Lecture 25: Free Particle Motion

Monday, October 14, 2019

The behavior of a particle in a potential is described by

$$H = \frac{P^2}{2m} + V(x)$$

In free motion, $V(x) \to 0$, so our states are eigenstates of the momentum states:

$$P\,|p\rangle = p\,|p\rangle$$

and

$$H\,|p\rangle = \underbrace{\frac{P^2}{2m}}_{E(p)}|p\rangle$$

If we want to see the time evolution, we use the Schrödinger equation:

$$\imath\hbar\frac{\partial}{\partial t}\,|\varphi\rangle = H\,|\varphi\rangle$$

$$|p\rangle\,(t) = e^{-\imath E(p)t/\hbar}\,|p\rangle\,(t=0)$$

We can also look at the state in terms of a wave packet:

$$|\varphi\rangle = \int \mathrm{d}p\,\tilde{\varphi}(p)\,|p\rangle$$

$$\varphi(x) = \langle x|\,|\varphi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int \mathrm{d}p\,\tilde{\varphi}(p)e^{\imath px/\hbar}$$

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int \mathrm{d}p\,\tilde{\varphi}(p,t=0)e^{\imath(px-E(p)t)/\hbar}$$

Let's write the momentum as $p = \hbar k$ such that k is the wave number. We will also rescale $E = \hbar\omega$, where $\omega = \omega(k)$. Finally, we will define $\sqrt{\hbar}\tilde{\varphi}(\hbar k) \equiv A(k)$. Finally, we have

$$\varphi(x,t) = \frac{1}{2\pi}\int \mathrm{d}k\,A(k)e^{\imath(kx-\omega t)}$$

If the phase is zero, $x(t) = \underbrace{\frac{\omega}{k}}_{v_p} t$ where v_p is the phase velocity. Let's construct a wave packet by making $A(k)$ a Gaussian about $\bar{k}$. In real space, this means that $x(t)$ is a wave of a frequency $\bar{k}$ with a Gaussian envelope. The wave inside the envelope moves at the phase velocity, but the envelope itself will move at v_g, the group velocity. To see what this means, let's imagine $A(k)$ is two δ functions, $\delta(\bar{k}+\delta k)+\delta(\bar{k})-\delta k$. Now the Fourier transform is relatively simple. We will expand $\omega(\bar{k}+\delta k) \approx \omega(\bar{k}) + \delta k\,\frac{\mathrm{d}\omega}{\mathrm{d}k}\big|_{\bar{k}} = \bar{\omega} + \delta\omega$:

$$\begin{aligned}\varphi(x,t) &= \frac{1}{2}e^{\imath[(\bar{k}+\delta k)x-(\bar{\omega}+\delta\omega)t]} + \frac{1}{2}e^{\imath[(\bar{k}-\delta k)x-(\bar{\omega}-\delta\omega)t]}\\ &= e^{\imath(\bar{k}x-\bar{\omega}t}\cos(\delta kx - \delta\omega t)\end{aligned}$$

The crests of the wave (the exponential) move at phase velocity $v_p = \omega/k$. However, the envelope (the cosine) moves at the group velocity $v_g \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k}\big|_{\bar{k}}$.

Recall that $\omega = E/\hbar = \frac{\hbar k^2}{2m}$ is nonlinear in k. Therefore, $v_p = \frac{\hbar k}{2m}$ and $v_g = \frac{\hbar k}{m}$. Notice that in general, these are not the same number. The fact that $v_p = v_p(k)$ will lead to a spreading of the group, which we call "wave packet spreading." Note that we are talking about particles with mass. For massive particles, there is this phenomenon of dispersion $\omega(k)$. Massive particles have nontrivial dispersion relations.

Recall from last week that

$$(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$$

and Ehrenfest's theorem:
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{1}{\imath\hbar}\langle [A,H]\rangle$$

If $H = \frac{P^2}{2m}$, $[X, P^2] = 2\imath\hbar P$. Also, $[X^2, P^2] = 2\imath\hbar\{X, P\}$ and $[\{X, P\}, P^2] = 4\imath\hbar P^2$.

$$\frac{\mathrm{d}\langle X\rangle}{\mathrm{d}t} = \frac{\langle P\rangle}{m} = v_0 \implies \langle X\rangle = v_0 t + \langle X\rangle_0$$

$$\frac{\mathrm{d}\left\langle X^2\right\rangle}{\mathrm{d}t} = \frac{\langle\{X,P\}\rangle}{m} \implies \frac{\mathrm{d}^2\left\langle X^2\right\rangle}{\mathrm{d}t^2} = \frac{1}{\imath\hbar m}\langle[\{X,P\},H]\rangle = \frac{2\left\langle P^2\right\rangle}{m^2}$$

Therefore
$$\frac{\mathrm{d}\left\langle X^2\right\rangle}{\mathrm{d}t} = \frac{2\left\langle P^2\right\rangle_0 t}{m^2} + \xi_0$$

where $\xi_0 \equiv \left.\frac{\mathrm{d}\left\langle X^2\right\rangle}{\mathrm{d}t}\right|_{t=0} \propto 2v_0x_0$ in the classical limit.

$$\left\langle X^2\right\rangle = \frac{\left\langle P^2\right\rangle t^2}{m^2} + \xi_0 t + \left\langle X^2\right\rangle_0$$

Finally we can write
$$(\Delta X)^2 = (\Delta v)_0^2 t^2 + 2\Delta(v_0x_0) + (\Delta X)_0^2$$

Taking the square root, we can get ΔX, which is like the width of the wave packet as a function of time. It rests on diagonal asymptotes with slope $\pm(\Delta v)_0$, and it intersects $t = 0$ at $(\Delta X)_0$. The initial slope at $t = 0$ is proportional to ξ_0.