Lecture 4: Laplace Equation

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0.1 Review

Dirichlet Problem:

$$G_D(x,x') = 0 \tag{1}$$

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int G_D(x,x')\rho(x') + \frac{1}{4\pi}\oint_\Sigma \frac{\partial G_D}{\partial n'_+}da'\Phi(x') \tag{2}$$

$$G_D(x,x') = G_D(x',x) \tag{3}$$

Neumann Problem:

We can't impose $\left.\frac{\partial G_N}{\partial n_-}\right|_\Sigma = 0$, so we will impose $\left.\frac{\partial G_N}{\partial n_-}\right|_\Sigma = -\frac{4\pi}{\text{Area}(\Sigma)}$:

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int G_N(x,x')\rho(x') + \langle\Phi\rangle_\Sigma + \oint_\Sigma G_N(x,x')\frac{\partial\Phi}{\partial n'_-}da' \tag{4}$$

$$G_N(x,x') = G_N(x',x) \tag{5}$$

If we only have conductors raised to potentials Φ_i (constants), then the charge in the jth conductor becomes:

$$Q_j = -\frac{1}{4\pi}\oint_{\Sigma_j}\oint_{\Sigma_i}\frac{\partial^2 G}{\partial n_+\partial n'_+}dada'\Phi_i \tag{6}$$

$$= \sum_j C_{ji}\Phi_i \tag{7}$$

Remark. For $\nabla^2\Phi = 0$, the potential satisfies this equation at charge free regions. In charge free regions, $\Phi(x)$ is given by an average over any sphere around x as long as the sphere is in the charge free region:

$$\Phi(x) = \frac{1}{4\pi b^2}\int_{S^2}\Phi(x+b\hat{\xi})da \tag{8}$$

where b is the radius of the sphere and $\hat{\xi}$ is the normal outwards. $da = b^2 d\Omega$ and

$$\frac{\partial}{\partial b}\langle\Phi\rangle_{S_b^2} = \frac{\partial}{\partial b}\frac{1}{4\pi}\oint\Phi(x+b\hat{\xi})d\Omega = \frac{1}{4\pi}\oint\nabla\Phi\cdot\hat{\xi}d\Omega = \frac{1}{4\pi}\int_V\nabla\cdot(\nabla\Phi)d^2x = 0 \tag{9}$$

since $\nabla \cdot (\nabla \Phi) = 0$. This implies Φ has no max or min apart from the charged regions or boundaries. Suppose there was a maximum at x_*. Take a small sphere around x_* and average it, all the values on the sphere will be less than $\Phi(x_*)$, so the average will be less than the "true" value. Therefore, there are no true stable equilibrium points in electrostatics.

0.2 Energy Considerations

In free space, if we have point charges,

$$W = \frac{1}{2} \sum_{i \neq j} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{|x_i - x_j|} \tag{10}$$

(Jackson uses "W" for energy). This is like the cost of bringing in charges from infinity. Alternatively, $W = \frac{1}{2}\varepsilon \int_{\text{everywhere}} E^2 d^3x$ for continuous charge distributions (for point charges, you get infinities).

Let us derive $W = \frac{1}{2}\varepsilon_0 \int E^2 d^3x$:

The work to add an infinitesimal charge $\delta\rho(x)$ to a continuous distribution is

$$\delta W = \int_\Omega \Phi(x)\delta\rho(x) d^3x \tag{11}$$

$$\nabla \cdot \delta E = \delta\rho / \varepsilon_0 \tag{12}$$

$$\delta W = \varepsilon_0 \int_\Omega \Phi(x) \nabla \cdot (\delta E) d^3x = \varepsilon_0 \int \nabla \cdot [\Phi(x)\delta E] d^3x - \varepsilon_0 \int_\Omega \nabla\Phi \cdot \delta E d^3x \tag{13}$$

$$= \varepsilon_0 \oint_\Sigma \Phi(x)\delta E \cdot d\vec{a}_- + \varepsilon \int_\Omega (-\nabla\Phi) \cdot \delta E d^3x \tag{14}$$

$$= \varepsilon_0 \sum_i \left(\oint_{\Sigma_i} \delta E \cdot da_- \right) \Phi_i + \varepsilon_0 \int E \cdot \delta E d^3x \tag{15}$$

$$\varepsilon_0 \sum_i \left(\oint_{\Sigma_i} \delta E \cdot da_- \right) \Phi_i = 0, \tag{16}$$

so

$$\delta W = \varepsilon \int_\Omega E \cdot \delta E d^3x = \delta \left(\frac{\varepsilon_0}{2} \int_\Omega E^2 d^3x \right) \tag{17}$$

The 1/2 here comes from pulling the δ out of the integral.

So $W = \frac{\varepsilon_0}{2} \int_\Omega E^2 d^3x + W_0$. $W \to 0$ as $|E| \to 0$ so $W_0 \equiv 0$.

In the presence of conductors,

$$W = \frac{1}{2}\int_{\Omega} \Phi \rho d^3x + \frac{1}{2}\sum_{k=1}^{N} Q_k \Phi_k \tag{18}$$

Remark. $\delta W = \sum_i (C^{-1})_{ik} Q_k \delta Q_i$, therefore $\delta W = \delta\left(\frac{1}{2}\sum Q_i (C^{-1})_{ik} Q_k\right)$.