33-755 Homework 8

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November 1, 2019

5. Particle Subject to a Constant Force

In a one-dimensional problem, consider a particle of potential energy $V(X) = -fX$, where f is a positive constant.

a. Write Ehrenfest's theorem for the mean values of the position X and the momentum P of the particle. Integrate these equations; compare with the classical motion.

Ehrenfest's theorem states that

$$\partial_t \langle X \rangle = \frac{1}{m} \langle P \rangle$$

and

$$\partial_t \langle P \rangle = \frac{1}{\imath\hbar} \langle [P, H] \rangle$$

In the second equation, $[P, H] = [P, V(X)] = -\imath\hbar V'(X) = \imath\hbar f$ so by Ehrenfest's theorem,

$$\langle P \rangle = \int f \, \mathrm{d}t = f\Delta t$$

Similarly,

$$\langle X \rangle = \frac{1}{m} \int \langle P \rangle \, \mathrm{d}t = \frac{1}{2m} f\Delta t^2$$

Classically, if $f = ma$, $f = \partial_t p$, and $x = \frac{1}{2}at^2$, assuming constant acceleration.

b. Show that the root-mean-square deviation ΔP does not vary over time.

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2}$$

Since $\left[P^2, H\right] = \left[P^2, V(X)\right] = \hbar^2 V''(X)$,

$$\partial_t \left\langle P^2 \right\rangle = \imath\hbar \left\langle V''(X) \right\rangle = 0$$

since $V(X) = fX$ so $V''(X) = 0$. We know that $\partial_t \langle P \rangle^2 = f^2$ from the previous section, and this is independent of t, so $\partial_t \Delta P = 0$.

c. Write the Schrödinger equation in the $\{|p\rangle\}$ representation. Deduce from it a relation between $\partial_t |\langle p|\psi(t)\rangle|^2$ and $\partial_p |\langle p|\psi(t)\rangle|^2$. Integrate the equation thus obtained; give a physical interpretation.

If we write

$$\imath\hbar\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

in the $\{|p\rangle\}$ representation, we get

$$\begin{aligned}\imath\hbar\partial_t\psi(p,t) &= \int \mathrm{d}p \, \frac{P^2}{2m} |p\rangle \langle p|\psi\rangle + fX |p\rangle \langle p|\psi\rangle \\ &= \frac{p^2}{2m}\psi(p,t) - \imath\hbar f \partial_p \psi(p,t)\end{aligned}$$

9.7.6 Reflection Delay

(a) Consider the reflection coefficient B for wave $e^{\imath kx}$ incident at a potential step of height V_0, where $E = \hbar^2 k^2/2m < V_0$. Show that $|B| = 1$ so we can write $B = e^{-\imath\phi}$ with ϕ a real quantity which you must determine.

If we normalize the incident wave to 1, the probability flux must be conserved, so $\frac{\hbar k}{m} = |B^2| \frac{\hbar k}{m}$, which means $|B^2| = 1 = |B|$. Next, we know that B is the reflection coefficient, and for this scenario it is equal to $B = -\frac{\kappa + \imath k}{\kappa - \imath k}$, with $\kappa^2 = \frac{2m(E-V_0)}{\hbar^2}$. Suppose $B = e^{-\imath\phi}$. We now have to find $\phi = \imath \ln \frac{\kappa - \imath k}{\kappa + \imath k}$. Note that

$$\arctan(z) = \frac{\imath}{2} \ln\left(\frac{\imath + z}{\imath - z}\right)$$

If $z = \frac{k}{\kappa}$, we find that $\phi = 2\arctan\left(\frac{k}{\kappa}\right)$ satisfies the equation for the proper reflection coefficient.

(b) Given the incident wave packet

$$\phi(x,t) = \int \mathrm{d}k \, \frac{A(k)}{\sqrt{2\pi}} e^{\imath(kx - \omega(k)t)}$$

determine the reflected wave packet and show that the reflection occurs with a delay $\tau = -\hbar\frac{\mathrm{d}\phi}{\mathrm{d}E} > 0$. Interpret your result in terms of motion in the classically forbidden region.

Noting the addition of the phase factor from part (a), we can write the reflected wave packet as

$$\phi_r(x,t) = \int \mathrm{d}k \, \frac{A(k)}{\sqrt{2\pi}} e^{\imath(-kx + \omega(k)t - \phi)}$$

If we set the phase to be a constant, in the incident wave packet, we find that the phase velocity comes from $kx - \omega t = 0 \implies x = \frac{\omega}{k}t$ where $v_p = \frac{\omega}{k}$ is the phase velocity. Equivalently, if the wave packet was Gaussian with a peak at $\overline{k}$, the group velocity would be $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}\big|_{\overline{k}}$. However, with the added phase in the reflected wave packet, the constant phase calculation gives $kx = \omega t - \phi$, or $x = \frac{\omega}{k}t - \frac{\phi}{k}$. If we look at the group velocity, we find that the peaks of the wave are delayed by a factor of $-\frac{\mathrm{d}\phi}{\mathrm{d}k} = \hbar\frac{\mathrm{d}\phi}{\mathrm{d}E}$. This occurs because the wave function can enter the forbidden region, and while the distance it goes depends on the energy, if we average over the wave packet, the packet enters some set distance into the classically forbidden region. The time it spends there is the delay τ which was found in this problem.