Lecture 18: Image Method in Mediums, Continued

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0.1 Image Method in Mediums, Continued

From last time, we had, from the continuity of the potential

$$q + q' = \frac{\varepsilon_1}{\varepsilon_2} q'' \tag{1}$$

and from continuity of D:

$$-\varepsilon_2 \frac{\partial \Phi}{\partial z}\bigg|_{z \to 0^-} = -\varepsilon_2 \frac{\partial \Phi}{\partial z}\bigg|_{z \to 0^+} \Rightarrow q'' = q - q' \tag{2}$$

Therefore,

$$q' = -\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2} q \tag{3}$$

and

$$q'' = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} q \tag{4}$$

In this sense, conductors can be thought of as dielectrics with $\varepsilon \to \infty$ limits. If we take the plane to be normal to $\hat{z}$ with region ε_1 in the positive direction, we find that

$$\vec{P}_2 \cdot \hat{n}_{12} + \vec{P}_2 \cdot \hat{n}_{21} = \vec{P}_2 \cdot \hat{z} - \vec{P}_1 \cdot \hat{z} \tag{5}$$

$$= \underbrace{\varepsilon_2 \vec{E}_2 \cdot \hat{z} - \varepsilon_1 \vec{E}_1 \cdot \hat{z}}_{=0 \quad \text{since} \quad \vec{D} \quad \text{is continuous}} + \varepsilon_0 [\vec{E}_1 \cdot \hat{z} - \vec{E}_2 \cdot \hat{z}] \tag{6}$$

so

$$\sigma_{\text{excess}} = \frac{1}{2\pi} \frac{\varepsilon_0}{\varepsilon_1} \left[\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right] \frac{qd}{[\rho^2 + d^2]^{3/2}} \tag{7}$$

0.1.1 Energy Considerations in Dielectrics

For a number of dielectrics in a space,

$$\delta W = \int_\Omega \delta \rho_{\text{free}} \cdot \Phi \, \mathrm{d}^3 x = \int_\Omega \boldsymbol{\nabla} \cdot (\delta \vec{D} \, \Phi \, \mathrm{d}^3 x = \underbrace{\sum_{k=1}^{N} \oint_{\Sigma_k \Phi \delta \vec{D}} \cdot \mathrm{d}\vec{a}}_{=0} + \int_\Omega \vec{E} \cdot \delta \vec{D} \, \mathrm{d}^3 x \tag{8}$$

so

$$\delta W = \int_\Omega E_i \varepsilon_{ij}(x) \, \delta E_j \, \mathrm{d}^3 x \tag{9}$$

or

$$W = \frac{1}{2} \int_\Omega \varepsilon_{ij} E_i E_j \, \mathrm{d}^3 x \tag{10}$$

In our special case for homogeneous dielectrics,

$$W = \frac{1}{2} \int \varepsilon E^2 \, \mathrm{d}^3 x \tag{11}$$

In the no dielectric case, we have $\frac{1}{2}\int \mathrm{d}^3x\,\vec{E}_0\cdot\vec{D}_0$. When a dielectric is inserted, we can look at the change in energy, or

$$\Delta W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,\mathrm{d}^3x - \frac{1}{2}\int \vec{E}_0\cdot\vec{D}_0\,\mathrm{d}^3x \tag{12}$$

$$= \frac{1}{2}\left[\int \vec{E}\cdot\vec{D}_0\,\mathrm{d}^3x - \int \vec{E}_0\cdot\vec{D}\,\mathrm{d}^3x + \int(\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0)\,\mathrm{d}^3x\right] \tag{13}$$

The final term here is

$$\int(\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0)\,\mathrm{d}^3x = -\int \boldsymbol{\nabla}(\Phi+\Phi_0)\cdot(\vec{D}-\vec{D}_0)\,\mathrm{d}^3x \tag{14}$$

$$= \underbrace{-\int \boldsymbol{\nabla}\cdot[(\vec{D}-\vec{D}_0)(\Phi+\Phi_0)]\,\mathrm{d}^3x}_{-\left\{\sum_k \oint_{\Sigma_k}\vec{D}(\Phi+\Phi_0)\mathrm{d}\vec{a} - \oint_{\Sigma_k}\vec{D}_0(\Phi+\Phi_0)\mathrm{d}\vec{a}\right\}=0} \tag{15}$$

$$+ \underbrace{\int[\overbrace{\boldsymbol{\nabla}\vec{D}}^{\rho_{\text{free}}} - \overbrace{\boldsymbol{\nabla}\vec{D}_0}^{\rho_{\text{free}}}](\Phi+\Phi_0)\,\mathrm{d}^3x}_{=0} \tag{16}$$

$$= 0 \tag{17}$$

so

$$\Delta W = -\frac{1}{2}\int \vec{P}\cdot\vec{E}_0\,\mathrm{d}^3x \tag{18}$$

Again, the field will be $\vec{E}_0$ if there were no dielectrics.

We can find the force due to this dielectric:

$$F_i = -\left.\frac{\partial W}{\partial\xi^i}\right|_{Q_k=\quad\text{fixed}} \tag{19}$$

where $\vec{\xi}$ is some displacement of the dielectric.

We also know that $W = \frac{1}{2}\int\rho\Phi$ so

$$\delta W = \frac{1}{2}\int(\delta\rho\,\Phi + \rho\,\delta\Phi)\,\mathrm{d}^3x = \int\delta\rho\,\Phi\,\mathrm{d}^3x \tag{20}$$

so if you have batteries keeping the dielectrics at constant potential, they will do some work on the system $\sum_k \delta Q_k\,\Phi_k$ which will have to be accounted for.