33-761 Homework 9

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1 Coaxial Cable

(a) Consider a coaxial cable with uniform cylindrical cross section. Assume that the inner thin cylinder has radius a and the outer one has radius b. Current I goes through one and returns from the other. Calculate the self-inductance L per unit length of this cable. In a similar way we can calculate the capacitance C per unit length, then verify the formula $CL = \epsilon_0\mu_0$.

Using an Ampereian loop, we can find the magnetic field inside the coaxial cable to be $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$. Now let's imagine taking a rectangle which goes from the inside to the outside conductors with length $\mathrm{d}l$ in the direction of the cable. The magnetic flux through such a rectangle would give us the flux per unit length. Because the flux is given by

$$\Phi = \int \vec{B} \cdot \mathrm{d}\vec{a}$$

and $a = r\,\mathrm{d}l$ (which we integrate from a to b to get the full flux), the integral becomes

$$\Phi = \int_a^b B(r)\,\mathrm{d}r = \frac{\mu_0 I}{2\pi}\ln\left(\frac{b}{a}\right)$$

The inductance is defined as $\frac{\Phi}{I}$ so

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$$

For the capacitance, we first find the electric field between the two conductors, which is just $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$ from the usual Gauss's law, assuming the Gaussian surface contains charge λ per unit length. Therefore, the change in voltage is just the integral

$$\Delta V = \int_a^b E(r)\,\mathrm{d}r = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

The capacitance is defined as $\frac{\lambda}{\Delta V}$ so

$$C = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

so

$$CL = \mu_0\epsilon_0$$

(b) Assume that the coaxial cable has an arbitrary cross section, show that we can verify the relation $CL = \epsilon_0\mu_0$ in this case as well, even though we cannot compute them explicitly.

I'll work backwards from the last problem:

$$CL = \frac{\lambda}{\Delta V}\frac{\Phi}{I} = \frac{\lambda}{I}\frac{\int \vec{B}\cdot \mathrm{d}\vec{a}_1}{\int \vec{E}\cdot \mathrm{d}\vec{a}_2}$$

Where a_1 is a surface which is perpendicular to the conductors and a_2 is a surface which is parallel to the conductors. We can no longer use cylindrical symmetry to pull out factors of 2π, but in general an integral over the B-field should be proportional to the enclosed current times μ_0 and the integrating over the same path over the E-field should give the enclosed charge (per unit length) divided by ϵ_0:

$$CL = \frac{\lambda}{I}\frac{I}{\lambda}\mu_0\epsilon_0 \frac{\iint \vec{b}\,\mathrm{d}\gamma\cdot \mathrm{d}\vec{a}_1}{\iint \vec{e}\,\mathrm{d}\gamma\cdot \mathrm{d}\vec{a}_2} = \mu_0\epsilon_0$$

Here $\vec{b}$ and $\vec{e}$ are functions of position and should depend on the path γ in the same way, making this fraction equal to 1.

2 Jackson 5.21

Note that the terms of the form $\int \mathrm{d}^3x\,\vec{M}\cdot\vec{M}$ are constant and can be ignored.

A magnetostatic field is due entirely to a localized distribution of permanent magnetization.

(a) Show that

$$\int \vec{B}\cdot\vec{H}\,\mathrm{d}^3x = 0$$

provided the integral is taken over all space.

Because of the first condition, we can safely assume there are no free currents, so $\boldsymbol{\nabla}\times\vec{H} = 0$. Next, we expand the magnetic field in terms of the vector potential:

$$\begin{aligned}\int \vec{B}\cdot\vec{H}\,\mathrm{d}^3x &= \int (\boldsymbol{\nabla}\times\vec{A})\cdot\vec{H}\,\mathrm{d}^3x \\ &= \int \boldsymbol{\nabla}\cdot\left(\vec{A}\times\vec{H}\right)\mathrm{d}^3x + \int A\cdot\underbrace{(\boldsymbol{\nabla}\times\vec{H})}_{=0}\mathrm{d}^3x \\ &= \int_{S(\infty)} (\vec{A}\times\vec{H})\,\mathrm{d}\vec{a} = 0\end{aligned}$$

by divergence theorem. If the first integral was over all space, the surface is a surface at infinity, and since the magnetization is local, H must vanish at infinity.

(b) From the potential energy (5.72) of a dipole in an external field, show that for a continuous distribution of permanent magnetization the magnetostatic energy can be written

$$W = \frac{\mu_0}{2}\int \vec{H}\cdot\vec{H}\,\mathrm{d}^3x = -\frac{\mu_0}{2}\int \vec{M}\cdot\vec{H}\,\mathrm{d}^3x$$

apart from an additive constant, which is independent of the orientation or position of the various constituent magnetized bodies.

The energy from a single dipole in an external magnetic field is $W = -\vec{m}\cdot\vec{B}$. If one dipole is brought into the presence of another, we know the dipole moments will interact with the magnetic fields generated by the other dipoles, so for finite dipoles, $W = -\frac{1}{2}\sum_{i\neq j}\vec{m}_i\cdot\vec{B}_j$, where the $\frac{1}{2}$ avoids double counting the energy of dipole a in magnetic field b and dipole b in magnetic field a. For a continuous distribution of magnetization, this becomes an integral over infinitesimal magnetic moments:

$$W = -\frac{1}{2}\int \vec{B}\cdot \mathrm{d}\vec{m}$$

Integrating over these dipoles is the same as integrating the magnetization over space:

$$W = -\frac{1}{2}\int \vec{M}\cdot\vec{B}\,\mathrm{d}^3x$$

We can expand the magnetic field as $\vec{B} = \mu_0(\vec{M}+\vec{H})$:

$$W = -\frac{\mu_0}{2}\int \vec{M}\cdot\vec{M}\,\mathrm{d}^3x - \frac{\mu_0}{2}\int \vec{M}\cdot\vec{H}\,\mathrm{d}^3x$$

We ignore the first term since it is just an additive constant and does not depend on the distribution:

$$W = -\frac{\mu_0}{2}\int \vec{M}\cdot\vec{H}\,\mathrm{d}^3x$$

Expanding $\vec{M}$ as $\vec{M} = \frac{1}{\mu_0}\vec{B} - \vec{H}$ we find

$$W = -\frac{1}{2}\int \vec{B}\cdot\vec{H}\,\mathrm{d}^3x + \frac{\mu_0}{2}\int \vec{H}\cdot\vec{H}\,\mathrm{d}^3x$$

We showed in (a) that the first term is zero so

$$W = \frac{\mu_0}{2}\int \vec{H}\cdot\vec{H}\,\mathrm{d}^3x$$

as long as there are no free currents.

3 Jackson 5.23 (a) and (b) only

Two identical circular loops of radius a are initially located a distance R apart on a common axis perpendicular to their planes.

(a) From the expression $W_{12} = \int \mathrm{d}^3x\,\vec{J}_1\cdot\vec{A}_2$ and the result for A_ϕ from Problem 5.10b, show that the mutual inductance of the loops is

$$M_{12} = \mu_0\pi a^2\int_0^\infty \mathrm{d}k\,e^{-kR}J_1^2(ka)$$

We are given the potential of a current loop of radius a as

$$A_\phi(\rho,z) = \frac{\mu_0 I a}{2}\int_0^\infty \mathrm{d}k\,e^{-k|z|}J_1(ka)J_1(k\rho)$$

We also know that

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij}I_iI_j$$

so

$$W_{12} = M_{12}I_1I_2$$

or

$$M_{12} = \frac{W_{12}}{I_1 I_2}$$

Using our formula for W_{12}, we know that the current is proportional to I and is always in the radial direction, so

$$W_{12} = \int_0^{2\pi} a\,\mathrm{d}\phi\, I_1 A_\phi(a, R) = \mu_0 \pi a^2 \int_0^\infty \mathrm{d}k\, e^{-kR} J_1^2(ka)$$

(b) Show that for $R > 2a$, M_{12} has the expansion,

$$M_{12} = \frac{\mu_0 \pi a}{2}\left[\left(\frac{a}{R}\right)^3 - 3\left(\frac{a}{R}\right)^5 + \frac{75}{8}\left(\frac{a}{R}\right)^7 + \cdots\right]$$

We are basically just assuming a is small, so we can expand around ka in the integral. If we expand $J_1^2(ka)$ as a Taylor series about 0, we find that the first few terms are

$$J_1^2(ka) = \frac{(ka)^2}{4} - \frac{(ka)^4}{16} + \frac{5(ka)^6}{768} - \cdots$$

so the integral becomes

$$M_{12} = \mu_0 \pi a^2 \left[\int_0^\infty \mathrm{d}k\, e^{-kR}\left(\frac{a^2}{4}k^2 - \frac{a^4}{16}k^4 + \frac{5a^6}{768}k^6 - \cdots\right)\right]$$

Integrals from 0 to ∞ of exponentials multiplied by polynomials are well defined, and evaluating this expression gives

$$M_{12} = \mu_0 \pi a^2 \frac{1}{2}\left[\frac{a^2}{R^3} - \frac{3a^4}{R^5} + \frac{75}{8}\frac{a^6}{R^7} - \cdots\right]$$

Distributing an a gives the desired answer.

4 Jackson Section 5.18 Part B

Go through the details of Section 5.18 part B of Jackson and verify the answer given at equation (5.176). This is a self-study exercise, it is nice to work it out and see how the field gradually decreases in the sample.

We begin by defining the current density to be $J_y = H_0[\delta(z + a) - \delta(z - a)]$. Suppose $H_x(z, t) = \int_0^\infty e^{-pt}\overline{h}(p, z)\,\mathrm{d}p$. Plugging this into the diffusion equation, $\nabla^2 \vec{H} = \mu\sigma\partial_t \vec{H}$ gives us just the z-derivative, so we have

$$\int_0^\infty \partial_z^2 e^{-pt}\overline{h}(p, z)\,\mathrm{d}p + \int_0^\infty \underbrace{\mu\sigma p}_{k^2}\, e^{-pt}\overline{h}(p, z)\,\mathrm{d}p = 0$$

This means $\overline{h}$ satisfies the equation

$$\partial_z^2 \overline{h} + k^2 \overline{h} = 0$$

since $t = 0 \implies e^{-pt} = 1$ and the integrands follow the above equation.
Symmetry apparently suggests $\overline{h} \propto \cos(kz)$, so

$$H_x(z, t) = \int_0^\infty e^{-\frac{k^2}{\mu\sigma}t} h(k)\cos(kz)\,\mathrm{d}k$$

By the Laplace transform of the initial current distribution, we have

$$\int_0^\infty h(k)\cos(kz)\,\mathrm{d}k = H_0(\Theta(z+a) - \Theta(z-a))$$

We can split the cosine on the left side into two exponentials:

$$\int_0^\infty h(k)\cos(kz)\,\mathrm{d}k = \int_0^\infty h(k)\frac{1}{2}e^{-\imath kz}\,\mathrm{d}k + \int_0^\infty h(k)\frac{1}{2}e^{\imath kz}\,\mathrm{d}k = \int_{-\infty}^\infty h(k)e^{\imath kz}\,\mathrm{d}k$$

This only works assuming $h(k)$ is even about zero. It probably is, since Jackson says so and also because we defined k to be $k^2 = \mu\sigma p$ so changing the sign of k shouldn't mess with the constant determined by initial conditions.
Next, let's look at the right side. By definition, the Heaviside functions really look like

$$H_0(\Theta(z+a) - \Theta(z-a)) = H_0\left[\int_{-\infty}^{z+a}\delta(s)\,\mathrm{d}s - \int_{-\infty}^{z-a}\delta(s)\,\mathrm{d}s\right] = H_0\int_{-a}^{a}\delta(s)\,\mathrm{d}s$$

so we can invert the Fourier transform (don't forget the 2π factor) to get

$$h(k) = \frac{2H_0}{2\pi}\int_{-a}^{a} e^{-\imath kz}\,\mathrm{d}z = \frac{2H_0}{\pi k}\sin(ka)$$

Plugging this back into the original equation for H_x we get

$$H_x(z,t) = \int_0^\infty e^{-\frac{k^2}{\mu\sigma}t}\left(\frac{2H_0}{\pi k}\right)\sin(ka)\cos(ka)\,\mathrm{d}k = \frac{2H_0}{\pi}\int_0^\infty e^{-\nu t\kappa^2}\frac{\sin(\kappa)}{\kappa}\cos\!\left(\frac{z}{a}\kappa\right)\mathrm{d}\kappa$$

making the substitutions $\kappa = ka$, $\mathrm{d}\kappa = a\,\mathrm{d}k$, and $\nu = \frac{1}{\mu\sigma a^2}$.
Next, we are told to evaluate this integral using

$$\Phi(\xi) = \frac{2}{\pi}\int_0^\infty e^{-\frac{x^2}{4\xi^2}}\frac{\sin(x)}{x}\,\mathrm{d}x$$

If I split up the cosine as previously, it gives the nice factor of $\frac{1}{2}$ that is required to get this problem in the correct form, but unfortunately puts an interesting exponent into the problem. Now I have

$$e^{-\nu t\kappa^2 \pm \imath\frac{z}{a}\kappa}$$

inside the integral but working backwards from the answer, I can't see how this is supposed to be equal to

$$e^{-\frac{1}{2}\left(\nu t\left(1\pm\frac{z}{a}\right)^{-2}\right)\kappa^2}$$

It must be true because Jackson deems it so, but unfortunately I can't figure out why.