Lecture 43: Rotation of Observables

Friday, November 22, 2019

Let's imagine we have some observable $\hat{\mathbf{A}}$ with eigenstates $\hat{\mathbf{A}}\left|u_n\right\rangle = a_n \left|u_n\right\rangle$. We first define the rotated eigenvector: $\left|u_n'\right\rangle = \hat{\mathbf{R}}\left|u_n\right\rangle$, and next, we define $\hat{\mathbf{A}}'\left|u_n'\right\rangle = a_n\left|u_n'\right\rangle$. Therefore

$$\hat{\mathbf{A}}'\hat{\mathbf{R}}\left|u_n\right\rangle = a_n\hat{\mathbf{R}}\left|u_n\right\rangle \quad \forall\left|u_n\right\rangle$$

so

$$\hat{\mathbf{A}}' = \hat{\mathbf{R}}\hat{\mathbf{A}}\hat{\mathbf{R}}^{-1}$$

We've discussed some types of observables. Scalar observables commute with all components of angular momentum: $\left[\vec{\mathbf{J}}, \hat{\mathbf{A}}\right] = 0$. For example, $\hat{\mathbf{H}}$, $\left|\vec{\mathbf{R}}\right|^2$, $\left|\vec{\mathbf{P}}\right|^2$, $\vec{\mathbf{R}}\cdot\vec{\mathbf{P}}$, etc.

Additionally, we can have vector observables: $\vec{\mathbf{A}} = (\hat{\mathbf{A}}_x, \hat{\mathbf{A}}_y, \hat{\mathbf{A}}_z)$.

$$\mathscr{R}_{\hat{\mathbf{x}}}(\mathrm{d}\alpha) \implies \begin{cases} \hat{\mathbf{x}} \to \hat{\mathbf{x}}' = \hat{\mathbf{x}} \\ \hat{\mathbf{A}}_x \to \hat{\mathbf{A}}_x' = \hat{\mathbf{x}}'\cdot\vec{\mathbf{A}} = \hat{\mathbf{A}}_x \\ \left[\hat{\mathbf{J}}_x, \hat{\mathbf{A}}_x\right] = 0 \end{cases}$$

However,

$$\mathscr{R}_{\hat{\mathbf{y}}}(\mathrm{d}\alpha) \implies \begin{cases} \hat{\mathbf{x}} \to \hat{\mathbf{x}}' = \hat{\mathbf{x}} + \mathrm{d}\alpha\,\hat{\mathbf{y}}\times\hat{\mathbf{x}} = \hat{\mathbf{x}} - \mathrm{d}\alpha\,\hat{\mathbf{z}} \\ \hat{\mathbf{A}}_x \to \hat{\mathbf{A}}_x' = \hat{\mathbf{A}}_x - \mathrm{d}\alpha\,\hat{\mathbf{A}}_z \\ \left[\hat{\mathbf{J}}_y, \hat{\mathbf{A}}_x\right] = -\imath\hbar\hat{\mathbf{A}}_z \end{cases}$$

and similarly

$$\left[\hat{\mathbf{J}}_z, \hat{\mathbf{A}}_x\right] = \imath\hbar\hat{\mathbf{A}}_y$$

Since $\hat{\mathbf{R}} = e^{-\frac{\imath}{\hbar}\mathrm{d}\alpha\vec{\mathbf{J}}\cdot\hat{\mathbf{u}}} \approx 1 - \frac{\imath}{\hbar}\,\mathrm{d}\alpha\,\vec{\mathbf{J}}\cdot\hat{\mathbf{u}}$ so $\hat{\mathbf{A}}' = \hat{\mathbf{A}} - \frac{\imath}{\hbar}\,\mathrm{d}\alpha\left[\vec{\mathbf{J}}\cdot\hat{\mathbf{u}}, \hat{\mathbf{A}}\right]$.

We can see that vector observables are things that commute with angular momentum in a way similar to angular momentum's commutation relations: $\vec{\mathbf{J}}$, $\vec{\mathbf{L}}$, $\vec{\mathbf{S}}$, $\vec{\mathbf{R}}$, $\vec{\mathbf{P}}$, etc.

Example. 2-D Harmonic Oscillator: $\hat{\mathbf{V}}(x,y) = \frac{1}{2}m\omega^2(x^2+y^2)$, and $\hat{\mathbf{H}} = \frac{\left|\vec{\mathbf{P}}\right|^2}{2m} + \frac{1}{2}m\omega^2\left|\vec{\mathbf{R}}\right|^2 = \hat{\mathbf{H}}_x + \hat{\mathbf{H}}_y$. We can write the eigenstates as

$$\left|n_x n_y\right\rangle = \left|n_x\right\rangle \otimes \left|n_y\right\rangle = (\hat{\mathbf{a}}_x^\dagger)^{n_x}(\hat{\mathbf{a}}_y^\dagger)^{n_y}\left|00\right\rangle$$

Note that we can write the Hamiltonian as $\hat{\mathbf{H}} = (\hat{\mathbf{N}}_x + \hat{\mathbf{N}}_y + 1)\hbar\omega$, so

$$E_{n_x n_y} = (n_x + n_y + 1)\hbar\omega$$

Therefore, E_{00} is the ground state, $E_{10} = E_{01}$ are degenerate first excited states, and $E_{20} = E_{11} = E_{02}$ are degenerate second excited states. Because of this degeneracy, we know there must be another observable to add to our complete set of commuting observables (CSOCO) to get all the "good" quantum numbers to sufficiently distinguish the states of the system. The angular momentum operator $\hat{\mathbf{L}}_z = \hat{\mathbf{X}}\hat{\mathbf{P}}_y - \hat{\mathbf{Y}}\hat{\mathbf{P}}_x = \imath\hbar(\hat{\mathbf{a}}_x\hat{\mathbf{a}}_y^\dagger - \hat{\mathbf{a}}_x^\dagger\hat{\mathbf{a}}_y)$ commutes with the Hamiltonian. Our number states are not eigenstates of the angular momentum, but we can define some new raising and lowering operators (with French suffixes, of course):

$$\hat{\mathbf{a}}_d = \frac{1}{\sqrt{2}}(\hat{\mathbf{a}}_x - \imath\hat{\mathbf{a}}_y)$$

(droite/right)

$$\hat{\mathbf{a}}_g = \frac{1}{\sqrt{2}}(\hat{\mathbf{a}}_x + \imath\hat{\mathbf{a}}_y)$$

(gauche/left)

$$\left[\hat{\mathbf{a}}_{(d,g)}, \hat{\mathbf{a}}^\dagger_{(d,g)}\right] = 1$$

We can also see that

$$\hat{\mathbf{a}}_d \left|n_x n_y\right\rangle = (\cdots)\left|n_x - 1, n_y\right\rangle + (\cdots)\left|n_x, n_y - 1\right\rangle$$

$\hat{\mathbf{a}}_d$ and $\hat{\mathbf{a}}_g$ are lowering operators which take us between linear combinations of energy states. Our previous operators $\hat{\mathbf{a}}_{(x,y)}$ act on linearly polarized states which are only in the x or y direction. However, these operators act on linear combinations of linearly polarized states (which give us circularly or elliptically polarized states), the "droite" operator acting on right-polarized states and "gauche" acting on left-polarized states.

This allows us to express the Hamiltonian as

$$\hat{\mathbf{H}} = \left(\hat{\mathbf{a}}_d^\dagger \hat{\mathbf{a}}_d + \hat{\mathbf{a}}_g^\dagger \hat{\mathbf{a}}_g + 1\right)\hbar\omega = (\hat{\mathbf{N}}_d + \hat{\mathbf{N}}_g + 1)\hbar\omega$$

Additionally,

$$(\hat{\mathbf{N}}_d - \hat{\mathbf{N}}_g)\hbar = \hat{\mathbf{L}}_z$$

We can now create simultaneous eigenstates of these number operators:

$$\left|n_d n_g\right\rangle = \frac{1}{\sqrt{n_d! n_g!}}(\hat{\mathbf{a}}_d^\dagger)^{n_d}(\hat{\mathbf{a}}_g^\dagger)^{n_g}\left|00\right\rangle$$

so

$$\hat{\mathbf{H}}\left|n_d n_g\right\rangle = (n_d + n_g + 1)\hbar\omega\left|n_d n_g\right\rangle$$

and

$$\hat{\mathbf{L}}_z\left|n_d n_g\right\rangle = (n_d - n_g)\hbar\left|n_d n_g\right\rangle$$

Therefore, we now have a similar ladder of degenerate energy states: One ground state $\chi_{00} \sim e^{-r^2/2}$, two first-excited states, $\chi_{01} \sim re^{-r^2/2}e^{-\imath\varphi}$, $\chi_{10} \sim re^{-r^2/2}e^{\imath\varphi}$, and three second-excited states, $\chi_{02} \sim r^2 e^{-r^2/2}e^{-2\imath\varphi}$, $\chi_{11} \sim (r^2-1)e^{-r^2/2}$, and $\chi_{20} \sim r^2 e^{-r^2/2}e^{2\imath\varphi}$. $\diamond$