LECTURE 15:
Wed Sep 25 2019

0.1 Finding Potentials for Continuous Charge Densities

Suppose we have a charge density $\rho(\vec{x}')$ and a ball about $\vec{0}$. We know that $\int_{\text{ball}} \vec{E}(\vec{x}) d^3x = -\oint_{\text{sphere}} \Phi d\vec{a}$.

We could also imagine that the charge density is inside the sphere.

$$
\begin{aligned}
-\oint \Phi d\vec{a} &= -\frac{1}{4\pi\epsilon_0} \oint \int \frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} R^2 d\Omega \hat{x} \\
&= -\frac{1}{4\pi\epsilon_0} \int d^3x' \rho(\vec{x}') R^2 \int d\Omega \hat{x} \sum_{l=0}^{\infty} \left(\frac{r_<^l}{r_>^{l+1}} \right) P_l(\cos\gamma) \\
&= -\frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') d^3x' R^2 \sum_{l=0}^{\infty} \left(\frac{r_<^l}{r_>^{l+1}} \right) \int d\Omega \hat{x} P_l(\cos\gamma)
\end{aligned}
$$

We can perform this final integral. If we rotate so that our $\vec{x}$ is the new z-axis, we can see that, due to the orthogonal condition on P_l, the only nonzero term is $\hat{x} \to (\hat{x}')P_1(\cos\gamma)$, so the final answer is

$$
\int_{\text{ball}} \vec{E} d^3x = -\frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') d^3x' R^2 \hat{x}' \frac{4\pi}{3} \frac{r_<}{r_>^2}
$$

where $r_< = \min(|\hat{x}'|, R)$ and $r_> = \max(|\hat{x}'|, R)$. So

$$
\int_{\text{ball}} \vec{E} d^3x = \begin{cases} \frac{4\pi}{3} R^3 \int \frac{\rho(\vec{x}')[-\hat{x}']d^3x'}{4\pi\epsilon_0 |\vec{x}'|^2} = \frac{4\pi}{3} R^3 \vec{E}(0) & \text{charge outside sphere} \\ -\int \frac{\rho(\vec{x}')\vec{x}' d^3x'}{3\epsilon_0} = -\frac{\vec{p}}{3\epsilon_0} & \text{charge inside sphere} \end{cases}
$$

0.1.1 Ideal Point Dipoles

What is the immediate application? What is the ideal point dipole? Naïvely, we would think

$$
\Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{x}}{r^2}
$$

$$
\vec{E} = -\nabla \Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{x})\hat{x}x\vec{p}}{|\vec{x}|^3}
$$

but this implies that the average electric field in a small ball around the dipole is zero, which contradicts our previous result!We fix this by adding the term by hand:

$$
\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{x})\hat{x} - \vec{p}}{|\vec{x}|^3} - \frac{\vec{p}}{3\epsilon_0} \delta(\vec{x})
$$

0.1.2 Energy Calculations

Problem: Calculate energy for a charge distribution immersed into the field of an external charge distribution. We assume our distribution $\rho(\vec{x})$ is centered somewhere and we have some external charges generating some fields far away.

$$W = \int \rho(x)\Phi_{\text{ext}}(x)d^3x$$

Suppose the length scale of our distribution is L. If $\frac{|\nabla\Phi_{\text{ext}}|}{L} \ll 1$,

$$W = \int \rho(x)[\Phi_{\text{ext}}(0) + x^i\partial_{x^i}\Phi_{\text{ext}}\Big|_0 + \frac{1}{2}x^ix^j\partial_{x^i}\partial x^j\Phi_{\text{ext}}\Big|_0 + \dots]d^3x$$

so

$$W = \left(\int \rho(x)d^3x\right)\Phi_{\text{ext}}(0) - \left[\int \rho(x)x^2d^3x\right]\overbrace{[-\partial_{x^i}\Phi_{\text{ext}}]}^{\vec{E}_{\text{ext}}(0)} + (-)\int \frac{d^3x\rho(x)}{3}\left[\frac{3}{2}x_ix_j - \frac{1}{2}r^2\partial_{ij}\right]\partial_{x^j}E_i\Big|_0$$

so

$$W \approx Q\Phi_{\text{ext}}(0) - \vec{p}\cdot\vec{E}_{\text{ext}}(0) - \frac{1}{3\cdot 2}Q_{ij}\partial_{[i}E^{\text{ext}}_{j]}(0) + \dots$$

0.1.3 Dipole-Dipole Interactions

Suppose we have two dipoles now, with $\hat{n}_{21}$ is the vector pointing from the first to the second.

$$W = -p_1\left[\frac{3(p_2\cdot\hat{n}_{12})\hat{n}_{12} - \vec{p_2}}{|\vec{x}_1 - \vec{x}_2|}\right] = \frac{\vec{p_1}\cdot\vec{p_2} - 3(\vec{p_2}\cdot\hat{n}_{12})(\vec{p_1}\cdot\hat{n}_{12})}{|\vec{x}_1 - \vec{x}_2|^3}$$

There is, of course, the dipole correction term, but it has a delta function in it. Our dipoles never overlap, so this term drops out.