Lecture 26: The Feynman Path Integral

Wednesday, October 16, 2019

0.1 Feynman Path Integrals

Recall the double slit experiment. If we add up the possible paths the particle could take to get to a particular point, labeling the distance from the source to the slits as L_1 and L_2 respectively, priming the lengths after the slits, we find that at a point x on the screen, $L_j + L_j'(x) \implies A_j(x) = e^{\imath(k(L_j+L_j'(x)))}$ and $A(x) = \sum_j A_j(x)$. For a massive particle, we suppose we have a unitary time operator $U(t)$:

$$U(t)\colon\ |\psi(t=0)\rangle \to |\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

$$U(x,t;x_0) \equiv \langle x|\,U(t)\,|x_0\rangle$$

We define the "propagator" as:

$$\psi(x,t) = \int \mathrm{d}x_0\, U(x,t;x_0)\psi_0(x)$$

since

$$\langle x| \left(|\psi(t)\rangle = \int \mathrm{d}x_0\, U(t)\,|x_0\rangle\langle x_0|\,|\psi(t=0)\rangle \right)$$

Let

$$H = \frac{P^2}{2m} + V(x) \implies U(t) = e^{-\imath Ht/\hbar}$$

Let us separate the time axis into N discrete portions ($\epsilon = t/N$). Doing this, we can write the unitary time operator in the following form:

$$U(t) = \overbrace{e^{-\imath H\epsilon/\hbar} e^{-\imath\epsilon/\hbar} \cdots e^{-\imath H\epsilon/\hbar}}^{N}$$

Next, we insert the identity:

$$U(t) = \underbrace{e^{-\imath H\epsilon/\hbar} e^{-\imath\epsilon/\hbar}}_{I_{N-1}} = \overbrace{\int \mathrm{d}x_{N-1}\,|x_{N-1}\rangle\langle x_{N-1}| \cdots \underbrace{e^{-\imath H\epsilon/\hbar} e^{-\imath H\epsilon/\hbar}}_{I_1 = \int_{-\infty}^{\infty} \mathrm{d}x_1 |x_1\rangle\langle x_1|}}^{N}$$

This is similar to a unitary history. We start at x_0. At time $t=1$ we could be anywhere in space, so we evolve unitarily in time from time 0 to 1, projecting into a generic state $|x_1\rangle$. From here, we find

$$\langle x|\,U(t)\,|x_0\rangle = \int \prod_{j=1}^{N-1} \mathrm{d}x_j\, \langle x_j|\,U_\epsilon\,|x_{j-1}\rangle$$

is the propagator. For each term in this product, what is

$$\langle x_j|\, e^{-\imath\left(\frac{P^2}{2m}+V\right)\epsilon/\hbar}\,|x_{j-1}\rangle?$$

Operator Exponentials

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

We can make use of this identity, supposing that, with a potential V and kinetic term K,

$$e^{V+K} = e^V e^K e^{-\frac{1}{2}[V,K]}$$

To order $\mathcal{O}(\epsilon^1)$ (we will later take $\epsilon \to 0$ we can ignore the last term:

$$\langle x_j| \, e^{-\imath H\epsilon/\hbar} \, |x_{j-1}\rangle = e^{-\frac{\imath}{\hbar}V(x_j)} \, \langle x_j| \, e^{-\frac{\imath}{\hbar}\frac{P^2}{2m}\epsilon} \, |x_{j-1}\rangle$$

We insert the identity (using now the momentum basis):

$$e^{-\frac{\imath}{\hbar}V(x_j)} \, \langle x_j| \left(I = \int \mathrm{d}p \, |p\rangle\!\langle p| \right) e^{-\frac{\imath}{\hbar}\frac{P^2}{2m}\epsilon} \, |x_{j-1}\rangle = e^{-\frac{\imath}{\hbar}V(x_j)} \int \mathrm{d}p \, \frac{1}{2\pi\hbar} e^{\frac{\imath}{\hbar}\left[p(x_j - x_{j-1}) - \frac{p^2\epsilon}{2m}\right]}$$

Now we need to evaluate this integral. It looks like a Gaussian integral, and we can evaluate it by completing the square. Recall that if (in the exponent) we have the following form: $ap^2 + bp = (\sqrt{a}p + \sqrt{c})^2 - c$ where $c = b^2/4a$. Let $a = \epsilon/2m$ and $b = (x_j - x_{j-1})$

$$\implies e^{\imath m(x_j - x_{j-1})^2/(2\epsilon\hbar)} \int \mathrm{d}p \, e^{(\sqrt{a}p + \sqrt{c})^2}$$

where the integral evaluates to

$$\sqrt{\frac{m}{2\pi\imath\hbar\epsilon}}$$

Finally, we can write out our short time propagator as

$$U(x_j, \epsilon, x_{j-1}) = \sqrt{\frac{m}{2\pi\imath\hbar\epsilon}} e^{\frac{\imath}{\hbar}L_j\epsilon}$$

where

$$L_j = \frac{1}{2}m\dot{x}_j^2 - V(x_j)$$

where

$$\dot{x}_j \equiv \frac{x_j - x_{j-1}}{\epsilon}$$

which we will call the velocity, so we see that L_j is the Lagrangian.

Now let's get rid of the ϵ terms.

$$U(x, t; x_0) = \int \mathscr{D}[x(t)] e^{\frac{\imath}{\hbar}\underbrace{\int_0^t \mathrm{d}t' \, L(x(t'), \dot{x}(t'))}_{\text{action } S[x(t)]}}$$

and

$$\sqrt{\frac{m}{2\pi\imath\hbar\epsilon}} \to \sqrt{\frac{m}{2\pi\imath\hbar t}}$$

where we are integrating over all possible positions at all possible times (which we've done) in a continuum of both (which we haven't).

Example. Free Particle ($v = 0$): Around the path of least action, the complex exponentials will not cancel against each other, and the dominant feature will result from this path. The path in general is $x(t)\colon x(t=0) = x_0$, $x(t_{\text{final}}) = x$, $x = x_0 + vt$ where $v = \frac{x - x_0}{t}$.

$$U(x, t; x_0) = \sqrt{\cdots} e^{\frac{\imath}{\hbar}m(x - x_0)^2/(2t)}$$

Using this, we can work out the wave function at any position by

$$\psi(x, t) = \int \mathrm{d}x_0 \, U(x, t; x_0)\psi_0(x)U(x, t; x_0)$$

As time goes to 0, the propagator becomes the δ function at x_0, and as time goes toward ∞, the propagator spreads like a Gaussian. ◇