Lecture 16: Electrostatics of Dielectrics

Friday Sep 27 2019

0.1 Microscopic vs. Macroscopic Structure

The micro scale is $\propto 10^{-9} \to 10^{-8}$ meters, while the macro scale is $\propto 10^{-6}$ meters. We can look in the range right between these to average out these microscopic fields. In this range, $\vec{B}_{\text{micro}} \approx \vec{0}$. Microscopic electric fields may be induced, and averaging over these can be modeled by a macroscopic dipole density $\vec{P}(\vec{x})$. This is our working, unjustified assumption to be discussed further by some models. If we believe this assumption, we can write down the potential as:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')}{\|\vec{x}-\vec{x}'\|} \, \mathrm{d}^3x + \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{p}(\vec{x}' \cdot (\vec{x}-\vec{x}')}{\|\vec{x}-\vec{x}'\|^3} \, \mathrm{d}^3x' \tag{1}$$

This is equivalent to

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')}{\|\vec{x}-\vec{x}'\|} \, \mathrm{d}^3x + \frac{1}{4\pi\varepsilon_0} \int \frac{\boldsymbol{\nabla} \cdot' \vec{P}(x')}{\|\vec{x}-\vec{x}'\|} \, \mathrm{d}^3x' + \int_\Omega \frac{-\boldsymbol{\nabla}' \vec{P}}{\|\vec{x}-\vec{x}'\|} \, \mathrm{d}^3x' \tag{2}$$

The numerator of the second term here is the bound surface charge of the medium. We can think of the measured field as $\vec{E} = \vec{E}^{\text{ext}} + \vec{E}_{\text{micro}}$. We have a working hypothesis that $\boldsymbol{\nabla} \times \vec{E} = 0$, because the microscopic magnetic field does not change in time, so $\vec{E} = -\boldsymbol{\nabla} \cdot \Phi$. $\vec{P}$ is a function of "local" $\vec{E}$ for the static case. We assume the linear term is the dominant contribution (it can be nonlinear, a simple model of permanent dipoles depends non-linearly on temperature, for example).

$$\vec{P} = \varepsilon_0 \chi \tag{3}$$

Isotropic materials have $\chi_{ij} = \chi\delta_{ij}$. Homogeneous materials have $\chi(\vec{x}) = \chi$. We can therefore show that

$$\rho_{\text{bound}} = -\boldsymbol{\nabla} \cdot \vec{P} \tag{4}$$

so

$$\boldsymbol{\nabla} \cdot \vec{E} = \frac{\rho_{\text{free}}}{\varepsilon_0} - \frac{\boldsymbol{\nabla}\vec{P}}{\varepsilon_0} \tag{5}$$

or

$$\boldsymbol{\nabla} \underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\vec{D}} = \rho_{\text{free}} \tag{6}$$

If we assume $\vec{P} = \varepsilon_0 \chi \vec{E}$,

$$D = \varepsilon_0 (1+\chi) \vec{E} \tag{7}$$

where $\varepsilon_0(1+\chi) \equiv \varepsilon$. This brings us the familiar Poisson equation on the potential:

$$\varepsilon \nabla^2 \Phi = -\rho \tag{8}$$

Charge free regions still satisfy $\nabla^2\Phi = 0$, and we can use boundary conditions to determine solutions.

0.1.1 Boundary Conditions

If we take a Gaussian pillbox around a boundary, we know that $\boldsymbol{\nabla} \cdot \vec{D} = \rho_{\text{free}}$, so

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{12} = 0 \tag{9}$$

Also, the normal component of $\vec{D}$ is continuous in a linear material, since $\vec{D} = \varepsilon\vec{E}$, so

$$\varepsilon_1(\vec{E}_1)_n = \varepsilon_2(\vec{E})_n \tag{10}$$

Additionally, the tangential components of $\vec{E}$ are continuous:

$$(\vec{E}_1 - \vec{E}_2)_{\text{tangent}} = \vec{0} \tag{11}$$

or

$$(\vec{E}_1 - \vec{E}_2) \times \hat{n}_{12} = \vec{0} \tag{12}$$