Lecture 29: Faraday's Law

Wednesday, October 23, 2019

Again, for a surface Σ with boundary Γ,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \vec{B}\cdot\hat{n}\,\mathrm{d}a = \mathscr{E}$$

where $\mathscr{E}$ is the electromotive force, the net energy gain after a unit charge moves around the loop. In the rest frame of the loop Γ, the $\vec{E}$-field does the work, so

$$\mathscr{E} = \oint_\Gamma \vec{E}'\cdot\mathrm{d}\vec{l}$$

If we fix the loop, we can bring the time derivative inside the integral, so

$$\oint_\Gamma \vec{E}\cdot\mathrm{d}\vec{l} = -\int_\Sigma \frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{a}$$

where $\vec{E}' = \vec{E}$ for a fixed loop. By Stokes theorem,

$$\int_\Sigma \boldsymbol{\nabla}\times\vec{E}\cdot\mathrm{d}\vec{a} + \int_\Sigma \frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{a} = 0$$

so

$$\boldsymbol{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Digression

$$\vec{B} = \boldsymbol{\nabla}\times\vec{A}$$

now implies that

$$\boldsymbol{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$$

or

$$\vec{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

What happens if the loop does move? Let's assume rigid motion (no deformation of the loop itself, just translation in space). Suppose the loop moves with velocity $\vec{v}$ and $\vec{B}$ is constant in time but could vary in space. If we imagine connecting a surface Σ_0 and Σ_{dt}, we can find the flux:

$$\int_{\Sigma_0}\vec{B}\cdot\mathrm{d}\vec{a} + \int_{-\Sigma_{dt}}\vec{B}\cdot\mathrm{d}\vec{a} + \int_{\text{sides}}\vec{B}\cdot\mathrm{d}\vec{a} = 0$$

since there is no divergence of the $\vec{B}$ field. The flux through the opposite orientation can be found by just negating the middle integral. $\vec{v} \times \mathrm{d}\vec{l} = \mathrm{d}\vec{a}$ on the sides, so

$$\int_{\Sigma_{dt}} \vec{B} - \int_{\Sigma_0} \vec{B} = \oint_\Gamma (\vec{v} \times \mathrm{d}\vec{l}) \cdot \vec{B} dt$$

so

$$\mathscr{E} = -\oint_\Gamma (\vec{v} \times \mathrm{d}\vec{l}) \cdot \vec{B} = \oint_\Gamma (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{l}$$

Notice how similar this is to the magnetic force on the charges. Technically, the motion of the charges includes a drift velocity along the loop in addition to the motion of the loop itself, but because that velocity is parallel to $\mathrm{d}\vec{l}$, its contribution is zero. By relating this equation to our loop frame, we find that, non-relativistically,

$$\vec{E}' = \vec{v} \times \vec{B}$$

so

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

This is the general formulation for changing reference frames in electromagnetism (for $v << c$). This is how we "make" magnetic fields do work. If the charges are constrained, like on a loop, we can move them and use a magnetic field to do work on them.

0.1 Energy Stored in Magnetic Fields

Let's look at a loop Γ upon which we are establishing a current. As we increase the flux, there will be a back-emf generated due to the loop's own magnetic field.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -I \cdot \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t}$$

where $\mathcal{F}$ is the flux. Therefore

$$\mathrm{d}W = I\mathrm{d}\mathcal{F}$$

We lose the minus sign because this is the back-emf, so the current for it is going in the opposite direction. If we suppose the flux contains some geometric factor L (self-inductance) $\mathcal{F} = L \cdot I$,

$$\mathrm{d}W = IL\mathrm{d}I$$

so

$$W = \frac{1}{2}LI^2$$

Now let's generalize to some current density $\vec{J}$ with $\boldsymbol{\nabla} \cdot \vec{J} = 0$. Imagine we perform this adiabatically, such that $\frac{\partial \rho}{\partial t} \approx 0$. If we look at a small cross-section $\mathrm{d}\vec{\sigma}$, we have $\underbrace{\vec{J} \cdot \mathrm{d}\vec{\sigma}}_{\mathrm{d}I} \underbrace{\int_S \delta\vec{B} \cdot \mathrm{d}\vec{a}}_{\delta\mathcal{F}}$ where S is the area of the cross-section, so

$$\delta(\mathrm{d}W) = \mathrm{d}I \int_S \delta\vec{B} \cdot \mathrm{d}\vec{a} = \mathrm{d}I \int_S \boldsymbol{\nabla} \times \delta\vec{A} \cdot \mathrm{d}\vec{a} = \mathrm{d}I \oint_\Gamma \delta\vec{A} \cdot \mathrm{d}\vec{l}$$

so

$$\delta(\mathrm{d}W) = \oint \delta\vec{A} \cdot \mathrm{d}\vec{I}\, \mathrm{d}l = \oint \delta\vec{A} \cdot \vec{J}\, \mathrm{d}\sigma\, \mathrm{d}l$$

If we sum over all of these segmented loops (all $\mathrm{d}\sigma\, \mathrm{d}l$), we say that this becomes a volume integral over the region.

$$\delta W = \int_{\Omega} \vec{J} \cdot \delta\vec{A}\, \mathrm{d}^3 x$$