Lecture 7: Sturm-Liouville Problems with Periodic Functions

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The other case where Sturm-Liouville still works is when the function is periodic.

Example. Periodic Functions: Suppose $\mathcal{D} = -\frac{d^2}{dx^2}$. The spectrum here is $\mathcal{D}\sin\frac{n\pi x}{a} = \frac{n\pi^2}{a^2}\sin\frac{n\pi x}{a}$. $f_n \equiv \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ for odd functions which are periodic on $[-a, a]$.

If you let the intervals become $(-\infty, +\infty)$, operators like $\imath\frac{d}{dx} \to \frac{e^{\imath kx}}{\sqrt{2\pi}}$, $k \in \mathbb{R}$. The eigenvalues are no longer discrete, but $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath kx}e^{\imath k'x}dx = \delta(k-k')$ still (a generalized orthonormality condition). Also, $\frac{1}{2\pi}\int dk e^{\imath k(x-x')} = \delta(x-x')$ as a generalized completeness theorem. ◇

0.1 Cylindrical Symmetry

Take concentric cylinders with inner radius R_1, outer radius R_2, and given potentials on the edges of each cylinder. We will use cylindrical coordinates (ρ, ϕ, z). First, we will deal with the 2D version in which Φ has no z-dependence. In the example in Jackson, we have two large conducting sheets joined at an angle β at an insulated corner. To find the solution inside the wedge, given the potentials on each sheet, we can also use cylindrical coordinates.

What is the Laplacian for cylindrical coordinates?

$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$ or $d\vec{x} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$.

$$\nabla\Phi \cdot d\vec{x} = d\Phi = \partial_\rho \Phi d\rho + \cdots \tag{1}$$

$$\nabla\Phi = [\frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} + \frac{\partial\Phi}{\partial z}\hat{z}] \cdot d\vec{x} \tag{2}$$

We can use this to find the Laplacian:

$\int(\nabla\Phi)^2 d^3x = -\int \Phi\nabla^2\Phi d^3x$. The volume element is $\rho d\rho d\phi dz$. We can solve this by integrating by parts. The Laplacian is therefore

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

If you have z-dependence, you have to use Bessel functions. Let's avoid that for now. With no z-dependence, we have,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi = 0 \tag{4}$$

, assuming no inside charge.

Assume separation of variables, $\Phi = R(\rho)\Psi(\phi)$. Now we have

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}R+\frac{1}{\rho^2}\frac{1}{\Psi}\frac{d^2}{d\phi^2}\Psi=0\Rightarrow\frac{1}{R}\left(\rho\frac{d}{d\rho}\right)\left(\rho\frac{d}{d\rho}\right)R+\frac{1}{\Psi}\frac{d^2}{d\phi^2}\Psi=0=+\nu^2-\nu^2 \tag{5}$$

(they must be constants because we can vary ϕ and ρ independently but they still add up to 0).

$$\Psi_\nu = A_\nu \sin \nu\phi + B_\nu \cos \nu\phi \tag{6}$$

$$R_n u = a_\nu \rho^\nu + b_\nu \rho^{-\nu} \tag{7}$$

If $\nu = 0$, the logarithm also solves the original equation; $R_0 = a_0 + b_0 \ln \rho$. We can't have a logarithm of a dimension-full thing, so the a_0 must also have some dimensional log term to divide it out.

The general solution by superposition can be written:

$$\Phi = (A_0 + B_0\phi)(a_0 + b_0 \ln \rho) + \int d\nu (a_\nu \rho^\nu + b_\nu \rho^{-\nu})(\sin(\nu\phi + \alpha_\nu)) \tag{8}$$

In the angle problem, we restrict $\Phi\Big|_{\phi=0} = V_0$, so $\alpha_\nu = 0$, $b_0 = 0$, and $b_\nu = 0$ (if we assume finite fields at $\rho = 0$). With $\Phi\Big|_{\phi=\beta} = V_0$, we can say $A_0 = V_0$, $B_0 = 0$. $\sin \nu\beta = 0$ would mean that killing the term at the boundary requires a discrete $\nu = \frac{m\pi}{\beta}$. Therefore

$$\Phi = V_0 + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta} + \phi\right). \tag{9}$$

However, the power term is infinite as $m \to \infty$.