that these fields must carry some information about angular momentum. This can't be derived from our current expansions of $\vec{B}$ and $\vec{E}$. There is a more "transparent" expansion, but of course, it requires a "roundabout" way of doing the expansion. In the special case of

$\vec{J} = 0$, we find that $\boldsymbol{\nabla} \times \vec{B} = 0$ so $\vec{B} = -\boldsymbol{\nabla} \cdot \Phi_M$, where Φ_m is some scalar potential for the magnetic field. There is a problem with this. If we were to look at some path of current and integrate over a path overlapping it (passing through x_0),

$$\oint_\Gamma \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I$$

this would imply that

$$\int \boldsymbol{\nabla}\Phi_m \cdot \mathrm{d}\vec{l} = \Phi_m(\vec{x}_0) - \Phi_m(\vec{x}_0) = 0$$

unless we allow the potential to be multivalued (which we shouldn't).

If we look at $\vec{x} \cdot \vec{B}$ instead, we see that

$$\vec{x} \cdot \nabla^2 \vec{B} = \nabla^2(\vec{x} \cdot \vec{B}) - 2\cancelto{0}{\boldsymbol{\nabla} \cdot \vec{B}}$$

so

$$\nabla^2 \vec{B} = -\mu_0 \boldsymbol{\nabla} \times \vec{J} \implies \vec{x} \cdot \nabla^2 \vec{B} = -mu_0 \vec{x} \cdot \boldsymbol{\nabla} \times \vec{J} = \nabla^2(\vec{x} \cdot \vec{B})$$

We can now start playing with this expression:

$$\begin{aligned} \nabla^2(\vec{x} \cdot \vec{B}) &= -\mu_0 \vec{x} \cdot \boldsymbol{\nabla} \times \vec{J} \\ &\to x_i \epsilon_{ijl} \partial_j J_l \\ &= \epsilon_{ijl} x_i \partial_j J_l \\ &= (x \times \nabla) \cdot \vec{J} \end{aligned}$$

so

$$\begin{aligned} \nabla^2(\vec{x} \cdot \vec{B}) &= -\mu_0 (\vec{x} \times \nabla) \cdot \vec{J} \\ &= -\imath\mu_0 (\underbrace{-\imath\vec{x} \times \nabla}_{\mathbb{L}}) \cdot \vec{J} \\ &= -\imath\mu_0 \vec{\mathbb{L}} \cdot \vec{J} \end{aligned}$$

so

$$\vec{x} \cdot \vec{B} = \frac{\imath\mu_0}{4\pi} \int \frac{\vec{\mathbb{L}} \cdot \vec{J}(x')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

We now expand the denominator in terms of our spherical harmonics:

$$\begin{aligned} \vec{x} \cdot \vec{B} &= \frac{\imath\mu_0}{4\pi} \int (\vec{\mathbb{L}} \cdot \vec{J})(x') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi) \frac{r^l_<}{r^{l+1}_>} \, \mathrm{d}^3 x' \\ &= \frac{\imath\mu_0}{4\pi} = \int \sum_{l,m} Y^*_{lm}(\Omega') \vec{\mathbb{L}} \cdot \vec{J}(\Omega', r') \, \mathrm{d}\Omega' \, \mathrm{d}r' \, \frac{4\pi}{2l+1} \frac{r^l_<}{r^{l+1}_>} \, \mathrm{d}^3 x' \, Y_{lm}(\Omega) \end{aligned}$$

It turns out that we can also express

$$\vec{x} \cdot \vec{B} = -r \frac{\partial \Phi_m}{\partial r}$$

so

$$
\begin{aligned}
\frac{\partial \Phi_m}{\partial r} &= \frac{-\imath\mu_0}{4\pi}\frac{1}{r}\sum_{l,m}\int \frac{4\pi}{2l+1}Y^*_{lm}(\Omega')(\vec{\mathbb{L}}\cdot\vec{J})(\Omega',r')r'^{l}\,\mathrm{d}\Omega'\,\mathrm{d}r'\,\frac{Y_{lm}(\Omega)}{r^{l+1}} \\
&= \left(\frac{-\imath\mu_0}{4\pi}\right)\sum_{l,m}\left(\frac{4\pi}{2l+1}\int Y^*_{lm}(\Omega)\vec{\mathbb{L}}\cdot\vec{J}\,\mathrm{d}\Omega'\,r'^{l}\,\mathrm{d}r'\right)\frac{Y_{lm}}{r^{l+1}} \\
&= \Phi_M = \frac{\imath\mu_0}{\sqrt{l+1}}\sqrt{l}\sum_{l,m}\left(\frac{4\pi}{2l+1}\right)\left\{\int \frac{\vec{\mathbb{L}}Y^*_{lm}}{\sqrt{l(l+1)}}\cdot\vec{J}\,\mathrm{d}\Omega'\,r'^{l}\,\mathrm{d}r'\right\}\frac{Y_{lm}}{r^{l+1}}
\end{aligned}
$$

where $\vec{\mathbb{L}}Y^*_{lm}$ are the vector spherical harmonics