Lecture 25: Time-Dependent Perturbation Theory: Unstable States
Wednesday, March 25, 2020

0.1 Unstable States

So far we've been calculating probabilities to transition to another state ($i \to n$). Now we want to talk about the probability of staying in a particular state ($i \to i$). Recall that at second order in perturbation theory, we found that

$$c_n^{(2)} = \frac{\imath}{\hbar} \sum_m \frac{V_{nm} V_{mi}}{E_m - E_i} \int_0^t \left(e^{\imath \omega_{mi} t'} - e^{\imath \omega_{nm} t'} \right) \mathrm{d}t'$$

Notice this value is singular when $E_m = E_i$, which means this formula isn't very useful to calculate probabilities of $i \to i$. The reason for this is because we derived this formula assuming the interactions turn on instantaneously, which is never possible. Instead, let's adiabatically turn on the potential:

$$V(t) = e^{\eta t} V$$

and take the limit as $\eta \to 0$.

$$\begin{aligned} c_n^{(1)} &= -\frac{\imath}{\hbar} \lim_{t_0 \to -\infty} \int_{t_0}^t e^{\eta t'} e^{\imath \omega_{ni} t'} V_{ni} \, \mathrm{d}t' \\ &= -\frac{\imath}{\hbar} \frac{V_{ni}}{\eta + \imath \omega_{ni}} e^{\eta t + \imath \omega_{ni} t} \end{aligned}$$

so

$$\left| c_n^{(1)} \right|^2 = \frac{1}{\hbar^2} \frac{|V_{ni}|^2 e^{2\eta t}}{(\eta^2 + \omega_{ni}^2)}$$

$$\lim_{\eta \to 0} \mathrm{d}t \left| c_n^{(1)} \right|^2 = \frac{2\eta e^{2\eta t} |V_{ni}|^2}{\hbar^2 (\eta^2 + \omega_{ni}^2)}$$

We would imagine this would go to 0, but in reality,

$$\lim_{\eta \to 0} \frac{\eta}{\eta^2 + \omega_{ni}^2} = \pi \delta(\omega_{ni})$$

so

$$\lim_{\eta \to 0} \mathrm{d}t \left| c_n^{(1)} \right|^2 = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i)$$

Again, we regain the Fermi Golden Rule, and η is irrelevant using this η derivation. Now let's examine the second-order:

$$\begin{aligned} c_n^{(2)}(t) &= \langle n | \left(-\frac{\imath}{\hbar} \right)^2 \int_{t_0}^t \mathrm{d}t' \int_{t_0}^{t'''} \mathrm{d}t'' \, V_I(t') V_I(t'') \, | i \rangle \\ &= \sum_m \langle n | \left(-\frac{\imath}{\hbar} \right)^2 \int_{t_0}^t \mathrm{d}t' \int_{t_0}^{t''} \mathrm{d}t'' \, V_I(t') \, | m \rangle \langle m | \underbrace{V_I}_{= e^{\imath H_0 t} V e^{-\imath H_0 t}} (t'') \, | i \rangle \\ &= \sum_m \left(-\frac{\imath}{\hbar} \right)^2 \int_{t_0}^t \mathrm{d}t' \int_{t_0}^{t''} \mathrm{d}t'' \, e^{\imath \omega_{nm} t' + \eta t'} V_{nm} \int_{t_0}^{t'} \mathrm{d}t'' \, e^{\imath \omega_{mi} t'' + \eta t''} V_{mi} \\ &= \sum_m \left(-\frac{\imath}{\hbar} \right)^2 \int_{t_0}^t \mathrm{d}t' \, e^{\imath \omega_{nm} t' + \eta t'} (-\imath)(V_{nm})(V_{mi}) \frac{e^{\imath \omega_{mi} t' + \eta t'}}{(\omega_{mi} - \imath \eta)} \\ (n = i) &= \sum_m \left(-\frac{\imath}{\hbar} \right)^2 (-\imath) \int_{t_0}^t \mathrm{d}t' \, \frac{e^{2\eta t'}}{(\omega_{mi} - \imath \eta)} \end{aligned}$$

$$= \left(\frac{\imath}{\hbar^2}\right)\int_{t_0}^{t}\frac{e^{2\eta t'}|V_{ii}|^2}{(-\imath\eta)}\,\mathrm{d}t' + \frac{\imath}{\hbar^2}\int_{t_0}^{t}\sum_{m\neq i}\frac{e^{2\eta t'}|V_{im}|^2}{(\omega_{mi}-\imath\eta)}\,\mathrm{d}t'$$

$$c_i(t) = 1 - \frac{\imath}{\hbar}\frac{V_{ii}}{\eta}e^{\eta t} - \left(-\frac{\imath}{\hbar}\right)^2\frac{|V_{ii}|^2 e^{2\eta t}}{2\eta^2} + \frac{\imath}{\hbar^2}\sum_{m\neq i}\frac{|V_{im}|^2 e^{2\eta t}}{(2\eta)(\omega_{mi}-\imath\eta)}$$

We want to know about the stability of this state, so let's take the time derivative:

$$\begin{aligned}
\dot{c_i(t)} &= -\frac{\imath}{\hbar}V_{ii} + \frac{1}{\hbar^2}|V_{ii}| + \frac{\imath}{\hbar^2}\sum_{m\neq i}\frac{|V_{im}|^2}{(\omega_{mi}-\imath\eta)}\\
\frac{\dot{c}_i}{c_i} &= \frac{-\frac{\imath}{\hbar}V_{ii} + \frac{1}{\hbar^2}|V_{ii}| + \frac{\imath}{\hbar^2}\sum_{m\neq i}\frac{|V_{im}|^2}{(\omega_{mi}-\imath\eta)}}{\left(1-\frac{\imath}{\hbar^2}\frac{V_{ii}}{\eta}\right)}\\
&= -\frac{\imath}{\hbar}V_{ii} + \frac{\imath}{\hbar}\sum_{m\neq i}\frac{|V_{im}|^2}{(\omega_{mi}-\imath\eta)}
\end{aligned}$$

so

$$c_i(t) = e^{-\imath\Delta_i t/\hbar}$$

where

$$\Delta = V_{ii} + \frac{\imath}{\hbar}\sum_{i\neq m}\frac{|V_{im}|^2}{(\omega_{mi}-\imath\eta)} = \underbrace{V_{ii}}_{\Delta^{(1)}} - \underbrace{\frac{\imath}{\hbar}\sum_{m\neq i}\frac{|V_{im}|^2}{(\omega_i-\omega_m+\imath\eta)}}_{\Delta^{(2)}}$$

so to first order, $\Delta \to \Delta E_i = \langle i|V|i\rangle$ and we obtain the typical result

$$c_i(t) = e^{-\frac{\Delta E_i t}{\hbar}}$$

What happens with the second-order term?

$$\begin{aligned}
\Delta^{(2)} &= \sum_{m\neq i}\frac{|V_{im}|^2}{(E_i-E_m+\imath\eta)}\\
&= \sum_{m\neq i}\mathrm{Pr}\left(\frac{|V_{im}|^2}{E_i-E_m}\right) - \imath\pi\delta(E_i-E_m)|V_{im}|^2
\end{aligned}$$

where here, by $\mathrm{Pr}(\,\cdot\,)$ we mean the principal part:

$$\mathrm{Pr}(f(x)) = \lim_{\delta\to 0}\int_{-\infty}^{\delta} f(x)$$

«Ira scrolls around too fast and my internet lags him and allows him to draw at superhuman speed for short bursts. I will finish this when the lectures are available online»

$$|c_i|^2 = e^{2\Delta_I t/\hbar} = e^{-\Gamma t/\hbar}$$

where Γ is the width of the state:

$$\tau = \frac{\hbar}{\Gamma}$$

What this is telling us is if we have some initial state and Δ has an imaginary part, we will see spontaneous emission from this state down to a lower state.

If we have a particle, even a fundamental particle, the mass of the particle has some uncertainty because of the uncertainty in the energy, $\Delta E\tau = 0$.