Lecture 2: Electrostatics

September 19, 2019

0.1 The Electric Field

Start with the first equation:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

Assume there is no $\vec{B}$ field, or at least $\vec{B}$ is not changing in time (electrostatics). Also, $\vec{E}$ won't change with time, and there will be no currents, so the only equations left are the first one and the equation corresponding to the magnetic source ($\nabla \cdot \vec{B} = 0$), as well as $\nabla \times \vec{E} = 0$.

0.1.1 Integral Form

For a stationary surface Σ with charges inside, the divergence equation says that:

$$\int_V (\nabla \cdot \vec{E}) dv = \frac{Q_{\text{enclosed}}}{\varepsilon_0} = \oint_\Sigma \vec{E} \cdot d\vec{a} \tag{2}$$

Consider a stationary point charge q. Take a spherical shell around the charge (S^2 sphere) of radius r with outward normal vector $\hat{r}$. By symmetry, $\vec{E} = E(r)\hat{r}$ since the curl is zero. Gauss's law tells us

$$\oint \vec{E} \cdot d\vec{a} = E(r) \oint_{S^2} \hat{r} \cdot d\vec{a} = E(r) 4\pi r^2 = \frac{q}{\varepsilon_0}. \tag{3}$$

Therefore, $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ when q is at the center of the sphere.

$$\vec{E}(\vec{x}) = \frac{q}{4\pi\varepsilon_0 |\vec{x} - \vec{x}'|^2} \cdot \left[\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}\right] = \frac{q}{4\pi\varepsilon_0 |\vec{x} - \vec{x}'|^3} \cdot [\vec{x} - \vec{x}'] \tag{4}$$

for an arbitrary position. Here, $\vec{x}'$ is the vector pointing from the origin to the charge and $\vec{x}$ is the vector pointing to the position of observation.

In general:

$$\vec{E} = \int_\Omega \frac{\rho(\vec{x}')dv'}{4\pi\varepsilon_0 |\vec{x} - \vec{x}'|^3} \cdot [\vec{x} - \vec{x}'] \tag{5}$$

for some charge distribution in a volume Ω.

Let's look at $-\nabla \left[\frac{1}{|\vec{x} - \vec{x}'|}\right]$. If you were to expand out the denominator and take the gradient, you would get

$$-\frac{1}{2} \frac{1}{\text{something}^{3/2}} 2(x_i - x_i')\hat{e}_i \tag{6}$$

so

$$-\nabla \left[\frac{1}{|\vec{x} - \vec{x}'|}\right] = \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}. \tag{7}$$

Therefore, we can use

$$\vec{E}(\vec{x}) = -\nabla\left[\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')dv'}{|\vec{x}-\vec{x}'|}\right], \tag{8}$$

where the piece in the brackets is a scalar, called the scalar potential (only when there are no boundaries around). If $\vec{E} = -\nabla\Phi$ then $\nabla\times\vec{E} = \vec{0}$. Boundaries would mean some materials exist in the problem, so the properties of these materials will complicate the problem.

Remark.

$$\nabla\cdot\frac{\vec{r}}{r^3} \Rightarrow \partial_i\frac{x_i}{r^3} = \frac{3}{r^3} - 3\frac{x_i}{r^4}\frac{x_i}{r} = \frac{3}{r^3} - \frac{3r^2}{r^5} = 0. \tag{9}$$

However, this is not exactly true. It is true as long as $r \neq 0$. However, if it is, these derivatives are not justified.

$$\int_{S^3}\frac{\vec{r}}{r^3}dv = \oint_{S^2}\frac{\vec{r}}{r^3}\hat{r}d\vec{a} = 4\pi. \tag{10}$$

Therefore,

$$\nabla\cdot\frac{\vec{r}}{r^3} = 4\pi\delta(\vec{r}). \tag{11}$$

This is very useful, as we can show that,

$$\nabla\frac{q(\vec{x}-\vec{x}')}{4\pi\varepsilon_0|\vec{x}-\vec{x}'|^3} = \frac{1}{\varepsilon_0}q\delta(\vec{x}-\vec{x}'). \tag{12}$$

Say we have a charge in an electric field moving from point A to point B. The change in kinetic energy is the integral of the work done, or

$$\Delta(\text{KE}) = -\int_A^B q\vec{E}\cdot d\vec{r} \Rightarrow \frac{1}{2}mv_B^2 + q\Phi(\vec{x}_B) = \frac{1}{2}mv_A^2 + q\Phi(\vec{x}_A) \tag{13}$$

0.1.2 Ideal Conductors

They are "ideal" meaning they have a sufficient number of charges such that in static equilibrium, $\vec{E} = \vec{0}$ inside an ideal conductor. This automatically means $\rho = 0$ inside— there is only surface charge σ on an ideal conductor. The electric field is zero inside, and $\vec{E} \parallel \vec{n}$ outside (perpendicular to the surface). $\vec{E} = -\nabla\Phi$ is perpendicular to Φ-constant surfaces. This implies Φ is constant on the surface of the conductor. $\vec{E} = 0$ on the inside implies conductors are equipotential regions.

On the surface, to calculate anything, we take a small Gaussian pillbox with a thickness $\delta \to 0$ across the boundary. The electric field will therefore be outwardly perpendicular to the surface (away from the conductor): $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}_+$.

Let us formulate a problem in an electrostatic system in the presence of a conductor. Either you put charges inside conductors or you put the conductors at certain potentials. Pretend we can keep the conductors at a certain constant potential with an "idealized" cable connected to a battery. We could also put charges on them, such that the total charge on a conductor is, say, Q. Maybe we'd have some ρ outside and ask what the potential is at a given point in space. Every vector field can be decomposed into a pure curl and pure gradient part. If we knew the surface charge distributions on all the conductors, we could write down the solution easily:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} + \sum_i \oint_{\Sigma_i} \frac{\sigma_i(\vec{x}')da'}{4\pi\varepsilon_0|\vec{x}-\vec{x}'|}. \tag{14}$$

However, we don't know the σ_is. we can write down some equation $\varepsilon_0[-\nabla\Phi\hat{n}] = \sigma$, or

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} + \sum_i \oint_{\Sigma_i} \frac{[-\nabla\Phi](\vec{x}')da'}{4\pi|\vec{x}-\vec{x}'|}. \tag{15}$$

This is not the most practical way to solve the problem. Typically, you turn this "integral" equation into a "differential" equation:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \nabla \cdot [-\nabla\Phi] = \nabla^2\Phi \tag{16}$$

Either Φ is given on the boundaries (Dirichlet Problem for the Poisson Equation), or $\partial_t\Phi$ is given (Neumann Problem).