Lecture 33: Harmonic Oscillator
Wednesday, November 06, 2019

0.1 The Harmonic Oscillator

If our potential is $V = \frac{1}{2}kx^2$, we can write our Hamiltonian as

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}k\mathbf{X}^2 = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{X}^2$$

where $\omega = \sqrt{\frac{k}{m}}$. We expect the eigenfunctions should have definite parity, since $[\mathbf{H}, \mathbf{\Pi}] = 0$ so $\mathbf{\Pi}\left|\varphi\right\rangle = \pm\left|\varphi\right\rangle$. We also know $[\mathbf{X}, \mathbf{P}] = \imath\hbar$ and $\mathbf{H}\left|\varphi\right\rangle = E\left|\varphi\right\rangle$. If we were to imagine differentiating the Schrödinger equation from $-\infty$, only a few miraculous values of E will solve this equation so that it vanishes at $+\infty$. We can make live a bit easier by getting rid of every quantity with physical dimensions. Let's introduce $\hat{\mathbf{X}} = \sqrt{\frac{m\omega}{\hbar}}\mathbf{X}$ and $\hat{\mathbf{P}} = \sqrt{\frac{1}{m\hbar\omega}}\mathbf{P}$ such that $\left[\hat{\mathbf{X}}, \hat{\mathbf{P}}\right] = \imath$. Therefore

$$\hat{\mathbf{H}} = \frac{1}{\hbar\omega}\mathbf{H} = \frac{1}{2}\left(\hat{\mathbf{P}}^2 + \hat{\mathbf{X}}^2\right)$$

We solve this by introducing two new operators, called "raising" and "lowering" operators:

$$\mathbf{a} \equiv \frac{1}{\sqrt{2}}\left(\hat{\mathbf{X}} + \imath\hat{\mathbf{P}}\right)\mathbf{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{X}} - \imath\hat{\mathbf{P}}\right)$$

so $\left[\mathbf{a}, \mathbf{a}^\dagger\right] = 1$ and we define $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2}\left(\hat{\mathbf{P}}^2 + \hat{\mathbf{X}}^2 - 1\right)$. Therefore

$$\hat{\mathbf{H}} = \mathbf{a}^\dagger\mathbf{a} + \frac{1}{2} = \mathbf{N} + \frac{1}{2}$$

so $[\mathbf{H}, \mathbf{N}] = 0$.

$$\mathbf{N}\left|\varphi_\nu^{(i)}\right\rangle = \nu\left|\varphi_\nu^{(i)}\right\rangle$$

and

$$\hat{\mathbf{H}}\left|\varphi_\nu^{(i)}\right\rangle = \left(\nu + \frac{1}{2}\right)\left|\varphi_\nu^{(i)}\right\rangle$$

where (i) is an additional degree of freedom that we will find is not important.

$\nu \geq 0$

$$\nu = \nu\left\langle\varphi_\nu|\varphi_\nu\right\rangle = \left\langle\varphi_\nu\right|\mathbf{N}\left|\varphi_\nu\right\rangle = \left(\left\langle\varphi_\nu\right|\mathbf{a}^\dagger\right)\left(\mathbf{a}\left|\varphi\nu\right\rangle\right) = \left\|\mathbf{a}\left|\varphi_\nu\right\rangle\right\|^2 \geq 0$$

$\nu = 0$

$$\implies \mathbf{a}\left|\varphi\nu\right\rangle = 0$$

$\nu > 0$

$$\implies \mathbf{N}\mathbf{a}\left|\varphi_\nu\right\rangle = (\nu - 1)\mathbf{a}\left|\varphi_\nu\right\rangle$$

This is because $[\mathbf{N}, \mathbf{a}] = -\mathbf{a}$, so $\mathbf{N}\left(\mathbf{a}\left|\varphi_\nu\right\rangle\right) = \mathbf{a}\mathbf{N}\left|\varphi_\nu\right\rangle - \mathbf{a}\left|\varphi_\nu\right\rangle = (\nu - 1)\mathbf{a}\left|\varphi_\nu\right\rangle$.

$\mathbf{a}^\dagger\left|\varphi_\nu\right\rangle \neq 0$

$\mathbf{N}\mathbf{a}^\dagger\left|\varphi_\nu\right\rangle = (\nu + 1)\mathbf{a}^\dagger\left|\varphi_\nu\right\rangle$

ν is a non-negative integer Assume $n < \nu < n+1$. $\mathbf{a}^{n+1} |\varphi_\nu\rangle = 0$, therefore $\nu - (n+1) = 0$ so $\nu \in \mathbb{Z}$.

$|\varphi_\nu\rangle$ is non-degenerate

$|\varphi_0\rangle$ Lowering this state must give us zero, so

$$\mathbf{a} |\varphi_0\rangle = 0 = \frac{1}{\sqrt{2}} \left(\hat{\mathbf{X}} + \imath \hat{\mathbf{P}} \right) |\varphi_0\rangle$$

In x-space,

$$\left(x + \frac{\mathrm{d}}{\mathrm{d}x} \right) \varphi_0(x) = 0 \implies \varphi_0(x) = C_0 e^{-\frac{x^2}{2}}$$

$|\varphi_n\rangle$ non-degenerate implies $|\varphi_{n+1}\rangle$ is non-degenerate

$$\begin{aligned}
\mathbf{a}^\dagger [\mathbf{a} \left|\varphi_{n+1}^{(i)}\right\rangle &= C^{(i)} |\varphi_n\rangle] \\
\mathbf{N} \left|\varphi_{n+1}^{(i)}\right\rangle &= (n+1) \left|\varphi_{n+1}^{(i)}\right\rangle = C^{(i)} \mathbf{a}^\dagger |\varphi_n\rangle \\
\left|\varphi_{n+1}^{(i)}\right\rangle &= \frac{C^{(i)}}{n+1} \mathbf{a}^\dagger |\varphi_n\rangle
\end{aligned}$$

0.1.1 Eigenfunctions of the Harmonic Oscillator

We start by normalizing the ground state wave function:

$$\varphi_0(x) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{x^2}{2}}$$

The other eigenfunctions can be found by raising the ground state:

$$|\varphi_n\rangle = \frac{1}{\sqrt{n!}} (\mathbf{a}^\dagger)^n |\varphi_0\rangle$$

so

$$\varphi_1(x) = \sqrt[4]{\frac{4}{\pi}} x e^{-\frac{x^2}{2}}$$

and

$$\varphi_2(x) = \sqrt[4]{\frac{1}{4\pi}} \left[2x^2 - 1 \right] e^{-\frac{x^2}{2}}$$

where the polynomials in front of the exponential are the Hermite polynomials $H_n(x)$. The energy levels are evenly spaced by $\hbar\omega$ (so that the energy difference between the energy of the ground state is $\hbar\omega$ away from the first state, and the same with the first and second state). The space between the ground state and the x-axis is $\frac{1}{2}\hbar\omega$, so the energy eigenvalues are $\hbar\omega(n + \frac{1}{2})$.