Lecture 22: Infinite Dimension Hilbert Spaces, Continued

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Recall the position basis $\{|x\rangle\}$ and the position operator $X|x\rangle = x|x\rangle$. We are going to use these to represent an arbitrary function on the real line:

$$|\varphi\rangle = \int \mathrm{d}x\, \varphi(x)\, |x\rangle \tag{1}$$

or

$$\varphi(x) = \langle x|\, |\varphi\rangle \tag{2}$$

Let us now introduce a translation operator U. There could be many different transformations, so we will label ours $U(\alpha)$ such that $U(\alpha)|x\rangle = |x+\alpha\rangle$. We have chosen to symbolize this with a capital "U" because we suspect it's unitary. Unitary transformations transform orthonormal bases to orthonormal bases. $\{|x\rangle\}$ is orthonormal, and the operator maps to states $\{|x+\alpha\rangle\} = \{|x\rangle\}$.

Let us translate the state $|\varphi\rangle$ from above. We want to evaluate some $\varphi(x)$ so we put the bra for the x states on each side. This shows that $U\colon \varphi(x) \to \varphi(x-\alpha)$:

$$\begin{aligned}\langle x|U(\alpha)\,|\varphi\rangle &= \langle x| \int \mathrm{d}x'\, \varphi(x')\, |x' + alpha\rangle && (3)\\ &= \int \mathrm{d}x'\, \varphi(x')\delta(x-(x'+\alpha)) && (4)\\ &= \varphi(x-\alpha) && (5)\end{aligned}$$

What if we want to translate an operator? We say that $A\colon\, |\varphi\rangle \to |\chi\rangle = A|\varphi\rangle$. We want the following to happen

$$A'\colon U|\varphi\rangle \to U|\chi\rangle = UA|\varphi\rangle \tag{6}$$

We know for a fact (from one line above) that $A'\colon U|\varphi\rangle \to A'U|\varphi\rangle$. These must be equal, so

$$A' = UAU^\dagger \tag{7}$$

Now let us consider "infinitesimal" transformation. We consider a small δ such that $U(\delta)$ can be written as some Taylor series:

$$\begin{aligned}U(\delta) &\approx U(0) + \delta \left.\frac{\partial U(\alpha)}{\partial \alpha}\right|_{\alpha=0} && (8)\\ &= I - \imath\delta T && (9)\end{aligned}$$

where

$$T \equiv \imath \left.\frac{\partial U}{\partial alppha}\right|_{\alpha=0} \tag{10}$$

Additionally,

$$U^\dagger = I + \imath\delta T^\dagger + \cdots \tag{11}$$

Together

$$UU^\dagger = I = I + \imath\delta(T^\dagger - T) + \mathcal{O}\big(\delta^2\big) \tag{12}$$

This order of δ must vanish, so $T = T^\dagger$, or T is hermitian.

Let us combine a finite and an infinitesimal transformation:

$$U(\alpha+\delta) = U(\delta)U(\alpha) = (I - \imath\delta T)U(\alpha) \tag{13}$$

Therefore,

$$\frac{\partial U}{\partial \alpha} = -\imath T U(\alpha) \tag{14}$$

We can solve this:

$$U(\alpha) = e^{-\imath\alpha T} \tag{15}$$

Now consider the infinitesimal operator acting on an arbitrary function

$$U(\delta)\varphi(x) = \varphi(x-\delta) \approx \varphi(x) - \delta\varphi'(x) \tag{16}$$

so

$$T = -\imath\frac{\mathrm{d}}{\mathrm{d}x} \tag{17}$$

so in general,

$$U(\alpha) = e^{-\alpha\frac{\mathrm{d}}{\mathrm{d}x}} \tag{18}$$

We say that the derivative is the "generator" for the transformation group. This can be brought into three dimensions. Define $\vec{P} = -\imath\hbar\nabla$ such that $E^{-\imath\vec{\alpha}\cdot\vec{P}/\hbar}$.

$$X \xrightarrow{U} X' \Rightarrow X' = (I - \imath\delta P/\hbar)X(I + \imath\delta P/\hbar) \tag{19}$$

$$= X + (\imath\delta/\hbar)(XP - PX) + \mathcal{O}\big(\delta^2\big) \tag{20}$$

$$= X - \delta I \Rightarrow [X, P] \equiv XP - PX = \imath\hbar I \tag{21}$$

or by components,

$$[\vec{R}_j, \vec{P}_k] = \imath\hbar I\delta_{jk} \tag{22}$$

Theorem 1. Ehrenfest Theorem: Property $A = A(t)$ in $|\varphi(t)\rangle$. We want to look at

$$\langle A\rangle_\varphi(t) = \langle\varphi(t)|\, A(t)\, |\varphi(t)\rangle \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\varphi|\, A\, |\varphi\rangle = \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\varphi|\right) A\,|\varphi\rangle + \langle\varphi|\,\frac{\mathrm{d}}{\mathrm{d}t}A\,|\varphi\rangle + \langle\varphi|\, A\frac{\mathrm{d}}{\mathrm{d}t}\,|\varphi\rangle \tag{24}$$

This is equivalent to

$$\frac{1}{\imath\hbar}\langle\varphi|\,[A,H]\,|\varphi\rangle + \langle\varphi|\,\frac{\mathrm{d}}{\mathrm{d}t}A\,|\varphi\rangle \tag{25}$$