1 Decomposing Tensor Products into Irreps

$$j \otimes j' = |j + j'| \oplus \cdots \oplus |j - j'|$$

For example,

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

When we write $j \otimes j'$, we really mean $|j, m\rangle\, |j', m'\rangle$, and these transform under rotations using the Wigner matrices:

$$|j, m\rangle\, |j', m'\rangle \to D^{j}_{mm'} D^{j'}_{m'', m'''}\, |j, m'\rangle\, |j', m''\rangle$$

We can think of this like a two-index object:

$$A_{ij} = R_{ia} R_{jb} A_{ab}$$

or in general,

$$A_{i_1 \ldots i_n} = R_{i_1 a_1} \cdots R_{i_n a_n} A_{a_1 \ldots a_n}$$

We can decompose any tensor into a symmetric and antisymmetric part:

$$A_{ij} = A^{S}_{ij} + A^{A}_{ij}$$

Additionally, the trace is preserved under rotations, so

$$A^{S}_{ij} = \left[A^{S}_{ij} - \frac{1}{3} \delta_{ij} \operatorname{Tr}(A) \right] + \frac{1}{3} \delta_{ij} \operatorname{Tr}(A)$$

Using these symmetries, we have broken up the original into the following pieces:

$$A_{ij} = A^{(STF)}_{ij} + A^{(A)}_{ij} + \frac{1}{3} \delta_{ij} \operatorname{Tr}(A)$$

where the first part is symmetric trace free, so by the number of degrees of freedom, we can call it an $l = 2$ irrep. The second part is $l = 1$ and the third part is $l = 0$ for the same reason.