Lecture 46: The Helmholtz Equation in Spherical Coordinates

Monday, November 25, 2019

From last lecture, we showed that

$$\frac{e^{\imath k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = \sum_{l,m} (\imath k) j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}(\Omega) Y_{lm}^*(\Omega')$$

Recall that our vector potential was

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi} \int \frac{e^{\imath k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')\,\mathrm{d}^3x'$$

so in the far field, we can expand this as

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0 \imath k}{4\pi} \left[\int j_l(kr') Y_{lm}^*(\Omega') \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')\,\mathrm{d}^3x' \right] h_l^{(1)}(kr) Y_{lm}(\Omega)$$

Of course, we can expand the spherical Bessel functions, but in general it won't decouple the equation nicely. We can expand $h^{(1)}$ in the radiation zone ($\frac{r}{\lambda} >> 1$), but this doesn't solve any problems on the inside of the integral, because of the vector components of $\vec{\mathbf{J}}_\omega$.

Instead, we have to find another (not so obvious) expansion. If we are away from the source region, the equations which we are solving are technically source-less:

$$\left(\nabla^2 + k^2\right) \begin{pmatrix} \vec{\mathbf{E}}_\omega \\ \vec{\mathbf{H}}_\omega \end{pmatrix} = 0$$

Let's take a step back and solve the Helmholtz equation in this region, away from the source: $(\nabla^2 + k^2)\psi = 0$. We want a vector solution, not a scalar. Note that $\vec{\mathbb{L}} = \frac{1}{\imath}\vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}}$ commutes with the Laplacian because the Laplacian is a scalar operator. This tells you that if you had a scalar solution, that solution would also satisfy

$$(\nabla^2 + k^2)\vec{\mathbb{L}}\psi = \vec{\mathbf{0}}$$

and $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbb{L}}\psi = 0$.

Also note that, through a few substitutions, $\vec{\mathbf{H}}_\omega = -\frac{\imath}{kZ_0}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}}_\omega$, so if

$$\vec{\mathbf{E}}_\omega = \vec{\mathbb{L}}\psi$$

then

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{kZ_0}\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\psi$$

We could also do this the other way around, where

$$\vec{\mathbf{H}}_\omega = \vec{\mathbb{L}}\chi$$

so

$$\vec{\mathbf{\nabla}} \times \vec{\mathbf{H}}_\omega = \epsilon_0(-\imath\omega)\vec{\mathbf{E}}_\omega$$

so

$$\vec{\mathbf{E}}_\omega = \frac{\imath}{k} Z_0 \vec{\mathbf{\nabla}} \times \vec{\mathbf{H}}_\omega$$

or

$$\vec{\mathbf{E}}_\omega = \frac{\imath}{k} Z_0 \vec{\mathbf{\nabla}} \times \vec{\mathbb{L}}\chi$$

The addition of these solutions is indeed the general solution:

$$\vec{\mathbf{E}}_\omega = \frac{\imath}{k} Z_0 \vec{\mathbf{\nabla}} \times \vec{\mathbb{L}}\chi + \vec{\mathbb{L}}\psi$$

and

$$\vec{\mathbf{E}}_\omega = \vec{\mathbb{L}}\chi - \frac{\imath}{kZ_0} \vec{\mathbf{\nabla}} \times \vec{\mathbb{L}}\psi$$

Solutions to the source-less Helmholtz equation can be expanded as

$$\psi = \sum \underbrace{\left[A_{lm}^{(1)} h_l^{(1)}(kr) + A_{lm}^{(2)} h_l^{(2)}(kr)\right]}_{f_{lm}} Y_{lm}(\Omega)$$

and

$$\chi = \sum \underbrace{\left[B_{lm}^{(1)} h_l^{(1)}(kr) + B_{lm}^{(2)} h_l^{(2)}(kr)\right]}_{g_{lm}} Y_{lm}(\Omega)$$

so

$$\vec{\mathbf{E}}_\omega = \sum_{lm} \left[f_{lm}(kr) \underbrace{\vec{\mathbb{L}} Y_{lm}}_{\sim \vec{\mathbb{L}}_{lm}} + \frac{\imath Z_0}{k} \vec{\mathbf{\nabla}} \times \left(g_{lm}(kr) \vec{\mathbb{X}}_{lm}\right) \right]$$

where $\vec{\mathbb{X}}_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{\mathbb{L}} Y_{lm}$ are the vector spherical harmonics, and

$$\vec{\mathbf{H}}_\omega = \sum_{lm} \left[-\frac{\imath}{kZ_0} \vec{\mathbf{\nabla}} \times \left(f_{lm}(kr) \vec{\mathbb{X}}_{lm}\right) + g_{lm}(kr) \vec{\mathbb{X}}_{lm} \right]$$

If we only want outgoing solutions, we can just look at the $h^{(1)}$ terms and expand them as $(-\imath)^{l+1} \frac{e^{\imath kr}}{kr}$. In practice, we only do this to the $\vec{\mathbf{H}}$ field and then use Maxwell's equations to get the $\vec{\mathbf{E}}$ field. Suppose we absorb Z_0 into f_{lm} in the equation for $\vec{\mathbf{H}}_\omega$:

$$\vec{\mathbf{E}}_\omega = \sum_{lm} \left[f_{lm}(kr) \vec{\mathbb{X}}_{lm} + \frac{\imath}{k} \vec{\mathbf{\nabla}} \times \left(g_{lm} \vec{\mathbb{X}}_{lm}\right) \right]$$

and

$$\vec{\mathbf{H}}_\omega = \sum_{lm} \left[-\frac{\imath}{k} \vec{\mathbf{\nabla}} \times \left(f_{lm}(kr) \vec{\mathbb{X}}_{lm}\right) + g_{lm} \vec{\mathbb{X}}_{lm} \right]$$

Recall that we showed (on a homework) that

$$\imath \vec{\mathbf{\nabla}} \times \vec{\mathbf{L}} = \vec{\mathbf{x}} \nabla^2 - \vec{\mathbf{\nabla}} \times \left[1 + \vec{\mathbf{x}} \cdot \vec{\mathbf{\nabla}}\right]$$