Lecture 38: The Kramers-Krönig Relations

Monday, November 11, 2019

Recall from last lecture that, since the real part of $\epsilon(\omega)$ is an even function and the imaginary part is odd, we find the Kramers-Krönig relations:

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}] - 1 = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Im}[\frac{\epsilon(\omega')}{\epsilon_0}]}{\omega' - \omega}\,\mathrm{d}\omega'$$

$$\mathrm{Im}[\frac{\epsilon(\omega)}{\epsilon_0}] = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Re}[\frac{\epsilon(\omega')}{\epsilon_0} - 1]}{\omega' - \omega}\,\mathrm{d}\omega'$$

By splitting this into separate integrals at 0, we can show that these are equivalent to

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}] - 1 = \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{\omega'\,\mathrm{Im}[\epsilon(\omega')]}{\omega'^2 - \omega^2}\,\mathrm{d}\omega'$$

and

$$\mathrm{Im}[\frac{\epsilon(\omega)}{\epsilon_0}] = -\frac{2\omega}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{\mathrm{Re}[\frac{\epsilon(\omega')}{\epsilon_0} - 1]}{\omega'^2 - \omega^2}\,\mathrm{d}\omega'$$

0.0.1 Region of Transparency

If $\mathrm{Im}[\epsilon(\omega)] \approx 0$ over a range $[\omega_1, \omega_2]$, such that $n(\omega) \sim \sqrt{\epsilon_R(\omega)}$ and $n_I(\omega) \approx 0$ in the region of transparency. What does this imply? In the Kramers-Krönig relations, we know that

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0} - 1] \simeq \frac{2}{\pi\epsilon_0}\mathcal{P}\int_{0}^{\omega_1} \frac{\omega'\,\mathrm{Im}[\epsilon(\omega')]}{\omega'^2 - \omega^2}\,\mathrm{d}\omega' \frac{2}{\pi\epsilon_s n}\mathcal{P}\int_{\omega_2}^{\mathrm{inf}} \frac{\omega'\,\mathrm{Im}[\epsilon(\omega')]}{\omega'^2 - \omega^2}\,\mathrm{d}\omega'$$

These are convergent integrals, and we therefore don't actually need the principle values because ω' never comes near ω. This allows us to take the derivatives of these expressions.

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0} - 1] = \frac{2}{\pi\epsilon_0}\int_{0}^{\omega_1} \frac{\omega\omega'\,\mathrm{Im}[\epsilon(\omega')]}{(\omega'^2 - \omega^2)^2}\,\mathrm{d}\omega' + \frac{2}{\pi\epsilon_0}\int_{\omega_2}^{\infty} \frac{\omega\omega'\,\mathrm{Im}[\epsilon(\omega')]}{(\omega'^2 - \omega^2)^2}\,\mathrm{d}\omega' > 0$$

We know that $n^2(\omega) \simeq \mathrm{Re}[\epsilon(\omega)]$ so

$$2n(\omega)\frac{\mathrm{d}}{\mathrm{d}\omega}n \simeq \frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Re}[\epsilon(\omega)] > 0$$

so

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} > 0$$

Therefore, the sky is blue because of causality (neat).

0.1 Transmission of Waves and Propagation in an Arbitrary Region of Frequency

Suppose we have an $x = 0$ boundary and a material to the right with $n(\omega)$ and vacuum to the left. We send a signal to the left, which hits the boundary at $t = 0$. We want to describe what happens after this.

$$u(x,t) = \int_{-\infty}^{\infty} \left[A(\omega)e^{\imath kx - \imath\omega t} + B(\omega)e^{-\imath kx - \imath\omega t}\right] \frac{\mathrm{d}\omega}{2\pi}$$

in the $x \leq 0$ region and

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega)e^{\imath k(\omega)x - \imath\omega t} \frac{\mathrm{d}\omega}{2\pi}$$

inside the material. These functions are real, and we can use time reversal symmetry (taking the complex conjugate) to find a relation between A and B:

$$A^*(-\omega) = B(\omega)$$
$$B^*(-\omega) = A(\omega)$$

Suppose we know what the incoming wave looks like, so we therefore know what $u(0,t)$ and $\left.\frac{\partial u}{\partial x}\right|_{x=0}(t)$, so

$$\left\{\begin{Bmatrix} A(\omega) \\ B(\omega) \end{Bmatrix}\right\} = \frac{1}{2}\int_{-\infty}^{\infty} \left[u(0,t) \pm \frac{c}{\imath\omega}\left.\frac{\partial u}{\partial x}\right|_{x=0}\right] e^{\imath\omega t}$$

The wave function and its time derivative must be continuous across the boundary, so

$$F(\omega) = \frac{2}{1+n(\omega)}A(\omega)$$

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e won't prove the following, but it turns out that $|n(\omega)| \to 1$ as $|\omega| \to \infty$ in the upper-half-plane. Also, $\epsilon_R(\omega)$ is never negative or zero if we assume $\epsilon_I(\omega) \geq 0$.

This implies the following interesting thing. $n^2(\omega) = \epsilon(\omega)\mu_0$, but if ϵ cuts the negative axis somewhere, the square root will not be uniquely defined (the square root has a branch cut along the negative real line). Therefore, if ϵ is always well-defined and never cuts this region, $n(\omega)$ becomes an analytic function when ω is in the upper-half-plane. From this, we know that $F(\omega)$ is analytic since $A(\omega)$ is analytic in the upper-half-plane because $u(x,t)$ is real. Therefore, the integral which defines $u(x,t)$ in the material can be written as a contour integral which evaluates to 0 minus the half-circle at infinity, so

$$\int_{-\infty}^{\infty} F(\omega) \mapsto \oint \frac{2}{1+n(\omega)}A(\omega)e^{\imath\left(\frac{\omega x}{c} - \omega t\right)\left(\frac{x}{c} - t\right)} > 0$$

so $x \leq ct$. Even though we can't use the group velocity here, we still see that the speed of propagation doesn't exceed c.