33-761 Homework 8

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1 Verify the Following Identity

$$\boldsymbol{\nabla} \times \vec{L} = -\imath\vec{x}\nabla^2 + \imath\boldsymbol{\nabla}\left[1 + \vec{x} \cdot \nabla\right]$$

$$\vec{L} - \imath\vec{x} \times \nabla$$

so we want to show that

$$\boldsymbol{\nabla} \times x \times \nabla = x\nabla^2 - \boldsymbol{\nabla}(1 + \vec{x} \cdot \nabla)$$

The ith component of this is

$$\epsilon^{ijk}\partial_j L_k = \epsilon^{ijk}\epsilon^{klm}\partial_j x_l \partial_m$$

We can expand this using the contracted epsilon identity:

$$\begin{aligned}
\epsilon^{ijk}\epsilon^{klm}\partial_j x_l \partial_m &= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\partial_j x_l \partial_m \\
&= \partial_j x_i \partial_j - \partial_j x_j \partial_i \\
&= x_i \partial_j^2 + \delta_{ij}\partial_j - \partial_j x_j \partial_i
\end{aligned}$$

Since $\boldsymbol{\nabla} \times \vec{x} = 0$, we can write $\partial_i x_j = \partial_j x_i$. However, $x_j\partial_i = 2\delta_{ij} + x_i\partial_j$, so

$$x_i\partial_j^2 + \delta_{ij}\partial_j - \partial_j x_j \partial_i = x_i\partial_j^2 + \partial_i - 2\partial_i - \partial_i x_j \partial_j = \vec{x}\nabla^2 - \boldsymbol{\nabla}[1 + \vec{x} \cdot \nabla]$$

2 Magnetic Dipole Moment

Show that for a current circuit represented by a (regular non-self-intersecting) curve in space, the magnetic dipole moment is given by

$$\vec{m} = I \int_\Sigma \mathrm{d}\vec{a}$$

where I is the current in the circuit and Σ is any (regular) surface admitting the circuit as its boundary (in the plane this becomes the familiar $m = I \cdot A$ formula).

Jackson says that in general, for a current confined to a path,

$$\vec{m} = \frac{I}{2} \int_\Gamma x \times \mathrm{d}\vec{l}$$

In our first homework, we showed that

$$\int_\Gamma \lambda\,\mathrm{d}\vec{l} = -\int_\Sigma (\boldsymbol{\nabla}\lambda)\times \mathrm{d}\vec{a}$$

Writing this in index notation, we find that the kth element is:

$$\int_\Gamma \lambda\,\mathrm{d}l_k = -\int_\Sigma \epsilon^{klm}\partial_l\lambda\,\mathrm{d}a_m$$

If we write the magnetic moment in index notation, we find that

$$m_i = \frac{I}{2}\int_\Gamma \epsilon^{ijk}x_j\,\mathrm{d}l_k$$

so if we set $\lambda = \epsilon^{ijk}x_j$, we find

$$\begin{aligned}
m_i &= -\frac{I}{2}\int_\Sigma \epsilon^{ijk}\epsilon^{klm}\partial_l x_j\,\mathrm{d}a_m \\
&= -\frac{I}{2}\int_\Sigma (\delta_{il}\delta_{jm} - \delta_\imath\delta_{jl})\partial_l x_j\,\mathrm{d}a_m \\
&= -\frac{I}{2}\left(\int_\Sigma \partial_i x_j\,\mathrm{d}a_j - \int_\Sigma \partial_j x_j\,\mathrm{d}a_i\right) \\
&= \frac{I}{2}\left(-\int_\Sigma \delta_{ij}\partial_i x_j\,\mathrm{d}a_j + \int_\Sigma \partial_j x_j\,\mathrm{d}a_i\right) \\
\rightarrow \vec{m} &= \frac{I}{2}\int_\Sigma (3-1)\,\mathrm{d}\vec{a} = I\int_\Sigma \mathrm{d}\vec{a}
\end{aligned}$$

3 General Force between Current Distributions

Show that the force acting on a localized current distribution in a region Ω_1 due to a localized current distribution in a region Ω_2 in the magnetostatic approximation is given by

$$\vec{F}_{12} = \lim_{|\vec{a}_1|\to 0}\frac{\mu_0}{4\pi}\nabla_{\vec{a}_1}\int_{\Omega_1}\mathrm{d}^3x_1\int_{\Omega_2}\mathrm{d}^3x_2\,\frac{\vec{J}_1(\vec{x}_1)\cdot\vec{J}_2(\vec{x}_2)}{|\vec{x}_1+\vec{a}_1-\vec{x}_2|}$$

We start with

$$\vec{F}_{12} = \int \vec{J}^{(1)}(\vec{x_1})\times\vec{B}^{(2)}(\vec{x_1})\,\mathrm{d}^3x_1$$

We are imagining the force acting on the currents $J^{(1)}$ from external currents which make a magnetic field. We can write this field in terms of the curl of its vector potential, which has an associated current density integral:

$$\begin{aligned}
F_i^{(12)} &= \int_{\Omega_1}\epsilon^{ijk}J_j^{(1)}B_k^{(2)}\,\mathrm{d}x_i^{(1)} \\
&= \int_{\Omega_1}\epsilon^{ijk}J_j^{(1)}\epsilon^{klm}\partial_l\frac{\mu_0}{4\pi}\int_\Omega J_m^{(2)}\frac{1}{|\vec{x}_1-\vec{x}_2|}\,\mathrm{d}x_m^{(2)} \\
&= \frac{\mu_0}{4\pi}(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\int_{\Omega_i}J_j^{(1)}\partial_l\,\mathrm{d}x_j^{(1)}\int_{\Omega_2}J_m^{(2)}\frac{1}{|\vec{x}_1-\vec{x}_2|}\,\mathrm{d}x_m^{(2)} \\
&= \frac{\mu_0}{4\pi}\left[\int_{\Omega_i}J_j^{(1)}\partial_i\,\mathrm{d}x_i^{(1)}\int_{\Omega_2}J_j^{(2)}\frac{1}{|\vec{x}_1-\vec{x}_2|}\,\mathrm{d}x_j^{(2)} - \int_{\Omega_i}J_j^{(1)}\partial_j\,\mathrm{d}x_i^{(1)}\int_{\Omega_2}J_i^{(2)}\frac{1}{|\vec{x}_1-\vec{x}_2|}\,\mathrm{d}x_i^{(2)}\right]
\end{aligned}$$

Acting the differential over the current in the first term cancels the second term, so we are left with

$$\frac{\mu_0}{4\pi}\int_{\Omega_i}\mathrm{d}x_i^{(1)}\int_{\Omega_2}\mathrm{d}x_j^{(2)}\,J_j^{(1)}J_j^{(2)}\partial_i\frac{1}{|\vec{x}_1-\vec{x}_2|}$$

Because

$$\frac{\mathrm{d}f}{\mathrm{d}x}=\lim_{a\to 0}\frac{\mathrm{d}}{\mathrm{d}a}f(x+a)$$

we can write this as

$$F_{12}=\lim_{|\vec{a}_1|\to 0}\frac{\mu_0}{4\pi}\nabla_{\vec{a}_1}\int_{\Omega_1}\mathrm{d}^3x_1\int_{\Omega_2}\mathrm{d}^3x_2\,\frac{\vec{J}_1(\vec{x}_1)\cdot\vec{J}_2(\vec{x}_2)}{|\vec{x}_1+\vec{a}_1-\vec{x}_2|}$$

4 Complete the discussion presented in Section 5.12

While I'm not entirely sure what there is to complete about the discussion presented in Jackson, I guess I will just fill in the intermediate calculations. If we suppose

$$\Phi_M=\begin{cases}-H_0r\cos(\theta)+\sum_{l=0}^{\infty}\frac{\alpha_l}{r^{l+1}}P_l(\cos(\theta)) & b<r\\ \sum_{l=0}^{\infty}\left(\beta_l r^l\gamma_l\frac{1}{r^{l+1}}\right)P_l(\cos(\theta)) & a<r<b\\ \sum_{l=0}^{\infty}\delta_l r^l P_l(\cos(\theta)) & r<a\end{cases}$$

we can now look at the boundary conditions. Jackson outlines them as the H field in the θ direction (tangent) is continuous across the boundary, and the B field in the radial direction (perpendicular) must also be continuous. By orthogonality of the $P_l(\cos(\theta))$ we only have the $l=1$ terms because the field outside has some proportionality to $\cos(\theta)$ which is $P_1(\cos(\theta))$. If we look at the boundary at $r=a$, we have

$$\begin{aligned}\partial_\theta\Phi(a_+)&=\partial_\theta\Phi(a_-)\\ -\delta_1 a&=-\beta_1 a-\frac{\gamma_1}{a^2}\\ -\delta_1a^3+\beta_1a^3+\gamma_1&=0\end{aligned}$$

Using the B-field condition,

$$\begin{aligned}\mu_0\partial_r\Phi(b_+)&=\mu\partial_r\Phi(b_-)\\ \mu_0(-H_0+-2\frac{\alpha_1}{b^3})&=\mu(\beta_1-2\frac{\gamma_1}{b^3})\\ 2\alpha_1+\mu' b^3\beta_1-2\mu'\gamma&=-b^3H_0\end{aligned}$$

Similarly, at the boundary $r=b$, we have

$$\begin{aligned}\partial_\theta\Phi(b_+)&=\partial_\theta\Phi(b_-)\\ H_0b\sin(\theta)-\frac{\alpha_1}{b^2}\sin(\theta)&=\left(-\frac{\gamma_1}{b^2}-\beta_1b\right)\sin(\theta)\\ H_0b^3-\alpha_1+\gamma_1+\beta_1b^3&=0\end{aligned}$$

and finally,

$$2\alpha_1+\mu'b^3\beta_1-2\mu'\gamma_1+b^3H_0=0$$

comes from continuity of B at $r=b$.
Jackson solves for α_1 and δ_1, and while I'm pretty good at Mathematica, I could not quite get it in the same form, so I'll just assume he's correct and take the limits as $\mu>>\mu_0$ or $\mu'\to\infty$.

$$\alpha_1=\left[\frac{(2\mu'+1)(\mu'-1)}{(2\mu'+1)(\mu'+2)-2\frac{a^3}{b^3}(\mu'-1)^2}\right](b^3-a^3)H_0$$

$$\delta_1 = 0\left[\frac{9\mu'}{(2\mu'+1)(\mu'+2) - 2\frac{a^3}{b^3}(\mu'-1)^2}\right]H_0$$

We can see that the equation for α_1 is of leading order 2 in μ' in both the numerator and denominator, so the limit as $\mu' \to \infty$ will be proportional to the leading order terms:

$$\lim_{\mu'\to\infty} \alpha_1 = \frac{2\mu'^2}{2\mu'^2 - 2\frac{a^3}{b^3}\mu'^2}(b^3 - a^3)H_0 = b^3 H_0$$

For δ_1, the numerator is only of order 1, so the resulting limit will yield

$$\delta_1 - \frac{9H_0}{2\mu'^2 - 2\frac{a^3}{b^3}\mu'^2} = \frac{0\mu_0}{2\mu\left(1 - \frac{a^3}{b^3}\right)}H_0$$

Because the inner field is inversely proportional to μ, having a large μ means the inner field is small. The field inside is proportional to $\boldsymbol{\nabla}\sum_{l=0}^{\infty}\delta_l r^l P_l(\cos(\theta))$.

5 Jackson 5.20 (a)

Starting from the force equation (5.12) and the fact that a magnetization $\vec{M}$ inside a volume V bounded by a surface S is equivalent to a volume current density $\vec{J}_M = \boldsymbol{\nabla} \times \vec{M}$ and a surface current density $\vec{M} \times \hat{n}$, show that in the absence of macroscopic conduction currents the total magnetic force on the body can be written

$$\vec{F} = -\int_V (\boldsymbol{\nabla}\cdot\vec{M})\vec{B}_e \, \mathrm{d}^3x + \int_S (\vec{M}\cdot\hat{n})\vec{B}_e \, \mathrm{d}a$$

where $\vec{B}_e$ is the applied magnetic induction (not including that of the body in question). The force is now expressed in terms of the effective charge densities ρ_M and σ_m. If the distribution of magnetization is not discontinuous, the surface can be at infinity and the force given by just the volume integral.

We begin with equation (5.12) which states

$$\vec{F} = \int_V \vec{J} \times \vec{B} \, \mathrm{d}^3x$$

In our case, we split up the current into a surface term and a volume term:

$$\vec{F} = \int_V \vec{J}_M \times \vec{B}_e \, \mathrm{d}^3x + \int_S \vec{K}_M \times \vec{B}_e \, \mathrm{d}a$$

We know what these are in terms of the magnetization:

$$\vec{F} = \int_V (\boldsymbol{\nabla}\times\vec{M}) \times \vec{B}_e \, \mathrm{d}^3x + \int_S (\vec{M}\times\hat{n}) \times \vec{B}_e \, \mathrm{d}a$$

In the volume integral, we apply the identity

$$(\boldsymbol{\nabla}\times\vec{M})\times\vec{B}_e = -\vec{B}_e\times(\boldsymbol{\nabla}\times\vec{M}) = (\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M} + \vec{M}(\underbrace{\boldsymbol{\nabla}\times\vec{B}_e}_{J_{\text{cond}}=0}) - \boldsymbol{\nabla}(\vec{M}\cdot\vec{B}_e)$$

By divergence theorem, the final term $\int_V \boldsymbol{\nabla}(\vec{M}\cdot\vec{B}_e)\,\mathrm{d}^3x = \int_S (\vec{M}\cdot\vec{B}_e)\hat{n}\,\mathrm{d}a$, so we now have

$$\vec{F} = \int_V (\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M}\,\mathrm{d}^3x + \int_S (\vec{M}\times\hat{n})\times\vec{B}_e - (\vec{M}\cdot\vec{B}_e)\hat{n}\,\mathrm{d}a$$

Next, we use the identity

$$(\vec{M}\times\hat{n})\times\vec{B}_e = -\vec{B}_e\times(\vec{M}\times\hat{n}) = (\vec{B}_e\cdot\hat{n})\vec{M} - (\vec{B}_e\cdot\vec{M})\hat{n}$$

so

$$\vec{F} = \int_V (\vec{M} \cdot \nabla)\vec{B}_e + (\vec{B}_e \cdot \nabla)\vec{M} \, \mathrm{d}^3x - \int_S (\vec{B}_e \cdot \hat{n})\vec{M} \, \mathrm{d}a$$

Next, we use the identity

$$\int_V (\vec{B}_e \cdot \nabla)\vec{M} \, \mathrm{d}^3x = -\int_V (\boldsymbol{\nabla} \cdot \vec{B}_e)\vec{M} + \int_S (\hat{n} \cdot \vec{B}_e)\vec{M} \, \mathrm{d}a$$

We now have

$$\vec{F} = \int_V (\vec{M} \cdot \nabla)\vec{B}_e - (\underbrace{\boldsymbol{\nabla} \cdot \vec{B}_e}_{0})\vec{M} \, \mathrm{d}^3x$$

Applying the identity once more, we have

$$\vec{F} = \int_V (\vec{M} \cdot \nabla)\vec{B}_e = -\int_V (\boldsymbol{\nabla} \cdot \vec{M})\vec{B}_e \, \mathrm{d}^3x + \int_S (\vec{M} \cdot \hat{n})\vec{B} \, \mathrm{d}a$$