Lecture 11: Spherical Symmetry, Continued

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We will restrict $a \le r \le b$. To form the Green's Function, we put an imaginary point charge somewhere with the normalization condition of $-4\pi \to (-4\pi)\delta(\vec{x}-\vec{x}')$:

$\nabla'^2 G = -4\pi\delta(\vec{x}-\vec{x}')$ which is equivalent to $\nabla^2 G$ in this case, since the Green's function is symmetric. Also, the Green's function must vanish on the boundaries.

$$\nabla^2 = \frac{1}{r}\partial_r^2 r - \frac{\mathbb{L}^2}{r^2} \tag{1}$$

Additionally, we can use

$$\sum_l \sum_{-l\le m\le l} Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \tag{2}$$

. We can use this to write

$$\delta(\vec{x}-\vec{x}') = \delta(\phi-\phi)\delta(\cos\theta-\cos\theta')\frac{\delta(r-r')}{r^2} \tag{3}$$

Let's suppose

$$G = \sum_{l,m} g_l(r,r')Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \tag{4}$$

so

$$\sum_{l,m}\left[\frac{1}{r}\frac{d^2}{dr^2}r\frac{l(l+1)}{r^2}\right]g_l(r,r')Y_{lm}Y^*_{lm} = (-4\pi)\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')\frac{\delta(r-r')}{r^2} \tag{5}$$

comes from acting the Laplacian on G.

For $a \le r < r' < b$ or $a < r' < r \le b$, we have $\frac{1}{r}\frac{d^2}{dr^2}rg_l - \frac{l(l+1)}{r^2}g_l = 0$

Suppose $g_l = A_l r^l + B_l r^{-(l+1)}$

If $r = a$,

$$A_l a^l + B_l a^{-(l+1)} = 0 \Rightarrow B_l = -A_l a^{2l+1} \tag{6}$$

so for $r < r'$,

$$g_l = A_l\left[r^l - \frac{a^{2l+1}}{r^{l+1}}\right] = y^{(1)} \tag{7}$$

On the other boundary, $r = b$, we get that for $r' < r$,

$$g_l = E_l\left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right] = y^{(2)} \tag{8}$$

Because of the symmetric nature of the Green's function, our complete solution must be formed from these two solutions.

$$g_l(r,r') = C_l\left[r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right]\left[\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right] \tag{9}$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$. Apparently this is related to the product space.

What happens when $r = r'$?

$$\int_{r'-\varepsilon}^{r'+\varepsilon} \frac{1}{r}\frac{d^2}{dr^2} r g_l - \frac{l(l+1)}{r^2} g_l = \int_{r'-\varepsilon}^{r'+\varepsilon} -4\pi\frac{\delta(r-r')}{r^2} = -4\pi\frac{1}{r'} \tag{10}$$

On the right side, we assume $\frac{g_l}{r^2} \to 0$ so we are left with

$$\frac{d}{dr}(rg_l)\bigg|_{r'-\varepsilon}^{r'+\varepsilon} = -\frac{4\pi}{r'}\frac{d}{dr}[rg_l]\bigg|_{r'+\varepsilon>r'} - \frac{d}{dr}[rg_l]\bigg|_{r'-\varepsilon<r'} \tag{11}$$

so we are taking

$$C_l\frac{d}{dr}\left(r\left[r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right]\left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right]\right)_{r\to r'} \tag{12}$$

and similar for the case where $r' > r$. Taking the derivatives and limits will tell us what C_l must be.

$$C_l = \frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \tag{13}$$

Finally, we can write our general spherical Green's function:

$$G(r,\theta,\phi,r',\theta',\phi') = \sum_{l,m}\frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right]\left[\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right]Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi') \tag{14}$$

1. As $a \to 0$ and $b \to \infty$, we get back the original $G = \frac{1}{|\vec{x}-\vec{x}'|}$
2. As $a \neq 0$ and $b \to \infty$, $G = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a/x'}{|\vec{x}-\frac{a^2}{x'^2}\vec{x}'|}$ from our method of images (this will not look the same if you just write out these limits, but it can be found through some careful algebra).
3. As $a = 0$ and b is finite and say $\rho(x') = 0$, we have $G = \sum \frac{4\pi}{2l+1}[r_<^l]\left[\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right]Y_{lm}Y_{lm}^*$. As we approach the boundary, $r' > r$

$$\partial_{r'}G\bigg|_{r'\to b} = \sum\frac{4\pi}{2l+1}r^l\left[-\frac{(l+1)}{r'^{l+2}} - l\frac{r'^{l-1}}{b^{2l+1}}\right]Y_{lm}Y_{lm}^*\bigg|_{r'\to b} \tag{15}$$

. There's some more to do here but we basically get

$$\Phi(\vec{x}) = \sum \frac{r^l}{b^{l+2}} Y_{lm}(\theta, phi) \int Y_{lm}^*(\theta', \phi') V(\theta', \phi') b^2 d\Omega'. \tag{16}$$