Lecture 42: Radiation in the Far Field

Monday, November 18, 2019

Wait, that's the only kind of radiation.

Recall our three regimes:

- Far Field: $d << \lambda << r$
- Intermediate $d << \lambda \sim r$
- Near Field $d << r << \lambda$ (static limit)

If we expand our solutions in the near field,

$$e^{\imath \frac{2\pi}{\lambda}|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \sim 1 + \imath \frac{2\pi}{\lambda}|\vec{\mathbf{x}}-\vec{\mathbf{x}}'| + \cdots$$

where 1 represents the static point.

In the radiation zone, let's expand the exponential in the vector potential:

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')e^{\imath kr\left(1-\frac{2\vec{\mathbf{x}}\cdot\vec{\mathbf{x}}'}{r^2}+\frac{\vec{\mathbf{x}}'^2}{r^2}\right)^{\frac{1}{2}}}}{r\left[1-\frac{2\vec{\mathbf{x}}\cdot\vec{\mathbf{x}}'}{r^2}+\frac{\vec{\mathbf{x}}'^2}{r^2}\right]^{\frac{1}{2}}}\,\mathrm{d}^3x'$$

However, $r\frac{\vec{\mathbf{x}}\cdot\vec{\mathbf{x}}'}{r^2} \to k\hat{\mathbf{n}}\cdot\vec{\mathbf{x}}'$ is on the order of $\mathcal{O}\left(\frac{d}{\lambda}\right)$. In the radiation zone, $\frac{d}{r} << \frac{d}{\lambda}$. The next term also has a vanishing order.

Let's try ignoring both of these terms. We find, to zeroth order, that

$$\vec{\mathbf{A}}_\omega \simeq \frac{\mu_0}{4\pi}\int \frac{\mathrm{d}^3x'\,\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')e^{\imath kr}}{r}$$

In the radiation zone, we find

$$\vec{\mathbf{A}}_\omega \simeq \frac{\mu_0}{4\pi}\left[\int \mathrm{d}^3x'\,\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')e^{\imath k\hat{\mathbf{n}}\cdot\vec{\mathbf{x}}'}\right]\frac{e^{\imath kr}}{r}$$

We can expand this last term in the intermediate range as

$$\frac{e^{\imath k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = \imath k\sum_{l,m} j_l(kr')h_l^{(1)}(kr)Y_{lm}(\Omega)Y^*_{lm}(\Omega')$$

where

$$j_l(kr) = \frac{J_{l+1/2}(kr)}{\sqrt{r}}$$

and $h_l^{(1)}$ is a Hankel function of the first kind. We can expand this in the far field to get the radiation effects, since the Hankel function will look like an exponential for large r. We'll derive all of this later.

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi}\frac{e^{\imath kr}}{r}\int \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')\,\mathrm{d}^3x'$$

Now let's look at the divergence of $\vec{\mathbf{A}}_\omega$:

$$\int_\Omega \partial_j(x_iJ_j)\,\mathrm{d}^3x = \int_\Omega \delta_{ij}J_j + x_i\partial_jJ_j$$

so

$$\int_\Omega J_i = -\int x_i\partial_jJ_j$$

Recall that $\partial_t\rho + \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{J}} = 0$, so

$$\int_\Omega J_i = -\int \imath\omega\rho_\omega(\vec{\mathbf{x}}')x'\,\mathrm{d}^3x'$$

This term is actually the dipole moment of the charge distribution!

$$\vec{\mathbf{A}}_\omega = -\frac{\imath\mu}{4\pi}\omega\left[\int \underbrace{\mathrm{d}^3x'\,\rho_\omega(\vec{\mathbf{x}}')\vec{\mathbf{x}}'}_{\vec{\mathbf{p}}_\omega}\right]\frac{e^{\imath kr}}{r}$$

Therefore,

$$\vec{\mathbf{B}}_\omega = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}_\omega = +\frac{\imath\mu_0\omega}{4\pi}\vec{\mathbf{p}}_\omega\times\vec{\boldsymbol{\nabla}}\cdot\left(\frac{e^{\imath kr}}{r}\right)$$

and

$$\vec{\boldsymbol{\nabla}}e^{\imath kr} = \imath k\left(\frac{\vec{\mathbf{x}}}{r}\right)e^{\imath kr}$$

so

$$\vec{\mathbf{B}}_\omega = \left(\frac{\imath(\imath k)\mu_0\omega}{4\pi}\vec{\mathbf{p}}(\omega)\times\hat{\mathbf{n}}\right)\frac{e^{\imath kr}}{r} + \cdots$$

and

$$\vec{\mathbf{E}}_\omega = \frac{\imath c}{k}\vec{\boldsymbol{\nabla}}\times\left[-\frac{k^2\mu_0c}{4\pi}\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}}\right]\frac{e^{\imath kr}}{r} = \frac{\imath c^2k\mu_0}{4\pi}\left[(\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\times\imath k\hat{\mathbf{n}}\right]\frac{e^{\imath kr}}{r}$$

so

$$\vec{\mathbf{E}}_\omega = \frac{-c^2k^2}{4\pi}\mu_0\left[(\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right]\frac{e^{\imath kr}}{r}$$

Now lets look at the power. We will calculate $\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \mathrm{Re}[\left\langle\vec{\mathbf{S}}\right\rangle\cdot r^2\hat{\mathbf{n}}]$, the change in power as a function of solid angle. This morning we found that the time average of the Poynting vector was something like $\frac{1}{2}(\vec{\mathbf{E}}_\omega\times\vec{\mathbf{B}}^*_\omega)$, which will get rid of the $e^{\imath kr}$ terms, so we will get something like

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}\sim\left[\frac{c^2k^2}{4\pi}\mu_0\frac{ck^2\mu_0}{4\pi}\right]\hat{\mathbf{n}}\cdot\left(\left[(\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right]\times(\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\right)$$

We can rewrite it using some vector identities, so its similar to

$$\sim k^4\left((\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right)\cdot\left((\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right)^*$$