33-765 Homework 6

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20. Inexact Differentials, Integrating Factors, and Singular Cases

1. Show that for all $\beta \in \mathbb{R}$, with one exception, the differential $\text{đ}f = \sqrt{\frac{y}{x}}\,\mathrm{d}x - \beta\sqrt{\frac{x}{y}}\,\mathrm{d}y$ is not closed. What is the exception?

First, let's go along one path through $(x, 0)$:

$$\int_a^x \sqrt{\frac{y}{x}}\,\mathrm{d}x\,\bigg|_{y=0} - \int_0^y \beta\sqrt{\frac{x}{y}}\,\mathrm{d}y\,\bigg|_{x=x} = -2\beta\sqrt{xy}$$

Along a path through $(0, y)$, we find

$$-\beta\int_0^y \sqrt{\frac{x}{y}}\,\mathrm{d}y\,\bigg|_{x=0} + \int_0^x \sqrt{\frac{y}{x}}\,\mathrm{d}x\,\bigg|_{y=y} = 2\sqrt{xy}$$

so clearly these are not the same unless $\beta = -1$.

2. Show that for all $\beta \in \mathbb{R}$, with one exception, there is an $\alpha \in \mathbb{R}$ for which $r(x, y) = \left(\frac{x}{y}\right)^\alpha$ becomes an integrating factor of $\text{đ}f$. What is the exception?

$$\mathrm{d}G = \left(\frac{x}{y}\right)^\alpha \left[\sqrt{\frac{y}{x}}\,\mathrm{d}x - \beta\sqrt{\frac{x}{y}}\,\mathrm{d}y\right] = \left(\frac{x}{y}\right)^\alpha \sqrt{\frac{y}{x}}\,\mathrm{d}x - \beta\left(\frac{x}{y}\right)^\alpha \sqrt{\frac{x}{y}}\,\mathrm{d}y$$

Now we require this new function to be closed, so

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{x}{y}\right)^\alpha \sqrt{\frac{y}{x}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\beta\left(\frac{x}{y}\right)^\alpha \sqrt{\frac{x}{y}}\right)$$

$$\left(\frac{1}{2x}\frac{x}{y}\right) = -\frac{\left(\alpha + \frac{1}{2}\right)\beta}{y}$$

$$\implies \alpha = -\left(\frac{1}{2\beta} + \frac{1}{2}\right)$$

We can see here that α is not an integrating factor when $\beta = 0$, since this expression would not make sense in that case.

21. Three Very Simple Legendre Transforms

The Legendre transform $f^*(p)$ of a function $f(x)$ is defined as $\min_x\{f(x) - px\}$, if $f(x)$ is convex, and $\max_x\{f(x) - px\}$, if $f(x)$ is concave. Calculate the Legendre transform $f^*(p)$ and its derivative $f^{*\prime}(p) = \frac{\partial f^*(p)}{\partial p}$ for the following functions:

1. $f(x) = e^x$

Since e^x is convex, we want to minimize $e^x - px$. To do this, we first take the derivative and set it equal to zero:

$$0 = e^{x_0} - p \implies x_0 = \ln p$$

Then we insert x_0 back into the minimization:

$$\begin{aligned} f^*(p) &= \min_x\{f(x) - px\} \\ &= f(x_0) - px_0 \\ &= e^{\ln p} - p\ln p \\ &= p - p\ln p \\ &= p(1 - \ln p) \end{aligned}$$

Finally, the derivative is

$$f^{*\prime}(p) = (1 - \ln p) + p\left(-\frac{1}{p}\right) = -\ln p$$

2. $f(x) = \log(x)$

The logarithm is concave, so we now want to maximize:

$$\max_x\{\log(x) - px\} \implies 0 = \frac{1}{x_0} - p \implies x_0 = \frac{1}{p}$$

Therefore,

$$f^*(p) = f(x_0) - px_0 = \log\frac{1}{p} - 1$$

$$f^{*\prime}(p) = -\frac{1}{p}$$

3. $f(x) = \cosh(x)$

The hyperbolic cosine is convex, since it is defined by $\cosh(x) = \frac{1}{2}(e^{-x} + e^x)$, so for the positive reals, it behaves like an exponential, since the second term dominates, and in the negative reals, it also behaves like an exponential where the first term dominates. Therefore, we want to minimize:

$$\min_x\{\cosh(x) - px\} \implies \sinh(x_0) - p = 0 \implies x_0 = \sinh^{-1}(p)$$

Therefore,

$$f^*(p) = \cosh\big(\sinh^{-1}(p)\big) - p\sinh^{-1}(p) = \sqrt{1+p^2} - p\ln\Big(p + \sqrt{1+p^2}\Big)$$

$$f^{*\prime}(p) = \frac{p}{\sqrt{p^2+1}} - \frac{p\left(\frac{p}{\sqrt{p^2+1}} + 1\right)}{\sqrt{p^2+1} + p} + \ln\Big(\sqrt{p^2+1} + p\Big) = -\sinh^{-1}(p)$$

22. One Slightly Less Simple (but Slightly More Instructive) Legendre Transform

Same notation and tasks as in problem 21, but now we have $f(x) = \frac{x^2}{1+|x|}$. Plot both $f(x)$ and $f^*(p)$.

First, this function is convex, since both x^2 and $1 + |x|$ are convex. I will take the hint to look at both the positive and negative domains for x, since this will make the derivative easy to take. For $x > 0$,

$$\min_x \{\frac{x^2}{1+x} - px\} \implies \frac{2x_0}{x_0+1} - \frac{x_0^2}{(x_0+1)^2} = p \implies x_0 = \frac{1 \pm \sqrt{1-p} - p}{p-1}$$

The + solution is clearly smaller when $p < 1$, and the requirement that $x > 0$ means that $x_0 > 0$, so $p > 0$.

$$f^*(p < 1) = \frac{\left(\frac{1+\sqrt{1-p}-p}{p-1}\right)^2}{1 + \frac{1+\sqrt{1-p}-p}{p-1}} = \frac{p - 2(1 + \sqrt{1-p})}{\sqrt{1-p}}$$

For $x < 0$,

$$\min_x \{\frac{x^2}{1-x} - px\} \implies \frac{x_0^2}{(1-x_0)^2} + \frac{2x_0}{1-x_0} = p \implies x_0 = \frac{1 \pm \sqrt{1+p} + p}{1+p}$$

Now the $-$ solution is smaller for $p > -1$, and requirement that $x < 0$ means $p < 0$.

$$f^*(p > -1) = \frac{\left(\frac{p-\sqrt{1+p}+1}{1+p}\right)^2}{1 - \frac{p-\sqrt{1+p}+1}{1+p}} = \frac{p - 2(1 + \sqrt{1+p})}{\sqrt{1+p}}$$

$$f^*(p) = \begin{cases} \frac{p-2(1+\sqrt{1+p})}{\sqrt{1+p}} = p + 2\sqrt{1-p} - 2 & 1 > p > 0 \\ \frac{p-2(1+\sqrt{1-p})}{\sqrt{1-p}} = -p + 2\sqrt{1+p} - 2 & -1 < p < 0 \end{cases}$$

and

$$f^{*\prime}(p) = \begin{cases} \frac{p}{2(1+p)^{3/2}} & 1 > p > 0 \\ -\frac{p}{2(1-p)^{3/2}} & -1 < p < 0 \end{cases}$$

See Figure 0.1 for the plots of these functions.

23. One Nontrivial (but Enormously Instructive) Legendre Transform

Consider the function $f(x)$, which contains a parameter $a \in \mathbb{R}$, and its Legendre transform $f^*(p)$:

$$f(x) = -\frac{1}{2}ax^2 + \frac{1}{4}x^4 \qquad f^*(p) = \min_x \{f(x) - px\}.$$

Studying the Legendre transform $f^*(p)$ is nontrivial, because depending on the value of a, the function $f(x)$ is not everywhere convex. Always keep in mind that the value of a might qualitatively change the results, so be careful about this.

1. Find any minima, maxima, and inflection points of $f(x)$. For which values of a is the function always convex? Sketch $f(x)$ for typical representative cases.

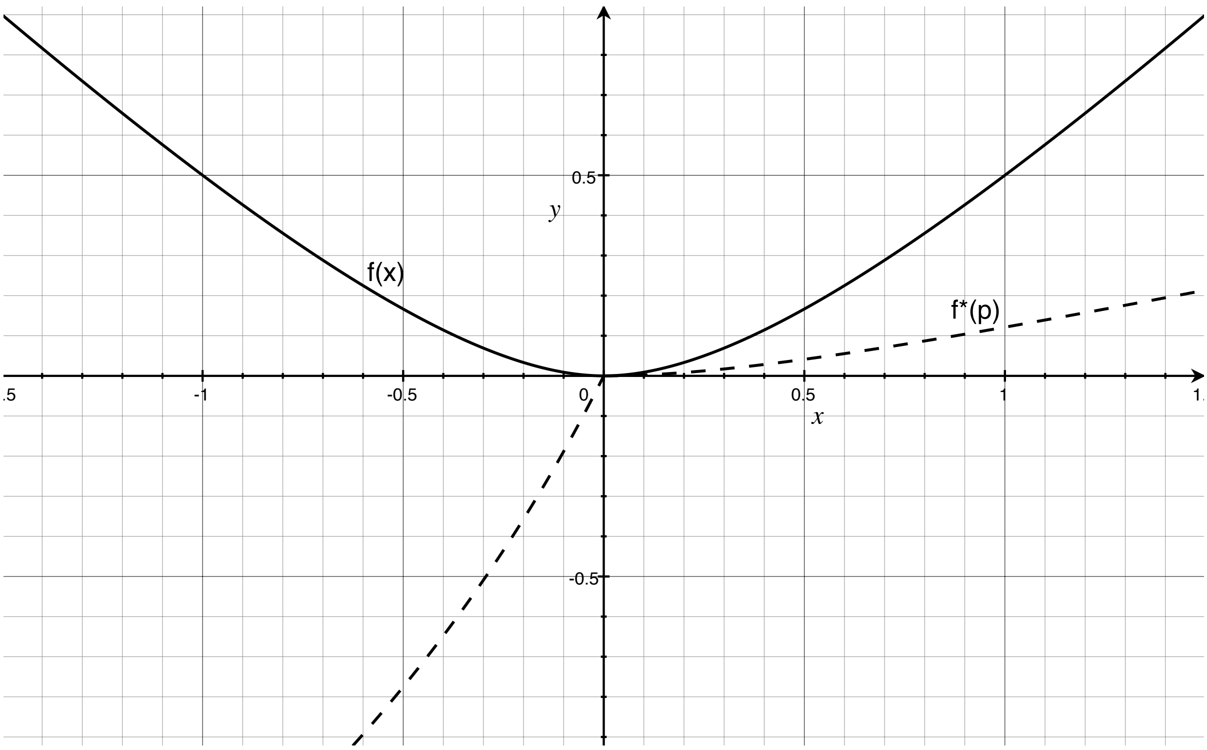

Figure 0.1: Plot for Problem 22

> The derivative of the function is $x^3 - ax$, so there are three critical points which we should examine, which are the roots $x = 0$ and $x = \pm\sqrt{a}$. The second derivative is $3x^2 - a$, so plugging these extrema in, we can see that $3\left(\begin{cases}\pm\sqrt{a}\\ 0\end{cases}\right)^2 - a = \begin{cases}2a\\ -a\end{cases}$. When $a > 0$, $\pm\sqrt{a}$ are minima, while 0 will be a maximum. When $a < 0$, 0 will now be a minimum, and the two extrema at $\pm\sqrt{a}$ will become imaginary. Additionally, the two inflection points of $f(x)$ are the roots of the second derivative: $\pm\sqrt{\frac{a}{3}}$ and they also vanish for $a < 0$. It's clear that $a < 0$ makes the function always convex, since it will only have a single minimum and no inflection points. However, $a > 0$ will create two maxima, and the points between the inflection points will be concave while the rest of the function will be convex. See Figure 0.2 for some representative graphs.

2. In order to actually perform the Legendre transform, you need the equation that links p and x. Find it.

> I already did the derivative above. To minimize $f(x) - px$, we require an x_0 such that
>
> $$x_0^3 - ax_0 = p$$

3. The graph of $f^*(p)$ is the collection of points $\{p, f^*(p)\}$. Neglecting for a moment the "min" prescription in the Legendre transform, let us consider the collection of points $\{p(x), f(x) - p(x)x\}$, which you could view as a parametric representation of the graph of $f^*(p)$ (with x being the parameter). Using your favorite plotting program, provide plots of that graph for representative values of a. What happens when you tune a such that $f(x)$ ceases to be convex? Which bits of the (possibly funny-looking) graph of $f^*(p)$ will survive after applying the "min" in the Legendre transform that we have ignored so far? What therefore happens to the Legendre transform $f^*(p)$ once $f(x)$ deviates locally from convexity?

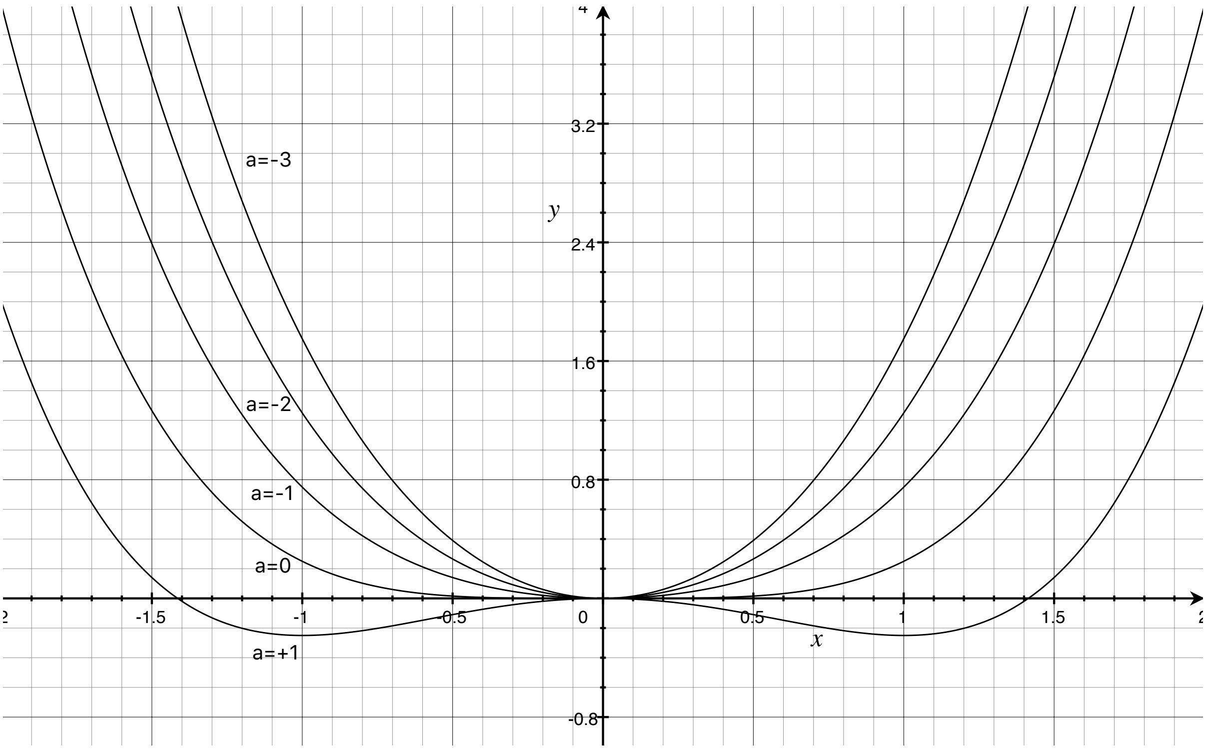

Figure 0.2: Plot for Problem 23.1

While my previous couple of plots have not used it, my favorite plotting program is Python. See Figure 0.3 for first, a plot of this parametric curve for a few values of a, and second, for a demonstration of what makes the cut if you take the minimum or maximum values of the curve (although note that I did not faithfully represent the maximum for values of x outside the region where it's the same as the minimum, I couldn't think of a clever way to show it visually). When the function deviates from convexity, a maximizing transformation must be made rather than a minimization. This area corresponds to the region between the inflection points.

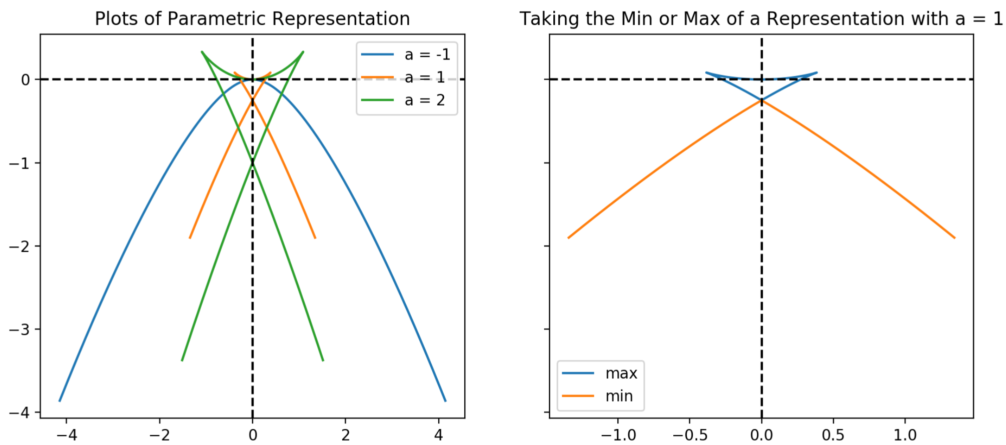

Figure 0.3: Plots for Problem 23.3

4. To get $f^*(p)$, we need to solve the equation linking p and x for x. Defining $r^2 = \frac{4a}{3}$ and $\cos(3\alpha) = \frac{4p}{r^3}$, show (without using Mathematica or relatives!) that the three solutions $\{x_0, x_1, x_2\}$ can be written as $x_k = r\cos\left(\alpha + \frac{2\pi}{3}k\right)$.

$$\begin{aligned} p(x) &= x^3 - ax \\ \frac{r^3\cos(3\alpha)}{4} &= x^3 - \frac{3r^2}{4}x \\ \frac{r^3}{4}\left(4\cos^3(\alpha) - 3\cos(\alpha)\right) &= \\ (r\cos(\alpha))^3 - \frac{3r^2}{4}r\cos(\alpha) &= x^3 - \frac{3r^2}{4}x \end{aligned}$$

Clearly, $x = r\cos(\alpha)$ is a solution. If we factor it out, we will be left with some arbitrary quadratic:

$$(x - r\cos(\alpha))(x^2 + \beta x + \gamma) = x^3 - \frac{3r^2}{4}x$$

Matching like powers of x, we see that $\beta = r\cos(\alpha)$ and $\gamma = r^2\cos^2(\alpha) - \frac{3}{4}r^2$. The roots of this expression are

$$x = r\left(-\frac{1}{2}\cos(\alpha) \pm \frac{\sqrt{3}}{2}\sin(\alpha)\right) = r\left(\cos\left(\frac{2\pi}{3}\right)\cos(\alpha) \pm \sin\left(\frac{2\pi}{3}\right)\sin(\alpha)\right) = \cos\left(\alpha \mp \frac{2\pi}{3}\right)$$

by periodicity in the cosine, we can see that the given solutions are correct.

5. Identify the three solutions with the three interesting branches of the Legendre transform which show up once $f(x)$ is no longer convex. Feel free to use your favorite plotting program to do that; no formal proof is required.

If we solve our solutions for p and a, we find that:

$$\alpha = \pm\frac{1}{3}\left(\arccos\left(\frac{3\sqrt{3}p}{2a^{3/2}}\right) + 2\pi c\right), \quad c \in \mathbb{Z}$$

Picking the positive solution and setting $c = 0$, we can then plug this into the three solutions and get regions corresponding to the graph in Figure 0.4. Taking the other solution or a different value of c just cyclically rotates which region corresponds to the given solution. Additionally, in solving this, I took the positive root for r but I could have just as easily taken the negative root which would have introduced a negative sign into the argument of the arccosine. However, this reduces back to the original solutions if we choose a different branch of the arccosine, which is defined on the whole complex plane.

6. What is $f^*(0)$? And what is $\lim_{p\to 0^+} f^{*\prime}(p)$ and $\lim_{p\to 0^-} f^{*\prime}(p)$?

Clearly from the parametric plot there are two solutions at $p = 0$. One is where $f^*(0) = 0$ and the other (solving our solutions for $p = 0$ to get $x_i = 0, \pm\sqrt{a}$, is $f^*(0) = -\frac{a^2}{4}$. If we approach from the left, we use the first solution $f^{*\prime}(0) = -x_0(0) = -\sqrt{a}$. From the right, I think we have to use $f^{*\prime}(0) = -x_2(0) = 0$, so I'm not quite sure what this is supposed to tell us, since I would expect both of these solutions to correspond to moving on the long arcs which give a zero at $-\frac{a^2}{4}$ rather than the top solution. Technically we could also use the x_1 solution which would give $\sqrt{\alpha}$ as the derivative, but I'm not sure which solution we're supposed to take as we approach from either side. Does this not depend on our choice of branch?

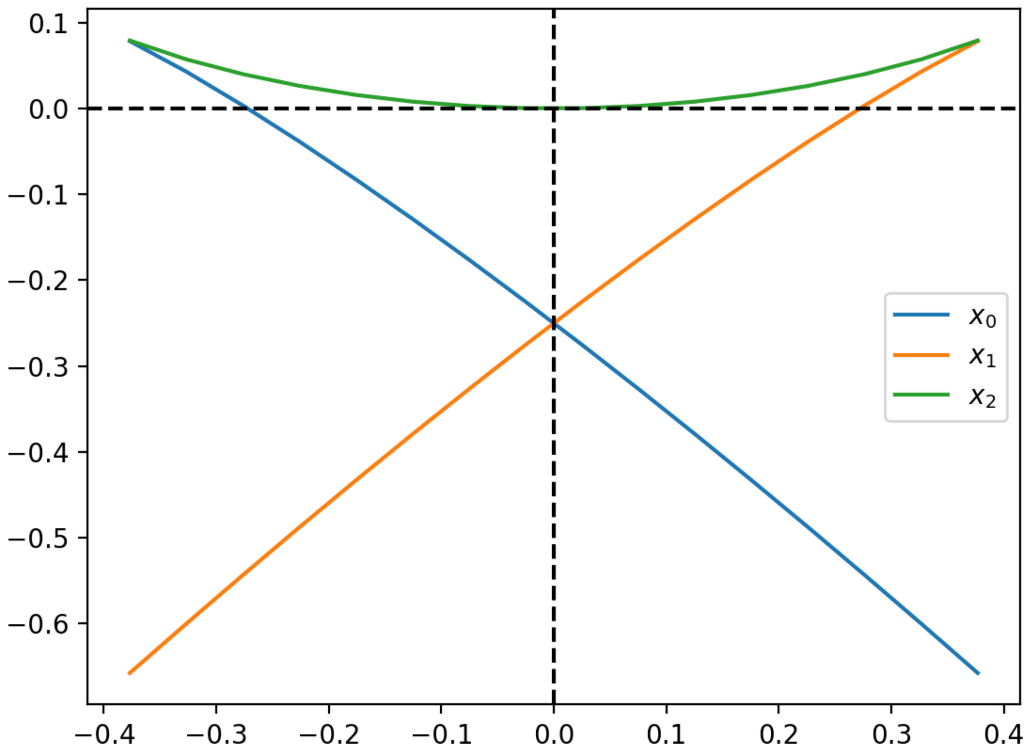

Figure 0.4: Plot for Problem 23.5 ($a = 1$)