33-761 Homework 6

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October 9, 2019

1. Uniqueness of General Linear Dielectrics

Let us consider general linear dielectrics. Recall that in principle we could have $D_i(x) = \varepsilon_{ij}(x)E_j(x)$ where summation over repeated indices are implied. If the medium has some specified free charge distribution inside as well as some boundary conditions, we then need to solve the following equation for the potential ϕ

$$\frac{\mathrm{d}}{\mathrm{d}x^i}\left(\varepsilon_{ij}\frac{\mathrm{d}}{\mathrm{d}x^j}\phi\right) = -\rho. \tag{1}$$

Assume that we have a bunch of conductors immersed into the medium with fixed charges on them as well as some other boundary surfaces with potentials specified on them. Using these boundary conditions show that the above equation has a unique solution (assuming there is a solution). For this, recall the Green identity in the following form

$$\int_\Omega \mathrm{d}^3x\,[\partial_i(\varepsilon_{ij}\partial_j U)U + \varepsilon_{ij}\partial_j U\partial_i U] = \sum_a \oint_{S_a} U\varepsilon_{ij}\partial_j U\,\mathrm{d}a_i \tag{2}$$

where Ω is the domain of interest, union of S_a's is the boundary of the domain Ω, and U is a scalar field.

First, let's suppose we have two different solutions, ϕ_1 and ϕ_2 and define $U = \phi_1 - \phi_2$. The first thing we should note is that the right-hand side of the equation for the Green identity is now zero, since

$$\sum_a \oint_{S_a} U\varepsilon_{ij}\partial_j U\,\mathrm{d}a_i = \sum_a \oint_{S_a} (\phi_1 - \phi_2)\varepsilon_{ij}\partial_j U\,\mathrm{d}a_i = 0 \tag{3}$$

and the potential on any given surface should be the same under both solutions. Next, we can examine the first part of the identity:

$$\partial_i(\varepsilon_{ij}\partial_j U) = 0 \tag{4}$$

since the Laplacian of the potentials should give the charge in the domain, but this charge should be the same for any given conductor, so the difference in the charge, which would be given by the Laplacian over U, should vanish. Finally, we are left with

$$\int_\Omega \mathrm{d}^3x\,\varepsilon_{ij}\partial_j U\partial_i U = 0 \tag{5}$$

This means that either the dielectric tensor is zero (which would imply a rather boring problem), or the gradient of U is zero, which means that ϕ is unique up to a constant.

2. Solve Jackson Problem 4.5 part (a) only.

A localized charge density ρ is placed in an external electrostatic field described by potential $\Phi^{(0)}$. The external potential varies slowly in space over the region where the charge density is different from zero.

From first principles calculate the total force acting on the charge distribution as an expansion in multipole moments times derivatives of the electric field, up to and including the quadrupole moments. Compare this to the expansion of the energy W. What is its connection to F?

We can begin with the statement that

$$F = \int \mathrm{d}^3x\, \rho(x) E^{(0)}(x) = \int \mathrm{d}^3x\, \rho(x) \sum_i E_i^{(0)} \hat{x}_i \tag{6}$$

Now I will Taylor expand around $x = 0$ in $E^{(0)}$:

$$E_i(x) = \left[E_i(x') + \sum_j x_j \partial'_j E_i(x') + \frac{1}{2} \sum_{jk} x_j x_k \partial'_j \partial'_k E_i(x') \right] \Bigg|_{x' \to 0} \tag{7}$$

Next, we plug this into the force equation (I am leaving off the superscript $E^{(0)}$):

$$F = \sum_i \hat{x}_i E_i(x') \int \mathrm{d}^3x\, \rho(x) \tag{8}$$

$$+ \sum_{ij} \hat{x}_i \partial'_j E_i(x') \int \mathrm{d}^3x\, x_j \rho(x) \tag{9}$$

$$+ \frac{1}{2} \sum_{ijk} \hat{x}_i \partial'_j \partial'_k E_i(x') \int \mathrm{d}^3x\, x_j x_k \rho(x) \tag{10}$$

Next, we can use the fact that $\nabla \times \vec{E} = 0 \Rightarrow \partial_j E_i = \partial_i E_j$:

$$F = qE(x') \tag{11}$$

$$+ \sum_{ij} \hat{x}_i \partial'_i E_j(x') \int \mathrm{d}^3x\, x_j \rho(x) \tag{12}$$

$$+ \frac{1}{2} \sum_{ijk} \hat{x}_i \partial'_i \partial'_k E_j(x') \int \mathrm{d}^3x\, x_j x_k \rho(x) \tag{13}$$

Now that we are summing over the same derivative components, we can simplify the summations as ∇'s. Additionally, we can recognize the integrals as the dipole and quadrupole moments. The quadrupole moment requires us to take the fact that the field is changing slowly, which implies $\nabla \cdot \vec{E} \approx 0$ so we can subtract $\frac{1}{6}r^2\nabla \cdot \vec{E}(x')$ to get the proper form:

$$F = qE^{(0)}(0) \tag{14}$$

$$+ \nabla[p \cdot E^{(0)}(x')] \tag{15}$$

$$+ \frac{1}{6}\nabla[\sum_{jk} \partial'_k E_j^{(0)}(x') Q_j k] \tag{16}$$

where $x' \to 0$.
Since $E^{(0)} = \nabla\Phi^{(0)}$, we can rewrite this whole expression, factoring the gradient:

$$F = \nabla q\Phi(0) - p \cdot E(0) - \frac{1}{6}\sum_{ij} Q_{ij}\partial_i E_j(0) \tag{17}$$

or simply

$$\int F dx = W \tag{18}$$

3. Nonisotropic Medium

Suppose we have a nonisotropic but otherwise uniform dielectric medium of infinite extend with dielectric tensor ε_{ij}. Let us put a point charge in it, show that the resulting Coulomb potential can be written as

$$\phi(\vec{x}) = \frac{q}{4\pi\sqrt{\det(\varepsilon)(\varepsilon_{ij}^{-1}x_i x_j)}}. \tag{19}$$

Hint: Go to a coordinate system in which ε_{ij} becomes diagonal, since this tensor is symmetric and positive definite, there are three positive eigenvalues ε^1, ε^2, ε^3 and the associated mutually orthogonal directions.

We can diagonalize ε_{ij} by writing it as

$$\varepsilon_{ij} = (P^T)_{ik}\Lambda_{ke}P_{ej} = P_{ki}\Lambda_{ke}P_{ej} \tag{20}$$

where Λ is a diagonal matrix of the eigenvalues of ε. Our charge density is at a point, so

$$\partial_i \varepsilon_{ij}\partial_j\Phi = -\delta(\vec{x})q \tag{21}$$

We will transform

$$\partial_i = (P^T)_{im}\partial'_m \tag{22}$$

such that

$$\varepsilon_{ij}\partial_i\partial_j = \Lambda_{ij}\partial'_i\partial'_j = \sum_i \varepsilon_i\partial_i\partial_i \tag{23}$$

We now make a substitution $y = \frac{1}{\sqrt{\varepsilon_i}} x_i'$ such that

$$\frac{\partial}{\partial y_i} \frac{\partial}{\partial y_i} \Phi(y) = \delta(\vec{x}) q \tag{24}$$

We can equivalently scale the x on the left-hand side:

$$\delta(\vec{x}) = |\varepsilon| \delta(y) \tag{25}$$

Now we can solve it as if it were a regular point charge in free space and then transform back by the scaling factor and then by the rotation. I'm not quite sure how to do this, but the scaling factor would give the determinant, since $\mathrm{d}y = \frac{\mathrm{d}x}{\det(\varepsilon)}$ since $\det(\varepsilon) = \varepsilon_1 \varepsilon_2 \varepsilon_3$.

4. Solve Jackson Problem 4.7 part (a) only.

$$\rho(r) = \frac{1}{64\pi} r^2 e^{-r} \sin^2 \theta \tag{26}$$

Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

We first need to calculate the multipole moments:

$$q_{lm} = \int Y_{lm}^*(\theta', \varphi') r'^l \rho(x') \, \mathrm{d}^3 x' \tag{27}$$

By inserting the charge density into this equation, we find

$$q_{lm} = \frac{1}{64\pi} \int Y_{lm}^*(\theta', \varphi') r'^{l+4} e^{-r} \sin^3 \theta' \, \mathrm{d}r' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \tag{28}$$

We know from the charge distribution that the system is azimuthally symmetric, so only the $m = 0$ terms survive. Next, we can complete the φ integral and reduce the spherical harmonics to Legendre polynomials:

$$q_{l0} = \frac{1}{32} \sqrt{\frac{2l+1}{4\pi}} \int_0^{2\pi} \mathrm{d}\theta' \, P_l(\cos\theta') \sin^3 \theta' \int 0^\infty r'^{l+4} e^{-r'} \, \mathrm{d}r' \tag{29}$$

Next, I will focus on the first integral over θ'. If we make the substitution $x = \cos(\theta')$, $\mathrm{d}x = -\sin(\theta') \, \mathrm{d}\theta'$, and $\sin^2(\theta') = 1 - x^2$, we can reduce it to the form

$$\int_{-1}^{1} P_l(x)(1 - x^2) \, \mathrm{d}x \tag{30}$$

We can then rewrite $1 - x^2$ in terms of two of the first few Legendre polynomials:

$$1 - x^2 = \frac{2}{3}(P_0(x) - P_2(x)) \tag{31}$$

Now we have

$$\int_{-1}^{1} \mathrm{d}x\, P_l(x) \frac{2}{3}(P_0(x) - P_2(x)) \tag{32}$$

By orthogonality of the Legendre polynomials, only the $l = 0$ and $l = 2$ terms are nonzero. Additionally,

$$\int_{-1}^{1} P_l(x)^2\, \mathrm{d}x = \frac{2}{2l+1} \tag{33}$$

so

$$q_{00} = \frac{2}{3} \frac{1}{32} \sqrt{\frac{1}{\pi}} \int_0^\infty \mathrm{d}r'\, r'^4 e^{-r'} = \sqrt{\frac{1}{4\pi}} \tag{34}$$

and

$$q_{20} = -\frac{2}{3} \frac{1}{32} \sqrt{\frac{1}{5\pi}} \int_0^\infty \mathrm{d}r'\, r'^6 e^{-r'} = -6\sqrt{\frac{5}{4\pi}} \tag{35}$$

The multipole expansion can be given as

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \tag{36}$$

Now, all but two of these terms are zero, and what remains is

$$\frac{1}{4\pi\varepsilon_0} \left(4\pi \sqrt{\frac{1}{4\pi}} \frac{Y_{00}}{r} - \frac{4\pi}{5} 6 \sqrt{\frac{5}{4\pi}} \frac{Y_{20}}{r^3} \right) \tag{37}$$

Using the closed form of these two spherical harmonics, the problem can be reduced to

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \left[\frac{1}{r} - \frac{3}{r^3} \left(3\cos^2\theta - 1\right) \right] \tag{38}$$