Lecture 48: Radiation Review, Continued

Monday, December 02, 2019

Putting everything from the last few lectures together, we can write the frequency decomposed elements of the fields in the radiation zone ($kr >> 1$) as

$$\vec{\mathbf{H}}_\omega \mapsto (-\imath)^{l+1}\frac{e^{\imath kr}}{kr}\sum_{l,m}\left[a_E(l,m)\vec{\mathbb{X}}_{lm} + a_M(l,m)\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{lm}\right]$$

and

$$\vec{\mathbf{E}}_\omega \mapsto Z_0\vec{\mathbf{H}}_\omega \times \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}} = \frac{\vec{\mathbf{x}}}{r}$.

If we now have this expansion in the radiation zone, how do we find the radiated power in some solid angle far away?

$$\frac{\mathrm{d}P_\omega}{\mathrm{d}\Omega} = r^2\hat{\mathbf{n}}\frac{1}{2}\left(\vec{\mathbf{E}}_\omega \times \vec{\mathbf{H}}^*_\omega\right)$$

where it is implied that we are taking the real part of this expression (which often turns out to be real anyway).

Therefore,

$$\begin{aligned}\frac{\mathrm{d}P_\omega}{\mathrm{d}\Omega} = \frac{Z_0}{k^2}\frac{1}{2}\hat{n}\cdot\sum_{l,m,l',m'}&\left[a_E(l,m)\vec{\mathbb{X}}_{lm} + a_M(l,m)\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{lm}\right]\\ &\times\left(\left[a^*_E(l',m')\vec{\mathbb{X}}^*_{l'm'} + a^*_M(l',m')\hat{\mathbf{n}}\times\vec{\mathbb{X}}^*_{l'm'}\right]\times\hat{\mathbf{n}}\right)\end{aligned}$$

This is not exactly the most appealing form for this equation. We can rewrite

$$\hat{\mathbf{n}}\cdot\left[\left(\vec{\mathbf{H}}_\omega\times\hat{\mathbf{n}}\right)\times\vec{\mathbf{H}}^*_\omega\right] = \vec{\mathbf{H}}_\omega\cdot\vec{\mathbf{H}}^*_\omega$$

Doing this will still give us double summations, but we can integrate this expression over the sphere. To do this, the following identity is useful:

Lemma 0.0.1.

$$\vec{\mathbb{X}}^*_{l'm'}\cdot\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{lm}\right)\mathrm{d}\Omega = 0$$

Therefore,

$$\begin{aligned}P_\omega = \frac{1}{2}\frac{Z_0}{k^2}\sum_{l,m,l',m'}\int\mathrm{d}\Omega[&a^*_E a_E\vec{\mathbb{X}}^*_{lm}\vec{\mathbb{X}}_{l'm'}\\ &+ \left(a^*_E a_M + a_M a^*_E\right)\vec{\mathbb{X}}^*_{lm}\cdot\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{l'm'}\right)\\ &+ a^*_M a^*_M\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{lm}\right)\cdot\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}^*_{l'm'}\right)]\end{aligned}$$

The integral over the first term reduces to δ-functions, the middle term vanishes, and the final term also reduces to δ-functions, so

$$P_\omega = \frac{1}{2}\frac{Z_0}{k^2}\sum_{l,m}\left[|a_E(l,m)|^2 + |a_M(l,m)|^2\right]$$

Recall Maxwell's equations in this region:

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}}_\omega = \frac{\rho_\omega}{\epsilon_0}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{H}}_\omega = \vec{\mathbf{J}}_\omega - \epsilon_0 \imath\omega\vec{\mathbf{E}}_\omega$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}}_\omega - \imath k Z_0\vec{\mathbf{H}}_\omega = \vec{\mathbf{0}}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{H}}_\omega + \frac{\imath k}{Z_0}\vec{\mathbf{E}}_\omega = \vec{\mathbf{J}}_\omega$$

since

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{J}}_\omega = \imath\omega\rho_\omega$$

Therefore we can write

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}}_\omega = -\frac{1}{\imath\omega\epsilon_0}\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{J}}_\omega \implies \vec{\boldsymbol{\nabla}}\cdot\underbrace{\vec{\mathbf{E}}_\omega + \frac{1}{\imath\omega\epsilon_0}\vec{\mathbf{J}}_\omega}_{\vec{\mathbf{E}}'_\omega} = \vec{\mathbf{0}}$$

where

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}}'_\omega = \vec{\mathbf{0}} = \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{H}}_\omega$$

Therefore, we find that

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{H}}_\omega = \frac{\imath k}{Z_0}\left[\vec{\mathbf{E}}'_\omega - \frac{\imath Z_0}{k}\vec{\mathbf{J}}_\omega\right] = \vec{\mathbf{J}}_\omega$$

so

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{H}}_\omega + \frac{\imath k}{Z_0}\vec{\mathbf{E}}'_\omega = \vec{\mathbf{0}}$$

and

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}}'_\omega - \imath k Z_0\vec{\mathbf{H}}_\omega = \frac{\imath Z_0}{k}\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{J}}_\omega$$

Why are we doing this? We want to be able to determine a_M and a_E from the source components. If we take the curl of the previous equations, we find

$$-\nabla^2\vec{\mathbf{H}}_\omega + \frac{\imath k}{Z_0}\left[\imath k Z_0\vec{\mathbf{H}}_\omega + \frac{\imath Z_0}{k}\vec{\boldsymbol{\nabla}}\times JVec_\omega\right] = \vec{\mathbf{0}}$$

or

$$\left(\nabla^2 + k^2\right)\vec{\mathbf{H}}_\omega = -\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{J}}_\omega$$

From the other equation, we find

$$\left(\nabla^2 + k^2\right)\vec{\mathbf{E}}'_\omega = -\frac{\imath Z_0}{k}\vec{\boldsymbol{\nabla}}\times\left(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{J}}_\omega\right)$$

Observe that

Lemma 0.0.2.

$$\nabla^2(\vec{\mathbf{x}} \cdot \vec{\mathbf{F}}) = 2\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{F}} + (\nabla^2\vec{\mathbf{F}}) \cdot \vec{\mathbf{x}}$$

If we apply this to our fields, which are divergence free, we find that

$$(\nabla^2 + k^2)(\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega) = -\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega$$

and

$$(\nabla^2 + k^2)(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}'_\omega) = -\frac{\imath Z_0}{k}\vec{\mathbf{x}} \cdot \left(\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega)\right)$$

We can rewrite the second equation as

$$(\nabla^2 + k^2)(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}'_\omega) = \frac{Z_0}{k}\vec{\mathbb{L}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega)$$

We can solve these equations:

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega = \frac{1}{4\pi}\int \frac{e^{\imath k|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}$$

which we can expand as

$$\sum_{l,m}(\imath k) j_l(kr') h_l^{(1)}(kr) Y_{lm}(\Omega) Y^*_{l'm'}(\Omega')$$

so

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega = \sum_{l,m}(\imath k)\int j_l(kr') Y^*_{lm}(\Omega')[-\imath\vec{\mathbb{L}} \cdot \vec{\mathbf{J}}_\omega](\vec{\mathbf{x}}')\,\mathrm{d}^3x'\, h_l^{(1)}(kr) Y_{lm}(\Omega)$$

and

$$Z_0 a_E(l,m) h_l^{(1)}(kr) = -\frac{k}{\sqrt{l(l+1)}}\int Y^*_{lm}(\Omega')(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega)\,\mathrm{d}\Omega$$