Lecture 17: Dielectrics

Mon Sep. 30 2019

0.1 Dielectrics

In isotropic and homogeneous materials, we said that

$$\vec{P} = \varepsilon_0 \chi \vec{E} \tag{1}$$

so

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{2}$$

and

$$\boldsymbol{\nabla} \cdot \vec{D} = \rho_{\text{free}} \tag{3}$$

Of course, we also must satisfy

$$\boldsymbol{\nabla} \times \vec{E} = \vec{0} \Rightarrow \vec{E} = -\boldsymbol{\nabla}\Phi \tag{4}$$

so

$$\vec{D} = \underbrace{(\varepsilon_0 + \varepsilon_0 \chi)}_{\varepsilon} \vec{E} \tag{5}$$

so

$$\varepsilon \nabla^2 \Phi = -\rho_{\text{free}} \tag{6}$$

In more general cases,

$$D_i = \varepsilon_{ij}(x) E_j \tag{7}$$

so

$$\partial_i(\varepsilon_{ij}(x)\partial_j \Phi) = -\rho_{\text{free}} \tag{8}$$

which is in general pretty hard to solve.

If we recall our boundary conditions:

> **Quote**
>
> "Do I have enough c's in 'across'? You know in French there are always more letters than you think there should be."

$$\vec{E}_t \quad \text{is continuous across a boundary} \tag{9}$$

and

$$\vec{D}_n \quad \text{is continuous} \tag{10}$$

Example. For a dielectric ($\varepsilon \neq 0$ sphere inserted into a uniform electric field),

$$\vec{E}_0 = E_0 \hat{z} \tag{11}$$

There are no free charges and symmetry around z

$$\Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) \tag{12}$$

The potential cannot go to zero at infinity, since there is an electric field there, so

$$\Phi_{\text{out}} = -E_0 z + B_0 + \sum_{l=0}^{\infty} C_l r^{-(l+1)} P_l(\cos(\theta)) \tag{13}$$

We can set $B_0 = 0$ because the potential is invariant up to a constant. We know that Φ must be continuous across the $r = a$ boundary, so

$$\Phi(a)_{\text{out}} = -E_0 a \underbrace{P_1(\cos(\theta))}_{\cos(\theta)} + \sum_{l=0}^{\infty} C_l a^{-(l+1)} P_l(\cos(\theta)) \tag{14}$$

so for $l = 1$

$$A_1 = -E_0 + \frac{C_1}{a^3} \tag{15}$$

and for $l \neq 1$

$$A_l = \frac{C_l}{a^{2l+1}} \tag{16}$$

We know that $D = \varepsilon E$ and we know the inside and outside permittivity, so to maintain continuity,

$$(-\varepsilon \boldsymbol{\nabla} \Phi) \cdot \hat{n}\Big|_{r \to a^-} = (-\varepsilon_0 \boldsymbol{\nabla} \Phi) \cdot \hat{n}\Big|_{r \to a^+} \tag{17}$$

or

$$-\varepsilon \frac{\partial \Phi_{\text{in}}}{\partial r}\bigg|_{r=a} = -\varepsilon_0 \frac{\partial \Phi_{\text{out}}}{\partial r}\bigg|_{r-a} \tag{18}$$

so

$$\varepsilon E_0 P_1(\cos(\theta)) - \varepsilon \sum_{l=0}^{\infty} [-(l+1) a^{-(l+2)} C_l P_l(\cos(\theta)) = -\varepsilon_0 \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos(\theta)) \tag{19}$$

so when $l \neq 1$:

$$\varepsilon (l+1) C_l a^{-(l+2)} = -\varepsilon_0 l A_l a^{l-1} \tag{20}$$

and when $l = 1$:

$$\varepsilon E_0 + \varepsilon (1 + l) a^{-3} C_1 = -\varepsilon_0 A_1 \tag{21}$$

From this and the previous boundary condition, we see that all of the $l \neq 1$ terms are 0, and solving between the remaining $l = 1$ terms, we see that

$$A_1 = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \tag{22}$$

and

$$C_1 = \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} a^3 E_0 \tag{23}$$

so

$$\Phi_{\text{in}} = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 r \cos(\theta) \tag{24}$$

and

$$\Phi_{\text{out}} = -E_0 r \cos(\theta) + \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \frac{a^3 E_0}{r^2} \cos(\theta) \tag{25}$$

Taking the proper derivatives, we see that

$$\vec{E}_{\text{in}} = \frac{3\varepsilon_0 E_0}{\varepsilon + 2\varepsilon_0}\hat{z} \tag{26}$$

and if we say that

$$\vec{p} = (4\pi a^3)\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\varepsilon_0 E_0 \hat{z} \tag{27}$$

$$\vec{E}_{\text{out}} = E_0\hat{z} + \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}\right] \tag{28}$$

We see here that the field inside is a reduction of the constant field outside, and the field outside has been amplified by the inclusion of the dielectric (unless the material is "active"), $\varepsilon > \varepsilon_0$. It is also clear here that $\vec{P} = \vec{D} - \varepsilon_0\vec{E}$. ◇

Quote

> "What do we want to do with this example? Of course, we don't want to do anything with it - minimum action principle."

Example. Imagine two media which meet at a straight boundary. On the left side, we have ε_2 and on the right we have ε_1. Imagine placing a charge q a distance d from the boundary. We know that maintaining the boundary condition on the interface (the Green's function must vanish) must create some sort of image charge at $-d$. On the right side, $z > 0$,

$$\Phi = \frac{q}{4\pi\varepsilon_1\sqrt{\rho^2 + (z-d)^2}} + \frac{q'}{4\pi\varepsilon_1\sqrt{\rho^2 + (z+d)^2}} \tag{29}$$

and on the other side,

$$\Phi = \frac{1}{r\pi\varepsilon_2}\frac{q''}{\sqrt{\rho^2 + (z-d)^2}} \tag{30}$$

where q'' is some "blurred" charge seen from the left side of the boundary. However, since Φ is continuous across the boundary, we know that

$$\frac{q}{4\pi\varepsilon_1\cancel{\sqrt{\rho^2 + d^2}}} + \frac{q'}{4\pi\varepsilon_1\cancel{\sqrt{\rho^2 + d^2}}} = \frac{q''}{4\pi\varepsilon_2\cancel{\sqrt{\rho^2 + d^2}}} \tag{31}$$

so

$$q + q' = \frac{\varepsilon_1}{\varepsilon_2}q'' \tag{32}$$

By taking $\vec{D}$ having no jump across the boundary,

$$q - q' = q'' \tag{33}$$

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