Lecture 25:

Monday, October 14, 2019

Recall

$$\nabla \times B = \mu_0 \vec{J}$$

and we used this last lecture to show that

$$\nabla^2 \vec{B} = -\mu_0 \nabla \times \vec{J}$$

which we solved to find

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{(\nabla \times \vec{J}')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

We also did the same formulation with $\vec{x} \cdot \vec{B}$:

$$\nabla^2 \vec{x} \cdot \vec{B} = -\mu_0 \vec{x} \cdot \nabla \times \vec{J}$$

$$\vec{x} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \frac{(\vec{x}' \cdot \nabla \times \vec{J}')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1} Y^*_{lm}(\Omega') Y_{lm}(\Omega)$$

If we assume

$$\vec{B} = -\nabla \Phi_M$$

we found

$$\vec{x} \cdot \nabla \Phi_M = r \frac{\mathrm{d}}{\mathrm{d}r} \Phi_m$$

so

$$\Phi_m = \sum_{l,m} \frac{4\pi}{2l+1} \sqrt{\frac{l}{l+1}} \frac{1}{\sqrt{(l+1)l}} \left[\int r'^l (\vec{\mathbb{L}} Y^*_{lm}) \cdot \vec{J} \, \mathrm{d}\Omega' \, r'^2 \, \mathrm{d}r' \right] \frac{Y_{lm}(\Omega)}{r^{l+1}}$$

where

$$\frac{1}{\sqrt{l(l+1)}} \vec{\mathbb{L}} Y_{lm} = \vec{\mathbb{X}}_{lm}$$

are the vector spherical harmonics.

$$\begin{aligned} \int \vec{\mathbb{X}}^*_{lm} \cdot \vec{\mathbb{X}}_{l'm'} \, \mathrm{d}\Omega &= \int Y^*_{lm} \vec{\mathbb{L}} \cdot \vec{\mathbb{L}} Y_{l'm'} \, \mathrm{d}\Omega \, \frac{1}{\sqrt{l(l+1)l'(l'+1)}} \\ &= \frac{l(l+1)}{l(l+1)} \delta_{ll'} \delta_{mm'} = \delta_{ll'} \delta_{mm'} \end{aligned}$$

so the vector spherical harmonics are an orthonormal basis. Our expansion is now

$$\Phi_M = \sum_{l,m} \frac{4\pi}{2l+!} \imath \sqrt{\frac{l}{l+1}} \left[\int r'^l r'^2 \, \mathrm{d}r' \, \mathrm{d}\Omega' \, \vec{\mathbb{X}}^*_{lm} \cdot \vec{J}(\vec{x}') \right] \frac{Y_{lm}(\Omega)}{r^{l+1}}$$

The idea is, we want to turn this into an expansion for $\vec{A}$, the vector potential for the magnetic field. We want something like $\boldsymbol{\nabla} \times \vec{A}$ since $\vec{B} = -\boldsymbol{\nabla}\Phi_M = \boldsymbol{\nabla} \times \vec{A}$. We use the following identity:

$$\boldsymbol{\nabla} \times \vec{\mathbb{L}} = -\imath\vec{x}\nabla^2 + \imath\boldsymbol{\nabla}(1 + \vec{x}\cdot\nabla)$$

In spherical coordinates, $\vec{x}\cdot\nabla = r\frac{\mathrm{d}}{\mathrm{d}r}$. Additionally recall that,

$$\nabla^2\left(\frac{Y_{lm}}{r^{l+1}}\right) = 0$$

Let us then write

$$\boldsymbol{\nabla} \times \vec{\mathbb{L}}\left(\frac{Y_{lm}}{r^{l+1}}\right) = -\imath\vec{x}\cancelto{0}{\nabla^2\left(\frac{Y_{lm}}{r^{l+1}}\right)} + \imath\boldsymbol{\nabla}\cdot\underbrace{\left[1 + r\frac{\mathrm{d}}{\mathrm{d}r}\right]\left(\frac{Y_{lm}}{r^{l+1}}\right)}_{(1-l-1)\frac{1}{r^{l+1}}Y_{lm}}$$

Therefore

$$\boldsymbol{\nabla} \times \left[\frac{1}{l\imath}\right]\vec{\mathbb{L}}\left(\frac{Y_{lm}}{r^{l+1}}\right) = -\boldsymbol{\nabla}\cdot\left(\frac{Y_{lm}}{r^{l+1}}\right)$$

Using this, we see that

$$-\boldsymbol{\nabla}\Phi_M = -\nabla\sum_{l,m} B_{lm}\frac{Y_{lm}}{r^{l+1}} = \sum_{l,m} B_{lm}\left(-\nabla\frac{Y_{lm}}{r^{l+1}}\right) = \sum_{l,m} B_{lm}\left[\frac{1}{\imath l}\boldsymbol{\nabla}\times\vec{\mathbb{L}}\frac{Y_{lm}}{r^{l+1}}\right]$$

Therefore, we see that

$$\boldsymbol{\nabla} \times \sum_{l,m}\left[\frac{B_{lm}}{\imath l}\frac{\vec{\mathbb{L}}Y_{lm}}{r^{l+1}}\right] = \boldsymbol{\nabla} \times \vec{A}$$

$$\begin{aligned}\vec{B} &= \boldsymbol{\nabla} \times \left[\sum_{l,m}\frac{4\pi}{2l+1}\frac{\imath}{\imath l}\sqrt{\frac{l}{l+1}}\left(\int \mathrm{d}^3x'\, r'^l\vec{\mathbb{X}}^*\cdot\vec{J}\right)\frac{\vec{\mathbb{L}}Y_{lm}}{r^{l+1}}\right] \\ &= \boldsymbol{\nabla} \times \left[\sum_{l,m}\frac{4\pi}{2l+1}\left(\int \mathrm{d}^3x'\, r'^l\vec{\mathbb{X}}^*_{lm}\cdot\vec{J}\right)\frac{\vec{\mathbb{X}}_{lm}}{r^{l+1}}\right]\end{aligned}$$

Therefore, the true multipole expansion for the vector potential is

$$\vec{A} = \left[\sum_{l,m}\frac{4\pi}{2l+1}\left(\int \mathrm{d}^3x'\, r'^l\vec{\mathbb{X}}^*_{lm}\cdot\vec{J}\right)\frac{\vec{\mathbb{X}}_{lm}}{r^{l+1}}\right]$$

Example. Let us have an example of using Φ_M. It can be useful in some situations and is not just an operational trick. This is the homework problem for a rotating sphere. We have a charged sphere rotating with angular velocity ω with a surface current density $\vec{J} = \sigma\omega a\sin(\theta)\hat{\varphi}\delta(r-a)$. We want to find the B field. The homework is to solve for $\vec{A}$. However, there are two regions that are free from currents, the inside of the sphere and the outside. Φ_M works when there are no currents, so we could just glue these regions together using Φ_M. Recall that on the surface, the normal component of B in the two regions should be equal and continuous because $\boldsymbol{\nabla}\cdot\vec{B} = 0$. If we had a surface current, the tangential components must jump by the surface current (not a volume current) across the boundary.

In the outside region,

$$\nabla \cdot \vec{B} = 0 \implies \nabla^2 \Phi_M = 0$$

$$\Phi_M = \begin{cases} \sum A_l r^l P_l(\cos(\theta)) & r < a \\ \sum \frac{B_l}{r^{l+1}} P_l(\cos(\theta)) & r > a \end{cases}$$

additionally, the continuity of the field across the boundary implies

$$-\frac{\partial \Phi_M}{\partial r}\bigg|_{r\to a^- = r\to a^+} \implies A_l = -\frac{l+1}{l}\frac{B_l}{a^{2l+1}}$$

Our other boundary condition tells us

$$B_\theta^{\text{outside}} - B_\theta^{\text{inside}} = k_\varphi = \sigma a \omega \sin(\theta)$$

so

$$-\frac{1}{r}\frac{\partial \Phi_M}{\partial \theta}\bigg|_{r\to a^+} + \frac{1}{r}\frac{\partial \Phi_M}{\partial r}\bigg|_{r\to a^-} = \sigma a \omega \sin(\theta)$$

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