Particle Astrophysics

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April 30, 2021

Contents

Lecture 1: Cosmology

Tuesday, January 19, 2021

0.1 The Big Picture

The current understanding of our Universe is summarized in *two* Standard Models: The ΛCDM or concordance model and the Standard Model of Particle Physics. The first of these describes the universe's expansion in terms of free parameters (Λ, the cosmological constant). These models intertwine the large (galactic and larger) and small (particle) scales.

The ΛCDM model assumes that on the largest scales (100Mparsec), the universe is homogeneous (everywhere is the same) and isotropic (in all directions). Galaxies and smaller structures are formed from the gravitational collapse of small inhomogeneities in the background density with the amplitude of these fluctuations being $\sim \frac{\Delta\rho}{\rho_{\text{background}}} \sim 10^{-5}$ where $\rho_{\text{background}} \sim 10^{-29}\text{g/cm}^3$. We can see these inhomogeneities in temperature fluctuations in the Cosmic Microwave Background ($\frac{\Delta T}{T} \sim 10^{-5}$), which were first detected in the 1990s but predicted much earlier. The universe is undergoing Hubble expansion, and in our time, this expansion is accelerating. There is growing evidence which suggests that most of the matter in the universe is in the form of "dark matter" which only interacts gravitationally. Additionally, this model includes "dark energy" to describe the cause of the universe's accelerated expansion. The amount of the universe's energy comprised of these forms is as follows:

Λ (dark energy) $\sim 72\%$

CDM (dark matter) $\sim 23\%$

Ordinary matter $\sim 5\%$

The Standard Model of Particle Physics includes and unifies the strong, electromagnetic, and weak interactions. All of these interactions (including gravity) are described by gauge theories. In this model, there are six quarks, six leptons, eight flavors of gluons (to mediate the strong force), one photon (to mediate the electromagnetic force), and three additional massive bosons, $W^{\pm}$ and Z^0 (to mediate the weak interactions). "Quantized gravity" would predict gravitons, which are quantized gravitational waves. Finally, the Higgs boson gives mass to the quarks, leptons, and massive bosons. However, none of these particles can explain dark matter.

If we take this theory along with statistical mechanics, we can predict several phase transitions in the early universe with symmetry breaking at each phase. At $T \sim 10^{29}$K, the strong force separates from the electroweak force. At $T \sim 10^{15}$K, the electromagnetic and weak forces separate, and the weak bosons become massive. At $T \sim 10^{12}$K, quark-gluon confinement causes hadrons to form. Finally, at $T \sim 10^{9}$K, light elements form (Big Bang Nucleosynthesis).

The remainder of this semester will be filling in this "big picture" by trying to understand the connections between gravity (GR), particle physics, and (quantum) statistical mechanics. On the Planck scale ($M_{Pl} = \sqrt{\frac{\hbar c}{G}} \sim 1.2 \times 10^{19}\text{GeV}/c^2$), it is conjectured that gravity is unified with the other three interactions.

0.1.1 Astrophysical scales

$$\begin{aligned} R_\odot &\sim 7 \times 10^8 \text{m} \\ M_\odot &\sim 2 \times 10^{33} \text{g} \\ R_{\text{MW}} &\sim 15 \text{kparsec} \\ M_{\text{MW}} &\sim 10^{11} M_\odot \\ M_{\text{Clusters}} &\sim 10^{14} M_\odot \\ M_{\text{Visible Universe}} &\sim 10^{23} M_\odot \end{aligned}$$

0.1.2 Strengths and Ranges of Forces

Strong: $\alpha_{\text{S}} \sim 1$ —$\sim$ 1fm.

Electromagnetic: $\alpha_{\text{EM}} \sim \frac{1}{137}$ —$\sim \infty$.

Weak: (missed this)

Gravity: "self gravity of particles" causes the "self energy" of a mass to be $\frac{GM^2}{R}$ for radius R. The energy over the rest energy is therefore $\frac{GM^2}{RMc^2}$. If $R \sim \lambda_c \sim \frac{\hbar}{MC}$ (the Compton wavelength), then $\frac{GM^2}{\hbar c} \sim 5 \times 10^{-40}$ for $M \sim m_p$. In this respect, gravity is negligible on the scales of particle physics, but dominates on the larger scales of galaxies.

0.2 The Observed Universe and Hubble's Expansion

Non-relativistic treatment: In 1929, Hubble observed red-shifts in spectral lines from distant galaxies with the red-shift amount correlated with the luminosity distance to the galaxy. If we consider a source that emits photons isotropically with intrinsic luminosity of $L =$
$texterg/$s, which is the light energy released per unit time, then this radiation spreads uniformly. If we look at the flux of radiation on a surface at a distance d_L from the source, $F = \frac{L}{4\pi d_L^2}$ where d_L is called the "luminosity distance". F is measured by measuring the intensity of light captured by a telescope. If L is known, d_L can be extracted. To know L, we need "standard candles" which are nearby objects whose intrinsic luminosity is well known.

Hubble used (Classical) Cepheid variable stars. These stars undergo periodic pulsations of their atmospheres due to interplay between their pressure and opacity (the mean free path of photons in the star). The period is well-defined by thermodynamics and observations such that $\tau \propto L^{0.8}$. Using these periods, one can establish the intrinsic luminosity of these stars. The process of identifying these stars and finding their intrinsic luminosities and luminosity distances is referred to as "establishing rungs on a distance ladder".

Hubble then measured the difference between the expected spectral lines of these stars (based on their elemental composition) and compared it to the luminosity distances of the standard candles in these galaxies. If a moving light source is measured as it moves away, the observer will see the wavelength as $\lambda_0 = \lambda_e\sqrt{\frac{1+v/c}{1-v/c}}$. The red-shift z is defined by $z \equiv \frac{\lambda_0-\lambda_e}{\lambda_e}$ or $1+z = \frac{\lambda_0}{\lambda_e}$. For $v/c << 1$, the recession velocity is given by $\frac{\lambda_0}{\lambda_e} \sim 1 + v/c$, so $v \cong cz$. Hubble plotted the red-shift z against luminosity distance d_L and found a straight line correlation.

$$v = H_0 r \qquad \text{(Hubble's Law of Universal Expansion)}$$

where H_0 is Hubble's constant, v is the velocity of recession, and $r = d_L$ is the distance. This law only holds on very large scales, and if we look on smaller scales, stars are subject to the gravitational pull of other stars and obtain velocities which are not described by Hubble's law, which we call peculiar velocities:

$$v = H_0 r + v_p$$

These peculiar velocities were an early indicator of the presence of dark matter.

Hubble's expansion can be thought of like inflating a balloon. Observers at rest on the surface of the balloon are called "comoving" and maintain a constant angular separation as the balloon inflates. If the radial expansion is determined by $R(t)$, then the distance which separates the observers ($d(t)$) is

$$d(t) = R(t)\theta$$

and their recession velocity is

$$\frac{\mathrm{d}d(t)}{\mathrm{d}t} = \dot{R}(t)\theta$$

so

$$v(t) = H(t)d(t)$$

where $H(t) = \frac{\dot{R}(t)}{R(t)}$.

If the observers are not comoving, then

$$v = H(t)d(t) + \underbrace{R(t)\dot{\theta}(t)}_{v_p}$$

Note that $H(t)$ has units of 1/time, so the time scale of expansion is $\tau \sim \frac{1}{H_0} \sim 1.4 \times 10^{10}$years. This tells us that the universe today is around 13.5 billion years old. The Hubble constant looks constant on our small human time scale, but from the above analysis, it should have some time dependence. We'll learn more about its time-evolution later.

Hubble's expansion can be made manifest by writing $R(t) = R_0 \frac{a(t)}{a(t_0)}$ where $a(t)$ is a scale factor, a dimensionless variable that scales lengths as a function of time describing the expansion (t_0 is an arbitrary reference time). By this formulation, $H(t) = \frac{\dot{a}(t)}{a(t)}$, so

$$v(t) = \frac{\dot{a}(t)}{a(t)} d(t) + v_p$$

According to Hubble's laws, all length scales stretch by the scale factor, so for example, the wavelength of light would be $\lambda(t) = a(t)\lambda_c$ (where λ_c is the wavelength of a comoving emitter). If a signal is emitted at t_e and observed at t_0 with wavelengths λ_e and λ_0 respectively, then

$$\lambda_e = a(t_e)\lambda_c \qquad \lambda_0 = a(t_0)\lambda_c$$

so

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)} \equiv \lambda_c \implies \lambda_0 = \lambda_e \frac{a(t_0)}{a(t_e)}$$

If t_0 is "today" and t_e is the emission time in the past, then $\frac{\lambda_0 - \lambda_e}{\lambda_e} = z_e$. Therefore, the scale factor at time of emission is related to the red-shift by

$$a(t_e) = \frac{a(t_0)}{1 + z_e}$$

For a source at red-shift z_e at a luminosity distance $d_e = \frac{z_e c}{H_0}$, the Universe at time of emission was smaller by a factor of $\frac{1}{1+z_e}$. In Hubble's expansion lies the intrinsic idea that the universe was smaller and therefore hotter in the past. In the next class, we will look at the evolution of this scale factor solely based on Newtonian cosmological dynamics. Once we have a Classical understanding of this, we can start to incorporate the effects of General Relativity.

Lecture 2: Time Evolution of Hubble's Law

Thursday, January 21, 2021

If we take a symmetric shell of radius $R(t)$, we find that it should have mass

$$M(R) = \frac{4\pi}{3}\rho R$$

where ρ is the density of the shell.

Then the total energy of this shell is conserved since the shell is at R and the mass inside is $M(R)$, so

$$E = \frac{1}{2}mv^2 - \frac{GM(R)m}{R}$$

where $v = \dot{R}$. Let's now combine this with Hubble's Law:

$$v(t) = H(t)R(t), \qquad H(t) = \frac{\dot{a}(t)}{a(t)}, \qquad R(t) = R_0 a(t)$$

The third ingredient is conservation of mass. The total mass in the volume is constant, even though the volume is expanding:

$$\rho(t)V(t) = \text{const.}$$

but

$$V(t) = V_0 \frac{a^3(t)}{a^3(t_0)}$$

and

$$V_0 = \frac{4\pi}{3} R_0^3$$

so

$$\rho(t) = \rho_0 \left(\frac{a(t_0)}{a(t)} \right)^3$$

This is not a surprise; the expansion increases the volume, but dilutes the mass, so the density should decrease. Note that if $E < 0$, R increases and v^2 decreases and reaches 0 at some fixed R_{max}. We call such a universe "closed" or "bounded".

If $E > 0$, then v^2 is not zero as $R \to \infty$. The universe is "unbounded" and "open" with $R \leq \infty$, and expansion continues forever.

Finally, if $E = 0$, $v^2 \to 0$ as $R \to \infty$, so the universe is still unbounded, but the expansion comes to a halt as $t \to \infty$. We will call this case "flat".

From the conservation of mass and energy, we find

$$\begin{aligned} E &= \frac{1}{2} m H^2(t) R^2(t) - \frac{4\pi}{3} G \rho(t) R^2(t) m \\ &= \frac{1}{2} m \left[H^2(t) a^2(t) R_0^2 - \frac{8\pi}{3} G \rho(t) R^2(t) \right] \end{aligned}$$

We define a constant κ by

$$E = \frac{1}{2} m c^2 R_0^2 \kappa$$

where κ has units of $1/(\text{length})^2$. From this we find

$$H^2(t) = \frac{8\pi}{3} G \rho(t) - \frac{\kappa c^2}{a^2(t)} \qquad \text{(Friedmann's Equation)}$$

The energy is negative when κ is negative, positive when κ is positive, and zero when κ is zero, so we can think of κ as the curvature (we will show this more definitively when we study GR).

For now, just take it for granted that this equation does describe the dynamics of $H(t)$ beyond Newtonian dynamics. We will later derive the same equation in GR.

From the First Law of Thermodynamics, $\mathrm{d}U = -P\,\mathrm{d}V + \mathrm{d}Q$ where U is the internal energy, P is pressure, and Q is heat exchange with the environment. However, the universe, by definition, doesn't have an outside environment; it contains everything. There is no heat exchange with any environment, and we treat the universe as a closed system. This implies the expansion of the universe is adiabatic:

$$\mathrm{d}U = -P\,\mathrm{d}V$$

Defining energy as $U = \rho c^2 V$, then

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -P \frac{\mathrm{d}V}{\mathrm{d}t} \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c^2 V\right) = -P \frac{\mathrm{d}V}{\mathrm{d}t}$$

so

$$\dot{\rho} c^2 V + \rho c^2 \dot{V} + P \dot{V} = 0$$

but

$$\dot{V} = 3 \frac{\dot{a}(t)}{a(t)} (a^3 V_0) = 3 H V(t)$$

so

$$\dot{\rho} + 3H(t) \left[\rho + \frac{P}{c^2} \right] = 0$$

This is energy conservation in an expanding cosmology from a thermodynamic perspective. We will later see in GR that this is a covariant conservation equation.

0.3 Equation of State

A gas of photons in equilibrium at temperature T has an energy $U = \sigma T^4 V$ where σ is proportional to the Stephan-Boltzmann constant and $P = \frac{1}{3}\frac{U}{V}$ with $U \equiv \rho c^2 V$, so

$$P = \frac{1}{3}\rho c^2$$

This is the equation of state of radiation. Then our conservation equation from the previous section tells us that

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a}$$

$$\rho_R(t) = \rho_{0,R}\left(\frac{a_0}{a(t)}\right)^4 \implies \rho_R(t)a^4(t) = \text{const.}$$

The entropy of a photon gas is $S \propto T^3 V$. For an adiabatic process, $S = \text{const.}$ so $T^3(t)V_0\left(\frac{a(t)}{a_0}\right)^3 =$ const.. Therefore,

$$T(t) = \frac{T_0 a_0}{a(t)} \sim \frac{1}{a(t)}$$

and

$$\rho c^2 = \frac{U}{V} \propto T^4 = \frac{T_0^4 a_0^4}{a^4(t)}$$

Next, let's look at non-relativistic matter ($\frac{1}{2}mv^2 \ll mc^2$ or $\frac{v^2}{c^2} \ll 1$).

In an ideal gas, the equation of state is $PV = Nk_BT$ and the mass density is $\frac{N}{V}m \equiv mn = \rho$. Then

$$P = \frac{N}{V}m\frac{k_BT}{m} = \rho\frac{k_BT}{m}$$

Recall that by the equipartition theorem in an 3D ideal gas, $\frac{1}{2}m\left\langle v^2\right\rangle = \frac{3k_BT}{2}$, so

$$P = \frac{1}{3}\rho\left\langle v^2\right\rangle$$

Then,

$$\frac{P}{\rho c^2} = \frac{1}{3}\frac{\left\langle v^2\right\rangle}{c^2} \ll 1$$

For non-relativistic matter, this implies

$$\dot{\rho} + 3H\rho = 0$$

and

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}$$

so

$$\rho_M(t) = \frac{\rho_{0,M}a_0^3}{a^3(t)}$$

The third possible state of matter is related to the cosmological constant, matter at a constant energy density $\rho_\Lambda(t)c^2 \equiv \frac{\Lambda c^2}{8\pi G}$. Einstein introduced this constant such that

$$\rho_{\text{tot}} = \rho_M + \rho_\Lambda$$

and

$$H^2 = \frac{8\pi G}{3}\rho_M + \frac{1}{3}\Lambda - \frac{\kappa c^2}{a^2} = 0$$

Of course, we know this isn't observationally zero. Using our conservation equation,

$$\rho_\Lambda(t) \equiv \rho_0\Lambda$$

$$P_\Lambda = -\rho_\Lambda c^2$$

By observation, Λ is positive, so whatever it is, it needs to have negative pressure, whatever that means.

LECTURE 3:
Thursday, January 28, 2021

[Lorentz transforms of covariant and contravariant 4-vectors]

0.3.1 Contravariant 4-Velocity

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

where x^μ is contravariant and τ is invariant $(c^2\tau^2 = c^2\tau^2 - \vec{\mathbf{x}}^2)$. Under LT, $x'^\mu = \Lambda^\mu_\nu x^\nu$ and $\tau \to \tau$ so

$$u'^\mu = \Lambda^\mu_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \lambda^\mu_\nu u^\nu$$

In components, $u^\mu = \left(c\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}\tau}\right)$. Therefore the scalar product is $u^\mu u_\mu = \eta_{\mu\nu}u^\mu u^\nu = c^2$. This is an invariant under LT.

From the expression for proper time,

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \frac{\vec{\mathbf{v}}^2}{c^2}} = \frac{\mathrm{d}t}{\gamma}$$

or

$$\mathrm{d}t = \gamma\,\mathrm{d}\tau$$

$\mathrm{d}\tau$ is the time measured by an observer at rest, so

$$u^\mu = \gamma\left(c, \frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t}\right) = \gamma(c, \vec{\mathbf{v}})$$

0.3.2 Contravariant 4-Momentum

$$p^\mu = mu^\mu = m\gamma(c, \vec{\mathbf{v}})$$
$$p^\mu p_\mu = m^2u^\mu u_\mu = m^2c^2$$

Therefore,

$$p^\mu = \gamma(mc, m\vec{\mathbf{v}}) \equiv \left(\frac{E}{c}, \vec{\mathbf{p}}\right)$$

where

$$E = \gamma mc^2 \qquad \vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}}$$

Then $p^\mu p_\mu = \frac{E^2}{c^2} - \vec{\mathbf{p}}^2 = m^2c^2$, so we get

$$E = \sqrt{\vec{\mathbf{p}}^2c^2 + m^2c^4} \tag{Dispersion Relation}$$

0.3.3 Contravariant Force

We define the force as

$$F^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = m\underbrace{\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}}_{a^\mu}$$

where a^μ is the 4-acceleration.

0.4 General 4-Vectors

Contravariant 4-vectors are vectors V^μ which transform under LT as $V'^\mu = \Lambda^\mu_\nu V^\nu$. Covariant 4-vectors transform under an inverse Lorentz transform.

0.4.1 Tensors

Tensors are 2nd rank (two index) objects which transform as

$$T'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$$

contravariantly. They transform as the direct product of two contravariant 4-vectors. Similarly, covariant tensors transform as

$$T_{\prime\mu\nu} = \tilde{\Lambda}^\alpha_\mu \tilde{\Lambda}^\beta_\nu T_{\alpha\beta}$$

where $\tilde{\Lambda} = \Lambda^{-1}$.

If a 4-vector vanishes or is constant in one frame, it is constant in any inertial frame related by that frame with a Lorentz transformation. This is the principle of covariance; the equations of motion maintain the same form in different frames.

0.5 Conservation Laws in Particle Kinematics

Consider collisions $A + B \to C + D$.

$$F^\mu = 0 \implies p^\mu_A + p^\mu_B = p^\mu_C + p^\mu_D$$

This is energy-momentum conservation. This brings us to an important concept of "threshold energy". In Classical mechanics, the minimum energy for a reaction like in the center of mass is the threshold energy. Energy isn't covariant, but in the center of mass, $\vec{\mathbf{p}}_A = -\vec{\mathbf{p}}_B \equiv \vec{\mathbf{p}}$. At threshold, the daughter particles are produced at rest: $\vec{\mathbf{p}}_C = \vec{\mathbf{p}}_D = 0$. Then the threshold condition says $E_A(p) + E_B(p) = (m_C + m_D)c$.

Photons are massless with energy $\hbar\omega(k) = \hbar c\left|\vec{\mathbf{k}}\right|$, so the momentum is

$$p^\mu = \left(\frac{\hbar\omega}{c}, \hbar\vec{\mathbf{k}}\right)$$

and $p^\mu p_\mu \equiv 0$, so this is a "null" 4-vector.

Principle of Covariance (elaborated): Because we assume physics works the same in all inertial frames, physical laws (equations of motion) must be written in terms of tensors, 4-vectors, etc.

An astrophysical application is the Greisen-Zatsepin-Kuzmin (GZK) cut-off in UHECR (astro-ph 0309027). Observations of "air showers" produced by cosmic rays hitting the upper atmosphere generate protons with energies around 10^{19}–10^{20}eV (like a baseball at 100km/h). In the 70's, GZK said that UHE protons would lose energy as they scatter off CMB radiation:

$$p + \gamma_{\text{CMB}} \to \Delta^+ \to p + \pi^0$$

or

$$p + \gamma_{\text{CMB}} \to \Delta'^+ \to n + \pi^+$$

The Lorentz factor is $\gamma = \frac{E}{mc^2} \sim \frac{10^{20}}{10^9} \sim 10^{11}$ so $\frac{v_p}{c} \sim 1 - 5 \times 10^{-24}$. The threshold energy for proton-pion production can be calculated as $E_{\text{th}} = (m_p + m_\pi)c^2 \simeq 1.14 \times 10^9$eV.

$$(p^\mu_p + p^\mu_\gamma) = \left(\frac{E_{\text{th}}}{c}, 0\right)$$

The scalar product of this should be $-(m_p+m_\pi)^2c^2$. We can calculate these invariants in the "rest frame" of the CMB. There is no rest frame for an individual photon, but we can think of the CMB as a fluid with a Boltzmann-Einstein distribution function. In this frame, the energy density is σT^4, so

$$p_p^\mu = \left(\frac{E_p^{\text{CMB}}}{c}, \vec{\mathbf{p}}\right)$$

and

$$p_\gamma^\mu = \frac{E_\gamma^{\text{CMB}}}{c}\left(1, \hat{p}_\gamma\right)$$

since $|\vec{\mathbf{p}}_\gamma| = \frac{E_\gamma}{c}$. Assume consistently that $|\vec{\mathbf{p}}_p| \gg m_p c$ (very high-energy) so $E_p^{\text{CMB}} \simeq c|\vec{\mathbf{p}}_p|$. Then

$$p_p^\mu p_{\gamma\mu} = \frac{E_p^{\text{CMB}} E_\gamma^{\text{CMB}}}{c^2}\left(1 - \hat{p}_p \cdot \hat{p}_\gamma\right)$$

For a head-on collision, $\hat{p}_p \cdot \hat{p}_\gamma = -1$. Given the average CMB photon energy, we can see that $E_p^{\text{CMB}} \sim 3\times 10^{20}$eV. This is the threshold energy required for this collision to happen.

One can measure the cross-section for proton-photon scattering to be $\sigma \sim 10^{-28}\text{cm}^2$. In a medium with n scatterers per unit volume, the mean free path is $\lambda = \frac{1}{\sigma n}$. In the CMB, $n = 422\text{cm}^{-3}$, so $\lambda \sim 3\times 10^{25}\text{cm} \sim 10\text{Mpc}$.

We have calculated that the energies of the protons for these reactions to happen need to be $\sim 10^{20}$eV. Knowing the typical mean free path, and knowing that in each collision the proton loses about 0.2 of its energy, we conclude that we should not see protons like this coming from further than around 50Mpc. This is called the GZK cutoff. This was confirmed in 2008. In the rest frame of the protons, the photons are coming at them with $\gamma \sim 10^{11}$, which means that the CMB photons are coming with about the energy of a pion, which explains the production.

0.6 Basics of G.R.

We begin by discussing the equivalence principle. We define the inertial mass through Newton's law:

$$\vec{\mathbf{F}} = m_I \vec{\mathbf{a}}$$

For the gravitational force,

$$\vec{\mathbf{F}} = -m_G \vec{\boldsymbol{\nabla}}\Phi = m_I \vec{\mathbf{a}}$$

where we say m_G is the gravitational mass and m_I is the inertial mass. From this,

$$\vec{\mathbf{a}} = -\left(\frac{m_G}{m_I}\right)\vec{\boldsymbol{\nabla}}\Phi$$

Experimentally, we find that all bodies fall in a gravitational field with the same acceleration, so m_G/m_I is the same for all bodies (constant). This constant can be absorbed into Newton's constant in the potential such that $m_G = m_I$.

Consider an elevator falling freely in the gravitational field of the earth. Inside the accelerator there is an observer and some other objects. The observer sees things moving either at rest or with a constant velocity relative to him in absence of external forces. Therefore, the observer describes the motion of the objects as if they were in an inertial frame.

For this observer in free fall, there is no gravitational field. This is a consequence of the equivalence principle. A gravitational field can be compensated by going to a freely falling frame.

0.6.1 Tidal Forces

This analysis isn't quite correct. It would be correct if the field was homogeneous, but we know that it isn't, since the field is generated by the earth. If the person is at the center of the elevator, objects around them will experience slightly less acceleration along z than the observer and will slowly accelerate towards the observer without any force acting on them. These apparent forces which are the result of an inhomogeneous gravitational field are called "tidal forces".

The Weak Equivalence Principle states that a freely falling observer experiences no gravitational field in a small neighborhood (small enough to neglect tidal forces). This is a consequence of $m_I = m_G$ for all bodies.

Einstein takes this a step further with the Strong Equivalence Principle. This states that the Weak E.P. holds for massless particles (photons).

LECTURE 4: GENERAL RELATIVITY
Tuesday, February 02, 2021

In the last lecture, we talked about how the Strong Equivalence Principle says that the Weak Equivalence Principle holds for massless objects. Before we understand why one would take such a principle as true, we should look at the consequences of this principle.

Imagine we have an emitter and detector inside our freely falling elevator, positioned on the floor and ceiling of the elevator respectively. If we think of the elevator as an inertial frame, then to an outside observer, the detector has gained a velocity gt where $t = h/c$, with h being the height of the elevator. Then the frequency measured by the detector is

$$\nu_d = \nu_e \left[\frac{1+v/c}{1-v/c}\right]^{1/2} \sim \nu_e \left(1 + \frac{v}{c}\right)$$

or

$$\frac{\nu_d - \nu_e}{\nu_e} = \frac{v}{c} = \frac{gh}{c^2}$$

This is just the Doppler effect. Both observers must agree in their observation. The Earth observer concludes that in climbing up the gravitational field, the photon must have lost energy (red-shift) which exactly cancels the blue-shift from the Doppler effect:

$$\left.\frac{\Delta\nu}{\nu}\right|_{\text{grav}} = -\frac{gh}{c^2}$$

with $g = \frac{GM_E}{R_E^2}$, so

$$\left|\left.\frac{\Delta\nu}{\nu}\right|_{\text{grav}}\right| = \frac{1}{2}\frac{h}{R_E}\frac{2GM_E}{c^2 R_E}$$

We label $\frac{2GM_E}{c^2}$ as the Schwarzschild radius of the Earth.

Let's estimate some of these numbers. Say the elevator has $h \sim 1\text{m}$ and $g \sim 10\text{m/s}^2$. This gives $\frac{\Delta\nu}{\nu} \sim 10^{-16}$. This red-shift was measured by looking at the red-shift of Fe^{57} with $h = 22.5\text{m}$ at a tower in Harvard.

The second effect of the Strong E.P. is light bending in a gravitational field (gravitational lensing). Consider again an observer in a freely falling frame, but now a photon is directed across this frame (perpendicular to the gravitational field). In the freely-falling frame, this photon follows a straight line. To an Earth observer, the detector has dropped at a velocity gt in the time it takes the photon to hit it, so the observer sees a bent path. The elevator has fallen a distance $\frac{1}{2}gt^2 = \frac{1}{2}g\frac{L^2}{c^2}$. The deflection angle can be found as

$$\tan(\varphi) \sim \varphi = \frac{1}{2}\frac{g\left(\frac{L^2}{c^2}\right)}{L} \sim \frac{1}{2}\frac{GM_E L}{c^2 R_E^2} \sim \frac{1}{4}\left(\frac{2GM_E}{c^2 R_E}\right)\frac{L}{R_E}$$

The Earth observer will explain the light trajectory as being bent by the gravitational field. The arc length, $L\varphi$, is about 5mm. If we allow the elevator to have a diameter of $L = 2R_E$, we find $\varphi \sim \frac{1}{2} \times 10^{-9}$rad.

0.7 Gravity as Geometry

0.7.1 Geodesics

In the gravitational field of the Earth, we have $z(t) = z_0 + v_0 t - \frac{1}{2}gt^2$. We can think of this as a coordinate transformation $z \to z' = z + \frac{1}{2}gt^2$. This is a general coordinate transformation $x^\mu \to x'^\mu(x)$ such that

$$\mathrm{d}x'^\mu = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}}_{\Lambda^\mu_\nu(x)} \mathrm{d}x^\nu$$

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \underbrace{\eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}}_{g_{\alpha\beta}} \mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta$$

Einstein reasoned that gravity can be removed locally using such a transformation, but not everywhere because of tidal forces. There is no single coordinate transformation that gets rid of gravity everywhere unless we consider a curved geometry of spacetime.

Consider two ants moving away from each other at constant velocity on the surface of a sphere in straight lines. As they walk away from each other, the distance between the ants does not increase uniformly, but rather appears as acceleration. Similarly consider a funnel. Particles sliding down the funnel at constant velocity will appear to be accelerating towards each other in the absence of any other forces. This is similar to the idea of tidal forces.

We can generalize our spacetime invariant $\mathrm{d}s^2 \equiv c^2\,\mathrm{d}\tau^2$ to $\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. Additional coordinate transforms can be absorbed into new metrics, so the metric transforms as a tensor under general coordinate transformations. This generalizes Lorentz transformations to general coordinate transformations. The proper distance between two points is

$$c\int_{\tau_A}^{\tau_B} \mathrm{d}\tau = \int \sqrt{g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}$$

Trajectories that minimize the proper distance in a geometry determined by the metric $g_{\mu\nu}(x)$ are called "geodesics". Rather than minimize the square root, we can use the Euler-Lagrange equations with $L = g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}L}{\mathrm{d}\dot{x}^\mu}\right] - \frac{\mathrm{d}L}{\mathrm{d}x^\mu} = 0$$

This implies that

$$\ddot{x^\mu} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \qquad \text{(Geodesic Equation)}$$

where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}\left(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha} - \partial_{\rho g_{\alpha\beta}}\right) \qquad \text{(Christoffel Symbols)}$$

where $g^{\mu\rho}g_{\rho\sigma} = \delta^\mu_\sigma$.

The solution to the geodesic equation gives trajectories parameterized in τ which describe paths that minimize proper distance.

Lecture 5: Geometry of Spacetime

Thursday, February 04, 2021

The equation of motion (Newton) of a particle in a gravitational field is $\frac{\mathrm{d}^2\vec{\mathbf{x}}}{\mathrm{d}t^2} = -\vec{\boldsymbol{\nabla}}\Phi$, so our geodesic equation should give us this in the non-relativistic limit. Also, in the local inertial frame, we don't see any curvature, so these Christoffel symbols must vanish in the local frame. These are derivatives of $g_{\mu\nu}$, so these derivatives must also vanish.

First, we consider that the weak field limit of $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ where $h \ll \eta$. Then in all Christoffel symbols, only derivatives of h matter, and the time derivative should vanish:

$$\Gamma^{\mu}_{\alpha\beta} \sim \frac{1}{2}\eta^{\mu\rho}(\partial_\alpha h_{\rho\beta} + \partial\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta})$$

We need $\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \left(c\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ for the weak field/non-relativistic limit ($c \gg v$). Setting the spatial part to zero, from special relativity, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \approx 1$, so only $\alpha = 0$ and $\beta = 0$ components and $\dot{x}^0 = c$:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\Gamma^{\mu}_{00}\underbrace{\dot{x}^0\dot{x}^0}_{c^2}$$

where

$$\Gamma^{\mu}_{00} = \frac{1}{2}\eta^{\mu\rho}\left(\cancelto{0}{\partial_0 h_{\rho 0}} + \cancelto{0}{\partial_0 h_{0\rho}} - \partial_\rho h_{00}\right)$$

$\partial_0 h_{00} = 0$, so

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^{22}} = -c^2 \underbrace{\Gamma^{i}_{00}}_{\frac{1}{2}\vec{\boldsymbol{\nabla}}^i h_{00}} = -\vec{\boldsymbol{\nabla}}^i\Phi$$

Therefore,

$$h_{00}(x) = +\frac{2\Phi}{c^2} = -\frac{2GM}{rc^2} \equiv -\frac{R_s}{r}$$

where R_S is the Schwarzschild radius. To get an idea of this, $R_S = \frac{2GM}{c^2} = \frac{2GM_\odot}{c^2}\left(\frac{M}{M_\odot}\right)$ where the Schwarzschild radius of the sun is 2.97km. The radius of the earth is about 5000km, so this h_{00} is incredibly small.

$$g_{00} = \eta_{00} + h_{00} \simeq \left(1 + \frac{2\Phi}{c^2}\right)$$

and

$$g_{ij} = \eta_{ij} = -\delta_{ij}$$

to leading order. Therefore

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = (c\,\mathrm{d}t)^2\left[1 + \frac{2\Phi}{c^2}\right] - \mathrm{d}\vec{\mathbf{x}}^2 + \cdots$$

0.8 Connection to Geometry

Gravitational potentials obey Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho$$

and outside a spherical distribution of mass M, $\Phi(r) = -\frac{GM}{r}$. We've shown that this potential has a direct relation to the metric and to geodesics in spacetime, so we can conclude that mass leads to geometry in spacetime.

0.9 Principle of General Covariance

Under general coordinate transformations, $x^\mu \to x'^\mu(x)$, the differentials,

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\mu \equiv \Lambda^\mu_\nu(x)\,\mathrm{d}x^\nu$$

follow the transformation laws of contravariant 4-vectors. covariant 4-vectors are formed as

$$x_\mu = g_{\mu\nu}(x)x^\nu$$

Tensors transform as

$$T^{\mu\nu}(x) \to T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}T^{\alpha\beta}(x)$$

Again, if a tensor vanishes in a coordinate system, it vanishes in all inertially related systems.

Christoffel symbols are **not** tensors. They may vanish in a local frame, but they do not vanish in every frame.

General covariance in GR says that physical laws must be written in terms of tensors. However, generally, equations of motion are differential equations, so we need derivatives that transform as tensors.

0.9.1 Covariant Derivatives

We will define the covariant derivative of a tensor $T^{\mu\nu}(x)$ as

$$T^{\mu\nu}_{;\alpha} = \frac{\partial T^{\mu\nu}(x)}{\partial x^\alpha} + \Gamma^\mu_{\alpha\sigma}T^{\sigma\nu} + \Gamma^\nu_{\alpha\sigma}T^{\mu\sigma}$$

The lack of covariance of the ordinary derivative gets exactly compensated by the two Christoffel symbols. This is similar to the way we define derivatives in a gauge theory. This is no coincidence; GR is a gauge theory, and the group of gauge transformations are the group of generalized transformations. In gauge theories, the Christoffel symbols would be vector potentials.

Lecture 6: Parallel Transport

Tuesday, February 09, 2021

Consider walking on a sphere following a particular geodesic from the north pole to the equator carrying a vector which points south. Continue walking along the equator, another geodesic, and then back to the north pole. The vector will now point in a different direction, and the difference is proportional to the Riemann curvature tensor:

$$R^\alpha_{\beta\rho\gamma} = \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\rho} - \frac{\partial \Gamma^\alpha_{\beta\rho}}{\partial x^\gamma} + \Gamma^\alpha_{\sigma\rho}\Gamma^\sigma_{\beta\gamma} - \Gamma^\alpha_{\sigma\gamma}\Gamma^\sigma_{\beta\gamma} \qquad \text{(Riemann Tensor)}$$

The first two terms do not vanish in a local inertial frame and are associated with tidal forces (the other two terms do). This is a rank 4 tensor, and due to symmetries, there are 20 independent components.

By the equivalence principle, in a (small) neighborhood, we can set $g_{\mu\nu} = \eta_{\mu\nu}$ such that the Christoffel symbols $\Gamma \sim \partial g \equiv 0$, but *not* all $\partial^2 g = 0$. Twenty of these components are nonzero, so these are curvature independent components which describe tidal forces. Covariant derivatives of the Riemann tensor obey the Bianchi identity which allows for a reduction in the number of components to 20. Further reduction can be created by various constructions:

$$g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} \equiv R_{\beta\delta} \qquad \text{(Ricci Tensor)}$$

$$g^{\beta\delta}R_{\beta\delta} = R \qquad \text{(Ricci Scalar)}$$

Finally, we have

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad \text{(Einstein Tensor)}$$

which is symmetric and covariantly conserved:

$$G_{\mu\nu;\nu} = 0$$

This tensor carries information about the geometry. We have also seen a relationship between mass distributions and this geometry. We can show that

$$G_{\mu\nu} = -\frac{8\pi G_N}{c^4} T_{\mu\nu}$$

where G_N is Newton's constant and $T_{\mu\nu}$ is the energy momentum tensor, which is also symmetric and covariantly conserved. It can also be shown that $g_{\mu\nu;\nu} = 0$.

0.10 The Energy Momentum Tensor

What is $T_{\mu\nu}$? Consider a fluid of particles of masses m_i at positions $\vec{\mathbf{x}}_i(t)$ where the total number of particles $N \gg 1$. We can describe this in terms of a continuum density:

$$\rho(\vec{\mathbf{x}}; t) = \sum_{i=1}^{N} m_i \delta^{(3)}(\vec{\mathbf{x}} - \vec{\mathbf{x}}_i(t))$$

so that

$$\int \mathrm{d}^3 x\, \rho(\vec{\mathbf{x}}; t) = \sum_{i=1}^{N} m_i \equiv M$$

the total mass. For each particle, we can introduce a momentum $\vec{\mathbf{p}}_i(t) = m_i \vec{\mathbf{v}}_i(t) = m_i \frac{\mathrm{d}\vec{\mathbf{x}}_i(t)}{\mathrm{d}t}$ and a momentum density

$$\vec{\mathbf{J}}(\vec{\mathbf{x}}; t) = \sum_{i=1}^{N} m_i \vec{\mathbf{v}}_i(t) \delta^{(3)}(\vec{\mathbf{x}} - \vec{\mathbf{x}}_i)$$

such that

$$\int \mathrm{d}^3 x\, \vec{\mathbf{J}}(\vec{\mathbf{x}}; t) = \vec{\mathbf{P}}$$

the total momentum. Note that

$$\begin{aligned} \frac{\partial \rho(\vec{\mathbf{x}}; t)}{\partial t} &= \sum_i m_i \frac{\partial}{\partial t} \delta^{(3)}(\vec{\mathbf{x}} - \vec{\mathbf{x}}_i(t)) \\ &= -\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} \end{aligned}$$

so

$$\frac{\partial \rho(\vec{\mathbf{x}}; t)}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}}(\vec{\mathbf{x}}; t) \equiv 0$$

This is a continuity equation for our fluid of particles. We can then introduce the idea of flow by starting with the average velocity of a set of particles as:

$$\vec{\mathbf{v}}_{\text{avg}} = \frac{\sum_i m_i \vec{\mathbf{v}}_i}{\sum_i m_i}$$

and thinking of the average velocity of the fluid of particles as

$$\vec{\mathbf{v}}(x; t) \equiv \frac{\vec{\mathbf{J}}(\vec{\mathbf{x}}; t)}{\rho(\vec{\mathbf{x}}; t)}$$

or

$$\rho(\vec{\mathbf{x}}; t) \vec{\mathbf{v}}(x; t) \equiv \vec{\mathbf{J}}(\vec{\mathbf{x}}; t)$$

Our continuity equation can now be written as

$$\frac{\partial\rho(\vec{\mathbf{x}};t)}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \rho(\vec{\mathbf{x}};t)\vec{\mathbf{v}}(\vec{\mathbf{x}};t) = 0$$

where this second term can now be thought of as the flow of the particle fluid.

Consider a fluid composed of cells. In each cell there is a local pressure, velocity, and momentum, along with any other local thermodynamic variables. Consider a point $\vec{\mathbf{x}}$ and a local quantity $Q(\vec{\mathbf{x}};t)$, the Eulerian time derivative corresponds to keeping the point fixed in space: $\left.\frac{\partial Q(\vec{\mathbf{x}};t)}{\partial t}\right|_{\vec{\mathbf{x}}}$. Now consider an observer moving along with the fluid, like a river and a boat which flows with the water (no paddling). The boat drifts along with the flow of the water, so they are at rest with the flow of water. If they now measure a property at time t and again at $t + \mathrm{d}t$, they have actually moved a distance $\vec{\mathbf{v}}\,\mathrm{d}t$ between measurements, or $Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\,\mathrm{d}t\,; t + \mathrm{d}t)$. Then we can define the Lagrangian derivative as

$$\left.\frac{Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\,\mathrm{d}t\,; t + \mathrm{d}t) - Q(\vec{\mathbf{x}};t)}{\mathrm{d}t}\right|_{\mathrm{d}t\to 0} = \frac{\partial Q}{\partial t} + (\vec{\mathbf{v}}(\vec{\mathbf{x}};t) \cdot \vec{\boldsymbol{\nabla}}_{\vec{\mathbf{x}}})Q(\vec{\mathbf{x}}) \equiv \frac{\mathrm{d}Q}{\mathrm{d}t}$$

This is the total derivative.

We can now define things like the total acceleration:

$$\vec{\mathbf{a}} = \frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t}a = \frac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}}$$

In components, this looks like

$$a_i = \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3}(v_j\frac{\partial}{\partial x_j})v_i$$

0.10.1 Newton's Laws for Fluids

Consider a small cell of mass Δm and volume ΔV with $\rho(\vec{\mathbf{x}}) = \frac{\Delta m}{\Delta V}$ immersed in a fluid. Then

$$\Delta m\vec{\mathbf{a}} = \vec{\mathbf{F}}_{\text{tot}} \equiv \Delta m\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t}$$

In the next lecture, we will talk about what this "total force" contains, including external forces such as gravity, with $\vec{\mathbf{F}}_{\text{grav}} = -\Delta m\vec{\boldsymbol{\nabla}}\varphi$ and the forces from pressure differences in nearby cells.

Lecture 7: Newton's Laws for Fluids

Thursday, February 11, 2021

In our last class, we ended by discussing the force acting on a cell of fluid:

$$\Delta m\vec{\mathbf{a}} = \vec{\mathbf{F}}_{\text{tot}} \equiv \Delta m\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t}$$

What goes into this $\vec{\mathbf{F}}_{\text{tot}}$? Consider a cubic cell and the forces along the z-axis. We have fluid on top of the cell exerting a pressure downwards ($P(z + \mathrm{d}z)$). There is also fluid below acting with a pressure upward ($P(z)$):

$$F_z = [P(z) - P(z + \mathrm{d}z)]\,\mathrm{d}x\,\mathrm{d}y$$

where $\mathrm{d}x\,\mathrm{d}y$ is the area perpendicular to the z-axis. In general, we can show that $\vec{\mathbf{F}} = -\vec{\boldsymbol{\nabla}}P\Delta V$. Adding gravity, we get

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = -\frac{\vec{\boldsymbol{\nabla}}P}{\rho} - \vec{\boldsymbol{\nabla}}\Phi \qquad \text{(Euler Equation)}$$

Again, we also have

$$\frac{\partial\rho}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \rho\vec{\mathbf{v}} = 0 \qquad \text{(Continuity Equation)}$$

If we include viscosity and neglect gravity, we would get the Navier-Stokes equations. We can recast these in a manner in which we can extract an energy-momentum tensor. Let's write them in terms of components:

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} = 0$$

and

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}$$

Premultiplying the first equation by v_i and the second by ρ and adding gives us:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\underbrace{(\rho v_i v_j + P\delta_{ij})}_{T_{ij}} = -\rho \frac{\partial \Phi}{\partial x_i}$$

We can then multiply by $\frac{c}{c}$:

$$\frac{1}{c}\frac{\partial}{\partial t}\underbrace{(\rho v_i c)}_{T^{0i}=T^{i0}} + \frac{\partial}{\partial x_j} T_{ij} = \underbrace{-\rho \frac{\partial \Phi}{\partial x_i}}_{\frac{F_{\mathrm{ext},i}}{V}}$$

This first term can then be rewritten

$$\frac{1}{c}\frac{\partial}{\partial t} T^{i0} + \frac{\partial}{\partial x_j} T^{ij} = \partial_\mu T^{i\mu}$$

So in the absence of gravity,

$$\partial_\mu T^{i\mu} = 0$$

Let's go back and define $T^{00} \equiv \rho c^2$, the rest energy density, such that

$$\int \mathrm{d}^3x\, T^{00} = Mc^2$$

Multiplying the continuity equation by c, we find that

$$\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot (\rho \vec{\mathbf{v}}) = \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\rho c^2}_{T^{00}} + \underbrace{\vec{\boldsymbol{\nabla}} \cdot (\rho \vec{\mathbf{v}} c)}_{\vec{\boldsymbol{\nabla}}_j T^{0j}} = 0$$

so in the absence of gravity, we have

$$\partial_\mu T^{\nu\mu} = 0$$

with

$$T^{00} = \rho c^2 \qquad T^{0i} = T^{i0} = \rho v^i c \qquad T^{ij} = \rho v^i v^j + P\delta^{ij}$$

We will show later that adding gravity back in will cause this tensor to be covariantly conserved.

For a fluid at rest,

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

$T^{\mu\nu}$ transforms as a tensor under Galilean transformation, so

$$T^{\mu\nu}(\vec{\mathbf{v}}) = \Lambda^\mu_\alpha(\vec{\mathbf{v}})\Lambda^\nu_\beta(\vec{\mathbf{v}}) T^{\alpha\beta}(0)$$

where under a Galilean transformation, $ct \to ct' = ct$ and $\vec{\mathbf{x}}' = \vec{\mathbf{x}} + \frac{\vec{\mathbf{v}}}{c}(ct)$.

Since Galilean transforms are the non-relativistic limit of special relativity, $T^{\mu\nu}$ transforms as a tensor in special relativity (again, in absence of gravity). We will then show (in homework) that

$$T^{\mu\nu} = -P\eta^{\mu\nu} + (P + \rho c^2)\frac{u^\mu}{c}\frac{u^\nu}{c}$$

where $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma(c, \vec{\mathbf{v}})$, the 4-velocity.

$T^{\mu\nu}$ is a symmetric, second-rank tensor, so both sides of this equation must also be that way.

$T^{\mu\nu}$ is also a tensor under general coordinate transforms:

$$T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}(x)$$

but

$$\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\eta^{\alpha\beta} \equiv g^{\mu\nu}(x')$$

and $u^\mu(x')$ also transforms as a 4-vector, so

$$Te^{\mu\nu}(x) = -P(x)g^{\mu\nu}(x) + (P(x) + \rho(x)c^2)\frac{u^\mu(x)u^\nu(x)}{c^2}$$

where $P(x)$ and $\rho(x)$ transform as scalars (for ideal fluids).

In general relativity, conservation laws are written in terms of covariant derivatives.

$$\partial_\mu T^{\nu\mu} = 0 = T^{\nu\mu}_{;\mu} = \partial_\mu T^{\mu\nu} + \Gamma^\nu_{\rho\alpha}T^{\rho\alpha} + \Gamma^\alpha_{\alpha\rho}T^{\nu\rho}$$

In the weak field limit, $\Gamma^0_{00} = 0$ and $\Gamma^i_{00} = \frac{1}{2c^2}\vec{\boldsymbol{\nabla}}^i\Phi$. The $\nu = 0$ component (continuity) remains the same, and we get precisely the Euler equation.

Again, consider Einstein's equations with $\Lambda = 0$:

$$G_{\mu\nu} = -\frac{8\pi G_N}{c^4}T_{\mu\nu}$$

We can also add a cosmological constant

$$T^{\mu\nu}_\Lambda = \frac{c^4\Lambda}{8\pi G_N}g^{\mu\nu}$$

since $g^{\mu\nu}_{;\nu} = 0$.

0.11 Solutions of Einstein's Equations

Let us begin with the vacuum solution of these equations. We are looking at a spherically symmetric, stationary, non-rotating distribution of matter and look at the metric *outside* this distribution (in the vacuum, where the pressure of the density vanishes). We are going to use a set of spherical coordinates (ct, r, θ, φ), which we will call the Schwarzschild coordinates. The invariant length element squared is

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \left[1 - \frac{2GM}{rc^2}\right](c\,\mathrm{d}t)^2 - \frac{\mathrm{d}r^2}{\left[1 - \frac{2GM}{c^2 r}\right]} + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2)$$

Again, $\frac{2GM}{c^2} \equiv R_S \approx 3\text{km}\frac{M}{M_\odot}$, the Schwarzschild radius. Notice that there is a singularity at $r = R_S$, which we will call an event horizon, but in other coordinates, there are no singularities. However, there is a true singularity at $r = 0$. As $r \to \infty$, we have $g_{\mu\nu} \to \eta_{\mu\nu}$.

LECTURE 8: EVENT HORIZONS AND SINGULARITIES
Tuesday, February 16, 2021

In the previous class, we discussed the Schwarzschild radius R_S. All geodesics inside R_S end up at $r = 0$. When $R_S \ll r$,

$$\mathrm{d}s^2 \simeq (c\,\mathrm{d}t)^2\left[1 + \frac{2\Phi}{c^2}\right] - \mathrm{d}r^2\left[1 - \frac{2\Phi}{c^2}\right] + r^2\,\mathrm{d}\Omega^2$$

in the weak-field limit, where $\Phi = -\frac{GM}{r}$. Centrifugal acceleration is balanced by the gravitational pull, so $v^2 = \frac{GM}{r}$, or

$$\frac{v^2}{c^2} = \frac{GM}{c^2 r} = \frac{1}{2}\frac{R_S}{r}$$

if $R_S \ll r$, then we are not only in the weak gravity limit, but also in the non-relativistic limit $v^2 \ll c^2$.

0.12 Gravitational Redshifts

The Schwarzschild coordinates describe a spacetime which is asymptotically flat at $r \to \infty$. Consider stationary observers (fixed r, θ, and φ):

$$\mathrm{d}\tau^2 = \left[1 - \frac{R_S}{r}\right] \mathrm{d}t^2 \equiv \left[1 + \frac{2\Phi}{c^2}\right] \mathrm{d}t^2$$

Since $\mathrm{d}t$ is the time measured by an observer at $r = \infty$,

$$\frac{\mathrm{d}\tau(r_1)}{\mathrm{d}\tau(r_2)} = \frac{\left[1 - \frac{R_S}{r_1}\right]}{\left[1 - \frac{R_S}{r_2}\right]}$$

so in the weak gravitational limit, $\frac{R_S}{r} \ll 1$,

$$\mathrm{d}\tau(r_1) = \left[1 + R_s\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right] \mathrm{d}\tau(r_2)$$

If $r_1 > r_2$ (higher), $\frac{1}{r_1} < \frac{1}{r_2}$ so $\mathrm{d}\tau(r_1) > \mathrm{d}\tau(r_2)$, and since the frequency is $\propto \frac{1}{\Delta\tau}$, $\nu(r_1) < \nu(r_2)$, which is a gravitational redshift. We can see also that $r \to R_S$ would mean the observer's proper time differential, $\mathrm{d}\tau(R_s) = 0$.

0.13 Gravitational Lensing

Imagine we have a very bright quasar and a large mass between it and the observer. If we allow β to be the observed angle between the lens and the source and D_d be the distance to the lens and D_{ds} to be the distance along that ray to meet perpendicular to the source (to form a triangle), then we find that

$$\theta_{\pm} = \frac{1}{2}\left[\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right]$$

where θ_E is Einstein's angle, $\theta_E = \left[\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_S}\right]$, where $D_S = D_{ds} + D_d$ describes the two angles at which we should see images of the source from the observer's perspective. In three-dimensions, this appears as a ring (Einstein ring) with an aperture angle of θ_E.

For typical galaxies with $M \sim 10^{11} M_\odot$ and distances in Gpc,

$$\theta_E \sim 0.9\text{"} \left(\frac{M}{10^{11} M_\odot}\right)^{1/2} \left(\frac{D_s}{1\text{Gpc}}\right)^{-1/2} \left(\frac{D_{ds}}{D_d}\right)^{1/2}$$

(quote mark for arcsecond).

0.14 Flat Rotation Curves

If we have stars in the outskirts of a galaxy, the halo of their rotational velocity is balancing the gravitational pull:

$$v(R) = \sqrt{\frac{GM(R)}{R}}$$

since $\frac{GM(R)}{R^2} = \frac{V^2(R)}{R}$. Consider a nearly constant density of matter *inside*: $M(R) = \frac{4\pi}{3} R^3 \rho$. Then inside the galaxy, $\frac{M(R)}{R} \sim R^2$ and $V(R) \propto R$, but a star outside the main density of the galaxy should have $V(R) \sim \frac{1}{\sqrt{R}} \sqrt{GM_{\text{tot}}}$.

LECTURE 9: COSMOLOGICAL GEOMETRY
Thursday, February 18, 2021

0.15 The Cosmological Principle

In early times, the universe was isotropic and homogeneous, properties reflected in the CMB and the large-scale distribution of galaxies. In other words, comoving observers see uniform P and ρ that only depend on time (and not space). A plane is a simple example of such a spacetime geometry, as is the 2D surface of a 3D sphere. Let's consider this example, a sphere of radius R: $x^2 + y^2 + z^2 = R^2$.

Here, the squared length element is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. We can reparameterize this in polar coordinates (in the x/y-plane):

$$\begin{aligned} &x = r\cos(\varphi) \qquad y = r\sin(\varphi) \qquad r = R\sin(\theta) \qquad z = R\cos(\theta) \\ &\implies x^2 + y^2 = r^2 = R^2\sin^2(\theta) \end{aligned}$$

In this parameterization,

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}r\cos(\varphi) - r\sin(\varphi)\,\mathrm{d}\varphi \\ \mathrm{d}y &= \mathrm{d}r\sin(\varphi) + r\cos(\varphi)\,\mathrm{d}\varphi \end{aligned}$$

For z, we can see that

$$z\,\mathrm{d}z = \frac{1}{2}\,\mathrm{d}(z^2) = -\frac{1}{2}\,\mathrm{d}(x^2) - \mathrm{d}(y^2) = -x\,\mathrm{d}x - y\,\mathrm{d}y$$

and

$$z = \sqrt{R^2 - r^2}$$

so

$$\mathrm{d}z = -\frac{r\,\mathrm{d}r}{\sqrt{R^2 - r^2}}$$

and

$$\mathrm{d}z^2 = \frac{r^2\,\mathrm{d}r^2}{R^2 - r^2}$$

Therefore, the length element squared is

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{r^2}{R^2}} + r^2\,\mathrm{d}\varphi^2 \equiv g_{rr}\,\mathrm{d}r^2 + g_{\varphi\varphi}\,\mathrm{d}\varphi^2$$

We can define the geometric curvature of a sphere by

$$\kappa = \frac{1}{R^2}$$

In 4-space, we can consider a 3D surface of a 4D sphere, introducing a fourth spatial coordinate w:

$$x^2 + y^2 + z^2 + w^2 = R^2$$

We can reparameterize this in spherical coordinates:

$$x = r\sin(\theta)\cos(\varphi) \qquad y = r\sin(\theta)\sin(\varphi) \qquad z = r\cos(\theta)$$

with

$$r = R\sin(\chi) \qquad w = R\cos(\chi)$$

From here, $w^2 = R^2 - r^2$, so $2w\,\mathrm{d}w = -2r\,\mathrm{d}r$ or

$$\mathrm{d}w = -\frac{r\,\mathrm{d}r}{\sqrt{R^2 - r^2}} \qquad \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = (\mathrm{d}r)^2 + r^2\,\mathrm{d}\Omega \qquad (\mathrm{d}w)^2 = \frac{r^2\,\mathrm{d}r^2}{R^2 - r^2}$$

Therefore,

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 = \frac{\mathrm{d}r^2}{1 - \frac{r^2}{R^2}} + r^2\,\mathrm{d}\Omega$$

where $\mathrm{d}\Omega \equiv \sin^2(\theta)\,\mathrm{d}\theta^2 + \mathrm{d}\varphi^2$. Again, the curvature of the 3D sphere is defined as $\kappa = \frac{1}{R^2}$.

0.16 Minkowski Spacetime

Now let's consider a hyperboloid in 4-space, where our fourth coordinate has a negative signature (or equivalently the other three do):

$$x^2 + y^2 + z^2 - w^2 = -R^2$$

Now

$$r = R\sinh(\chi) \qquad w = R\cosh(\chi)$$

$$2w\,\mathrm{d}w = 2r\,\mathrm{d}r \qquad w^2 = R^2 + r^2$$

Therefore

$$\begin{aligned}\mathrm{d}s^2 &= \underbrace{-\,\mathrm{d}w^2}_{-\frac{r^2\mathrm{d}r^2}{R^2+r^2}} + \underbrace{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}_{\mathrm{d}r^2 + r^2\mathrm{d}\Omega} \\ &= \frac{\mathrm{d}r^2}{1 + \frac{r^2}{R^2}} + r^2\,\mathrm{d}\Omega\end{aligned}$$

Both of these cases, the 3-sphere and 3-hyperboloid, can be summarized as

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - \kappa r^2} + r^2\,\mathrm{d}\Omega$$

where $\kappa = \pm\frac{1}{R^2}$ and σ is the spatial length element. This is the **most general** homogeneous and isotropic 3D geometry.

0.16.1 Hubble Expansion Geometry

The length element $\mathrm{d}\sigma$ describes the comoving distance on these geometries. We can include Hubble's expansion by adding a scale factor $a(t)$ and the physical distance $a(t)\,\mathrm{d}\sigma = \mathrm{d}\sigma_{\text{phys}}(t)$:

$$\mathrm{d}s^2 = (c\,\mathrm{d}t)^2 - (\mathrm{d}\sigma_{\text{phys}})^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1 - \kappa r^2} + r^2\,\mathrm{d}\Omega\right] \qquad \text{(Friedmann-Robertson-Walker Metric)}$$

(also known as FRW metric for short). In the comoving coordinates (measured by an observer at rest in the expanding cosmology) (ct, r, θ, φ), we can write the metric as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & -a^2 r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & -a^2 r^2 \end{pmatrix}$$

With $g_{\mu\nu}$, we can obtain $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ (note $G^{\mu\nu}_{;\nu} = 0$) with $T^{\mu\nu}$ of an ideal fluid:

$$T_F^{\mu\nu} = -P(t) g^{\mu\nu} + (P(t) + \rho(t) c^2) \frac{u^\mu u^\nu}{c^2}$$

with $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. For a fluid at rest in the comoving frame, there are no peculiar velocities, so $\frac{u^\mu}{c} = (1, 0, 0, 0)$.

We can include the cosmological constant:

$$T_\Lambda^{\mu\nu} = g^{\mu\nu} \frac{\Lambda c^4}{8\pi G}$$

such that $T^{\mu\nu} = T_F^{\mu\nu} + T_\Lambda^{\mu\nu}$:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

0.16.2 Newtonian Limit

Recall

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} [\rho_F + \Lambda] - \frac{\kappa c^2}{a^2} \qquad \text{(Friedmann Equation)}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho_F + 3\frac{P_F}{c^2}\right) + \frac{8\pi G}{3} \Lambda \qquad \text{(Acceleration Equation)}$$

and

$$r\dot{h}o_F + 3\frac{\dot{a}}{a} \left(\rho_F + \frac{P_F}{c^2}\right) = 0 \qquad \text{(Covariant Conservation)}$$

the last of which is a result of $T^{\mu\nu}_{;\nu} = 0$. Additionally, we know that the equation of state of the cosmological constant is $\rho_\Lambda = \Lambda = -\frac{P_\Lambda}{c^2}$.

We can see that the Newtonian limit yields the same result. We can obtain the acceleration equation from the Hubble expansion law combined with the Euler equation:

$$\vec{\mathbf{v}}(t) = H(t) \vec{\mathbf{r}}$$

where $\vec{\mathbf{r}}(t) = a(t) \vec{\mathbf{r}}_0$, where $\vec{\mathbf{r}}_0$ is comoving and time independent. Then

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} = \dot{H} \vec{\mathbf{r}}$$

$$(\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}}_r) = H(t) [x\partial_x + y\partial_y + z\partial_z]$$

and

$$(\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}}_r)(v_x; v_y; v_z) = H^2 [x\partial_x + y\partial_y + z\partial_z](x; y; z) \equiv H^2 \vec{\mathbf{r}}$$

since $\vec{\mathbf{v}} = H\vec{\mathbf{r}}$.

By the cosmological principle, P and ρ only depend on time, so we get a Poisson equation:

$$\nabla^2 \Phi = 4\pi G \rho \implies \Phi(\vec{\mathbf{r}}, t) = \frac{2\pi}{3} G \rho(t) \vec{\mathbf{r}}^2$$

or

$$-\vec{\boldsymbol{\nabla}}\Phi_r = -\frac{4\pi}{3}G\rho(t)\vec{\mathbf{r}}$$

Then

$$\left.\frac{\partial\vec{\mathbf{v}}}{\partial t}\right|_r + (\vec{\mathbf{v}}\cdot\vec{\boldsymbol{\nabla}}_r)\vec{\mathbf{v}} = -\cancelto{0}{\frac{\vec{\boldsymbol{\nabla}}P}{\rho}} \underbrace{-\vec{\boldsymbol{\nabla}}\Phi}_{-\frac{4\pi}{3}G\rho(t)\vec{\mathbf{r}}}$$

so

$$(\underbrace{\dot{H}}_{\frac{\partial\vec{\mathbf{v}}}{\partial t}} + \underbrace{H^2}_{v\,\cdot\,\vec{\boldsymbol{\nabla}}})r = -\frac{4\pi}{3}G\rho(t)\vec{\mathbf{r}}$$

The partial derivative on the left is taken at fixed $\vec{\mathbf{r}}$, so

$$\underbrace{(\dot{H}+H^2)}_{\frac{\ddot{a}}{a}} = -\frac{4\pi}{3}G\rho(t)$$

This acceleration equation is valid in the Newtonian limit $P/\rho c^2 \ll 1$. Note that all spatial quantities are physical: $\vec{\boldsymbol{\nabla}}_r = \frac{\partial}{\partial\vec{\mathbf{r}}}$ and $\vec{\mathbf{r}} \equiv \vec{\mathbf{r}}_0 a(t)$ where $\vec{\mathbf{r}}_0$ is comoving. Then the original acceleration equation is consistent with the Euler and Hubble equations in the non-relativistic limit $P/\rho c^2 \ll 1$ and $\Lambda = 0$.

Lecture 10: Standard Candles

Thursday, February 25, 2021

Consider a light source with intrinsic luminosity $L = \frac{\Delta E}{\Delta t}$ where Δt is measured in the rest frame of the source. In that flat Minkowski spacetime, the flux through a sphere of radius R is

$$F = \frac{L}{4\pi R^2}$$

For an expanding cosmology, there are several effects.

- A cosmological redshift in ΔE:
$$\Delta E_0 = \frac{\Delta E_e}{1+z_e}$$
because the frequency of the photons emitted is related by $\omega_0 = \frac{\omega_e}{1+z_e}$
- There is also a "time" redshift,
$$\Delta t_0 = \Delta t_e(1+z_e)$$
which of course is a consequence of the frequency shift since $\nu = \frac{1}{\Delta t}$, so
$$L_0 = \frac{L_e}{(1+z_e)^2}$$
gives the *observed* flux
$$F_0 = \frac{L_e}{4\pi(1+z_e)^2(a(t_0)R)^2}$$
since the physical distance between the observer and the sources is $a(t_0)R$ but we normalize $a(t_0) = 1$, so the luminosity distance is
$$d_L^2 = R^2\cdot(1+z_e)^2 \implies d_L = R(z_e)(1+z_e)$$
We now need to find $R(z_e)$. Light follows a null geodesic
$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \frac{a^2(\mathrm{d}r)^2}{1-\kappa r^2} = 0$$

which means $\frac{c\mathrm{d}t}{a(t)} = \frac{\mathrm{d}r}{\sqrt{1-\kappa r^2}}$, so

$$c\int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^R \frac{\mathrm{d}r}{\sqrt{1-\kappa r^2}}$$

Now we know that

$$\int \frac{\mathrm{d}t}{a} = \int \frac{\mathrm{d}t}{\mathrm{d}a}\frac{\mathrm{d}a}{a} = \int \frac{1}{Ha}\frac{\mathrm{d}a}{a}$$

so

$$c\int_{a_e}^{1} \frac{\mathrm{d}a}{a^2 H} = \int_0^R \frac{\mathrm{d}r}{\sqrt{1-\kappa r^2}} = F(R)$$

Now

$$a = \frac{1}{1+z} \implies \mathrm{d}a = -\frac{\mathrm{d}z}{(1+z)^2} = -a^2\,\mathrm{d}z \implies \frac{\mathrm{d}a}{a^2} = -\,\mathrm{d}z$$

so the left-hand side is

$$c\int_0^{z_e} \frac{\mathrm{d}z}{H(z)}$$

where

$$H(z) = H_0\left[\Omega_R(1+z)^4 + \Omega_M(1+z)^2 + \Omega_\Lambda + \Omega_\kappa(1+z)^2\right]^{1/2}$$

Finally,

$$F(R) = \underbrace{\frac{c}{H_0}}_{d_{H_0}} \int_0^{z_e} \frac{\mathrm{d}z}{\left[\Omega_R(1+z)^4 + \Omega_M(1+z)^3 + \Omega_\Lambda + \Omega_\kappa(1+z)^2\right]^{1/2}}$$

where $R \equiv R(z_e)$. For a flat ($\kappa = 0 = \Omega_\kappa$), $F(R) = R$ and the luminosity distance is $(1+z_e)R(z_e)$. Writing $a(t) = \frac{a(t_0)}{1+z} \equiv \frac{1}{1+z}$, we get

$$\int_{1/(1+z_e)}^{1} \frac{\mathrm{d}a}{a^2 H(a)} \equiv \int_0^R \frac{\mathrm{d}r}{\sqrt{1-\kappa r^2}} \implies R = R(z_e)$$

$R(z_e)$ is a function of the cosmological parameters Ω_X, which all must add up to 1. During most of "observable" ($z \lesssim 1$) life, we can neglect $\Omega_R \sim 10^{-5}$ and do a two-parameter fit over Ω_M and Ω_Λ. The strategy is to find standard candles, type I-a supeprnovae with known intrinsic L_e. Then, by measuring the flux and redshift,

$$d_L(z_e) = (1+z_e)\underbrace{R(z_e)}_{R(\Omega_X)}$$

A fit will give $\Omega_M \simeq 0.25$, $\Omega_\Lambda \simeq 0.75$, $\Omega_\kappa \simeq 0$ (flat cosmology), and Ω_R fixed by the CMB.

0.17 Horizons

An important concept is that of horizons. A particle horizon comes from the FRW metric:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}\vec{l}^2$$

where $\mathrm{d}\vec{l}^2$ is the comoving spatial distance squared, $\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\,\mathrm{d}\Omega$. A light ray follows a null geodesic $\mathrm{d}s^2 = 0$, so $c^2\,\mathrm{d}t^2 = a^2(t)\,\mathrm{d}\vec{l}^2$. The total comoving distance traveled since the time of the Big Bang ($t = 0$) is called the *comoving particle horizon*:

$$L_C = c\int_0^t \frac{\mathrm{d}t'}{a(t')}$$

whereas the *physical particle horizon* at time t is

$$L_p = ca(t)\int_0^t \frac{\mathrm{d}t'}{a(t')}$$

Particles which are farther away than this (L_C) distance are impossible for us to observe, because light couldn't travel that far in the time since the beginning of the universe.

0.18 Particle Physics

To proceed further, we need to understand what actually goes into ρ, P, and the equations of state that relate them. The microscopic distribution of matter is based on the Standard Model with three fundamental interactions (electromagnetism, weak, and strong), six quarks (fermions u, d, s, c, b, t with spin-1/2), six leptons (e, ν_e, μ, ν_μ, τ, ν_τ), various gauge vector bosons which mediate the interactions (γ for EM, $W^\pm$ (shared with EM) and Z^0 for weak, and eight gluons g for strong), and the scalar (spin-0) Higgs boson (H).

In non-relativistic limits, these particles obey the non-relativistic Schrödinger equation:

$$\imath\hbar\partial_t\psi = \frac{(-\imath\hbar\vec{\boldsymbol{\nabla}})^2}{2m}\psi \implies \psi(x,t) = e^{-\frac{\imath}{\hbar}Et}e^{\frac{\imath}{\hbar}\vec{\mathbf{p}}\,\cdot\,\vec{\mathbf{x}}}$$

where $E = \frac{\vec{\mathbf{p}}^2}{2m}$.

For relativistic dynamics, $E^2 = p^2c^2 + m^2c^4$, so

$$(\imath\hbar\partial_t)^2\,\psi = \left[(-\imath\hbar\vec{\boldsymbol{\nabla}})^2c^2 + m^2c^4\right]\psi$$

gives us

$$\frac{1}{c^2}\partial_t^2\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi \equiv 0 \qquad \text{(Klein Gordon Equation)}$$

The term $m^2c^2/\hbar^2$ has units of $1/\text{length}^2$, so we can define a wavelength

$$\lambda_C = \frac{\hbar}{mc} \qquad \text{(Compton Wavelength)}$$

of the particle, which is the length scale associated with its dynamics. In natural units, $c = \hbar = 1$, so m has units of inverse length and the Klein Gordon equation can be written

$$\partial_\mu\partial^\mu\psi + m^2\psi = 0$$

where

$$\partial_\mu\partial^\mu\psi = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\psi \equiv \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\psi \equiv \left(\partial_t^2 - \nabla^2\right)\psi$$

since $t = ct$. $\partial_\mu\partial^\mu \equiv \Box$ is an invariant under Lorentz transformations. By covariance, the wave function transforms as a scalar, which means we can't use it for particles with spin, but it does describe the dynamics of the Higgs boson.

$$\psi'(x',t') = \psi(x,t)$$

Solutions in natural units can be written

$$\psi(x,t) = A_k e^{-\imath E_k t}e^{\imath\vec{\mathbf{k}}\,\cdot\,\vec{\mathbf{x}}}$$

Plugging this into the Klein-Gordon equation gives $E_k^2 = \pm\sqrt{k^2+m^2}$. Therefore, the most general form of a solution is a linear superposition:

$$\psi(x,t) = \frac{1}{\sqrt{V}}\sum_k\left[A_k e^{-\imath E_k t}e^{\imath\vec{\mathbf{k}}\,\cdot\,\vec{\mathbf{x}}} + A_k^* e^{\imath E_k t}e^{-\imath\vec{\mathbf{k}}\,\cdot\,\vec{\mathbf{x}}}\right]$$

Where we define $E_k = +\sqrt{k^2+m^2}$ and V is the quantization volume.

0.19 Lagrangian and Hamiltonian Dynamics and Quantization

We want to find the equations of motion for the scalar field ψ from the variational principle provided by Special Relativity. We can introduce a Lagrangian function of ψ and $\partial_\mu\psi$ such that

$$I = \in_{t_i}^{t_f} L[\psi,\partial\psi]\,\mathrm{d}t$$

with $\psi \to \psi + \delta\psi$, $\partial_\mu\psi \to \partial_\mu\delta\psi$, and $\delta\psi\big|_{t_i}^{t_f} = 0$ yields the equation of motion

$$L = \int \mathrm{d}^3x \left[\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi\right]$$

and

$$I = \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}^3x \left[\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi\right]$$

Integrating by parts and discarding the surface turns ($\delta\psi$ vanishes at t_i, t_f, and the boundary of V), $I \to +\delta I$:

$$\delta I = -\int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}^3x \left[\frac{1}{2}\partial_\mu\psi\partial^\mu\psi + \frac{1}{2}m^2\psi\right]\delta\psi$$

Finally, requesting $\delta I = 0$, we get the Klein-Gordon equation. The Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi^2$$

privdes an action principle:

$$I = \int \mathrm{d}^4x\, \mathcal{L}[\psi, \partial\psi]$$

where $\mathrm{d}^4x = \mathrm{d}t\,\mathrm{d}^3x$ is invariant under Lorentz transformations:

$$\mathrm{d}t \to \gamma(\mathrm{d}t - \beta\,\mathrm{d}x) \qquad \mathrm{d}x \to \gamma(\mathrm{d}x - \beta\,\mathrm{d}t)$$

LECTURE 11: HAMILTONIAN FORMULATION AND QUANTIZATION

Tuesday, March 02, 2021

0.20 Hamiltonian Formulation

In single particle classical mechanics, the Lagrangian is given by

$$L[\dot{q}, q] = \frac{1}{2}m\dot{q}^2 - V(q)$$

We can then define the canonical momentum as

$$p = \frac{\mathrm{d}L}{\mathrm{d}\dot{q}} = m\dot{q} \implies \dot{q} = \frac{p}{m}$$

The Hamiltonian is

$$H(p, q) = p\dot{q} - L[\dot{q}, q] \to \frac{p^2}{2m} + V(q)$$

In a continuum (classical) theory with $\mathcal{L}[\psi, \dot{\psi}]$, the canonical momentum can be defined as $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}}$ with Lagrangian *density*

$$\mathcal{L}[\psi, \dot{\psi}] = \frac{\dot{\psi}^2}{2} - \frac{(\vec{\nabla}\psi)^2}{2} - \frac{1}{2}m^2\psi^2$$

which implies $\pi = \dot{\psi}$ and the Hamiltonian *density*

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L}[\psi, \dot{\psi}] = \frac{\pi^2}{2} - \frac{(\vec{\nabla}\psi)^2}{2} + \frac{m^2}{2}\psi^2$$

The total Hamiltonian can be found by

$$H = \int \mathrm{d}^3x\, \mathcal{H}[\pi, \psi] \equiv E$$

where E is the energy.

0.21 Quantization

We can write a general wave function:

$$\psi(\vec{\mathbf{x}}, t) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2E_k}} \left[a_k e^{-\imath E_k t} e^{\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}} + a_k^\dagger e^{\imath E_k t} e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}} \right]$$

where V is the quantization volume. This is a general linear superposition of plane wave solutions of the Klein-Gordon equation with some generalized Fourier coefficients a_k and $a_k^\dagger$.

Taking the time derivative, we find the canonical momentum,

$$\pi(\vec{\mathbf{x}}, t) = -\frac{\imath}{\sqrt{V}} \sum_k \sqrt{\frac{E_k}{2}} \left[a_k e^{-\imath E_k t} e^{\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}} - a_k^\dagger e^{\imath E_k t} e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}} \right]$$

Using the finite volume identity,

$$\int \mathrm{d}^3 x \, e^{\imath (\vec{\mathbf{k}} - \vec{\mathbf{k}}') \cdot \vec{\mathbf{x}}} = V \delta_{\vec{\mathbf{k}}, \vec{\mathbf{k}}'}$$

where $\vec{\mathbf{k}} = \frac{2\pi}{L}(m_x, m_y, m_z)$ for $m_i \in \mathbb{Z}$ and $V = L^3$. Plugging this into our Hamiltonian, we have

$$H = \int \mathrm{d}^3 x \, \frac{1}{2} \left[\pi^2 + (\vec{\boldsymbol{\nabla}} \psi)^2 + m^2 \psi^2 \right] = \sum_k \frac{E_k}{2} (a_k^\dagger a_k + a_k a_k^\dagger)$$

We define a and $a^\dagger$ as quantum operators. The quantization is achieved by imposing commutation relations on these operators:

$$\left[a_k, a_{k'}^\dagger \right] = \delta_{k,k'} \qquad [a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger \right] = 0$$

so

$$a_k a_k^\dagger = 1 + a_k^\dagger a_k$$

Therefore, we can write the Hamiltonian as

$$H = \sum_k E_k \left[a_k^\dagger a_k + \frac{1}{2} \right] = \sum_k E_k \left[n_k + \frac{1}{2} \right]$$

where we define n_k as the number of quanta. This is the equation for a collection of harmonic oscillators with frequency $E_k = \sqrt{k^2 + m^2}$ and $E_0 = \sum_k \frac{E_k}{2}$ zero-point energy.

If we take this finite sum and convert it into an integral $\left(\sum_k \to V \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \right)$, we can get the zero-point energy density,

$$\frac{E_0}{V} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{k^2 + m^2}$$

This quantity is *divergent*, which is a big problem in General Relativity where all sources of matter and energy contribute to the gravitational field in the energy-momentum tensor $T^{\mu\nu}$.

0.22 Fermions

Dirac showed that the Klein-Gordon equations can't describe electrons (or other fermions, particles with spin 1/2) because the Schrödinger equation leads to a positive semi-definite probability density:

$$\imath \hbar \partial_t \psi = 0 \frac{\hbar^2}{2m} \nabla^2 \psi \tag{a}$$

$$\implies -\imath \hbar \partial_t \psi^* = -\frac{\hbar^2}{2m} \nabla^2 \psi^* \tag{b}$$

so taking $\psi^*(a) - \psi(b)$, we get

$$\imath\hbar\partial_t(\psi^*\psi) = -\frac{\hbar^2}{2m}\vec{\boldsymbol{\nabla}} \cdot (\psi^*\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\psi^*)$$

This is like a continuity equation

$$\dot{\rho} + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0$$

with $\rho = \psi^*\psi$ being the probability density and $\vec{\mathbf{J}} = \frac{\hbar}{2m\imath}\left(\psi^*\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\psi^*\right)$ being the probability current. Conservation of probability implies that

$$\frac{\partial}{\partial t}\int \rho\, \mathrm{d}^3x = 0$$

if

$$\int \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}}\, \mathrm{d}^3x = 0$$

On the other hand, we can look at the Klein-Gordon equation:

$$\partial_t^2\psi = \nabla^2\psi - m^2\psi \tag{a}$$
$$\implies \partial_t^2\psi^* = \nabla^2\psi^* - m^2\psi^* \tag{b}$$

so $\psi^*(a) - \psi(b)$ gives us

$$\frac{\partial}{\partial t}\left[\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right] + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0$$

(with the same $\vec{\mathbf{J}}$ as in the Schrödinger equation). This $\rho = \psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*$ is *not* manifestly positive semi-definite because of the second time derivative.

Additionally, the wave function transforms as a scalar under rotations and Lorentz translations. However, an electron has spin 1/2. Dirac postulated that we require an equation with one time derivative (to get the same probability density as in the Schrödinger equation) but also only first-order spatial derivatives, to treat space and time on equal footing like in Special Relativity:

$$\imath\hbar\partial_t\psi = -\imath\hbar c\alpha ve \cdot \vec{\boldsymbol{\nabla}}\psi + \beta c^2 m\psi \equiv H\psi$$

the $\vec{\mathbf{a}} \cdot \vec{\boldsymbol{\nabla}}\psi$ term must be rotationally invariant, so $\vec{\alpha}$ must transform as vectors under rotations. However, there are no preferred vectors in Special Relativity, $\vec{\alpha}$ can't just be any arbitrary fixed vector, but $\vec{\alpha}$ and β must instead be Hermitian. How can we find these? Squaring both sides, we get the Klein-Gordon equation on the left, and on the right we have

$$\begin{aligned}(H\psi)^2 &= (-\imath\hbar c\alpha_i\vec{\boldsymbol{\nabla}}_i + \beta mc^2)(-\imath\hbar c\alpha_j\vec{\boldsymbol{\nabla}}_j + \beta mc^2)\psi \\ &= \left[-\hbar^2c^2\alpha_i\alpha_j\vec{\boldsymbol{\nabla}}_i\vec{\boldsymbol{\nabla}}_j + \beta^2m^2c^4 - \imath\hbar c\left(c^2\alpha_i\beta m\vec{\boldsymbol{\nabla}}_i + c^2\beta\alpha_i m\vec{\boldsymbol{\nabla}}_i\right)\right]\psi\end{aligned}$$

Since $\vec{\boldsymbol{\nabla}}_i\vec{\boldsymbol{\nabla}}_j$ is symmetric,

$$\alpha_i\alpha_j = \frac{1}{2}\underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{\text{symmetric}} + \frac{1}{2}\underbrace{(\alpha_i\alpha_j - \alpha_j\alpha_i)}_{\text{antisymmetric}}$$

so the antisymmetric part must cancel when multiplied by $\vec{\boldsymbol{\nabla}}_i\vec{\boldsymbol{\nabla}}_j$. In order to get the Klein-Gordon equation and the correct dispersion relation,

$$\frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i) = \delta_{ij} \qquad \beta^2 = 1 \qquad (\alpha_i\beta + \beta\alpha_i) = 0$$

These quantities cannot be simple numbers. The anticommutators can be shown to be

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad \{\alpha_i, \beta\} = 0$$

and $\beta^2 = 1$, $\alpha_i^2 = 1$ (by setting $i = j$). We need four matrices which square to 1, are Hermitian, and obey the anticommutation relations above. Also,

$$\alpha_i\beta = -\beta\alpha_i \quad \text{and} \quad \beta^2 = 1 \implies \beta\alpha_i\beta = -\alpha_i \implies \operatorname{Tr}\beta\alpha_i\beta = -\operatorname{Tr}\alpha_i = \operatorname{Tr}\alpha_i\beta^2$$

so

$$\operatorname{Tr}\alpha_i = 0$$

for $i = 1, 2, 3$ and since $\alpha_i^2 = 1$, $\alpha_i \beta \alpha_i = -\beta$ implies that $\operatorname{Tr}\beta = 0$. Therefore, we also require these matrices to be traceless. Dirac wrote down these matrices as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & \mathbb{I}_{2\times 2} \end{pmatrix}$$

where σ_i are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

All together, the Dirac equation can be written

$$\imath\hbar\partial_t\psi = \left[-\imath\hbar c\vec{\alpha} \cdot \vec{\boldsymbol{\nabla}} + \beta mc^2\right]\psi \tag{Dirac Equation}$$

Consider the plane wave solutions

$$\psi = \mathcal{A}e^{-\imath Et/\hbar}e^{\imath\vec{\mathbf{p}} \cdot \vec{\mathbf{x}}/\hbar}$$

where $\mathcal{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$ is a four-component spinor. Then

$$\begin{aligned}
E\mathcal{A} &= (c\vec{\alpha} \cdot \vec{\mathbf{p}} + \beta mc^2)\mathcal{A} \\
E^2\mathcal{A} &= (c\vec{\alpha} \cdot \vec{\mathbf{p}} + \beta mc^2)E\mathcal{A} \\
&= (c\vec{\alpha} \cdot \vec{\mathbf{p}} + \beta mc^2)(c\vec{\alpha} \cdot \vec{\mathbf{p}} + \beta mc^2)\mathcal{A} \\
&= \left(c^2 \underbrace{\alpha_i\alpha_j}_{\frac{1}{2}\{\alpha_i,\alpha_j\}=\delta_{ij}} p_i p_j + \underbrace{\beta^2}_{1} m^2c^4 + p_i \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{0}\right)\mathcal{A} \\
&= (p^2c^2 + m^2c^4)\mathcal{A} \\
E^2 &= p^2c^2 + m^2c^4
\end{aligned}$$

which is the correct dispersion relation. What exactly is this "spinor" component? If we go to the rest frame with $p = 0$,

$$E\mathcal{A} = mc^2\beta\mathcal{A}$$

This has four solutions,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies E = +mc^2 \qquad S = +\frac{\hbar}{2}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \implies E = +mc^2 \qquad S = -\frac{\hbar}{2}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \implies E = -mc^2 \qquad S = +\frac{\hbar}{2}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \implies E = -mc^2 \qquad S = -\frac{\hbar}{2}$$

where we define the spin operator by

$$\vec{\mathbf{S}} \equiv \begin{pmatrix} \frac{\hbar\vec{\sigma}}{2} & 0 \\ 0 & \frac{\hbar\vec{\sigma}}{2} \end{pmatrix}$$

These solutions have spin-1/2 and energy $\pm mc^2$ since $E^2 = m^2c^4$ at rest.

Lecture 12: The Dirac Equation, Continued

Thursday, March 04, 2021

We have just shown that solutions are 4-spinors with $E = \pm\sqrt{k^2 + m^2}$ and spin $S = \pm\hbar/2$. We can write a general solution as a linear superposition:

$$\psi(x,t) = \frac{1}{\sqrt{V}} \sum_k \sum_{\alpha=1,2} \left[b_{k,\alpha} u_{k,\alpha} e^{-\imath E_k t} e^{\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}} + d^\dagger_{k,\alpha} v_{k,\alpha} e^{\imath E_k t} e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}} \right]$$

with $E_k \equiv +\sqrt{k^2 + m^2}$ such that $E_k t - \vec{\mathbf{k}} \cdot \vec{\mathbf{x}} \equiv k^\mu x_\mu$ where $k^\mu \equiv (E_k, \vec{\mathbf{k}})$ (with $c = 1$ still). The 4-spinors $u_{k,\alpha}$ and $v_{k,\alpha}$ are obtained by requiring ψ to be the solution to the Dirac equation with $b_{k,\alpha}$ and $d^\dagger_{k,\alpha}$ as the Fourier coefficients. Plugging this into the Dirac equation, we get that

$$\begin{pmatrix} m - E_k & \vec{\sigma} \cdot \vec{\mathbf{k}} \\ \vec{\sigma} \cdot \vec{\mathbf{k}} & -(m + E_k) \end{pmatrix} u_{k,\alpha} = 0$$

We write $u_{k,\alpha} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$, each of these as 2-component spinors satisfying

$$(m - E_k)\varphi + \vec{\sigma} \cdot \vec{\mathbf{k}}\chi = 0 \tag{a}$$

and

$$\vec{\sigma} \cdot \vec{\mathbf{k}}\varphi - (m + E_k)\chi = 0 \tag{b}$$

From (b), we get

$$\chi = \frac{\vec{\sigma} \cdot \vec{\mathbf{k}}}{(E_k + m)}\varphi$$

so plugging this back into (a) and using $(\vec{\sigma} \cdot \vec{\mathbf{k}})(\vec{\sigma} \cdot \vec{\mathbf{k}}) = k^2$, we have

$$\left((m - E_k) + \frac{k^2}{m + E_k} \right) \varphi = 0$$

so for $\varphi \neq 0$, $E_k^2 = k^2 + m^2$. We choose

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which makes

$$u_{k,\alpha} = \mathcal{N}_{k,\alpha} \begin{pmatrix} \varphi_\alpha \\ \frac{\vec{\sigma} \cdot \vec{\mathbf{k}}}{E_k + m}\varphi_\alpha \end{pmatrix}$$

and

$$u^\dagger_{k,\alpha} = \begin{pmatrix} \varphi^\dagger_\alpha & \varphi^\dagger_\alpha \frac{\vec{\sigma} \cdot \vec{\mathbf{k}}}{E_k + m} \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} m+E_k & -\vec{\sigma}\cdot\vec{\mathbf{k}} \\ -\vec{\sigma}\cdot\vec{\mathbf{k}} & -(m-E_k) \end{pmatrix} v_{k,\alpha} = 0$$

gives us

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$v_{k,\alpha} = \tilde{\mathcal{N}}_{k,\alpha} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{\mathbf{k}}}{m+E_k}\chi_\alpha \\ \chi_\alpha \end{pmatrix}$$

As with u, we choose the normalization factor such that

$$v^\dagger_{k,\alpha} \cdot v_{k,\alpha'} = \delta_{\alpha,\alpha'}$$

0.23 Action Principle for the Dirac Equation

We want to derive the Dirac equation from an action principle,

$$I = \int \mathrm{d}^4x\, \mathcal{L}[\psi^\dagger, \psi]$$

Given independent variations of $\psi \to \psi + \delta\psi$, $\psi^\dagger \to \psi^\dagger + \delta\psi^\dagger$, we can find the variation $I \to I + \delta I$ to linear order in $\delta\psi, \delta\psi^\dagger$ and find the path where $\delta I = 0$:

$$\mathcal{L} = \psi^\dagger(\imath\partial_t\psi + \imath\vec{\alpha}\cdot\vec{\boldsymbol{\nabla}}\psi + \beta m\psi)$$

Under $\psi^\dagger \to \psi^\dagger + \delta\psi^\dagger$,

$$\delta\mathcal{L} = \delta\psi^\dagger \left(\imath\partial_t\psi + \imath\vec{\alpha}\cdot\vec{\boldsymbol{\nabla}}\psi + \beta m\psi\right)$$

and

$$\delta I = \int \mathrm{d}^4x\, \delta\mathcal{L} = 0$$

gives us

$$\imath\partial_t\psi = -\imath\vec{\alpha}\cdot\vec{\boldsymbol{\nabla}}\psi + \beta m\psi$$

Doing the same with $\delta\psi$ gives the Hermitian conjugate Dirac equation.

Define $\gamma^0 = \beta$, $\vec{\gamma} = \gamma^0\alpha = \beta\vec{\alpha}$ and $\bar{\psi} = \psi^\dagger\gamma^0$, where $(\gamma^0)^2 = \beta^2 = 1$. Then

$$\begin{aligned}\mathcal{L} &= \psi^\dagger \underbrace{\gamma^0\gamma^0}_{1} \left(\imath\partial_t\psi + \imath\vec{\alpha}\cdot\vec{\boldsymbol{\nabla}} - \gamma^0 m\right)\psi \\ &= \bar{\psi}\left(\underbrace{\imath\gamma_0\partial_t + \imath\vec{\gamma}\cdot\vec{\boldsymbol{\nabla}}}_{\equiv\imath\not{\partial}=\imath\gamma^\mu\partial_\mu} - m\right)\psi\end{aligned}$$

so

$$\mathcal{L} = \bar{\psi}(\imath\not{\partial} - m)\psi \qquad \text{(Dirac Lagrangian Density)}$$

Note that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

0.24 Dirac Hamiltonian

As in classical mechanics, $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \imath\psi^\dagger$ and $\mathcal{H} = \pi\dot{\psi} - \mathcal{L}$, so we can show that

$$\mathcal{H} = \psi^\dagger(-\imath\vec{\alpha} \cdot \vec{\boldsymbol{\nabla}} + \beta m)\psi \qquad \text{(Dirac Hamiltonian Density)}$$

with $H = \int \mathrm{d}^4 x\, \mathcal{H}$.

Using our generalized wave function from the beginning, we can go through the quantization process. We suppose $\frac{1}{V}\int \mathrm{d}^3 x\, e^{\imath(\vec{\mathbf{k}}-\vec{\mathbf{k}}') \cdot x} = \delta_{k,k'}$ and (need to prove in homework) $v^\dagger u = u^\dagger v = 0$, and we can show that

$$H = \sum_k \sum_\alpha E_k \left[b^\dagger_{k,\alpha} b_{k,\alpha} - d_{k,\alpha} d^\dagger_{k,\alpha}\right]$$

If we try to quantize using the canonical commutation relations (like in the Klein-Gordon theory for scalar particles),

$$\left[d_{k,\alpha}, d^\dagger_{k',\alpha'}\right] = \delta_{k,k'}\delta_{\alpha,\alpha'}$$

so

$$H = \sum_{k,\alpha} E_k \left[b^\dagger_{k,\alpha} b_{k,\alpha} - d^\dagger_{k,\alpha} d_{k,\alpha}\right] - \underbrace{2\sum_k E_k}_{\text{zero point energy}}$$

The minus sign in the terms implies that increasing the number of negative energy particles lowers the energy, so there is *no ground state*, since you can always have an arbitrary number of these negative energy particles. Because of this, Dirac instead theorized that we should quantize these particles with *anticommutation* relations:

$$\left\{b_{k,\alpha}, b^\dagger_{k',\alpha'}\right\} = \left\{d_{k,\alpha}, d^\dagger_{k',\alpha'}\right\} = \delta_{k,k'}\delta_{\alpha,\alpha'}$$

and

$$\{b, b\} = \{d, d\} = \left\{b^\dagger, b^\dagger\right\} = \left\{d^\dagger, d^\dagger\right\} = \{b, d\} = \cdots = 0$$

Using this quantization,

$$H = \sum_k \sum_\alpha E_k \left[b^\dagger_{k,\alpha} b_{k,\alpha} + d^\dagger_{k,\alpha} d_{k,\alpha}\right] - 2\sum_k E_k$$

In this theory (thinking only about electrons for now), $b^\dagger$ and b create/annihilate electrons with spin α, and $d^\dagger$ and d create/annihilate positrons (antiparticles with the same mass as electrons but opposite charge) with spin α. The total number of particles can be written as

$$\hat{N} = \sum_{k,\alpha} \left(\underbrace{b^\dagger_{k,\alpha} b_{k,\alpha}}_{n_{k,\alpha}} - \underbrace{d^\dagger_{k,\alpha} d_{k,\alpha}}_{\bar{n}_{k,\alpha}}\right)$$

where $n_{k,\alpha}$ represents the number of electrons and $\bar{n}_{k,\alpha}$ represents the number of positrons. This theory correctly predicts the number of spin degrees of freedom and the existence of antiparticles. The total charge is given by

$$\hat{Q} = e\hat{N}$$

so using the Dirac equation, it follows that the current

$$J^\mu = \bar{\psi}\gamma^\mu\psi \equiv (\overbrace{\psi^\dagger\psi}^{\rho}, \overbrace{\psi^\dagger\vec{\alpha}\psi}^{\vec{\mathbf{J}}})$$

is conserved ($\partial_\mu J^\mu = 0$). Assuming $\vec{\mathbf{J}}$ vanishes at spatial boundaries,

$$\frac{\partial}{\partial t}\int_V \rho\, \mathrm{d}^3 x + \int_V \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}}\, \mathrm{d}^3 x = 0$$

so

$$Q = \int \mathrm{d}^3x\, \psi^\dagger \psi$$

is a constant. Using the anticommutation relations and disregarding the overall constant term, we can write

$$Q = e \sum_{k,\alpha} b^\dagger_{k,\alpha} b_{k,\alpha} - d^\dagger_{k,\alpha} d_{k,\alpha}$$

0.24.1 Pauli Exclusion Principle

From $\left\{ b^\dagger_{k,\alpha}, b^\dagger_{k',\alpha'} \right\} = 0$, we see that, for $k = k'$ and $\alpha = \alpha'$, $2b^\dagger_{k,\alpha} b^\dagger_{k,\alpha} = 0$, so we cannot create two electrons with the same quantum numbers. Similarly, if $k = k'$ but $\alpha \neq \alpha'$, $b^\dagger_{k,\alpha} b^\dagger_{k,\alpha'} = -b^\dagger_{k,\alpha'} b^\dagger_{k,\alpha}$, so the wave function for a state with one electron with k, α and another with k, α' is antisymmetric under exchange of these particles. This is the Pauli exclusion principle. Note that $\left[\hat{N}, \hat{H} \right] = 0$. In quantum statistical mechanics, we need $Z = \operatorname{Tr} e^{-\beta(\hat{H} - \mu \hat{N})}$ where μ is the chemical potential and $\beta \equiv \frac{1}{k_B T}$. Then

$$\hat{H} - \mu \hat{N} = \sum_{k,\alpha} \left(b^\dagger_{k,\alpha} b_{k,\alpha} (E_k - \mu) + d^\dagger_{k,\alpha} d_{k,\alpha} (E_k + \mu) \right)$$

so antiparticles must have the opposite chemical potential from particles. The probability of finding a particle in a given state is

$$n_k = \frac{\operatorname{Tr} b^\dagger_{k,\alpha} b_{k,\alpha} e^{-\beta(H - \mu N)}}{\operatorname{Tr} e^{-\beta(H - \mu N)}} = \frac{1}{e^{\beta(E_k - \mu)} + 1}$$

and

$$\bar{n}_k = \frac{1}{e^{\beta(E_k + \mu)} + 1}$$

These are Fermi-Dirac statistics.

0.24.2 Spin-Statistics Connection

A profound result of QFT is that half-odd integer spins (1/2, 3/2, ...) must be quantized with anticommutation relations and therefore follow the Pauli exclusion principle and have antisymmetric wave functions under particle exchange. These are called fermions, and include leptons, quarks, and baryons. Particles with integer spin must be quantized with commutation relations, so they must have symmetric wave functions under particle exchange. These are called bosons, and include the scalar Higgs, the vector force-carrying particles like photons, weak bosons, and gluons, and composites like mesons.

LECTURE 13: MATTER-ANTIMATTER ASYMMETRY
Tuesday, March 09, 2021

0.25 Matter-Antimatter Asymmetry

We have $\hat{H} - \mu \hat{N} = \sum_{k,\alpha} \left(\hat{n}_{k,\alpha}(E_k - \mu) + \hat{\bar{n}}_{k,\alpha}(E_k + \mu) \right)$ with $\hat{n} = b^\dagger b$ and $\hat{\bar{n}} = d^\dagger d$ and $(\hat{n})^2 = \hat{n}$.

We previously showed that the statical expectation values should be

$$n_k = \frac{1}{e^{\beta(E_k - \mu)} + 1} \qquad \bar{n}_k = \frac{1}{e^{\beta(E_k + \mu)} + 1}$$

We can describe the asymmetry between antiparticles and particles by

$$\frac{1}{V}\sum_k (n_k - \bar{n}_k) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left(\frac{1}{e^{\beta(E_k-\mu)}+1} - \frac{1}{e^{\beta(E_k+\mu)}+1}\right)$$

This quantity is not equal to zero *iff* $\mu \neq 0$.

0.26 Electromagnetism

The photon field obeys Maxwell's equations:

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \qquad \vec{\nabla}\times\vec{\mathbf{B}} = \frac{1}{c}\partial_t\vec{\mathbf{E}} + \frac{4\pi}{c}\vec{\mathbf{J}} \qquad \vec{\nabla}\times\vec{\mathbf{E}} = -\frac{1}{c}\partial_t\vec{\mathbf{B}}$$

We can solve these by introducing scalar and vector potentials, Φ and $\vec{\mathbf{A}}$:

$$\vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}} \implies \vec{\nabla}\cdot\vec{\mathbf{B}} = 0$$

and

$$\vec{\mathbf{E}} = -\frac{1}{c}\partial_t\vec{\mathbf{A}} - \vec{\nabla}\Phi \implies \vec{\nabla}\times\vec{\mathbf{E}} = -\frac{1}{c}\partial_t\underbrace{(\vec{\nabla}\times\vec{\mathbf{A}})}_{\vec{\mathbf{B}}}$$

Setting $c = 1$, we can introduce the 4-potential $A^\mu = (\Phi, \vec{\mathbf{A}})$ and the strength tensor

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

with $\partial x_\mu = \left(\partial_t, -\vec{\nabla}\right)$. Note this tensor is antisymmetric in μ and ν.

$$F^{0i} = \partial_t A^i + \vec{\nabla}\Phi = -E^i = -F^{i0}$$

$$F^{xy} = -\frac{\partial A^y}{\partial x} + \frac{\partial A^x}{\partial y} = B^z$$

and so on. We can also introduce the 4-current $J^\mu = (\rho, \vec{\mathbf{J}})$ such that

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

with $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}\right)$. Because $F^{\mu\nu}$ is antisymmetric, $\partial_\nu\partial_\mu F^{\mu\nu} = 0 \implies \partial_\nu J^\nu = 0$, so J^μ is conserved.

0.26.1 Gauge Invariance

The description of electromagnetism in terms of the potential is redundant. $\vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}}$ is invariant under

$$\vec{\mathbf{A}} \to \vec{\mathbf{A}} - \vec{\nabla}\Lambda(x,t)$$

where Λ is some arbitrary function. Similarly, Φ is invariant under

$$\Phi \to \Phi + \frac{\partial\Lambda(x,t)}{\partial t}$$

or together,

$$A^\mu \to A^\mu + \partial^\mu\Lambda$$

and

$$F^{\mu\nu} \to F^{\mu\nu} + \underbrace{\partial^\mu\partial^\nu\Lambda - \partial^\mu\partial^\nu\Lambda}_{0}$$

This invariance is called gauge invariance and the transformations mentioned are gauge transformations. We can "fix" the gauge by choosing a function Λ which has the correct number of degrees of freedom.

0.26.2 The Coulomb Gauge

In this gauge, we define $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$, and this causes

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi\rho = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}\right) - \nabla^2\Phi$$

so

$$-\nabla^2\Phi = 4\pi\rho$$

We call Φ the Coulomb potential, and we can solve this equation as

$$\Phi(\vec{\mathbf{x}}) = \int \mathrm{d}^3x' \, \frac{\rho(x')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}$$

so $\Phi(x)$ is fixed by the charge distribution $\rho(\vec{\mathbf{x}})$, so $\Phi(x)$ is *not* an independent degree of freedom. We started with A^μ having four degrees of freedom. Introducing the Coulomb gauge reduced this to three degrees of freedom, and solutions for Φ tell us there are only two degrees of freedom.

0.26.3 Free Electromagnetic Waves

Suppose $J^\mu = 0$ (no sources), so $\Phi = 0$ and assume the Coulomb gauge ($\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$). Now Maxwell's equations have the form

$$\partial_\mu F^{\mu\nu} = 0 \implies \partial_\mu\partial^\mu\vec{\mathbf{A}} = 0$$

This can be written

$$\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{\mathbf{A}}(x,t) = 0$$

which has plane wave solutions,

$$\vec{\mathbf{A}} = \vec{\epsilon}(k)e^{-\imath\omega t}e^{\imath\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}}$$

$$\frac{\omega^2}{c^2} = \vec{\mathbf{k}}^2 \implies \omega = \pm c\left|\vec{\mathbf{k}}\right|$$

the dispersion relation of free electromagnetic waves. What is $\vec{\epsilon}(k)$? Using the Coulomb gauge condition on this plane wave solution, we find

$$\vec{\mathbf{k}} \cdot \vec{\epsilon}(k) = 0$$

so $\vec{\epsilon} \perp \vec{\mathbf{k}}$. This gives the transversality condition: We choose a right-handed triad of unit vectors $\hat{\mathbf{k}}$, $\hat{\epsilon}_1$, $\hat{\epsilon}_2$ with $\hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{\mathbf{k}}$. We can form this basis in a 3D sphere:

$$\sum_{\lambda=1}^{2}\hat{\epsilon}_i^\lambda\hat{\epsilon}_j^\lambda = \delta_{ij} - \hat{k}_i\hat{k}_j = \delta_{ij} - \frac{\vec{\mathbf{k}}_i\vec{\mathbf{k}}_j}{\left|\vec{\mathbf{k}}^2\right|}$$

Two transverse degrees of freedom describe electromagnetic waves. The most general solution of the wave equation under the Coulomb gauge is a superposition of waves:

$$\vec{\mathbf{A}}(\vec{\mathbf{x}},t) = \frac{1}{\sqrt{V}}\sum_k\sum_{\lambda=1}^{2}\frac{\vec{\epsilon}_\lambda(k)}{\sqrt{2\omega(k)}}\left[a_{k,\lambda}e^{-\imath\omega(k)t}e^{\imath\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}} + a^\dagger_{k,\lambda}e^{\imath\omega(k)t}e^{-\imath\vec{\mathbf{k}} \cdot \vec{\mathbf{x}}}\right]$$

0.27 Maxwell's Equations from Variational Principles

We can obtain Maxwell's equations from the variational principle with

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}F^{\mu\nu}\partial_\mu A_\nu = \frac{1}{2}(\vec{\mathbf{E}}^2 - \vec{\mathbf{B}}^2)$$

Under $A^\mu \to A^\mu + \delta A^\mu$,

$$\delta I = \int \mathrm{d}^4x\,(\partial_\mu F^{\mu\nu})\delta A_\nu = 0$$

Integration by parts gives us $\partial_\mu F^{\mu\nu}$. With $\pi = \frac{\partial \mathcal{L}}{\partial \dot{A}}$, the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}(\vec{\mathbf{E}}^2 + \vec{\mathbf{B}}^2)$$

With the solution for $\vec{\mathbf{A}}$ above, $\vec{\mathbf{E}} = \dot{\vec{\mathbf{A}}}$, and $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$ (and $\hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{\mathbf{k}}$) and the normalization condition

$$\frac{1}{V}\int \mathrm{d}^3x\, e^{\imath(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{x}}} = \delta_{k,k'}$$

we get

$$H = \frac{1}{2}\sum_k \sum_{\lambda=1}^{2} \omega(k)\left[a^\dagger_{k,\lambda}a_{k,\lambda} + a_{k,\lambda}a^\dagger_{k,\lambda}\right]$$

This is, again, a collection of harmonic oscillators, just like with the real scalar field, but now there are two transverse degrees of freedom. Using the commutation relations,

$$\left[a_{k,\lambda}, a^\dagger_{k',\lambda'}\right] = \delta_{k,k'}\delta_{\lambda,\lambda'} \qquad [a,a] = \left[a^\dagger, a^\dagger\right] = 0$$

we get

$$H = \sum_k \sum_{\lambda=1,2} \omega(k)\left[a^\dagger_{k,\lambda}a_{k,\lambda} + \frac{1}{2}\right]$$

where the zero-point energy is then $\sum \frac{\omega(k)}{2}$.

0.27.1 Thermodynamics of Photons

Note that the number of photons is not conserved, they can be absorbed or emitted, so $\mu = 0$ for photons. Then $Z = \mathrm{Tr}\, e^{-\beta H}$ gives us

$$n_{k,\lambda} = \frac{\mathrm{Tr}\, a^\dagger_{k,\lambda}a_{k,\lambda}e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} = \frac{1}{e^{\beta\omega(k)} - 1}$$

This is a Bose-Einstein distribution with $\mu = 0$.

LECTURE 14: QUANTUM STATISTICAL MECHANICS
Thursday, March 11, 2021

0.28 Quantum Statistical Mechanics

Let's begin with the partition function for the grand canonical ensemble

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$$

$$U = \left\langle \hat{H} \right\rangle = \frac{\mathrm{Tr}\, \hat{H}e^{-\beta(\hat{H}-\mu\hat{N})}}{Z}$$

and

$$N = \left\langle \hat{N} \right\rangle = \frac{\mathrm{Tr}\, \hat{N}e^{-\beta(\hat{H}-\mu\hat{N})}}{Z}$$

Furthermore, we have the equation of state for this ensemble,

$$TS = U + PV - \mu N$$

and
$$PV = k_B T \ln(Z)$$

We can write the energy density as
$$\frac{U}{V} = g \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \hbar E(k) \left[n_k + \bar{n}_k\right]$$

where g is the spin degrees of freedom, $g = (2s+1)$ for particles with spin, or the polarization degrees of freedom, $g = 2$ for photons, since there are two independent polarizations. The number density is then
$$\frac{N}{V} = g \int \frac{\mathrm{d}^3 k}{(2\pi)^3} (n_k - \bar{n}_k)$$

where n_k ($\bar{n}_k$) is the distribution function for particles (antiparticles). For photons, $g = 2$, and also $\bar{n}_k \equiv 0$, since there are no anti-photons. Additionally, for photons, $\mu = 0$, so
$$n_k = \frac{1}{e^{\beta\omega(k)} - 1}$$

For real scalar particles,
$$n_k = \frac{1}{e^{\beta E(k)} - 1}$$
where $E(k) = \sqrt{k^2 c^2 + m^2 c^4}$. In this case, $\mu = 0$ and $g = 1$.

For spin-1/2 fermions,
$$n_k = \frac{1}{e^{\beta(E(k)-\mu)} + 1} \qquad \bar{n}_k = \frac{1}{e^{\beta(E(k)+\mu)} + 1}$$
and $g = 2$. For particles with color or flavor, we also have to include g_c, the color or flavor degrees of freedom.

$$\rho c^2 \equiv \frac{U}{V}$$
so a general result of quantum statistical mechanics is that
$$P = \frac{g}{3} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \hbar k \bar{v}(k) [n_k + \bar{n}_k]$$

where $\bar{v}(k) = \frac{\mathrm{d}E(k)}{\mathrm{d}k}$ is the group velocity. For massless particles, $E(k) = c|k|$ and $\bar{v} = c$, so the energy is $\hbar c|k|$, which tells us that
$$P = \frac{1}{3}\rho c^2$$
the equation of state for a photon gas.

0.29 Gauge Interactions

We have been focusing on free field theories so far. Let's now consider interactions between electrons and photons, the basis of quantum electrodynamics, the coupling of charged particles to the electromagnetic force. We begin with classical mechanics. The Lorentz force is given by
$$m\ddot{\vec{\mathbf{x}}} = e\left(\vec{\mathbf{E}} + \frac{\vec{\mathbf{v}}}{c} \times \vec{\mathbf{B}}\right)$$

We can obtain this from a Hamiltonian:
$$\dot{\vec{\mathbf{x}}} = \frac{\partial H}{\partial \vec{\mathbf{p}}} \qquad \dot{\vec{\mathbf{p}}} = -\frac{\partial H}{\partial \vec{\mathbf{x}}}$$

If we take $H = \frac{p^2}{2m} + V(\vec{\mathbf{x}})$, then $\dot{\vec{\mathbf{x}}} = \frac{\vec{\mathbf{p}}}{m}$ and $\dot{\vec{\mathbf{p}}} = -\vec{\boldsymbol{\nabla}} V$.

Then, $\ddot{\vec{\mathbf{x}}} = \frac{\dot{\vec{\mathbf{p}}}}{m} = -\frac{\vec{\boldsymbol{\nabla}} V}{m}$. This means that the Hamiltonian itself cannot couple directly to $\vec{\mathbf{E}}$ or $\vec{\mathbf{B}}$, because then we would have time derivatives of these fields in the acceleration, which is not consistent with the Lorentz force. We need a Hamiltonian whose first time derivative has this generalized momentum and whose second time derivative has $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. The minimal coupling would be

$$\vec{\mathbf{p}} \to \vec{\mathbf{p}} - \frac{e}{c}\vec{\mathbf{A}}$$

such that

$$H = \frac{1}{2m}\left(\vec{\mathbf{p}} - \frac{e}{c}\vec{\mathbf{A}}\right)^2 + e\Phi$$

With this, Hamilton's equations will yield the Lorentz force. The Schrödinger equation will read

$$\imath\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-\imath\hbar\vec{\boldsymbol{\nabla}} - \frac{e}{c}\vec{\mathbf{A}}\right)^2\psi + e\Phi\psi$$

In Dirac theory, the Dirac equation becomes

$$\imath\hbar\frac{\partial\psi}{\partial t} = \left[c\vec{\alpha}\cdot\left(-\imath\hbar\vec{\boldsymbol{\nabla}} - \frac{e}{c}\vec{\mathbf{A}}\right) + \beta mc^2 + e\Phi\right]\psi$$

In terms of the 4-potential, in units where $\hbar = c = 1$,

$$\mathcal{L} = \bar{\psi}(\imath\partial\!\!\!/ - eA\!\!\!/ - m)\psi$$

where $A\!\!\!/ = \gamma^\mu A_\mu = \gamma^0\Phi + \vec{\gamma}\cdot\vec{\mathbf{A}}$. Under the gauge transformations $\psi \to e^{-\imath e\Lambda(x)}\psi$ and $\bar{\psi} \to \bar{\psi}e^{\imath e\Lambda(x)}$, $A^\mu \to A^\mu + \partial^\mu\Lambda(x)$, the Dirac Lagrangian is invariant:

$$\imath\partial_\mu(e^{-\imath e\Lambda}\psi) = e^{-\imath e\Lambda}\left(\imath\partial_\mu\psi + e\partial_\mu\Lambda\psi\right)$$

and

$$\begin{aligned}\gamma^\mu\left(\imath\partial_\mu - eA_\mu - e\partial_\mu\Lambda\right)\left(e^{-\imath e\Lambda}\psi\right) &= e^{-\imath e\Lambda}\gamma^\mu\left(\imath\partial_\mu + e\partial_\mu\Lambda - e\partial_\mu\Lambda - eA_\mu\right)\psi \\ &= e^{-\imath e\Lambda}\left(\imath\partial\!\!\!/ - eA\!\!\!/\right)\psi\end{aligned}$$

which implies

$$\bar{\psi}e^{\imath e\Lambda}e^{-\imath e\Lambda}(\imath\partial\!\!\!/ - eA\!\!\!/)\psi$$

is invariant. The coupling between electromagnetism and Dirac fields is

$$e\bar{\psi}\gamma^\mu\psi A_\mu \equiv eJ^\mu A_\mu$$

with $\partial_\mu J^\mu = 0$ by the Dirac equation. The complete QED Lagrangian is therefore

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\imath\partial\!\!\!/ - eA\!\!\!/ - m)\psi \\ &= \underbrace{\mathcal{L}_{\text{EM}}}_{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} + \underbrace{\mathcal{L}_{\text{D}}}_{\bar{\psi}(\imath\partial\!\!\!/ - m)\psi} + eJ^\mu A_\mu\end{aligned}$$

We quantized both the Dirac and EM fields in terms of creation and annihilation operators:

$$\vec{\mathbf{A}} = \frac{1}{\sqrt{V}}\sum\cdots(a_{k,\lambda} + a^\dagger_{k,\lambda}\cdots)$$

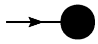
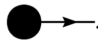
where a destroys and $a^\dagger$ creates,

$$\psi = \sum\cdots(b + d^\dagger\cdots)$$

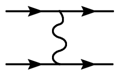
$$\bar{\psi} = \sum\cdots(b^\dagger + d\cdots)$$

where b and d destroy electrons and positrons and $b^\dagger$ and $d^\dagger$ create them.

0.29.1 Basic QED Feynman Rules

We can graphically depict electron/positron annihilation as  and creation as .

We can describe a process like Coulomb/Rutherford scattering graphically as



This is electron-electron scattering by exchange of a photon. The initial electron emits (creates) a photon which is absorbed by the other electron. We can write this as

$$\underbrace{b_{p_1} b_{p_3}}_{\text{destroy initial electrons}} \underbrace{a^\dagger a}_{\text{emit and absorb photon}} \underbrace{b^\dagger_{p_2} b^\dagger_{p_4}}_{\text{create end-state electrons}}$$

Photons are massless ($\omega = ck$) and electron scattering is long range, whereas the weak interactions are short ranged and the $W^\pm$ and Z^0 vector bosons are massive. They become massive through the process of spontaneous symmetry breaking.

0.30 Spontaneous Symmetry Breaking

Let's study the simplest framework, a real scalar field:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - V(\Phi)$$

For "free" massive scalars, (the Klein-Gordon equation) which we studied before, $V(\Phi) = \frac{1}{2}m^2\Phi^2$. Adding this simple mass term breaks gauge invariance. We can write the mass term of a vector boson the same way we write it for a scalar:

$$\frac{1}{2}m^2 A_\mu A^\mu$$

such that

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu$$

However, we no longer have gauge symmetry because $A^\mu \to A^\mu + \partial^\mu \Lambda$ is explicitly broken by this mass term. The solution is the Higgs mechanism.

If we consider the Hamiltonian density, which is now

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\Phi)^2 + V(\Phi)$$

where $\pi = \dot{\Phi}$, then if we consider a field $\bar{\Phi}$ which is constant in spacetime,

$$H[\bar{\Phi}] = \underbrace{V}_{\text{(volume)}} V(\bar{\Phi})$$

which gives us the energy density. With the potential we described, $V(\bar{\Phi}) = \frac{1}{2}m^2\bar{\Phi}^2$, so the minimum is at $\bar{\Phi} = 0$. However, now consider a different, more interesting potential:

$$V(\Phi) = \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \Phi^2\right)^2 = -\frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4 + \frac{\mu^4}{4\lambda}$$

This is a quartic equation with two minima at $\bar{\Phi}_\pm = \pm\frac{\mu}{\sqrt{\Lambda}}$. In this case, $\bar{\Phi} = 0$ is a maximum. The potential is symmetric under $\Phi \to -\Phi$, so the minima are degenerate ground states. We can expand around these minima, $\Phi = \Phi_\pm + \eta$:

$$V(\eta) = \frac{\lambda}{4}\left(2\Phi_\pm\eta + \eta^2\right)^2 = \mu^2\eta^2 + \lambda\Phi_\pm\eta^3 + \frac{\lambda}{4}\eta^4$$

This first term is related to the mass: $\mu^2\eta^2 = \frac{1}{2}m^2\eta^2 \implies m^2 = 2\mu^2$. Now the symmetry is $\eta \to -\eta$ and $\Phi_\pm \to \Phi_\mp$, so choosing one of the minima spontaneously breaks the symmetry. The Lagrangian is still symmetric, but when we quantize the field, we will break that symmetry. If we quantize harmonic oscillations around Φ_+, we have

$$V_+(\eta) = \frac{1}{2}m^2\eta^2 + \underbrace{\lambda\Phi_+\eta^3}_{\sqrt{\lambda}\mu\eta^3} + \frac{\lambda}{4}\eta^3$$

but around the other minimum,

$$V_-(\eta) = \frac{1}{2}m^2\eta^2 - \sqrt{\lambda}\mu\eta^3 + \frac{\lambda}{4}\eta^4$$

Essentially, choosing one of the vacua breaks the symmetry, and upon quantization, each vacua leads to a different Hilbert space, spaces which are orthogonal in the $V \to \infty$ limit.

0.31 Phase Transitions

The excitations around these vacua correspond to particles. If the system is at finite temperature T, there will be thermal fluctuations in addition to quantum fluctuations, so the quantization around the vacua treats $\frac{1}{2}m^2\eta^2$ as harmonic fluctuations and the linear terms η^3 and η^4 as perturbations, If we write $\Phi = \Phi_+ + \eta$, then $\langle\eta\rangle = 0$ (or $\langle\Phi\rangle = \langle\Phi_+\rangle$) at $T = 0$ is called the vacuum expectation value.

However, fluctuations of $\langle\eta^2\rangle \neq 0$, where $\sqrt{\langle\eta^2\rangle_T - \langle\eta^2\rangle_{T=0}}$ is of order $\mathcal{O}\left(2\mu/\sqrt{\lambda}\right)$, lead to field fluctuations where the original symmetry is absorbed (particles have enough energy to break out of double-well potential and move between wells). This is a phase transition which must occur at finite temperature T_c. For $T < T_c$, the symmetry is spontaneously broken and $\langle\Phi\rangle$ is either Φ_+ or Φ_-. We will continue this in the next lecture.

Lecture 15: Higgs Mechanism and Yukawa Coupling
Tuesday, March 16, 2021

For $T > T_c$ the symmetry is restored and for $\langle\Phi\rangle = 0$, we can estimate the critical temperature:

$$\left[\langle n^2\rangle_T - \langle n^2\rangle_{T=0}\right]^{1/2} \simeq \mu/\sqrt{\lambda}$$

If we suppose that

$$\eta(x,t) = \frac{1}{\sqrt{V}}\sum_k \frac{1}{\sqrt{2E_k}}\left[a_k e^{\imath kx}e^{-\imath E_k t} + a_k^\dagger e^{-\imath kx}e^{\imath E_k t}\right]$$

then the thermodynamic average is

$$\begin{aligned}\langle n^2\rangle_T &= \frac{1}{V}\sum_{k,k'}\left(\cancelto{0}{\left\langle a_k^\dagger a_{k'}^\dagger\right\rangle}(\cdots) + \cancelto{0}{\langle a_k a_{k'}\rangle}(\cdots) + \underbrace{\left\langle a_k a_{k'}^\dagger\right\rangle}_{\delta_{k,k'}(1+n_k)} + \underbrace{\left\langle a_k^\dagger a_{k'}\right\rangle}_{\delta_{k,k'}n_k}\right)\frac{1}{\sqrt{2E_k 2E_{k'}}} \\ &= \frac{1}{V}\sum_k \frac{1+2n_k}{2E_k}\end{aligned}$$

The $T = 0$ term is for $n_k = 0$, so we can subtract (and convert from a sum over k to an integral):

$$\langle n^2\rangle_T - \langle n^2\rangle_{T=0} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{E_k(e^{\beta E_k} - 1)}$$

with $\beta = \frac{1}{T}$ (setting $k_B = 1$) and $E_k = \sqrt{k^2+m^2}$. Writing $x = \frac{k}{T}$ and assuming (consistently) that $T_c \gg m$ or $m/T \ll 1$, we can expand this and integrate it to find that

$$\left\langle n^2 \right\rangle_T - \left\langle n^2 \right\rangle_{T=0} = cT^2 \implies cT_c^2 = \frac{\mu^2}{\lambda} \implies T_c = \tilde{c}\frac{m}{\sqrt{\lambda}}$$

where c is a constant. Again, this approximation is justified if $\lambda \ll 1$, which implies $T_c \gg m$.

0.32 Goldstone Bosons and Continuous Symmetry

Consider a complex scalar field $\Phi = \frac{1}{\sqrt{2}}\left(\Phi_R + \imath\Phi_I\right)$ where $\Phi_R, \Phi_I \in \mathbb{R}$. If we take the Lagrangian density to be

$$\mathcal{L} = \partial_\mu\Phi\partial^\mu\Phi^\dagger - V(\Phi^\dagger\Phi) = \frac{1}{2}\left(\partial_\mu\Phi_R\partial^\mu\Phi_R + \partial_\mu\Phi_I\partial^\mu\Phi_I\right) - V(\Phi_R^2 + \Phi_I^2)$$

then there is a continuous symmetry:

$$\Phi \to \Phi e^{\imath\theta} \qquad \Phi^\dagger \to \Phi^\dagger e^{-\imath\theta}$$

with some constant θ. This corresponds to a rotation in the Φ_R/Φ_I-plane. The prototype potential in the Standard Model is

$$V(\Phi^\dagger\Phi) = \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \Phi^\dagger\Phi\right)^2 = \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \frac{1}{2}\left(\Phi_R^2 + \Phi_I^2\right)\right)^2$$

This potential resembles a rotation of the previous scenario about the central axis, forming a "sombrero"/"Mexican hat"/"wine bottle" shape. It no longer contains just two minima, but rather a manifold of continuous, degenerate minima with

$$\Phi_R^2 + \Phi_I^2 = \frac{2\mu^2}{\lambda}$$

We can describe fluctuations perpendicular to this valley of degenerate minima as harmonic oscillators, but fluctuations along the valley (moving between degenerate states) require very little energy. Let's parameterize

$$\Phi = \frac{\rho(\vec{\mathbf{x}}, t)}{\sqrt{2}} e^{\imath\varphi(x,t)}$$

such that $\rho = \sqrt{\Phi^\dagger\Phi}$ is invariant under rigid rotation but $\varphi \to \varphi(x,t) + \theta$.

We can now write out the potential as

$$V(\Phi^\dagger\Phi) = \frac{\lambda}{4}\left[\frac{\mu^2}{\lambda} - \frac{\rho^2}{2}\right]^2$$

We can write out the entire Lagrangian density as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) + \frac{\rho^2}{2}(\partial_{\mu\varphi})(\partial^\mu\varphi) - \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \frac{\rho^2}{2}\right)^2$$

We can define the minimum of the potential to be at $\rho^2 = \frac{2\mu^2}{\lambda} \equiv \rho_0^2$ and write out fluctuations about $\rho = \rho_0 + h$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}m^2h^2 + \frac{1}{2}\rho_0^2(\partial_\mu\varphi)^2 + \rho_0 h\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}h^2\partial_\mu\varphi\partial^\mu\varphi - \frac{\lambda}{4}\rho_0 h^3 + \frac{\lambda}{16}h^4$$

where $m^2 = \frac{\lambda}{2}\rho_0^2$ by definition. We can further simplify this by defining $\eta = \rho_0\varphi$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}m^2h^2 + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \mathcal{L}_I$$

h is a real scalar with mass $m^2 = \frac{\lambda \rho_0^2}{2}$ corresponding to a harmonic oscillator mode perpendicular to the degeneracy. This is called the Higgs mode. On the other hand, η is a real *massless* mode corresponding to motion along the degenerate minima. The value $\rho_0 = \frac{\sqrt{2}\mu}{\sqrt{\lambda}}$ is called a "vacuum expectation value" and the η-field is a "Goldstone" excitation. It takes no energy for long wavelength distortions to propagate and it only interacts with gradients of the field. A particular (nearly) Goldstone particle, called the axion (yet to be discovered) is associated with a symmetry of the strong interaction and is a candidate for dark matter.

0.32.1 The Higgs Mechanism

Within the Standard Model, the Goldstone fields play the role of giving masses to the $W^\pm$ and Z^0 vector bosons in a process known as the Higgs mechanism.The rotational symmetry described above can be generalized to a spacetime dependent symmetry by coupling to a gauge field. Consider

$$\partial_\mu \Phi \to (\partial_\mu \Phi - \imath e A_\mu \Phi)$$

so that under gauge transformations $\Phi \to \Phi e^{\imath e \Lambda(x,t)}$ and $A_\mu \to A_\mu + \partial_\mu \Lambda(x,t)$,

$$\begin{aligned}\partial_\mu(\Phi e^{\imath e\Lambda}) - \imath e A_\mu \Phi(e^{\imath e\Lambda}) - \imath e(\partial_\mu \Lambda)(e^{\imath e\Lambda})\Phi &= e^{\imath e\Lambda}\left[\partial_\mu \Phi + \Phi \imath e \partial_\mu \Lambda - \imath e A_\mu \Phi - \imath e \Phi \partial_\mu \Lambda\right] \\ &= e^{\imath e\Lambda(\partial_\mu \Phi - \imath e A_\mu \Phi)}\end{aligned}$$

then $(\partial_\mu \Phi^\dagger + \imath e \Phi^\dagger A_\mu)(\partial^\mu \Phi - \imath e A^\mu \Phi)$ is invariant under gauge transformations. Therefore, so is $\Phi^\dagger \Phi$, so

$$\mathcal{L} = (\partial_\mu \Phi^\dagger + \imath e \Phi^\dagger A_\mu)(\partial^\mu \Phi - \imath e A^\mu \Phi) - V(\Phi^\dagger \Phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

is gauge invariant. If we write out our symmetry-breaking potential as before,

$$V(\Phi^\dagger \Phi) = \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \Phi^\dagger \Phi\right)^2$$

and $\Phi = \frac{\rho}{\sqrt{2}} e^{\imath\varphi}$, $\rho = \rho_0 + h$, then under a gauge transformation,

$$\Phi \to \Phi e^{\imath e\Lambda} \implies \rho \to \rho \quad \text{and} \quad \varphi \to \varphi + e\Lambda$$

Then,

$$V(\Phi^\dagger \Phi) = \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \rho^2\right)^2$$

$$\partial_\mu \Phi = e^{\imath\varphi}(\partial_\mu \rho + \imath\rho \partial_\mu \varphi)$$

and

$$\begin{aligned}\partial_\mu \Phi - \imath e A_\mu \Phi &\to e^{\imath\varphi}(\partial_\mu \rho + \imath\rho\partial_\mu \varphi - \imath e A_\mu \rho) \\ &\equiv e^{\imath\varphi}\left[\partial_\mu \rho - \imath e\left(A_\mu - \frac{1}{e}\partial_\mu \varphi\right)\rho\right]\end{aligned}$$

We can perform a similar transformation with the Hermitian conjugate such that

$$(\partial_\mu \Phi^\dagger + \imath e A_\mu \Phi^\dagger)(\partial^\mu \Phi - \imath e A^\mu \Phi) \equiv \frac{1}{2}\left[\partial_\mu \rho + \imath e\rho\left(A_\mu - \frac{1}{e}\partial_\mu \varphi\right)\right]\left[\partial^\mu \rho - \imath e\rho\left(A^\mu - \frac{1}{e}\partial^\mu \varphi\right)\right]$$

Note that $A^\mu - \frac{1}{e}\partial^\mu \varphi$ is just a gauge transformation of A^μ, and it is *also* gauge invariant. If we define

$$\tilde{A}^\mu \equiv A^\mu - \frac{1}{e}\partial^\mu \varphi$$

then

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\nu \equiv \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu \equiv \tilde{F}^{\mu\nu}$$

The Goldstone mode φ combines with A^μ to give the gauge invariant combo $\tilde{A}^\mu$:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \rho + \imath e \rho \tilde{A}_\mu\right)\left(\partial^\mu \rho - \imath e \rho \tilde{A}^\mu\right) - V(\rho) - \frac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}$$

Writing $\rho = \rho_0 + h$ where h is the Higgs particle, we get

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m^2 h^2 - \frac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} + \underbrace{\frac{1}{2}\rho_0^2 e^2 \tilde{A}^\mu \tilde{A}_\mu}_{\text{Mass term for } A^\mu - \frac{1}{e}\partial^\mu \varphi} + \text{interactions}$$

This generates a massive vector field with the equations of motion

$$\partial_\mu \tilde{F}^{\mu\nu} + M^2 \tilde{A}^\nu = 0$$

Note that $\partial_\nu \partial_\mu \tilde{F}^{\mu\nu} + M^2 \partial_\nu \tilde{A}^\nu = 0$ has only three degrees of freedom.

The Higgs mechanism is essential the spontaneous breaking of a continuous symmetry, which results in Goldstone bosons, which couple to gauge fields, which form a gauge-invariant combination of gauge fields and the Goldstone field, which forms a massive vector boson with three degrees of freedom. Two are transverse and one comes from the Goldstone boson. The gauge field "eats" the Goldstone boson, becoming a massive field.

0.32.2 Meissner Effect

In superconductivity, we see this arise as $\partial_\mu \tilde{F}^{\mu\nu} = -M^2 \tilde{A}^\mu$ where $M^2 \tilde{A}^\mu$ is called the "Meissner" current. Additionally, since $\partial_\mu \tilde{A}^\mu = 0$,

$$\Box \tilde{A}^\mu + M^2 \tilde{A}^\mu = 0$$

for the spatial component of static fields, and $-\nabla^2 \tilde{A} + M^2 \tilde{A} = 0$. Taking $\vec{\boldsymbol{\nabla}}\vec{\mathbf{x}} = -\nabla^2 \vec{\mathbf{B}} + M^2 \vec{\mathbf{B}} = 0$, consider only one spatial dimension:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\vec{\mathbf{B}} + M^2 \vec{\mathbf{B}} = 0$$

has one solution:

$$B(x) \propto e^{-Mx}$$

which has a characteristic length $\frac{1}{M}$ called the London penetration length. Magnetic fields cannot enter the sample, so the Meissner current screens the magnetic field. This is an analogue of the Higgs mechanism. In the Standard Model, the weak interaction bosons acquire mass through his mechanism, and the photon does *not* couple to the Goldstone mode and remains massless.

0.33 Yukawa Couplings and Fermion Masses

Fermions get mass through a similar mechanism. Consider the simpler case of a real scalar particle coupling to a Dirac field:

$$\mathcal{L} = \bar{\psi}(\imath \gamma^\mu \partial_\mu - Y\Phi)\psi + \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - V(\Phi)$$

where $Y\Phi$ is the Yukawa coupling term and $V(\Phi) = \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda} - \Phi^2\right)^2$. This potential again has two minima, $\Phi_\pm = \pm\frac{\mu}{\sqrt{\lambda}}$.

Examine the term $-Y\bar{\psi}\Phi\psi$, the Yukawa coupling between the Dirac fermion and the scalar. Writing $\Phi = \Phi_+ + h$, $-Y\bar{\psi}\Phi\psi \to -m\bar{\psi}\psi$ with $m = Y\Phi_+$. Similarly, if we choose Φ_- to expand around, we get $m = Y\Phi_-$. Note that $m^2 = Y^2\Phi_\pm^2$ is the same in both cases:

$$\mathcal{L} = \bar{\psi}(\imath \partial\!\!\!/ - m)\psi - \bar{\psi}Yh\psi + \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}M_h^2 h^2 - \text{interactions}$$

where $M_h^2 = 2\mu^2$.

In the Standard Model, all particles acquire their masses through Yukawa couplings to scalars with $m_i \equiv Y_i \Phi_\pm$. Since Φ is from the scalar field, it is the same for all fermionic species:

$$\frac{m_\alpha}{m_\beta} = \frac{Y_\alpha}{Y_\beta}$$

Spontaneous symmetry breaking gives masses to the weak bosons via the Higgs mechanism and it gives masses to all massive fermions via Yukawa couplings. However, the photon and neutrinos remain massless in this model.

Lecture 16: Helicity and Charge
Thursday, March 18, 2021

0.34 Helicity

From the last lecture, we mentioned that the neutrino remains massless in the Standard Model. It has no rest frame, so the only good quantum number is the helicity, the spin projection along the direction of motion, $\vec{\mathbf{S}} \cdot \hat{p}$. In nature, only negative helicity neutrinos exist. Weak interactions are maximally parity violating, and there are no right-handed neutrinos. Under parity, $\vec{\mathbf{x}} \to -\vec{\mathbf{x}}$ and $\vec{\mathbf{p}} \to -\vec{\mathbf{p}}$ but $\vec{\mathbf{S}} \to \vec{\mathbf{S}}$ since $\vec{\mathbf{S}}$ is a pseudovector. This was revealed in famous Cobalt-60 experiments by Wu. The gluons also remain massless, but quarks are confined into hadrons at distances greater than 10^{-15}m (femtometers).

0.35 Interaction Terms

We add an interaction term for charged particles and the photon field:

$$Q|e|\bar{\psi}\gamma^\mu \psi A_\mu$$

where $Q = 1$ for leptons and $Q = \pm 1/3, \pm 2/3$ for quarks. Only quarks interact with gluons with an interaction term

$$g\bar{\psi}\gamma^\mu G_\mu \psi$$

where g is the strong coupling factor. In electromagnetism, the fine structure constant describes the effective interaction strength:

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

For strong interactions, the equivalent is

$$\alpha_s = \frac{g^2}{4\pi\hbar c} \sim 1$$

For weak interactions, there are two kinds of interactions. Neutral currents:

$$g_1 \left(\bar{e}\gamma^\mu Z^0_\mu e + \bar{\nu}\gamma^\mu Z^0_\mu \nu + \bar{\mu}\gamma^\mu Z^0_\mu \mu + \cdots\right)$$

where

$$\frac{g_1^2}{4\pi\hbar c} \simeq \alpha_w = \frac{1}{30}$$

For charged currents:

$$g_2 \left(\bar{e}\gamma^\mu W^+_\mu \nu_e + \bar{\mu}\gamma^\mu W^+_\mu \nu_\mu + \bar{\tau}\gamma^\mu W^+_\mu \nu_\tau + \text{h.c.}\right)$$

with

$$\frac{g_2^2}{4\pi\hbar c} \simeq \frac{1}{30} \sim \alpha_w$$

However, $g_2 \neq g_1$ past leading order.

In the standard model, $M_W \sim M_Z \sim 90$GeV and the vacuum expectation value is around 120GeV.

0.36 Feynman Diagrams

Motivation: Scattering in the Early Universe establishes local thermal equilibrium (or not), and it is important to understand how that local thermal equilibrium is established in relation to time scales. If the scattering rate is larger than the expansion rate, then many scattering events occur during the time scale of expansion and particles reach local thermal equilibrium.

However, if the scattering rate is much less than the expansion rate, these processes cannot establish this local thermal equilibrium. For example, electrons and photons establish a fluid in local thermal equilibrium via Thomson/Compton scattering. We can describe the initial state as $|i\rangle = |\gamma_{k_i}, e_{p_i}\rangle$ and $|f\rangle = \left|\gamma_{k_f}, e_{p_f}\right\rangle$. The time evolution operator is the total Hamiltonian, so the transition amplitude is

$$\langle f| e^{-\frac{\imath}{\hbar}H(t_f - t_i)} |i\rangle$$

where $H = H_0 + H_I$, the free field Hamiltonian plus the interaction terms. In absence of interactions,

$$H \equiv H_0 \implies e^{-\imath H_0(t_f - t_i)} = e^{-\imath H_0 t_f} e^{\imath H_0 t_i}$$

These operate on the other states like $e^{\imath H_0 t_i} |i\rangle = e^{\imath E_i t_i} |i\rangle$ and $\langle f| e^{-\imath H_0 t_f} = \langle f| e^{-\imath E_f t_f}$. It is conveinient to write the interacting theory as separate:

$$e^{-\imath H(t_f - t_i)} = e^{-\imath H_0 t_f} U(t_f; t_i) e^{\imath H_0 t_i}$$

where $U(t_f; t_i)$ only contains the interactions:

$$U(t_f; t_i) = 1 - \imath \int_{t_i}^{t_f} H_I(t')\, \mathrm{d}t'$$

Then the transition amplitude will be

$$\mathcal{A}_{i\to f} = \langle f| e^{-\imath H(t_f - t_i)} |i\rangle = \underbrace{e^{-\imath(E_f t_f - E_i t_i)}}_{\text{phase}} \langle f| U(t_f; t_i) |i\rangle$$

The transition probability will only contain the interaction term, since the phase is complex:

$$\Pr_{i\to f} = |\langle f| U(t_f, t_i) |i\rangle|^2$$

We can further expand $U = 1 + U^{(1)} + U^{(2)} + \ldots$.

Fermi's Golden Rule states that if we conserve energy and momentum for $t_i \to -\infty$ and $t_f \to +\infty$, the total transition probability is proportional to $\Gamma_{i\to f}(t_f - t_i)$ where Γ is the transition probability per unit time. If we consider electron-photon scattering, then at first order we can either destroy or create a photon, but we need to destroy the initial photon *and* create the final, so this requires a second order term. The "fermionic legs" create/destroy, and we need to destroy the initial electron and create the final one. We also need to create and destroy an intermediate state. Similar to intermediate states in perturbation theory:

$$\sum_{m\neq n} \frac{|\langle m| H_I |n\rangle|^2}{E_m - E_n}$$

In QFT, the intermediate states are described by propagators. Defining $Q^\mu = p_i^\mu - k_f^\mu$, the momentum transfer of the intermediate state, we can write fermionic propagators as

$$\frac{\gamma^\mu Q_\mu + m}{(Q_\mu Q^\mu - m^2)}$$

and bosonic propagators as

$$\frac{1}{Q_\mu Q^\mu - m^2}$$

where m is the mass of the initial state particle.

0.36.1 Feynman Rules

1. Draw Feynman diagrams for each order in perturbation theory. 2. Conserve 4-momentum at each vertex. 3. Assign a propagator to each intermediate state. Then the transition amplitude (for e-γ scattering) is

$$\mathcal{M}_{i\to f} \propto \underbrace{e^2}_{second-order phase transition} \times \text{(initial electron spinor wave function)} \left(\frac{\gamma^\mu Q_\mu + m}{Q^2 - m^2}\right) \text{(final electron spinor wave fun}$$

For electron-electron scattering, we destroy *two* electrons and create two final state electrons, so we need second-order interactions. Because all the fermionic legs are used up, we are left with photon legs, so we must create/annihilate a photon intermediate state. The photon propagator is a massless boson: $\frac{1}{Q^\mu Q_\mu} = \frac{1}{Q^2}$, so

$$\mathcal{M}_{i\to f} \propto e^2 \frac{1}{Q^2}$$

0.36.2 Transition Probability

$$\Pr_{i\to f} = |\mathcal{M}_{i\to f}|^2$$

so for $e\gamma \to e\gamma$ (Thompson scattering),

$$\Pr_{i\to f} \propto \alpha^2 \left(\frac{\gamma^\mu Q_\mu + m}{Q^2 - m^2}\right)^2$$

For electron scattering,

$$\Pr_{i\to f} \propto \alpha^2 \left(\frac{1}{Q^2}\right)^2$$

We can also describe weak interactions, like neutral currents: $\bar{\nu}\nu \to \bar{\nu}\nu$. We must destroy the incoming and create the outgoing particles, so we need a second-order interaction. We use up the fermionic legs so we need to create and destroy a Z^0 boson intermediate state. With $Q = p_{1i} - p_{3f}$, the weak propagator has the form

$$\frac{\alpha_w}{Q^2 - M_z^2}$$

With $Q^2 \ll M_z^2$, $\mathcal{M}_{i\to f} \propto \frac{\alpha_W}{M_z^2} \equiv G_F \simeq 10^{-5}(\text{GeV})^{-2}$, called Fermi's constant. We can also consider charged currents, like $e\nu \to e\nu$, where a W boson is created with a propagator $\frac{\alpha_w}{Q^2 - M_W^2}$, which again has an interaction term similar to G_F. Both neutral and charged interactions in the low-energy limit have $\mathcal{M}_{i\to f} \sim G_F$. For weak interactions with $p_\mu p^\mu \ll M_{Z,W}^2$, Feynman diagrams collapse to a point-like vertex of strength G_F, and this is called Fermi's theory of weak interactions.

In the next lecture, we will discuss reaction rates and cross-sections, considering some incident particle a interacting with a target b and creating c and d final-state particles, $a + b \to c + d$.

Lecture 17: Scattering Cross Sections

Tuesday, March 23, 2021

Suppose a beam of incident particles has a density n_a particles per unit volume and are incident on a target with relative velocity v. The the number of particles incident upon the target in a time Δt is

$$\Delta N_a = n_a v \Delta t \Delta A$$

where $v\Delta t\Delta A$ is the volume of a cylinder of area ΔA and height $v\Delta t$. Then we can define the incident flux of particles as

$$\mathcal{F} \equiv n_a v = \frac{\Delta N_a}{\Delta A \Delta t}$$

The total number of reactions $a + b \to c + d$ per unit time is defined as

$$\frac{\Delta N_{ab\to cd}}{\Delta t} = \underbrace{\mathcal{F}}_{n_a v} \times \sigma$$

where σ is the total cross section. We call this the reaction rate, i.e. $\Gamma_{ab\to cd} \equiv \frac{\mathrm{d}N_{ab\to cd}}{\mathrm{d}t} \equiv n_a v \sigma$.

We can think of σ as the effective cross-sectional area that the target presents to the incident beam (note that it has dimensions of area).

In a statistical ensemble of particles, we find that

$$\Gamma_{ab\to cd} \equiv \langle \sigma |\vec{\mathbf{v}}| \rangle \, n_a$$

0.36.3 Examples of Cross Sections

Let's examine this in the case of Thompson scattering, $e\gamma \to e\gamma$. With $P_i^\mu \equiv p_i^\mu + k_i^\mu$, then we define

$$v = \frac{P_i^\mu P_{i\mu} - m_e^2}{P_i^\mu p_{i\mu} + m_e^2}$$

so

$$\sigma \equiv \frac{\pi\alpha^2}{m_e^2 v}(1-v)\left[\frac{4v}{1+v} + (v^2+2v-2)\ln\left(\frac{1+v}{1-v}\right) - 2v^3\frac{(1+2v)}{(1+v^2)}\right] \qquad \text{(Klein-Nishina Cross Section)}$$

For small photon energy $|k_i| \ll m_e$, $P_i^\mu \sim p_i^\mu \implies v \to 0$, so we get the Thompson scattering limit

$$\sigma_{\text{Th}} \equiv \frac{8\pi\alpha^2}{3m_e^2} = \frac{8\pi}{3}r_0^2 = 6.65 \times 10^{-25}\text{cm}^2$$

where $r_0^2 = \frac{\alpha^2}{m_e^2}$ is the "classical radius of the electron".

In the high energy limit with $|k_i| \gg m_e$, $P_i^\mu P_{i\mu} \gg m_e^2 \implies v \to 1$, so

$$\sigma \to \frac{\alpha^2}{P_i^\mu P_{i\mu}} \quad \underbrace{\to\sim \frac{\alpha^2}{m_e^2}}_{\text{low energy limit}} \quad \text{or} \quad \underbrace{\to\sim \frac{\alpha^2}{P^2}}_{\text{high energy limit}}$$

The moral of the story is that when the typical energy is much greater than the mass of the intermediate state,

$$\sigma \sim \frac{\alpha^2}{(\text{typical energy})^2}$$

and when $E \ll m$, we get

$$\sigma \sim \frac{\alpha^2}{m^2}$$

so σ has dimension of $[L]^2 \sim \frac{1}{[E]^2}$.

We can extend this to other processes. In general, we look at the Feynman diagram for the process, count the powers of α in the transition probability, which is the absolute square of the transition amplitude, and look at the intermediate state particle and use $\frac{1}{E^2}$ or $\frac{1}{m^2}$ depending on the energy regime. For weak interactions at low energies, Fermi theory tells us the exchange of a weak boson has an interaction strength like $g^2/M_{W,Z}^2 = G_F$, so the probability is proportional to $G_F^2 \sim \frac{10^{-10}}{(\text{GeV})^4}$. We then need two powers of energy in the numerator to get a cross section, so

$$\sigma_{WI} \sim G_F^2 E^2 \quad \text{for} \quad E \ll M_{W,Z}$$

0.37 Mean Free Path

The collision rate, or number of collisions per unit time is given by

$$\Gamma_{ab\to cd} = \sigma|v|n_a \qquad (= \langle\sigma|v|\rangle\, n_a)$$

In a time Δt, the total number of collisions is $\sigma|v|n_a\Delta t$. In this time, the particle travels a distance $L = |V|\Delta t$, so the average distance between collisions can be defined as

$$\lambda \equiv \frac{L}{\text{total number of collisions}} = \frac{|v|\Delta t}{\sigma|v|n_a\Delta t} = \frac{1}{\sigma n_a}$$

0.38 Thermal Equilibrium

In an expanding cosmology, thermal equilibrium is established if the average time between collisions is shorter than the expansion time scale $1/H$ (the collision rate is much greater than H). We can think of the average time between collisions as a relaxation time $\tau = \frac{1}{\Gamma}$ such that thermal equilibrium occurs when $\tau \ll \frac{1}{H}$. If $\Gamma < H$, the particle is *decoupled* from other species (it does not interact in an expansion time scale) and its distribution function "freezes out" and no longer adjusts to thermal equilibrium.

Why might this freeze-out happen? $\Gamma = \sigma|v|n$ where n is the particle density, so if the density is diluting upon expansion either as $\frac{1}{a^3}$ for matter or $\frac{1}{a?}$ (he left this blank in the notes, I'm not sure what it is) for radiation, ΓH at some point. Similarly, σ could depend on energy which could diminish (the weak force, for example) by the redshift of the energy. The full evolution of a distribution function is obtained through a Boltzmann equation:

$$\frac{\mathrm{d}f(P_{\text{phys}}(t);t)}{\mathrm{d}t} \equiv \mathbb{C}[f]$$

where $\mathbb{C}$ is some function of f and P_{phys} is the physical momentum of a particle, $\frac{2\pi}{\lambda_{\text{phys}}} = \frac{p_c}{a(t)}$. We can write the total time derivative as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \underbrace{\frac{\partial f(P_{\text{phys}(t)};t)}{\partial t}}_{derivative on explicit time dependence} + \frac{\mathrm{d}P_{\text{phys}}(t)}{\mathrm{d}t}\frac{\partial f}{\partial P_{\text{phys}}}$$

with $P_{\text{phys}} = \frac{p_c}{a(t)}$. p_c is comoving and time-independent, so $\frac{\mathrm{d}P_{\text{phys}}}{\mathrm{d}t} = -\frac{p_c\dot{a}}{a^2} = -P_{\text{phys}}(t)H(t)$. Therefore, the Boltzmann equation in an expanding cosmology is

$$\frac{\partial f(P_{\text{phys}};t)}{\partial t} - H(t)P_{\text{phys}}(t)\frac{\partial f(P_{\text{phys}};t)}{\partial P_{\text{phys}}} \equiv \mathbb{C}[f]$$

where $\mathbb{C}[f]$ is proportional to Γ, so when $\Gamma \ll H$, we can neglect this and set $\mathbb{C} = 0$. This is a collision-less "free-streaming" of particles.

We can do a back of the envelope estimate for reaction rates of ultra-relativistic particles with $T \gg m$. Assume near thermal equilibrium such that

$$n \sim g\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{e^{k/T}\pm 1}$$

Taking $k/T = x$, we can estimate this to be

$$n \sim \frac{gT^3}{2\pi^2}\underbrace{\int_0^\infty \mathrm{d}x\,\frac{x^2}{e^x \pm 1}}_{\mathcal{O}(1)\implies\text{a number}}$$

Then $n \sim T^3\mathbb{C}$ where $\mathbb{C}$ depends on g and the Bose/Fermi statistics of the particle. The typical energy of an ultra-relativistic particle (with $k_B = \hbar = c$) is $k \sim T$. For strong or electromagnetic interactions (or weak interactions with $T \gg M_{W,Z}$),

$$\sigma \sim \frac{\alpha^2}{E^2} \sim \frac{\alpha^2}{T^2}$$

with $v \sim 1$, and

$$n \sim T^3\mathbb{C} \implies \Gamma = \sigma n v \simeq \frac{\alpha^2 T^3}{T^2}\mathbb{C} \implies \Gamma \sim \alpha^2 T\mathbb{C}$$

and

$$\lambda \sim \frac{1}{\Gamma}$$

since $v \sim 1$.

For weak interactions with $T \ll M_{W,Z}$ but $T \gg m_{e,\mu,\nu,\ldots}$, we have $E \sim T$, so $\sigma \sim G_F^2 T^2$ and $\Gamma \sim G_F^2 T^5 \sim 10^{-10} T \left(\frac{T}{\text{GeV}}\right)^4$.

0.39 Equilibrium Thermodynamics

Now let's consider cases where $\Gamma \gg H$. For a species with g internal degrees of freedom, we have (in natural units)

$$n = \frac{g}{2\pi^2}\int_0^\infty \mathrm{d}p\, p^2 f(p)$$

$$\rho = \frac{g}{2\pi^2}\int_0^\infty \mathrm{d}p\, p^2 E(p) f(p)$$

and

$$P = \frac{1}{3}\frac{g}{2\pi}\int_0^\infty \mathrm{d}p\, p^2 \frac{p^2}{E(p)} f(p)$$

where $v(p) = \frac{p}{E(p)} = \frac{\mathrm{d}E(p)}{\mathrm{d}p}$ and $f(p) = (e^{\beta(E(p)-\mu)} \pm 1)^{-1}$ with $+$ for Fermions and $-$ for Bosons and $\beta = \frac{1}{T}$, $E(p) = \sqrt{p^2+m^2}$. It is usually convenient to change variables to E with $p\,\mathrm{d}p = E\,\mathrm{d}E$ and $p = \sqrt{E^2-m^2}$ and rescale $\frac{E}{T} = x$, $\frac{m}{T} = y$, and $\frac{\mu}{T} = \xi$. For species i (and for antiparticles, take $\xi \to -\xi$), we have

$$n_i = \frac{g_i T_i^3}{2\pi^2}\int_{y_i}^\infty \mathrm{d}x\, \frac{(x^2-y_i^2)^{1/2} x}{e^{x-\xi_i} \pm 1}$$

$$\rho_i = \frac{g_i T_i^4}{2\pi^2}\int_{y_i}^\infty \frac{x^2 (x^2-y_i^2)^{1/2}\,\mathrm{d}x}{e^{x-\xi_i} \pm 1}$$

and

$$P_i = \frac{1}{3}\frac{g_i T_i^4}{2\pi^2}\int_{y_i}^\infty \mathrm{d}x\, \frac{(x^2-y_i^2)^{3/2}}{e^{x-\xi_i} \pm 1}$$

In general these integrals must be performed numerically, but they simplify for ultra-relativistic particles with $T \gg m_i \implies y_i \to 0$ and also for the non-relativistic limit $T \ll m_i \implies y_i \to \infty$.

For a given species, the asymmetry between particles and antiparticles (for fermions) can be written as

$$n_i - \bar{n}_i = \frac{g_i T_i^3}{2\pi^2}\int_{y_i}^\infty \mathrm{d}x\, x(x_i - y_i)^{1/2}\left(\frac{1}{e^{x-\xi_i}+1} - \frac{1}{e^{x+\xi_i}+1}\right)$$

For photons, $g = 2$ and there are no antiparticles, so

$$n_\gamma = \frac{2T_\gamma^3}{2\pi^2}\underbrace{\int_0^\infty \frac{x^2\,\mathrm{d}x}{e^x - 1}}_{2\zeta(3)} = \frac{2}{\pi^2}T_\gamma^3 \zeta(3)$$

where $\zeta(3) = 1.20206$ is the Riemann zeta-function.

For ultra-relativistic fermionic species, $y_i \to 0$, and the particle-antiparticle asymmetry becomes

$$n_i - \bar{n}_i = \frac{g_i T_i^3}{6\pi^2}\left[\xi_i^3 + \pi^2 \xi_i\right]$$

and the asymmetry per photon is

$$\eta_i \equiv \frac{n_i - \bar{n}_i}{n_\gamma} = \frac{g_i \left(\frac{T_i}{T_\gamma}\right)^3}{12\zeta(3)} \left[\xi_i^3 + \pi^2 \xi_i\right]$$

Observations tell us that the baryon (and charged lepton) asymmetry per photon is $\eta \simeq 10^{-9}$, so the universe has much more matter than antimatter (but is electrically neutral, more on this later).

A species i may reach thermal equilibrium with members of its own species but might not with other species j, in which case the temperature $T_i \neq T_j$. The ultra-relativistic limit is important here. $T_i \gg m_i \implies y_i \to 0$, for which (with $\xi_i \to 0$),

$$\rho = g_i T_i^4 \frac{\pi^2}{30} \times \begin{cases} 1 & \text{Bose} \\ \frac{7}{8} & \text{Fermi} \end{cases}$$

$$n_i = \bar{n}_i = g_i T_i^3 \frac{\zeta(3)}{\pi^2} \times \begin{cases} 1 & \text{Bose} \\ \frac{3}{4} & \text{Fermi} \end{cases}$$

and $P_i = \rho_i/3$, $S_i = \frac{P_i + \rho_i}{T_i} = \frac{4}{3}\frac{\rho_i}{T_i}$. In the ultra-relativistic limit, each species acts like radiation with $\frac{P}{\rho} \equiv \frac{1}{3}$.

In the non-relativistic limit, $\frac{m_i}{T_i} \gg 1$, $E_i = m_i + \frac{p_i^2}{2m}$ gives us the Maxwell-Boltzmann limit:

$$\frac{1}{e^{\beta(E-\mu)} \pm 1} \to e^{-\frac{(m_i - \mu_i)}{T_i}} e^{-\frac{p_i^2}{2m_i T_i}}$$

If we write $\mu_i - m_i \equiv \bar{\mu}_i$ as the non-relativistic chemical potential, or the energy per particle measured from the rest energy, we have the classical result

$$n_i = g_i \left(\frac{m_i T_i}{2\pi}\right)^{3/2} e^{-\bar{\mu}_i/T_i}$$

$$\rho_i \simeq m_i n_i \qquad P_i \simeq n_i T_i \left(\text{from } P = \frac{N}{V} T\right)$$

so $\frac{P}{\rho} \sim \frac{T_i}{m_i} \ll 1$, so non-relativistic species behave as matter, and we can neglect pressure. Since $\eta \sim 10^{-9}$, let's take $\mu = 0$. Now accounting for all ultra-relativistic species of particles and antiparticles, we have

$$\rho_R = T^4 \sum_i \left(\frac{\rho_i}{T}\right)^4 \equiv \frac{\pi^2}{30} g_* T^4$$

where

$$g_* = \sum_{\text{Bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{Fermions}} g_i \left(\frac{T_i}{T}\right)^4$$

and

$$P_R = \frac{1}{3}\rho_R = \frac{\pi^2}{90} g_* T^4$$

$$S_R = \frac{4}{3} g_*^{(S)} T^3 \frac{\pi^2}{30}$$

where

$$g_*^{(S)} = \sum_{\text{Bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{Fermions}} g_i \left(\frac{T_i}{T}\right)^3$$

(difference is in the power of temperature).

Typically, $T \equiv T_\gamma$ if all species are in thermal equilibrium at the same $T_i \equiv T$, so we can neglect the T-dependent parts of g and $g_*^{(S)} = g_*$.

In the case of matter-radiation equality, $\frac{\Gamma_{0,R}}{a^4} = \frac{\Gamma_{0,M}}{a^3}$ with $\Omega_{0,R} \sim 10^{-5}$, $\Omega_{0,M} \sim 0.25$, we have $z_{\text{eq}} \sim 4000$ and $T_{\text{eq}} = T_{\text{CMB}} \times 4000 \sim 1\text{eV}$. For $T \gg 1\text{eV}$, the universe was *radiation dominated* and

$$H^2 = \frac{8\pi G}{3}\rho_R \equiv \frac{8\pi G}{3} g_* \left(\frac{\pi^2}{30}\right) GT^4$$

We can introduce the Planck mass,

$$M_{\text{pl}} = \frac{1}{\sqrt{G}} = 1.22 \times 10^{19}\text{GeV}$$

such that

$$H(T) \simeq 1.66 g_*^{1/2} \left(\frac{T^2}{M_{pl}}\right)$$

Since, for the radiation-dominated universe, $a(t) \sim t^{1/2} \implies H = \frac{1}{2t}$, then

$$t \sim \frac{0.3}{\sqrt{g_*}} \left(\frac{M_{pl}}{T^2}\right) \simeq 1\text{s} \left[\frac{T}{\text{MeV}}\right]^{-2} \frac{1}{\sqrt{g_*}}$$

and

$$S_R = \frac{2\pi^2}{45} g_*^{(S)} T^3$$

The effective degrees of freedom, g_* and $g_*^{(S)}$ depend on the temperature even if $T_i = T$ for all species i. This is because only the ultra-relativistic degrees of freedom contribute to the expansion, namely species for which $T \gg m_i$.

Lecture 18:
Thursday, March 25, 2021

For $T \gg 100\text{GeV}$, above the T_c for the SM phase transition at $T \sim 170\text{GeV}$, all W, Z, Higgs, and Fermions are ultrarelativistic. For $T \ll 100\text{GeV}$, they are non-relativistic (including the top quark). We can estimate the decoupling temperature at which $\Gamma \lesssim H$ by assuming self-consistently that species are in local thermal equilibrium in an radiation-dominated bath. For strong, EM, and weak interactions for $T \gg 100\text{GeV}$, $\sigma \sim \frac{\alpha^2}{E^2} \sim \frac{\alpha^2}{T^2}$ because $E \sim T$ and $\alpha_{\text{EM}} \sim 1/137$, $\alpha_{\text{W}} \sim 1/30$ and $\alpha_{\text{S}} \sim 0.1 - 1$. For ultrarelativistic particles, $v \sim 1$ and $n \sim T^3$, so

$$\Gamma = \sigma v n \sim \frac{\alpha^2}{T^2} T^3 \sim \alpha^2 T \times \mathcal{O}(1)$$

During the radiation-dominated universe, $H \simeq 1.66\sqrt{g_*}\left(\frac{T^2}{M_{pl}}\right)$ with $\sqrt{g_*} \lesssim 10$ and $M_{pl} \sim 10^{19}\text{GeV}$. Therefore

$$\frac{\Gamma}{H} \sim \frac{\alpha^2}{\sqrt{g_*}} \left(\frac{M_{pl}}{T}\right) \sim \frac{\alpha^2}{\sqrt{g_*}} \left(\frac{\text{GeV}}{T}\right) \times 10^{19}$$

Then for $\frac{1}{137} \lesssim \alpha \lesssim 1$ and $g_* \lesssim 100$ and $100\text{GeV} \ll T \ll M_{pl}$, *all* ultrarelativistic species are in local thermal equilibrium with $T_i \equiv T$ for all interactions, even non-relativistic particles with $\sigma \sim \frac{\alpha^2}{m^2}$ but $n \sim T^3$ are.

For $T \lesssim 100\text{GeV}$, QED and the strong interactions have $\sigma \sim \frac{\alpha^2}{T^2}$, but weak interactions have $\sigma \sim G_F^2 E^2 \sim G_F^2 T^2$ and for ultrarelativistic particles ($T \gg 1\text{MeV}$) like neutrinos and electrons,

$$\Gamma = \sigma n v \sim G_F^2 T^2 T^3 \sim G_F^2 T^5 \sim \frac{10^{-10}}{(\text{GeV})^4} T^5$$

Then

$$\frac{\Gamma}{H} \sim \frac{1}{\sqrt{g_*}} \left[\frac{T}{\text{MeV}}\right]^3$$

For $T \gg 1\text{MeV}$, $\frac{\Gamma}{H} \gg 1$, and for $T < 1\text{MeV}$, $\frac{\Gamma}{H} \lesssim 1$. Weak interactions decouple at $T \sim 1\text{MeV}$. Note that $m_n - m_p \simeq 1.2\text{MeV}$.

0.40 Thermal History of the Universe

- All particles with strong and EM interactions remain in local thermal equilibrium from $T \sim 10^{15}$GeV down to $T_{eq} \sim 1$eV
- The weak interactions freeze out/decouple at $T \sim 0.1 - 1$MeV ($t \sim 1$s), but the only particles that only interact with the weak interaction are neutrinos, so neutrinos fall out of local thermal equilibrium at $T \sim 1$MeV.
- Inflation: At $t \sim 10^{-35}$s the energy scale is somewhere around 10^{15} to 10^{16}GeV.
- Grand Unification (G.U.T.) transition occurs around $T \sim 10^{15}$GeV (again, very uncertain, more on this later).
- At $T_c \sim 175$GeV, the SM symmetry breaking phase transition occurs, above which all particles are massless, and below which the quarks, charged leptons, and weak bosons acquire masses via the Higgs and Yukawa mechanisms. This is called the Electroweak Phase Transition and occurs around $t \sim 10^{-11}$s after the Big Bang.
- At $T \sim 150$MeV ($t \sim 10\mu$s), the QCD interactions (strong interactions) have a phase transition. For $T > T_{\text{QCD}} \sim 150$MeV, quarks and gluons are free. Below this temperature, they are confined in hadrons like protons, neutrons, pions, etc. This is studied at LHC/RHiC by colliding heavy nuclei (gold/uranium).
- At $T \sim 1$MeV, the weak interaction freezes out and primordial nucleosynthesis begins at $t \sim 1$s and lasts around 3min. Below this temperature, e^+e^- pairs can no longer be in equilibrium with photons via the reaction $e^+e^- \leftrightarrow \gamma\gamma$, but instead they tend to annihilate and give up their entropy to the photon gas, which heats up. They also give a "lil' bit" of entropy to neutrinos via neutral and charged currents.
- At $T \sim 1$eV, matter and radiation equalize. Above this temperature, the universe is radiation dominated, and below it, the universe is matter dominated.
- At $T \sim 0.3$eV, electrons left from $e^+e^- \to 2\gamma$ (remember the matter/antimatter asymmetry, not all of them anihillate) combine with protons to form neutral hydrogen, and photons decouple at $t \sim 360,000$yr after the Big Bang. The CMB is "free" and the Universe becomes transparent to photons.

Lecture 19: Effective Potentials
Tuesday, March 30, 2021

We briefly studied phase transitions from symmetry breaking. Now we will cover them more generally and in detail. Begin with a discrete symmetry and a scalar theory with a real scalar field ϕ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)$$

where

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

which has minima at $\pm\phi_0 = \pm\frac{|\mu|}{\sqrt{\lambda}}$. This Lagrangian is invariant under the discrete symmetry $\phi \to -\phi$, and there are two degenerate minima for $\mu^2 > 0$. Picking one minimum to expand around breaks the symmetry spontaneously, as we've seen before: $\phi = +\phi_0 + \varphi(x)$ where $\varphi(x)$ are small harmonic oscillations. This is a classical description modified by quantum/thermal fluctuations. We can write the Hamiltonian using $\phi(x) = \dot{\phi}(x)$:

$$H = \int \mathrm{d}^3x \left[\frac{1}{2}\pi^2(x) + \frac{1}{2}(\vec{\boldsymbol{\nabla}}\phi)^2 + V(\phi)\right]$$

If ϕ is spacetime independent, $H \equiv \mathcal{V}V(\phi)$ where $\mathcal{V}$ is the total volume and $V(\phi)$ describes the energy density for the spacetime constant ϕ.

We can include fluctuations around ϕ (not just a minimum) by writing $\phi(x) = \phi + \varphi(x)$, since $\dot{\phi} = \vec{\boldsymbol{\nabla}}\phi = 0$ for a spacetime constant field. We can impose that $\langle\varphi(x)\rangle = 0$ as a constraint such that $\langle\phi(x)\rangle = \phi$ (this typically involves a Lagrange multiplier, but we don't need that level of detail). Expanding H in terms of the fluctuations, we have

$$H = \mathcal{V}V(\phi) + \int \mathrm{d}^3x \left[\frac{1}{2}\tilde{\pi} + \frac{1}{2}(\vec{\boldsymbol{\nabla}}\varphi)^2 + \varphi V'(\phi) + \frac{1}{2}\varphi^2 V''(\phi) + \cdots\right]$$

where $\tilde{\pi} = \frac{\partial\varphi}{\partial t}$ and V' and V'' correspond to ϕ-derivatives. We can neglect the $\varphi(x)V'(\phi)$ term since we have $\langle\varphi\rangle = 0$. Removing that linear term, we can write the Hamiltonian as

$$H = \mathcal{V}V(\phi) + \int \mathrm{d}^3x \left[\frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}(\vec{\boldsymbol{\nabla}}\varphi)^2 + \frac{1}{2}M^2(\phi)\varphi^2 + \cdots\right]$$

with $M^2(\phi) = V''(\phi) \equiv \frac{\mathrm{d}^2V(\phi)}{\mathrm{d}\phi^{22}}$. The second term is a free-field theory with effective mass-squared of $M^2(\phi)$. We can write the partition function as

$$Z[\beta] = \mathrm{Tr}\, e^{-\beta H} \equiv e^{-\beta F}$$

where F is the free energy,

$$F = -T\ln\left(\mathrm{Tr}\, e^{-\beta H}\right)$$

This is identified with a finite temperature *effective potential* (times the volume $\mathcal{V}$), or essentially a free energy density:

$$V_{\mathrm{eff}}(\phi; T) = -\frac{T}{\mathcal{V}}\ln\left(\mathrm{Tr}\, e^{-\beta H}\right)$$

Since $e^{-\beta H} = e^{-\mathcal{V}\frac{V(\phi)}{T}}e^{-\tilde{H}/T}$, we can write

$$V_{\mathrm{eff}}(\phi; T) = V(\phi) - \frac{T}{\mathcal{V}}\ln\left(Tre^{-\beta\tilde{H}}\right)$$

where

$$\tilde{H} = \int \mathrm{d}^3x \left[\frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}(\vec{\boldsymbol{\nabla}}\varphi)^2 + \frac{1}{2}M^2(\phi)\varphi^2 + \cdots\right]$$

which describes our free field theory. The higher order terms are treated in perturbation theory with higher powers of λ, the Lagrange multiplier.

We can quantize these fluctuations as harmonic oscillators:

$$\tilde{H} = \sum_k E_k\left[a_k^\dagger a_k + \frac{1}{2}\right]$$

where $E_k = \sqrt{k^2 + M^2(\phi)}$. This makes the second term in V_{eff} become

$$\underbrace{\int \frac{\mathrm{d}^3k}{(2\pi)^3}E_k}_{\text{Zero-point energy}} + \underbrace{\frac{T}{2\pi^2}\int_0^\infty k^2\ln\left(1 - e^{-E_k/T}\right)\mathrm{d}k}_{\text{Finite temperature contribution}}$$

All together,

$$V_{\mathrm{eff}}(\phi; T) = V(\phi) + \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{k^2 + M^2(\phi)} + \frac{T}{2\pi^2}\int \mathrm{d}k\, k^2\ln\left(1 - e^{-E_k/T}\right)$$

Examining the $T = 0$ quantum zero-point contribution, we can consider cutting off this integral at some finite radial value of k:

$$\frac{1}{2\pi^2}\int_0^\Lambda k^2\sqrt{k^2 + M^2}\,\mathrm{d}k \sim \Lambda^4 + \Lambda^2 + \ln(\Lambda) + \cdots$$

where Λ^4 is a constant, Λ^2 renormalizes the mass, and $\ln(\Lambda)$ renormalizes λ, the Lagrange constraint. If we absorb these renormalizations and focus on the $T \neq 0$ part at high T ($T \gg M^2$), we can change variables to $x = \frac{k}{T}$ and $\tilde{M} = \frac{M}{T}$. Then the $T \neq 0$ contribution is

$$\frac{T^4}{2\pi^2}\int_0^\infty x^2 \ln\left(1 - e^{-\sqrt{x^2+\tilde{M}^2}}\right) \mathrm{d}x = -\frac{\pi^2}{90}T^4 + \frac{M^2(\phi)T^2}{24} + \mathcal{O}\left(M^4/T^2\right) = \frac{\lambda}{8}\phi^2 T^2 + \text{constant} + \mathcal{O}\left(M^2/T^4\right)$$

Therefore, we can write the effective potential as

$$V_{\text{eff}}(\phi; T) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{8}\phi^2 T^2 + \frac{\lambda}{4}\phi^4 + \cdots \equiv \underbrace{\frac{\lambda}{8}(T^2 - T_c^2)}_{\frac{1}{2}m^2(T)}\phi^2 + \frac{\lambda}{4}\phi^4 + \cdots$$

(Landau-Ginzburg Free Energy Density)

with $T_c = \frac{4\mu^2}{\lambda}$. For $T > T_c$, the minimum is at $\phi = 0$, but for $T < T_c$, there are two degenerate minima (spontaneous symmetry breaking) at $\pm\phi_0 = \pm\sqrt{-\frac{m^2(T)}{\lambda}} = \pm\frac{1}{2}\sqrt{T_c^2 - T^2}$. Note that $\phi_0(T) \propto (T_c - T)^{1/2}$.

As $T \to T_c$ from below, $\phi_0(T) = \langle\phi\rangle$ vanishes continuously, which is the hallmark of a *second order* phase transition. For $T < T_c$, the original symmetry is spontaneously broken, but the resulting minima are degenerate.

Let us now introduce an explicit symmetry breaking term in a Lagrangian:

$$\mathcal{L} \to \mathcal{L} + h\phi$$

Now

$$V_{\text{eff}}(\phi; T; h) = \frac{1}{2}m^2(\phi)\phi^2 + \frac{\lambda}{4}\phi^4 - h\phi$$

For $T > T_c$, the minimum is no longer at 0 but rather at some $\bar{\phi}_h$, which approaches zero as $h \to 0$. For $h \neq 0$, the Lagrangian does not feature the discrete symmetry from before. For small h, the minimum is found to linear order in h by writing $\bar{\phi}_h = \chi(T)h$ and linearizing this to find $\chi(T) = \frac{1}{m^2(\phi)}$, defined as the susceptibility. As $T \to T_c^+$, $\chi(T) \propto \frac{1}{(T-T_c)^\alpha}$ where α is called the critical exponent ($\alpha = 1/2$ in the previous example but $\alpha = 1$ here. The susceptibility diverges as $T \to T_c$ from above. For $h > 0$, ϕ_+ is the absolute stable minimum, ϕ_- is a metastable point, and ϕ_M is the max. As $h \to 0^+$, both minima approach ϕ_0 and $-\phi_0$ respectively. For $h < 0$, the minima switch.

As $h \to 0^+$, the minimum approaches $+\phi_0$, but as $h \to 0^-$, the minimum approaches $-\phi_0$, so there is a discontinuity in the value of the absolute minimum of the free energy as a function of h, for $T < T_c$. The discontinuity in the minimum value of $\langle\phi\rangle$ as h crosses 0 is a *first order* phase transition.

Lecture 20: Solitons

Thursday, April 01, 2021

Suppose we slowly vary h across $h = 0$, like from $h < 0$ to $h > 0$. For $h \to 0^-$, the minimum is at ϕ_-, but as h crosses 0, the true minimum becomes ϕ_+, but the system remains trapped in the local metastable state at ϕ_- until it somehow "decays" to the true minimum. To do so, it must overcome a free-energy barrier.

Howe does it do this? Before we answer this, we must look at "topological" excitations associated with spontaneous symmetry breaking for $h = 0$ (the degenerate case).

Consider the potential $V(\phi) = \frac{\lambda}{4}\left[\phi^2 - \phi_0^2\right]^2$. In 1+1 dimensions, the Klein-Gordon equation is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2}{\partial\phi^2} + V'(\phi) = 0$$

so for static field configurations,

$$-\frac{\partial^2\phi}{\partial x^2} - \lambda\phi_0^2\phi + \lambda\phi^3 = 0$$

This features solutions that interpolate between the two minima at $\pm\phi_0$. If we define $\phi(x) = \phi_0 u(x)$ and $z = \sqrt{\lambda}\phi_0$, then

$$-\frac{\partial^2 u}{\partial z^2} - u + u^3 = 0$$

has solutions like $u(z) = \pm\tanh[\frac{z-z_0}{\sqrt{2}}]$. For the original equation, this means

$$\phi(x) = \pm\phi_0 \tanh[\sqrt{\frac{\lambda}{2}}\phi_0(x - x_0)]$$

These solutions are called solitons (+) or antisolitons (-) (or kinks (+) and anti-kinks (-)). x_0 is an arbitrary position by translational invariance. These excitations of the spontaneous symmetry breaking ground state are also known as *domain walls*. In 1+1 dimensions, these are topological. The topological current,

$$J^\mu = \varepsilon^{\mu\nu}\frac{\partial}{\partial\nu}\phi$$

with $\varepsilon^{01} = +1 = -\varepsilon^{10}$, the Levi-Civita symbol.

$$\partial_\mu J^\mu = 0 \implies Q = \int \mathrm{d}x\, J^0(x)$$

is conserved, but

$$Q = \int \mathrm{d}x\, \frac{\mathrm{d}\phi}{\mathrm{d}x} = \phi(\infty) - \phi(-\infty)$$

For solitons, $Q = 2\phi_0$, and for antisolitons, $Q = -2\phi_0$. This current is a topological current, since it is not related to any Noether symmetry.

The energy density of the soliton domain wall is localized around x_0. The width of this distribution is $\xi \sim \sqrt{\lambda}\phi_0$.

$$\mathcal{H}(x) = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi)$$

In 3+1 dimensions, how does a phase transition occur? Consider lowering the temperature from $T > T_c$ to $T < T_c$. The homogeneous single phase with $\langle\phi\rangle = 0$ breaks into domains with $\pm\phi_0$ separated by a domain wall. The domain wall features an energy per unit length given precisely by the soliton domain wall energy. Eventually these excitations relax as soliton-antisolitons collide (a soliton-antisoliton pair is "topologically trivial"). These concepts are useful to describe first-order phase transitions as nucleating "bubbles". In a first-order phase transition, the system is "trapped" in a local minimum and must "decay" to the one with the lowest free-energy. Consider a spontaneous thermal fluctuation that creates a bubble of radius R of the "true" phase in the "sea" of the "false" metastable phases. Inside the bubble, the phase with $\langle\phi\rangle = \phi_+$ has a lower free energy $-\Delta F$ than the phase outside the bubble, and the surface of the bubble is like a domain wall separating the phases. Such domain walls feature an energy density localized at the wall. The total change in free energy to create such a spontaneous bubble is:

$$\Delta F(R) = 4\pi R^2\sigma - \frac{4\pi R^3}{3}\Delta\mathcal{F}$$

where σ is the surface tension or energy per unit area, and $\Delta\mathcal{F}$ is the change in free energy density. If we plot this against R, we find a zero at $R^* = \frac{3\sigma}{\Delta\mathcal{F}}$. If $R < R^*$, the bubble shrinks because of the surface tension. If $R > R^*$, the bubble grows because the volume of the phase with lower free energy wins out. These bubbles grow gaining free energy and converting the metastable phase to the true stable phase. This process is called *nucleation* and is at the heart of the vapor chamber detector in particle physics. Here, energetic particles passing through a supersaturated water vapor "seed" the production of bubbles (bubble chamber). The system must overcome a free energy barrier $F[\phi(R^*)]$, a field configuration with critical radius. The nucleation process is suppressed by $e^{-\frac{F[\phi(R^*)]}{T}}$. For a radial configuration, ϕ obeys the static Klein-Gordon equation in spherical coordinates for spherical symmetry (spherical bubbles):

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}R^2} + \frac{2}{R}\frac{\mathrm{d}\phi}{\mathrm{d}R} - V'(\phi) = 0$$

Once such a supercritical bubble is formed, the physics is similar to a liquid-gas transition, a hallmark of first-order phase transitions driven strongly *out of equilibrium* by the growth of supercritical bubbles converting metastable phases to stable phases. In particle physics, the nature of the different transitions is till very much debated.

The electroweak theory might feature a first-order transition for a Higgs mass less than 80GeV but with $m_H = 125$GeV, the transition is either a smooth crossover (no transition but a continuous transformation) or a "mild" second-order transition between phases with broken/unbroken $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry. Such a transition occurs at $T \sim 100$GeV at a time $t \sim 10^{-11}$s after the Big Bang: these are known as sphalerons, which are also bubbles.

The QCD phase transition between the phase of deconfined and confined quarks occurs around $T \sim 150$MeV and $t \sim 10^{-5}$s. The evidence from lattice gauge theory (for the three lightest quarks) *suggests* a first-order transition to a "mixed" phase, but uncertainties remain. The important feature of QCD is that if it is a "mixed" phase of quarks, gluons, mesons, and baryons, the liquid-gas transition occurs along an isochore (equal pressure) line, which is a consequence of the Maxwell construction. In turn, this implies an anomalous small speed of sound which may result on the formation of black holes via Jean's instabilities (more on this later) with $M \approx M_\odot$.

Associated with these phase transitions, there emerge "topological excitations" like spontaneous symmetry breaking of discrete symmetries leads to solitons/domain walls (for example, in ferro magnets), SSB in gauge theories like the Standard Model lead to other excitations, strings, for instance, for $\mathrm{U}(1)$ symmetry. Sphalerons are configurations of critical bubbles associated with electroweak theory ($\mathrm{SU}(2) \times \mathrm{U}(1)$), and skyrmions are associated with baryons in the EFT for mesons/baryons in QCD ($\mathrm{SU}(3)$).

Lecture 21: Primordial Nucleosynthesis
Tuesday, April 06, 2021

0.41 Primordial Nucleosynthesis

To understand the formation of elements in the early universe, we must explore the concept of Nuclear Statistical Equilibrium. This has two parts: thermal equilibrium among species and chemical equilibrium among "reacting" species. Consider non-relativistic particles. Their distribution function is Maxwell-Boltzmann:

$$f(p) = e^{-(m-\mu)/T} e^{-p^2/2mT}$$

The number of species i in thermal equilibrium is

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{-(m_i - \mu_i)/T}$$

Consider a nuclear reaction in which Z protons and $A - Z$ neutrons combine to form a bound nucleus with atomic mass A and charge Z. Since n and p formed after the QCD confinement transition at $T_{\mathrm{QCD}} \sim 150$MeV and $m_p \sim m_n \sim 1$GeV, both are non-relativistic. If the temperature is large enough, the nucleus can dissociate into Zp and $(A - Z)n$. At this temperature,

$${}^{\mathrm{Z}}_{\mathrm{A}}\mathrm{N} \leftrightarrow Zp + (A - Z)n$$

is in chemical equilibrium when the nuclei are in constant coexistence with p and n and the reactions go both ways equally. Then

$$\mu_N = Z\mu_p + (A - Z)\mu_n$$

From the equation for the density for each species, we can move some stuff around to find

$$e^{\mu_N/T} = \frac{n_N}{g_N} \left(\frac{2\pi}{m_N T} \right)^{3/2} e^{m_N/T}$$

but

$$e^{\mu_N/T} = e^{Z\mu_p/T}e^{(A-Z)\mu_n/T} = \left[\frac{n_p}{g_p}\left(\frac{2\pi}{m_pT}\right)^{3/2} e^{m_p/T}\right]^Z \left[\frac{n_n}{g_n}\left(\frac{2\pi}{m_nT}\right)^{3/2} e^{m_n/T}\right]^{A-Z}$$

We know that $g_n = g_p = 2$ since they are spin-1/2 Fermions. We can approximate $m_n \simeq m_p$ such that $m_N \simeq Am_n$. Then

$$n_A \simeq \frac{g_A A^{3/2}}{2^A}\left(\frac{2\pi}{m_nT}\right)^{\frac{3}{2}(A-1)} n_p^Z n_n^{A-Z} e^{B_A/T}$$

where $B_A = m_pZ + m_n(A-Z) - m_N$ is the binding energy. For $n + p \leftrightarrow d + \gamma$, the deuteron binding energy is 2.2MeV $\ll$ 1GeV.

Nucleus	B_A(MeV)	g_A
$d/D = {}^2$H	2.22	3
$T = {}^3$H	6.92	2
^{3}He	7.72	2
^{4}He	28.3	1
^{12}C	92.2	1

The total nucleon density (protons and neutrons) is

$$n_N = n_p + n_n + \sum_i (An_A)_i$$

where the first two terms are free protons and neutrons, and the final term is the sum over *all* bound species i each with atomic number $(p+n) = A$, such that An_A is the number of nucleons bound in a nucleus of A.

The mass fraction of a nuclear species ${}^A_Z$N is $X_A = \frac{An_A}{n_N}$ so that by definition, $\sum_i X_i = 1$.

0.41.1 Baryon Asymmetry

We can write an asymmetry parameter

$$\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma}$$

which is time-dependent since all these terms scale as $a^{-3}(t)$. Observations reveal that there is no substantial amount of antimatter in the Universe, so $n_{\bar{B}} \sim 0$, otherwise there would be a large flux of hard (> 1MeV γ-rays from annihilations. We can estimate η today with $n_{\bar{B}} \sim 0$ and $\frac{\rho_B}{m_n} = n_B = \frac{\rho_B}{\rho_{0,c}}\frac{\rho_{0,c}}{m_n}$ with $\rho_{0,c} = 1.05h^2 \times 10^4 \frac{\text{eV}}{\text{cm}^3}$, $\frac{\rho_B}{\rho_{0,c} = \Gamma_B}$, and $n_\gamma = \frac{421}{\text{cm}^3}$. This will give

$$\eta = 2.68 \times 10^{-8}\underbrace{(\Omega_B h^2)}_{\sim 0.02}$$

This quantity is very important. There are around 10^9 photons per baryon. Using the expression for n_A above, we can write T in terms of n_γ and divide by n_N to get X_A:

$$X_A = A\frac{n_A}{n_N} = g_A[\zeta(3)]^{A-1}\pi^{\frac{1-A}{2}}2^{(3A-5)/2}A^{5/2}\left(\frac{T}{m_n}\right)^{\frac{3}{2}(A-1)} \times \eta^{A-1}X_p^Z X_n^{A-Z}e^{\frac{B_A}{T}}$$

where $\eta \sim n_N/n_\gamma$. This is the mass abundance ratio for a species A in nulear statistical equilibrium (both thermal and chemical). An important ratio is the neutron/proton ratio. Above $T >$ 1MeV, weak interactions maintain thermal equilibrium between n and p via

$$n \leftrightarrow p + e^- + \bar{\nu}_e$$
$$e^+ + n \leftrightarrow p + \bar{\nu}_e$$
$$\nu_e + n \leftrightarrow p + e^-$$

with $\mu_{e^+} = -\mu_{e^-}$ and $\mu_{\bar{\eta}} = -\mu_\eta$. With $\mu_n = \mu_p + \mu_{e^-} - \mu_{\eta_e}$, we can self-consitently assume that $\eta_\nu \equiv 0$. Remember that $\frac{\mu_e}{T} \equiv \zeta_e$ and

$$\frac{n_{e^-} - n_{e^+}}{n_\gamma} \propto (\zeta^3 + \zeta\pi)$$

But $n_{e^-} - n_{e^+} = n_B - n_{\bar{B}}$ by charge neutrality with $n_{\bar{B}} \sim 0$ and $\frac{n_B}{n_\gamma} \sim 10^{-9}$. We can then safely set $\zeta_e \equiv 0$ and $\mu_\nu = \mu_{e^-} = 0$, and $\mu_n \equiv \mu_p$ in equilibrium.

Then

$$\frac{n_n}{n_p} = \frac{X_n}{X_p} = e^{-Q/T}$$

where $Q = m_n - m_p = 1.293\text{MeV}$. n_n/n_p diminishes as T diminishes, and for $T \gg 1\text{MeV}$, $X_n/X_p \sim 1$. From the expression for X_A, the temperature T_A at which the mass fraction is $X_A \sim 1$ is given by

$$T_A \simeq \frac{1}{A-1} \frac{B_A}{\ln(1/\eta) + \frac{3}{2}\ln(m_n/T_A) + \mathbb{C}(A)}$$

where $\mathbb{C}(A)$ is some constant that depends on A. This can be solved numerically. The weak interactions freeze out at $T \sim 0.8\text{MeV}$, at which point the n/p ratio freezes to $X_n/X_p = e^{-1.29/0.8} = 0.2$. If the neutron was absolutely stable, this ratio would remain constant. However, the neutron has a decay with $\tau_n \sim 900\text{s}$.

As the temperature drops, the following chain of nuclear reactions takes place:

$$\begin{aligned} p + n &\leftrightarrow d + \gamma \\ d + d &\leftrightarrow {}^3\text{He} + n \\ {}^3\text{He} + d &\leftrightarrow {}^4\text{He} + p \end{aligned}$$

Note that the proton is restored. These reactions soak up all available neutrons into the ^{4}He state via another pathway:

$$\begin{aligned} d + d &\leftrightarrow {}^3\text{H} + p \\ {}^3\text{H} + d &\to {}^4\text{He} + n \end{aligned}$$

0.42 The Deuterium Bottleneck

The first step in that reaction chain is an electromagnetic reaction, $p + n \leftrightarrow d + \gamma$. Every single further step needs the deuterium to already be formed and will proceed whenever $X_D \sim 1$. From the expression for the criterion $X_A \sim 1$, assuming $X_n \sim X_p \sim 1$, we can see that

$$T_D \sim \frac{2.2\text{MeV}}{\ln(1/eta) + \frac{3}{2}\ln\left(\frac{m_n}{T_A} + \mathbb{C}_D\right)}$$

but $\eta \sim 10^{-10}$ and $\ln(1/\eta) \sim 10\ln(10) \sim 23$, so $T_D \sim 0.1\text{MeV}$. The temperature must get this low for deuterium to form with $X_D \sim 1$, and this is because at $T \sim B_D \sim 2.2\text{MeV}$, there are lots of photons that photodisintegrate the deuterium thatt is produced, which is a consequence of $\frac{n_\gamma}{n_B} \sim 10^{10}$.

However, once the deuteron is formed, the rest of the reactions happen very fast with large capture cross-sections. All the neutrons end up as Helium-4. Using the $T - t$ dictionary (our correspondence between temperature and time after the Big Bang from a few lectures ago), 0.1MeV comes to about $t \sim 3\text{min}$ after the Big Bang, since the weak freeze out is at $t \sim 1\text{s}$. Neutrons decay freely with $\tau_n \sim 900\text{s}$, and their abundence at t_D is suppressed by $e^{t_D/\tau_n} \sim 0.8$, since at freeze out, $X_n/X_p \sim 0.2$, so at t_D, $X_n/X_p = 0.2 \times 0.8 \sim 1/7$.

Lecture 22: Primordial Nucleosynthesis, Cont.

Thursday, April 08, 2021

After the deuteron is formed, the other reactions happen quickly with large neutron capture cross-sections, turning all neutrons into ^{4}He nuclei. Since each of those has four baryons, the mass fraction is

$$Y_{\text{He}} = \frac{4(n_n/2)}{n_n + n_p} = \frac{2\left(\frac{n_n}{n_p}\right)}{1+\left(\frac{n_n}{n_p}\right)} \sim \frac{2\,(1/7)}{1+(1/7)} \sim \frac{2}{8} \sim 1/4 \sim 0.25$$

Cosmological measurements in planetary nebula, globular clusters, interstellar gas, and other sources yield $Y_{\text{He}} \simeq 0.238 \pm 0.006$.

0.42.1 Heavier Nuclei

Once the deuteron is formed, a series of nuclear reactions lead to the production of lithium:

$$\begin{aligned} {}^4\text{He} + {}^3\text{H} &\leftrightarrow {}^7\text{Li} + \gamma \\ {}^4\text{He} + {}^3\text{He} &\leftrightarrow {}^7\text{Be} + \gamma \\ {}^7\text{Be} + n &\leftrightarrow {}^7\text{Li} + p \end{aligned}$$

Production of heavier nuclei is hindered by low density, the Coulomb barrier of heavier nuclei, and the lack of stable elements with $A = 5, 8$. Nucleosynthesis effectively stops at $t \sim 3$min with

$$\begin{aligned} Y_{\text{He}} &= \frac{{}^4\text{He}}{{}^4\text{He} + \text{H}} \simeq 0.24 \\ \frac{\text{Li}}{\text{H}} &\simeq 1.23 \times 10^{-10} \\ \frac{\text{D}}{\text{H}} &\simeq 2.6 \times 10^{-5} \\ \frac{{}^3\text{He}}{\text{H}} &\sim 10^{-6} \end{aligned}$$

While the abundance of ^{4}He is rather insensitive to η, the production of D critically depends on η and is considered a “baryometer”. The abundance of light elements in the primordial universe as predicted by Big-Bang-Nucleosynthesis are in remarkable agreement with observations that pinpoint $3 \lesssim \eta_{10} \lesssim 10$ ($\eta_{10} \equiv 10^{10}\eta$), leading to $\Omega_B h^2 \simeq 0.02$, an independent configuration of dark matter since $\Omega_B \ll \Omega_M$.

0.43 Photon Reheating, the CνB, and Dark Radiation

At the scale $T \sim 1$MeV, another important process occurs:

$$e^+e^- \to 2\gamma$$

as well as the inverse process. When this happens, the effective number of UR degrees of freedom changes. $g_*(T)$ jumps when species disappear (or when they become non-relativistic) via annihilations. At $T \gtrsim 1$MeV, only e^+, e^- (2 dof each), γ (2 dof), and ν (3 particles with 1 dof each) are ultrarelativistic. However, there are way more photons than baryons (and electrons) and $\gamma\gamma \to e^+e^-$ continues below 1MeV. A more detailed Boltzmann calculation shows that $e^+e^- \to 2\gamma$ at $T \leq 0.3$MeV, at which the neutrinos decouple from other species. The entropy is dominated by the ultrarelativistic ν and γ, whereas the entropy in baryons is negligible. For NR baryons,

$$S = N\left(\ln\left(\frac{V}{N}(mT)^{3/2}\right) + \frac{5}{2}\right) \qquad \text{(Sackur-Tetrode Equation)}$$

The entropy density is

$$\mathcal{S} = n_B \left\{ \ln\left(\frac{1}{n_B} (mT)^{3/2} \right) + \frac{5}{2} \right\}$$

but the entropy of an ultrarelativistic photon is

$$\mathcal{S}_{\text{UR}} \propto T^3 \propto n_\gamma$$

and $n_B/n_\gamma \sim 10^{-10}$, so

$$\frac{\mathcal{S}_B}{\mathcal{S}_{\text{UR}}} \sim 10^{-9}$$

At the temperature at which $e^+e^- \to 2\gamma$ (0.3MeV), the neutrinos are already decoupled. They were in equilibrium with all species at $T_\nu = T_\gamma$ until $T_D \sim 0.8$MeV and decoupled at this temperature where $T_D = T_\gamma$. However, in this reaction, the entropy of the electron-positron pairs is given off to photons, since the total entropy remains constant (adiabatic expansion requires this). Since

$$S = \text{const.} = \frac{4}{3} g_{\text{eff}} \frac{\pi^2}{30} T^3(t) \underbrace{V(t)}_{a^3(t)V_0}$$

we require

$$g_{\text{eff}}(T) \times (Ta)^3 \Big|_{\text{before annihilation}} = g_{\text{eff}}(T) \times (Ta)^3 \Big|_{\text{after annihilation}}$$

If we assume instantaneous annihilation so that $a_{\text{before}} = a_{\text{after}}$, and because the entropy of neutrinos is constant as they are decoupled from all other species (and cannot give or gain entropy), the only relevant entropies are those of e^+, e^-, and γ:

$$g_{\text{eff}}\Big|_{\text{before}} = 2 + \underbrace{4 \times \frac{7}{8}}_{\text{electrons + positrons}} = \frac{11}{2} \qquad g_{\text{eff}}\Big|_{\text{after}} = 2$$

Since the entropy is constant, we must then have

$$\left(\frac{11}{2}\right) T_\gamma^3 \Big|_{\text{before}} = 2\, T_\gamma^3 \Big|_{\text{after}}$$

or

$$\frac{T_\gamma\big|_{\text{after}}}{T_\gamma\big|_{\text{before}}} = \left(\frac{11}{4}\right)^{1/3}$$

Therefore, the photon gas *reheats* after electron-positron annihilation, but the neutrino gas decoupled with $T_\nu = T_\gamma\big|_{\text{before}}$ and redshifted along with the photons down to when the electrons and positrons annihilate, at which point the gas reheats by the factor above, so

$$\frac{T_\nu}{T_\gamma\big|_{\text{after}}} = \left(\frac{4}{11}\right)^{1/3}$$

The outcome of this is that today there is a cosmic neutrino background at $T_\nu \sim \left(\frac{4}{1}\right)^{1/3} T_\gamma = 1.95$K.

0.43.1 Dark Radiation

After photon reheating, the total energy density in radiation and ultrarelativistic species is

$$\rho_{\text{rel}} = g_* \frac{\pi^2}{30} T_\gamma^4$$

where $g_* = 2 + \frac{7}{8} \times 2 \times N_\nu \left(\frac{T_\nu}{T_\gamma}\right)^4$, where N_ν is the number of relativistic neutrino species. During the radiation-dominated period, N_ν effects the expansion history for $T < 0.3$MeV. This means g_* can be used to determine the number of ultrarelativistic neutrinos:

$$g_* = 2 + 2\frac{7}{8}N_{\text{eff}}\left(\frac{4}{11}\right)^{4/3}$$

If there are only 3 species of ultrarelativistic neutrinos *and* the instantaneous annihilation assumption is correct, then $N_{\text{eff}} = 3$. However, using a detailed Boltzmann analysis, we can show that the instantaneous annihilation assumption is in fact not exact but close. If $N_\nu = 3$, then it turns out N_{eff} as defined is $N_{\text{eff}} = 3.046$. There is a very small contribution to ν-reheating through $e^+e^- \to \nu\bar{\nu}$ via charged and neutral currents, but it is very small. From particle physics, the tighter constraint on the number of neutrinos emerges from the "invisible" width of the Z^0. The contributions from electron, muon, and quark pair productions are "visible", but $Z_0 \to \nu\bar{\nu}$ is "invisible" and causes $N_\nu = 2.96 \pm 0.04$.

Early measurements from WMAP suggested $N_{\text{eff}} > 4$, and also Big Bang Nucleosynthesis gave hints for a new species of neutrinos that contributes to radiation called "dark radiation". More recent analysis by WMAP, PLANCK, SPT, and ACT are consistent with $N_{\text{eff}} = 3.045$ (they find $N_{\text{eff}} = 3.3 \pm 0.27$). The analysis above also predicts the existence of a CνB at $T = 1.95$K, and proposals of large volume detectors to measure it have been advanced.

Lecture 23: Photon Decoupling and Recombination

Tuesday, April 13, 2021

Photons are in LTE with left-over electrons via Thompson scattering, but if we add electron-positron annihilation at $T \sim 0.3$MeV, lectrons become non-relativistic and the cross-section

$$\sigma_{Th} \sim \frac{\alpha_{EM}^2}{m_e^2} = 6.62 \times 10^{-25}\text{cm}^2$$

has a reaction rate $\Gamma_{e\gamma} = \sigma_{Th} n_e$ where n_e is the (free) electron density. However, the neutrality of the universe implies $n_e = n_p$ and for $T \leq T_H \sim 13.6$eV, neutral hydrogen forms and the number of free electrons drops dramatically, so photons can no longer scatter. They then decouple and their distribution function freezes when $\frac{\Gamma_{e\gamma}}{H} \ll 1$. At $T_H \sim 13.6$eV, electrons are non-relativistic and their distribution is Maxwell-Boltzmann. The "reaction" that stabilizes chemical equilibrium between e, p, and neutral Hydrogen is

$$e + p \leftrightarrow H + \gamma$$

Since $\mu_\gamma = 0$ and all the other particles are non-relativistic, $\mu_e + \mu_p \equiv \mu_H$ and

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\left(\frac{\mu_i - m_i}{T}\right)}$$

Then

$$n_H = g_H \frac{n_e n_p}{g_e g_p}\left(\frac{m_H T}{2\pi}\right)^{3/2}\left(\frac{4\pi^2}{m_e m_p T^2}\right)^{3/2} e^{-\underbrace{(m_H - m_e - m_p)}_{-B_H}/T}$$

where $B_H = 13.6$eV is the binding energy of hydrogen, the Rydberg energy.

Consider the ground state of hydrogen, where $g_H = 2 \times 2$ since $g_p = g_e = 2$ for each particle's spin (not considering antiparticles). By charge neutrality, $n_e = n_p$, so

$$n_H = n_e^2 \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T}$$

(assuming $m_H \sim m_p$)

We can consider the ionization fraction

$$X_e = \frac{n_e}{n_p + n_H} = \frac{n_e}{n_B}$$

where $n_B = n_p + n_H$ is the baryon density. Since the density of free protons n_p equals n_e by charge neutrality, we can write $X_e = \frac{n_p}{n_p+n_H} \implies 1 - X_e = \frac{n_H}{n_B}$. Then

$$n_H = n_B(1 - X_e) = n_e^2 \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T}$$

or

$$(1 - X_e) = \underbrace{\frac{n_e^2}{n_B^2}}_{X_e^2} \underbrace{\frac{n_B}{n_\gamma}}_{\eta} n_\gamma \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T}$$

with $n_\gamma = \frac{2\zeta(3)T^3}{\pi^2}$. Then

$$\left(\frac{1 - X_e}{X_e^2}\right) = \frac{4\zeta(3)\sqrt{2}}{\sqrt{\pi}}\eta \left(\frac{T}{m_e}\right)^{3/2} e^{B_H/T}$$

We know $\eta = 2.8 \times 10^{-8} \underbrace{(\Omega_B h^2)}_{0.02}$ and the temperature is redshifted to be $T = 2.73\text{K}(1 + z)$, where 2.73K is the temperature of the CMB today and $1 + z = \frac{1}{a}$ $(T = \frac{T_0}{a})$.

We can plot $X_e(z)$ to see that for $z \lesssim 1200$, $X_e \lesssim 0.1$, so around 90% of electrons are bound into neutral hydrogen. The equilibrium abundance of free electrons is found by setting $X_e\big|_{\text{eq}}$, or

$$X_e\Big|_{eq} \simeq 0.51\eta^{-1/2} \left(\frac{m_e}{T}\right)^{3/4} e^{-B_H/(2T)}$$

For $z \lesssim 1200$, the density of free electrons becomes very small and $\frac{\Gamma_{e\gamma}}{H} \ll 1$. Photons decouple at $z \sim 1100$ at $T_D \sim 3000\text{K} \sim 0.3\text{eV} \ll 13.6\text{eV}$. Again, the fact that $\eta \sim 10^{-10}$ implies that $T_D \ll B_H$ because there are an enormous amount of photons per baryon and the tail of the Big Bang spectrum is still capable of dissociating hydrogen atoms even for $T \ll B_H$. Since photons were in local thermal equilibrium until decoupling, we can estimate the time of decoupling from the following.

$$\Gamma_{e\gamma} = \sigma_{Th} n_e c$$

where c is the speed of photons, $\sigma_{Th} = 6.65 \times 10^{-25}\text{cm}^2$, and

$$n_e = X_e n_B = x_e \eta n_\gamma$$

and

$$n_\gamma = \frac{421\left(\frac{1}{\text{cm}}\right)^3}{a^3(t)} = \frac{421}{\text{cm}^3}(1 + z)^3$$

implies

$$n_e = X_e \underbrace{\Omega_B h^2}_{0.02}(1 + z)^3 \times 1.13 \times 10^{-5}\text{cm}^{-3}$$

so

$$\frac{\Gamma_{e\gamma}}{H} = 6.65 \times 0.02 \times 1.13 X_e \frac{10^{-30}}{\text{cm}} \times \frac{c}{H} \times (1 + z)^3$$

During matter domination,

$$H^2 = \frac{8\pi G}{3}\frac{\rho_M}{a^3} = H_0^2 \Omega_M (1 + z)^3 \implies H = H_0 \Omega_M^{1/2}(1 + z)^{3/2}$$

$$\frac{c}{H_0} = \text{Hubble radius} = 9.25 \times 10^{27}\frac{\text{cm}}{h}$$

so finally

$$\frac{\Gamma_{e\gamma}}{H} \simeq 0.15 X_e \times 9.25 \times \frac{10^{27}}{h} \times 10^{-30}(1 + z)^{3/2} \simeq 1.2 X_e \times 10^{-3}(1 + z)^{3/2}$$

For this to be $\lesssim 1$, we need $10^{-3} X_e (1 + z)^{3/2} \lesssim 1 \implies X_e \ll 1$.

From the SAHA equation with $X_e \ll 1$, we have

$$X_e \sim 0.51\eta^{-1/2}\left(\frac{m_e}{T_{CMB}}\right)^{3/4}(1+z)^{3/4}e^{-(B_H/T_{CMB})(1+z)}$$

Numerically, we can find $z_{\text{decouple}} \sim 1200$, which is $t_{\text{decouple}} \sim 400,000$ years with $\frac{1}{H_0} \sim 13.8$Gyr.

From this, we can consider their distribution as a black body, and after decoupling, the distribution becomes

$$f_\gamma(p_f;t) = \frac{1}{e^{p_f(t)/T_D(t)} - 1}$$

where $p_f(t) = \frac{p_c}{a(t)}$ and $T_D(t) = \frac{T_{CMB}}{a(t)}$ where T_{CMB} is the temperature of the CMB today, so

$$f_\gamma = \frac{1}{e^{p_c/T_{CMB}} - 1}$$

This $z \simeq 1100$ is called the "last scattering surface" (LSS) and corresponds to about $360,000$ years after the Big Bang.

0.44 Physics Beyond the Standard Model

0.44.1 Grand Unification–Supersymmetry

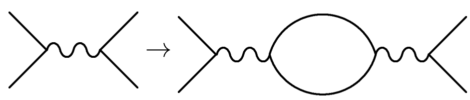
One consequence of quantum fluctuations is that the couplings α_{EM}, α_{QCD}, and α_W become energy dependent. We can see this when we talk about screening and anti-screening. Consider a metal. When a positive charge is deposited, electrons form a cloud around it. Dipole moments are induced that screen the original charge. From a (far) distance λ, the original charge appears smaller, and at a longer distance, or smaller wavevector, the total charge in the screening cloud cancels the original charge: the medium features a dielectric "constant". In a dielectric or polarizable medium, two charges do not interact with the "bare" Coulomb potential, but by a screened potential

$$V(r) = \frac{e^2}{r\varepsilon(r)}$$

where $\varepsilon(r)$ is the dielectric function. In terms of momentum (Fourier transform),

$$V(k) = \frac{e^2}{k^2\tilde{\varepsilon}(k)}$$

In QFT, the vacuum becomes a polarizable medium through the quantum fluctuations which materialize an e^+e^- pair during a short time scale, thanks to $\Delta E\Delta t \sim \hbar$ ($\Delta E \sim 2m_ec^2 \sim 1\text{MeV} \implies \Delta t \sim 10^{-23}$s). The vacuum polarization correction can be seen diagramatically by considering Rutherford scattering:



where the potential goes from $V(k) = \frac{e^2}{k^2} \sim \frac{\alpha_{EM}}{k^2}$ to $V(k) = \frac{\alpha_{EM}}{k^2\varepsilon(k^2)} \equiv \frac{\alpha_{EM}(k^2)}{k^2}$. We can calculate higher order loop corrections to show that $\alpha_{EM}(k^2)$ grows for large k^2 and diminishes for small k^2.

In QCD, the situation reverses. We still get fermionic loops like in QED, but due to the self-interaction of gluons, we also get gluon loops. Because of this, α_{QCD} decreases with k^2. It becomes very large as $k^2 \to 0$, which leads to quark-gluon confinement, but it is weak at large k^2, known as asymptotic freedom:

$$\alpha_{QED}(k^2) = \frac{\alpha_{QED}}{1 - \frac{\alpha_{QED}}{\pi}\ln\left(\frac{k^2}{m_e^2}\right)}$$

and

$$\alpha_{QCD}(k^2) = \frac{\alpha_{QCD}}{1 + \frac{7}{4\pi}\alpha_{QCD}\ln\left(\frac{k^2}{\Lambda^2}\right)} \equiv \frac{1}{\frac{7}{4\pi}\ln\left(\frac{k^2}{\Lambda^2_{QCD}}\right)}$$

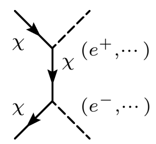
where $\Lambda_{QCD} \sim 150$MeV. $\alpha_W(k^2)$ has similar behavior to $\alpha_{QED}(k^2)$.

If we plot these curves against k^2, we find that they meet at various points. However, they don't quite meet at the same point. However, in supersymmetry theories, the couplings unify at $k^2 \sim (10^{16}\text{GeV})^2$, the Grand Unification scale. Supersymmetry implies an additional fermion for each boson and vice-versa, so for each neutrino (spin-1/2) there's a neutralino with spin-0 (electrons have selectrons and quarks have squarks). However, there are caveats to this theory.

Lecture 24: Grand Unified Theories
Thursday, April 15, 2021

0.45 Supersymmetry (SUSY) and Weakly Interacting Massive Particles (WIMPs)

In the previous lecture, we discussed the possibility of a theory with supersymmetric particles that could unify coupling constants at some energy around 10^{16}GeV. However, SUSY cannot be exact because there are no partners with the same masses. It is conjectured that SUSY must be broken at a scale just above the EW scale and the partners are split in mass by this symmetry breaking. One consequence of this is that the lightest SUSY particle, the photino or neutralino, is a neutral particle with a mass of nearly 100GeV, which could be a WIMP candidate for dark matter that is non-relativistic. Let's label this particle χ. It can annihilate into the typical Standard Model particles, but it cannot decay because it should be Dark Matter today, meaning it is the lightest *stable* SUSY particle. We could have interactions like



If χ is in local thermal and chemical equilibrium, $\chi\bar{\chi} \leftrightarrow e^+e^-$, and $\mu_\chi + \mu_{\bar{\chi}} = 0$. However, if it is a neutralino, the SUSY partner of the neutrino, it is charge neutral and a boson, so $\mu_\chi = 0$ (same if it is a photino, a Majorana SUSY partner). Then

$$n_\chi = \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

This particle decouples around $\Gamma/H = 1$ where $\Gamma = \langle\sigma\langle v\rangle\rangle n$, so in the radiation dominated universe,

$$H = 1.66 g_*^{1/2}\frac{T^2}{M_{pl}} \simeq \sqrt{g_*}\frac{T^2}{10^{19}\text{GeV}}$$

Setting this equal to Γ for some $T = T_D$, we get

$$m_\chi \left(\frac{m_\chi}{T_D}\right)^{1/2} e^{-\frac{m_\chi}{T_D}} = \frac{\sqrt{g_*}}{\langle\sigma|v|\rangle \times 10^{19}\text{GeV}}$$

We can rewrite this as

$$\left[\frac{m_\chi}{100\text{GeV}}\right]\left[\frac{m_\chi}{T_D}\right]^{1/2} e^{-\frac{m_\chi}{T_D}} = \left[\frac{\sqrt{g_*}}{10}\right]\frac{10^{-11}}{[\langle\sigma|v|\rangle \times 10^9(\text{GeV})^2]}$$

We then take the natural logarithm:

$$\frac{m_\chi}{T_D} \simeq \underbrace{25.34}_{11\ln(10)} + \ln\left(\frac{m_\chi}{100\text{GeV}}\right) + \frac{1}{2}\ln\left(\frac{m_\chi}{T_D}\right) + \ln\left(\frac{10}{\sqrt{g_*}}\right) + \ln\left(\frac{\langle\sigma|v|\rangle}{10^{-9}(\text{GeV})^{-2}}\right)$$

Then, a WIMP with $m_\chi \sim 100$GeV decouples at

$$\frac{m_\chi}{T_D} \sim 25 + \ln\left(\frac{\langle\sigma|v|\rangle}{10^{-9}(\text{GeV})^{-2}}\right)$$

If the σ is a weak interaction scale, $\sigma \sim \frac{\alpha_W^2}{m_\chi^2} \sim \frac{10^{-8}}{(\text{GeV})^2}$. For a Maxwell-Boltzmann particle (non-relativistic),

$$\frac{1}{2}m_\chi v^2 = \frac{3}{2}T \implies v \sim \sqrt{\frac{3T_D}{m}} \sim \frac{1}{3} \implies \langle\sigma|v|\rangle \sim \frac{(10^{-8} - 10^{-9})}{(\text{GeV})^2}$$

which gives $\frac{m_\chi}{T_D} \sim 25$.

If we call n_0 the WIMP number density today, then

$$\rho_{\text{WIMP},0} = n_0 m_\chi$$

Since

$$n(t) = \frac{n_D a^3(t_D)}{a^3(t)}$$

where n_D is the density at decoupling and

$$T(t) = \frac{T_0}{a(t)}$$

we can see that

$$n_0 = n_D\left(\frac{T_0}{T_D}\right)^3$$

We can see that

$$n_D \simeq \frac{\sqrt{g_*}T_D^2}{\langle\sigma|v|\rangle M_{pl}}$$

so

$$\rho_{\text{WIMP},0} = \frac{\sqrt{g_{*,D}}T_D^2}{\langle\sigma|v|\rangle\, 10^{19}(\text{GeV})}\left(\frac{T_0}{T_D}\right)^3 \simeq \left(\frac{m_\chi}{T_D}\right)\frac{\sqrt{g_{*,D}}T_0^3}{\langle\sigma|v|\rangle\, 10^{19}\text{GeV}}$$

If we have $T_0 \simeq 2.4 \times 10^{-13}$GeV and $\rho_{c,0} \simeq 8h^2 \times 10^{-47}(\text{GeV})^4$, then

$$\Omega_{\text{WIMP}} \approx \frac{1}{10}\left(\frac{m_\chi}{T_D}\frac{\sqrt{g_{*,D}}}{10}\right) \times \left[\frac{10^{-9}}{\langle\sigma|v|\rangle\,(\text{GeV})^2}\right]$$

Using the values of m_χ, σ, and $T_D \sim \frac{m_\chi}{25}$, we can estimate that $g_* \sim 30$–40 and $v \sim 1/3$, so $\langle\sigma|v|\rangle \sim \frac{10^{-9}}{(\text{GeV})^2}$. Given this, we find

$$\Omega_{\text{WIMP}} \lesssim 1$$

This is known as the WIMP miracle, sparking many experimental searches for these particles. Unfortunately, no evidence in their favor yet exists. In other dark matter candidates, the mechanism for production and decay is different, so these are not yet ruled out when WIMPs are.

0.46 Problems in the Standard Big Bang Cosmology and a Solution: Inflation

0.46.1 The Horizon Problem

Particle or causal horizons in a spatially flat FRW cosmology have

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}\vec{\mathbf{x}}^2$$

Photons have a blackbody distribution after decoupling, with a decoupling time/redshift of $z \sim 1100$. There are, however, small temperature anisotropies in the CMB of order $\Delta T/T_{\text{CMB}} \sim 10^{-5}$, the origin of which is associated with quantum fluctuations in the inflationary era.

0.46.2 Matter-Antimatter Asymmetry

There is more matter than antimatter, but what is the origin of this? Zakharov established three criteria for interactions that can cause this asymmetry:

- Baryon violating interactions
- C-violation (μ is C-odd) *and* CP-violation (or time reversal violation, where forward rates don't equal backward rates)
- Must occur out of equilibrium

The standard model has some of these ingredients. Baryon violation can happen through non-perturbative physics and anomalous conservation laws ($B+L$ for "sphaleron" transitions). Weak interactions violate C and P maximally (especially for neutrinos) and CP violation is present in the Cabibbo-Kobayashi-Maskawa (CKM) quark mass matrix (and possibly in themass matrix for neutrinos, which would be physics outside the SM). Non-equilibrium interactions would be present if the electroweak phase transition were first-order via bubble nucleation (like sphalerons). However, it is now clear that this transition is either a second-order or continuous (near equilibrium) crossover with $M_H \sim 125\text{GeV}$ and CP-violation in the CKM matrix is too small.

Perhaps the neutrino matrix violations could help solve this, but that is still outside the Standard Model, so it is likely that the solution to baryogenesis will lead to new physics. Possible extensions on higher symmetry group posit new particles like leptoquarks which could mediate these processes.

Lecture 25: Inflation
Tuesday, April 20, 2021

0.47 The Horizon Problem

In a spatially flat FRW cosmology, $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}\vec{\mathbf{x}}^2$. Photons travel along null geodesics:

$$\mathrm{d}s^2 = 0 \implies c\,\mathrm{d}t = a\,\mathrm{d}l$$

where $\mathrm{d}l$ is the spatial distance in comoving coordinates. Therefore, the comoving distance traveled by a photon between the time of the Big Bang ($t=0$) and time t is

$$L_c(t) = c\int_0^t \frac{\mathrm{d}t'}{a(t')}$$

and the physical distance

$$L_p(t) = ca(t)\int_0^t \frac{\mathrm{d}t'}{a(t')}$$

which is the physical particle horizon. In a radiation-dominated universe,

$$a(t) \propto t^{1/2}$$

so

$$L_p(t) = 2ct$$

The Hubble radius comes from $H(t) = \frac{\dot{a}}{a} = \frac{1}{2t}$, so

$$d_H(t) = \frac{c}{H(t)} = 2ct \implies L_p(t) = d_H(t)$$

in a radiation-dominated universe.

In a matter-dominated universe, $a(t) \propto t^{2/3}$, so

$$L_p(t) = 3ct$$

and

$$d_H(t) = \frac{3}{2}ct = \frac{1}{2}L_p(t)$$

In both cases, $L_p \propto d_H$, so we "see more" as time evolves. At the time of the last scattering, the causal (particle) horizon is about $L_p \sim 10^6$ly, which corresponds to a time of about 360,000yr or 0.3Mpc at $z_{\text{rec}} \sim 1100$. Today, this size would be $\sim 10^9$ly. However, the Hubble radius today is $\frac{c}{H_0} \sim 14 \times 10^9$ly, which is 14 times larger than the size of the horizon at the Last Scattering Surface extrapolated to today.

The CMB is homogeneous and isotropic to 1 part in 10^5 across the whole sky, which has $(14)^3 \sim 10000$ causally uncorrelated regions. So the question is:

> Why is the CMB so homogeneous and isotropic across the whole sky?

This is the Horizon Problem. This means that at the time of LSS, the CMB must have been correlated (homogeneous and isotropic) over a scale far larger than the particle horizon at the time.

Let's consider that the radiation-dominated era goes all the way back to the Planck time, $\sim 10^{-43}$s when the size of the particle horizon is the Planck length, $l_p \sim 10^{-33}$cm. Assuming the temperature at this time is similar to the Planck mass, 10^{19}GeV, the scale factor is $\frac{a(t_p)}{a(t_0)} \sim 10^{-32}$. The Planck-sized horizon today would be $l_p \times 10^{32} \sim 0.1$cm whereas today the horizon is 4000Mpc $\sim 10^{28}$cm!/ The problem is that in either era, $L_p \sim d_H$, so we need a cosmology during which $L_p \gg d_H$. Consider a Λ-dominated universe with $a(t) = e^{Ht}$ with $H = \sqrt{\frac{8\pi G}{3}\Lambda}$. Then

$$L_p = \frac{c}{H}(e^{Ht} - 1)$$

For $Ht \gg 1$, $L_p = e^{Ht} d_H \gg d_H$.

Then consider that before the radiation-dominated era, there was a Λ-dominated era during which the Hubble radius is $d_H = \frac{c}{H}$ but $L_p \sim e^{Ht} d_H$. Then if $d_H \sim l_p$ but $e^{Ht} \sim 10^{28} \sim e^{60}$, a Planck-sized region would encompass all the Universe today. If events are causally correlated in this Λ-era, they remain so during the radiation and matter-dominated eras.

0.48 The Flatness Problem

Observations clearly show that the universe today is spatially flat. From Friedmann's equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \implies \kappa = \frac{8\pi G}{3}(\rho - \rho_c)a^2 \quad \text{or} \quad \frac{\rho - \rho_c}{\rho} = \frac{3\kappa}{8\pi G}\frac{1}{a^2 \rho}$$

where $H^2 \equiv \frac{8\pi G}{3}\rho_c$.

Today, $\rho \sim \rho_c$, and for radiation domination, $\rho \sim \frac{1}{a^4}$, so $\frac{\rho - \rho_c}{\rho} \sim a^2 \sim t$. For matter domination, $\rho \sim a^{-3}$ so $\frac{\rho-\rho_c}{\rho} \sim a \sim t^{2/3}$. In either case, $\frac{\rho-\rho_c}{\rho}$ grows at large t. If $\frac{\rho-\rho_c}{\rho} \sim 0$ today, it must have been *extremely* small in the past! The flatness problem is that for regular fluids with $\frac{P}{\rho} = w > 0$

$$\rho \sim \frac{1}{a^{3(1+w)}} \quad \text{and} \quad \frac{\rho - \rho_c}{\rho} \sim a^{1+3w}$$

grow for $w > -\frac{1}{3}$ but shrink for $w < -\frac{1}{3}$. In particular, a Λ-dominated cosmology with $w = -1$ results in $\frac{\rho - \rho_c}{\rho} \to 0$ as a increases. Then a period of Λ domination makes the universe flatter, so both the Horizon and Flatness problems are solved by a period of *inflation*, a Λ-dominated cosmology with $a(t) = e^{Ht}$ and $H = \sqrt{\frac{8\pi G}{3}\Lambda}$, a de Sitter spacetime.

Note that during an inflationary era, a physical wavelength $\lambda_p(t) = \lambda_c a(t) = \lambda_c e^{Ht}$ grows like the particle horizon. If $\lambda < \frac{c}{H}$, it is always inside the particle horizon. If physical wavelengths were inside the particle horizon during inflation, then they are inside the particle horizon today, which means they are causally connected.

0.49 Implementing Inflation

We can examine a QFT of a scalar field in FRW cosmology. The action

$$S = \int \mathrm{d}t\, \mathrm{d}^3x_c\, a^3(t) \left\{ \frac{1}{2} g^{\mu\nu} \frac{\partial\phi}{\partial x^\mu} \frac{\partial\phi}{\partial x^\nu} - V(\phi) \right\}$$

and

$$\mathrm{d}s^2 = c^2\, \mathrm{d}t^2 - a^2(t)\, \mathrm{d}\vec{\mathbf{x}}^2$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/a^2 & 0 & 0 \\ 0 & 0 & -1/a^2 & 0 \\ 0 & 0 & 0 & -1/a^2 \end{pmatrix}$$

Then

$$\frac{1}{2} g^{\mu\nu} \frac{\partial\phi}{\partial x^\mu} \frac{\partial\phi}{\partial x^\nu} = \frac{1}{2} \left[\frac{\dot{\phi}^2}{c^2} - \frac{(\vec{\boldsymbol{\nabla}}\phi)^2}{a^2(t)} \right]$$

and

$$T^\mu_\nu = g^{\mu\alpha} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\nu} - \delta^\mu_\nu \mathcal{L}$$

Here ϕ defines the "inflaton" field. In QFT, $\phi(\vec{\mathbf{x}}, t)$ is an operator but the right-hand-side of Einstein's equation, $G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$ means that $T^{\mu\nu}$ must be a commuting number. For a semiclassical treatment assuming isotropy and homogeneity, there is a vacuum state $|0\rangle$ such that

$$\langle 0 | \hat{\phi}(\vec{\mathbf{x}}, t) | 0 \rangle \equiv \phi(t)$$

only depends on t and not $\vec{\mathbf{x}}$. Similarly

$$\langle 0 | \hat{T}^\mu_\nu | 0 \rangle \equiv \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & -P(t) & 0 & 0 \\ 0 & 0 & -P(t) & 0 \\ 0 & 0 & 0 & -P(t) \end{pmatrix}$$

with

$$\rho(t) = \frac{\dot{\phi}^2}{2} + V(\phi) \qquad P(t) = \frac{\dot{\phi}^2}{2} - V(\phi)$$

and

$$w = \frac{P}{\rho} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)}$$

Note that if $\dot{\phi} = 0$, $P = -\rho$ and $w = -1$. This would be the equation of state of the cosmological constant.

The equations of motion for the scalar field is obtained by the variational principle on the action:

$$S = \int \mathrm{d}t\, \mathrm{d}^3x_c\, a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\frac{\vec{\nabla}\phi}{a}\right)^2 - V(\phi) \right]$$

Using $\phi \to \phi + \delta\phi$, we have $\frac{1}{2}\dot{\phi}^2 \to \frac{1}{2}\dot{\phi}^2 + \dot{\phi}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\phi)$. Integrating this second term by parts, we pick up a term $\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\right) = 3\left(\frac{\dot{a}}{a}\right)a^3$. Setting $\delta S = 0$, we find

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \underbrace{\left(\frac{\vec{\nabla}\phi}{a}\right)^2}_{=0} = 0$$

where the final term can be neglected for isotropic and homogeneous universes.

The Friedmann equation reads

$$H^2 = \frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right]$$

and the Acceleration equation says

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[2\dot{\phi}^2 - 2V(\phi)\right] = \frac{8\pi G}{3}\left[V(\phi) - \dot{\phi}^2\right]$$

The continuity equation gives us

$$\dot{\rho} + 3H(\rho + P) = 0 \implies \dot{\phi}\left[\ddot{\phi} + 3H\dot{\phi} + V'(\phi)\right] = 0$$

We can compare this to a free field where $V(\phi) = \frac{1}{2}m^2\phi^2$, which makes

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$$

look like a damped harmonic oscillator. Consider $V(\phi) = \frac{\lambda}{4}(\phi^2 - \phi_0^2)^2$ and $\phi(t \sim 0)$ near the top with $V(\phi_i) \gg \dot{\phi}_i^2$.

$H^2 \simeq \frac{8\pi G}{3}V(\phi_i) \simeq$ constant, so taking the time derivative of the condition $V(\phi) \gg \frac{\dot{\phi}^2}{2}$, we get

$$\dot{\phi}V'(\phi) \gg \dot{\phi}\ddot{\phi} \implies V'(\phi) \gg \ddot{\phi}$$

so we can neglect $\ddot{\phi}$ in the equations of motion:

$$3H\dot{\phi} + V'(\phi) \approx 0$$

during inflation. In a Λ-cosmology, $H^2 \simeq \frac{8\pi G}{3}V(\phi)$, so taking the time derivative, we have

$$\begin{aligned} 2H\dot{H} &= \frac{8\pi G}{3}\dot{\phi}V'(\phi) \\ &= \frac{8\pi G}{3}\dot{\phi}(-3H\dot{\phi}) \\ \implies \dot{H} &\simeq -4\pi G\dot{\phi}^2 \\ \implies \frac{\dot{H}}{H^2} &\sim -\frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} \ll 1 \end{aligned}$$

so the Hubble parameter varies very slowly during inflation. This is known as the slow-roll condition: Using $\dot{\phi} = \frac{-V'(\phi)}{3H}$, we get

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\frac{(V'(\phi))^2}{9H^2V(\phi)} = -\frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv -\epsilon$$

where $\epsilon \ll 1$ is the "slow roll" parameter. Another related parameter is

$$\eta = \frac{1}{8\pi G}\left(\frac{V''(\phi)}{V}\right) \ll 1$$

Lecture 26: Inflation, Cont.

Thursday, April 22, 2021

$\epsilon \ll 1$ and $\eta \ll 1$ guarantee a slow roll, or enough e-folds so that $Ht_f \sim 60$ and $e^{Ht_f} \sim 10^{28}$, which is enough to inflate a Planck-sized region into today's Hubble radius. Measuring temperature anisotropies today measures a particular combination of ϵ and η, although they cannot be measured individually.

When $\phi(t)$ oscillates at the bottom of the potential, it creates (or decays into) particles that collide and produce a radiation dominated cosmology, ending inflation. This is called reheating and is not yet understood.

0.49.1 Generation of Fluctuations

Consider $\hat{\phi}(\vec{\mathbf{x}}, t) \equiv \underbrace{\phi(t)}_{\langle 0|\hat{\phi}(x,t)|0\rangle} + \delta\phi(x, t)$. Expanding in a Fourier transform,

$$\delta\phi(x, t) = \frac{1}{\sqrt{V}} \sum_k \delta\tilde{\phi}_k(t) e^{\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}}$$

Perturbations in this field lead to small perturbations in the metric:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \implies G^{(0)}_{\mu\nu} = \frac{8\pi G}{c^4} T^{(0)}_{\mu\nu}$$

and

$$\delta G_{\mu\nu} = \frac{8\pi G}{c^4} \delta T_{\mu\nu}$$

where $\delta T_{\mu\nu}$ comes from $\delta\phi$.

The perturbed metric acts like Newtonian perturbations around a Minkowski metric:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 \left[1 + 2\psi(x, t)\right] - a^2(t)\left[1 - 2\psi(x, t)\right] \mathrm{d}\vec{\mathbf{x}}^2$$

where $\psi(x, t)$ is like a gravitational potential. We can interpret this as a perturbation on the FRW metric, so $a(t) \to a(t) + \delta a(\vec{\mathbf{x}}, t)$. Then

$$(a + \delta a)^2 \simeq a^2 + 2a\delta a + \cdots = a^2 \left[1 + 2\frac{\delta a}{a} \cdots\right]$$

so

$$\psi(\vec{\mathbf{x}}, t) \simeq -\frac{\delta a(\vec{\mathbf{x}}, t)}{a(t)}$$

Now suppose the field $\phi(t) + \delta\phi(\vec{\mathbf{x}}, t)$ drives the cosmoogical expansion. $\delta\phi$ can be interpreted as small changes in the time at each spacetime point, like $t \to t + \delta t(\vec{\mathbf{x}}, t)$. At different spacetime positions, the observer's time changes slightly so that

$$\phi(t + \delta t(x)) \simeq \phi(t) + \dot{\phi}(t)\delta t(x) = \phi(t) + \delta\phi(x)$$

At each x, the value of ϕ changes slightly, effecting the expansion:

$$\delta t(x) \equiv \frac{\delta\phi(x)}{\dot{\phi}(x)}$$

Then we have

$$a(t + \delta t(x)) \equiv a(t) + \dot{a}(t)\delta t(x) = a(t) + \underbrace{Ha\delta t(x)}_{\delta a(x)}$$

so

$$\frac{\delta a(x)}{a(t)} = H(t)\delta t(x) \equiv -\psi(\vec{\mathbf{x}}, t)$$

or

$$\psi(x,t) = -\frac{H}{\dot{\phi}}\delta\phi(x)$$

so these small fluctuations in the inflaton field lead to small fluctuations in the metric in the form of a "Newtonian" gravitational potential.

Let's now examine the power spectrum of these fluctuations. Expanding

$$\delta\phi(\vec{\mathbf{x}}, t) \equiv \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{\imath kx} \delta\tilde{\phi}(\vec{\mathbf{k}}, t)$$

Then

$$P_{S\phi}[k;t] = \langle 0|\delta\tilde{\phi}(\vec{\mathbf{k}}, t)\delta\tilde{\phi} - \vec{\mathbf{k}}, t|0\rangle$$

To evaluate this, we would need to use QFT in curved spacetime. For $\lambda_{\text{phys}} \gg \frac{c}{H}$,

$$P_{S\phi}[k,t] = \left(\frac{H}{2\pi}\right)^2 \qquad \text{(Harrison-Zel'dovich)}$$

which is independent of k (scale-invariant).

Then

$$P_\psi[k,t] = \frac{H^2}{\dot{\phi}^2}\left(\frac{H}{2\pi}\right)^2$$

when $\lambda_{\text{phys}} \gg \frac{c}{d_H}$.

During slow roll, $3H\dot{\phi} = -V'(\phi)$ but $9H^2 = 3 \times 8\pi G V(\phi)$ and

$$\frac{\dot{\phi}^2}{H^2} = \frac{(V'(\phi))^2}{9H^4} = \frac{(V'(\phi))^2}{(8\pi)^2 G^2 V^2(\phi)} = \frac{M_{pl}^4}{6H\pi^2}\left(\frac{V'}{V}\right)^2 = \epsilon\frac{M_{pl}^2}{4\pi}$$

where ϵ is one of our slow roll parameters. Then

$$P_\psi(k) = \frac{H^2}{\pi M_{pl}^2 \epsilon} \qquad \lambda_{\text{phys}} \gg \frac{c}{H}$$

0.50 After Inflation

During inflation, the Hubble radius $\frac{c}{H}$ is constant, but the physical wavelength $\lambda_{\text{phys}} = \lambda_c e^{Ht}$ and the physical particle horizon $\lambda_p(t) = \frac{c}{H}e^{Ht} \implies \lambda_c < \frac{c}{H}$ are always inside the particle horizon (they are correlated). After inflation, in the radiation-dominated era, the particle horizon is approximately equal to the Hubble radius which is proportional to $t \sim a^2$. During matter domination, the Hubble radius scales like the particle horizon, or $t \sim a^{3/2}$.

Physical wavelengths that cross the Hubble radius during inflation re-enter the Hubble radius (the particle horizon) in the radiation or matter dominated eras. Those that re-enter during matter domination at the time of recombination (last scattering surface) have been *outside* the Hubble radius during the radiation-dominated era and therefore are *not* in causal contact with microphysics while outside. When the fluctuation in the metric (ψ) now re-enters during matter domination at the last scattering surface, theis effects the temperature of the CMB: $T(t) = \frac{T_0}{a(t)}$, but metric fluctuations like ψ lead to fluctuations in $a(t) \to a + \delta a$, or

$$T(t) \to \frac{T_0}{a}\left(1 - \frac{\delta a}{a}\right) = \frac{T_0}{a(t)} + \underbrace{\Delta T}_{\psi\frac{T_0}{a} = \psi T}$$

so
$$\left.\frac{\Delta T}{T}\right|_{LSS} \equiv \psi(\vec{\mathbf{x}}, t)$$

Additionally,
$$\left\langle \frac{\Delta T}{T}\frac{\Delta T}{T} \right\rangle \equiv P_T \simeq P_\psi(k) \simeq \frac{H^2}{\pi M_{pl}^2 \epsilon}$$

on large scales, like the LSS. Then
$$\left|\frac{\Delta T}{T}\right| \sim \frac{H}{M_{pl}\epsilon} \sim 10^{-5}$$

as measured by CMB observations. With $H \sim \sqrt{G}V^{1/2}(\phi)$, knowing ϵ can give us $V(\phi)$. Additionally, primordial gravitational waves are generated during inflation with a typical amplitude H/M_{pl}. Since $\rho_R \sim T^4$, we can see that $\frac{\Delta T}{T} \sim \frac{\delta\rho_R}{\rho_R}$, or $\delta\rho_R = 4T^3\Delta T$, so all together

$$\frac{delat\rho_R}{\rho_R} = 4\frac{\Delta T}{T} \sim \psi$$

However, since radiation and matter are strongly coupled through Thompson scattering, metric perturbations generated by quantum fluctuations of $\delta\phi$ during inflation *seed* inhomogeneities in the radiation and matter distribution with
$$\frac{\delta\rho_M}{\rho_M} \sim \frac{\Delta T}{T} \sim \psi \sim \frac{H}{\dot{\phi}}\delta\phi$$

Additionally, these small inhomogeneities *grow* under gravitational collapse. This is known as Jean's instability, and we will study it in the next lecture.

LECTURE 27: STRUCTURE FORMATION: JEANS INSTABILITY
Tuesday, April 27, 2021

0.51 Jeans Instability

In the derivation of $T^{\mu\nu}$ we arrived at the two main equations which describe fluids in the Newtonian non-relativistic limit:
$$\frac{\partial\rho}{\partial t} + \vec{\boldsymbol{\nabla}}(\rho\vec{\mathbf{v}}) = 0 \qquad \text{(Continuity)}$$
$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}P - \vec{\boldsymbol{\nabla}}\phi \qquad \text{(Euler)}$$

along with Poisson's equation for ϕ:
$$\nabla^2\phi = 4\pi G\rho$$

We will use this to understand the growth of structures under gravitational instabilities during the matter-dominated era. We'll later learn why this is the era when structure formation begins. We can study the evolution of perturbations in the matter/velocity distributions for non-relativistic fluids, since the speed of propagation of these perturbations is $c_s^2 \sim P/\rho \ll c^2$.

We will study the evolution of perturbations in stages:

- Minkowski Spacetime
  - Perfect Newtonian fluids without gravity
  - Adding viscosity
  - Adding gravity but removing velocity (this leads to Jeans Instability)
- Ideal self-gravitating fluids in an expanding (matter-dominated) cosmology

0.52 Minkowski Spacetime

0.52.1 Newtonian Fluids in Absence of Gravity

In the absence of gravity,

$$\partial_\mu T^{\mu\nu} = 0$$

implies the continuity equation above and the Euler equation without $-\vec{\boldsymbol{\nabla}}\phi$.

If we consider a closed system and adiabatic perturbations, we have

$$\mathrm{d}Q = \mathrm{d}U + P\,\mathrm{d}V = 0$$

For a fluid at rest ($\vec{\mathbf{v}} = 0$), $\rho(x,t) = \rho_0$ and $P(x,t) = P_0$, so the continuity and Euler equations are clearly satisfied. Consider a small perturbation

$$\vec{\mathbf{v}} = \underbrace{0}_{\vec{\mathbf{v}}_0} + \vec{\mathbf{v}}_1 \qquad \rho(x,t) = \rho_0 + \rho_1(x,t) \qquad P(x,t) = P_0 + P_1(x,t)$$

with $\rho_1 \ll \rho_0$ and $P_1 \ll P_0$. Then the continuity and Euler equations (to linear order in this perturbation) read

$$\frac{\partial\rho_1}{\partial t} + \vec{\boldsymbol{\nabla}}(\rho_0\vec{\mathbf{v}}_1) = \frac{\partial\rho_1}{\partial t} + \rho_0\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}_1 = 0$$

$$\frac{\partial\vec{\mathbf{v}}_1}{\partial t} = -\frac{1}{\rho_0}\vec{\boldsymbol{\nabla}}P_1$$

Unfortunately, this gives us three unknowns (the perturbed values) but only two equations, so we need an equation of state, or some equation relating P_1 as a function of ρ_1 (like $P_1(\rho_1)$). From $\mathrm{d}Q = 0 = \mathrm{d}U + P\,\mathrm{d}V$, we can take the ideal (non-relativistic) gas equation $P = \frac{N}{V}k_BT$ and $U = \frac{3}{2}PV$ from statistical mechanics to get

$$\mathrm{d}U = \frac{3}{2}\left(\mathrm{d}P\,V + P\,\mathrm{d}V\right) = -P\,\mathrm{d}V$$

so

$$\frac{\partial\vec{\mathbf{v}}_1}{\partial t} = -\frac{c_s^2}{\rho_0}\vec{\boldsymbol{\nabla}}\rho_1$$

Taking the time-derivative of the continuity equation, we get

$$\frac{\partial^2\rho_1}{\partial t^2} + \rho_0\vec{\boldsymbol{\nabla}}\left(\frac{\partial\vec{\mathbf{v}}_1}{\partial t}\right) = 0$$

Inserting this into the previous equation, we get a wave equation:

$$\frac{\partial^2\rho_1}{\partial t^2} - c_s^2\nabla^2\rho_1 = 0$$

or

$$\frac{1}{c_s^2}\frac{\partial^2\rho_1}{\partial t^2} - \nabla^2\rho_1 = 0$$

which describes waves with a propagation velocity c_s. We can get solutions via a Fourier transform:

$$\rho_1(x) = \tilde{\rho}_1(k)e^{-\imath\omega t}e^{\imath\vec{\mathbf{k}}\,\cdot\,\vec{\mathbf{x}}}$$

so

$$\omega^2 - c_s^2\vec{\mathbf{k}}^2 = 0 \implies \omega(k) = \pm c_s\left|\vec{\mathbf{k}}\right|$$

This is the dispersion relation of sound waves. Then

$$P_1(x,t) = c_s^2\rho_1(x,t) = c_s^2\tilde{\rho}(k)e^{-\imath\omega t}e^{\imath\vec{\mathbf{k}}\,\cdot\,\vec{\mathbf{x}}}$$

and from

$$\frac{3}{2}V\,\mathrm{d}P = -\frac{5}{2}P\,\mathrm{d}V$$

we get

$$\frac{\mathrm{d}P}{P} = -\frac{5}{3}\frac{\mathrm{d}V}{V}$$

Density is related to the volume: $\rho = \frac{Nm}{V} \implies \mathrm{d}\rho = -\frac{Nm}{V^2}\,\mathrm{d}V$, so

$$\frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}V}{V} \implies \frac{\mathrm{d}P}{P} = \frac{5}{3}\frac{\mathrm{d}\rho}{\rho} \implies P = P_0\left(\frac{\rho}{\rho_0}\right)^{5/3}$$

With P_0, ρ_0 constant, we get $\mathrm{d}P = P_1$ and $\mathrm{d}\rho = \rho_1$, so

$$\frac{P_0}{\rho_0} = \frac{k_B T}{m} \implies P_1 = c_s^2\rho_1 \qquad c_s^2 = \frac{5}{3}\frac{k_B T}{m}$$

Note that from the classical equipartition theorem, we have

$$\frac{1}{2}m\left\langle v^2\right\rangle = \frac{3}{2}k_B T \implies \left\langle v^2\right\rangle = \frac{3k_B T}{m} \implies c_s^2 = \frac{5}{9}\left\langle v^2\right\rangle$$

We can propose plane wave solutions to Euler's equation:

$$\vec{\mathbf{v}}_1(x,t) = \vec{\tilde{v}}(k)e^{-\imath\omega t}e^{\imath\vec{\mathbf{k}}\,\cdot\,\vec{\mathbf{x}}} \implies -\imath\omega\vec{\tilde{v}}(k) = -\imath\frac{c_s^2}{\rho_0}\vec{\mathbf{k}}\tilde{\rho}_1(k)$$

or $\vec{\tilde{v}}(k) = \frac{c_s}{\rho_0}\hat{k}\tilde{\rho}_1(k)$ is parallel to $\vec{\mathbf{k}}$, so this is a longitudinal perturbation.

0.52.2 Non-Ideal Fluids in Absence of Gravity

We can now add viscosity. Without proof, the energy-momentum tensor of viscous fluids is

$$T^{00} = \rho c^2 \qquad T^{0i} = T^{i0} = \rho c v^i \qquad T^{ij} = P\delta^{ij} + \rho v^i v^j - \eta\left[\partial_j v^i + \partial_i v^j - \frac{2}{3}\delta^{ij}\vec{\boldsymbol{\nabla}}\,\cdot\,\vec{\mathbf{v}}\right] - \xi\delta^{ij}\vec{\boldsymbol{\nabla}}\,\cdot\,\vec{\mathbf{v}}$$

where η is the sheer viscosity and ξ is the bulk viscosity, both of which are linearly proportional to the mean free path. The continuity equation is unchanged, but the Euler equation becomes

$$\frac{\partial(\rho v^i)}{\partial t} + \frac{\partial(\rho v^i v^j)}{\partial x^j} = -\frac{\partial P}{\partial x^i} + \eta\nabla^2 v^i + \left(\xi + \frac{\eta}{3}\right)\frac{\partial^2 v^j}{\partial x^i \partial x^j}$$

With

$$\frac{\partial(\rho v^i)}{\partial t} = \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial\rho}{\partial t}$$

we can combine this with the continuity equation to get

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}}\,\cdot\,\vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = -\frac{\vec{\boldsymbol{\nabla}}P}{\rho} + \frac{\eta}{\rho}\nabla^2\vec{\mathbf{v}} + \frac{\left(\xi + \frac{\eta}{3}\right)}{\rho}\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\,\cdot\,\vec{\mathbf{v}})$$

If we neglect thermal conductivity, we use an adiabatic equation of state:

$$\delta P = \frac{5}{3}\left(\frac{P}{\rho}\right)\delta\rho = c_s^2\delta\rho$$

with $c_s^2 = \frac{5}{3}\frac{k_B T}{m}$. If we expand in small perturbations and linearize as before, we can propose Fourier transforms for P_1, ρ_1, and $\vec{\mathbf{v}}_1$ as before with $\tilde{P}_1 = c_s^2\tilde{\rho}_1$. From the continuity equation, we get

$$-\imath\omega\tilde{\rho}_1 + \imath\rho_0 k\tilde{v}_\parallel = 0$$

where $\vec{\tilde{v}}_1 = \hat{k}\tilde{v}_\parallel + \tilde{v}_\perp$, so

$$\tilde{\rho}_1 = \rho_0\frac{k}{\omega}\tilde{v}_\parallel$$

From the Euler equation, we get

$$\imath\omega\tilde{v}_\perp = -k^2\frac{\eta}{\rho_0}\tilde{v}_\perp$$

along the perpendicular projection and

$$\omega^2 - c_s^2k^2 + \imath\omega\gamma_k = 0$$

for the parallel projection, where $\gamma_k = \frac{k^2}{\rho_0}\left(\xi + \frac{4}{3}\eta\right)$. this gives us a dispersion relation for the parallel component:

$$\omega_\pm = -\imath\frac{\gamma_k}{2} \pm \sqrt{c_s^2k^2 - \left(\frac{\gamma_k}{2}\right)^2}$$

Note that γ_k, the damping factor related to viscosity, vanishes in the long wavelength limit ($k \to 0$), which is the hallmark of hydrodynamic modes, long-lived in the long-wavelength limit.

0.52.3 Ideal Fluids with Gravity

Consider a spherical shell of mass Δm outside a uniform mass distribution of (fixed) radius r_0 and total mass M_0. The acceleration of the shell towards the inner mass by gravitational force is called the "gravitational collapse" onto the inner mass. This happens on a characteristic time scale, t_{ff}, the free-fall time. This is determined by energy conservation:

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{GM_0}{r} - \frac{GM_0}{r_i}$$

so that the shell begins to collapse with $v = 0$ at $r_i \gg r_0$ (r_i is the initial radius). Then

$$t_{ff} = -\int_{r_i}^{r_0}\frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r = -\int_{r_i}^{r_0}\frac{\mathrm{d}r}{\sqrt{\frac{2GM_0}{r} - \frac{2GM_0}{r_i}}}$$

Using $x = r/r_i$ and taking $r_0 \ll r_i$, we get

$$t_{ff} = \left[\frac{r_i^3}{2GM_0}\right]^{1/2}\int_0^1\left[\frac{x}{1-x}\right]^{1/2}\mathrm{d}x$$

Using $x = \sin^2(\theta)$, we get

$$t_{ff} \approx \frac{\pi}{2}\left[\frac{r_i^3}{2GM_0}\right]$$

If we consider the average density

$$\bar{\rho} = \frac{M_0}{\frac{4\pi}{3}r_1^3}$$

inside the initial ball, we get

$$t_{ff} \approx \sqrt{\frac{3\pi}{32}}\frac{1}{\sqrt{G\bar{\rho}}} \simeq \frac{1}{\sqrt{G\bar{\rho}}}$$

This is the time scale in absence of restoring forces. Note that an important feature is revealed in an expanding cosmology. If $H^2 = \frac{8\pi G}{3}\rho$, then $H \sim \sqrt{G\rho}$ or $t_{ff} \sim \frac{1}{H}$, which is the Hubble time scale.

0.52.4 Jeans Instability

Using the equations from the beginning of this lecture, along with the adiabatic equation of state $\mathrm{d}U = -P\,\mathrm{d}V \implies \delta P = c_s^2\delta\rho$, consider a perturbation around a homogeneous state (same as before but adding $\phi(x,t) = \phi_0 + \phi_1(x,t)$).

"Jeans Swindle": Technically, there is no homogeneous state because if $\rho = \rho_0$ then $\nabla^2\phi_0 = 4\pi G\rho_0$ implies ϕ_0 cannot be homogeneous, since it would imply $\phi_0 \sim \rho_0\vec{\mathbf{x}}^2$. Then with $\vec{\mathbf{v}} = 0$, we get $\frac{\vec{\boldsymbol{\nabla}}P}{\rho_0} = -\vec{\boldsymbol{\nabla}}\phi_0 \neq 0$, so P can't be homogeneous either. We will see later how this changes with expansion.

Let's argue that the gravitational forces will balance out and linearize around a homogeneous state:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}_1 = 0$$
$$\frac{\partial \vec{\mathbf{v}}_1}{\partial t} = -\frac{\vec{\boldsymbol{\nabla}} P_1}{\rho_0} - \vec{\boldsymbol{\nabla}} \phi_1$$
$$\nabla^2 \phi_1 = 4\pi G \rho_1$$
$$P_1 = c_s^2 \rho_1$$

Taking the time derivative of the first equation and using the second on the right-hand side, we can use the fourth equation to get

$$\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0$$

If we suppose the usual Fourier transform on ρ_1, we get a dispersion relation

$$\omega(k) = \pm\sqrt{c_s^2 k^2 - 4\pi G \rho_0}$$

For $c_s^2 k^2 > 4\pi G \rho_0$, $\omega(k)$ is real, so these will be oscillations like sound waves. Otherwise, ω is imaginary, so we have a constantly growing and a constantly decaying solution from $e^{\pm|\omega|t}$. For the growing solution, small density perturbations are *unstable* and grow, becoming large amplitude perturbations. This is the definition of Jeans instability and it leads to gravitational collapse.

In the next class, we'll look at the physics behind this, or what happens when the gravitational term $4\pi G \rho_0$ overpowers the $c_s^2 k^2$ term.

Lecture 28: Jeans Instability, Cont.

Thursday, April 29, 2021

Looking at the physical interpretation, consider the no-pressure case, where $c_s = 0 \implies P = 0$. Then

$$|\omega| = \sqrt{4\pi G \rho_0} = \sqrt{4\pi} \underbrace{\sqrt{G\rho_0}}_{\approx \frac{1}{t_{ff}}}$$

Then

$$\rho_1 \sim e^{\sqrt{4\pi} t / t_{ff}}$$

so the typical time scale for the growth of unstable perturbations is t_{ff}.

Now consider $c_s \neq 0$ with $k = \frac{2\pi}{\lambda}$. Then

$$c_s^2 k^2 - 4\pi G \rho_0 = 4\pi \left[\pi \frac{c_s^2}{\lambda^2} - \underbrace{G\rho_0}_{\approx \frac{1}{t_{ff}^2}} \right] = \frac{4\pi c_s^2}{\lambda^2} \left[\pi - \left(\frac{\lambda}{c_s t_{ff}} \right)^2 \right]$$

Now, $\lambda / c_s \equiv t_s$ is the time scale during which a a sound wave travels a distance λ. Therefore, $\omega^2 \sim \left[\pi - \left(\frac{t_s}{t_{ff}} \right)^2 \right]$, so if $t_s \gg t_{ff}$, $\omega^2 < 0$ so we have gravitational collapse. A sound wave cannot restore the pressure/equilibrium because it takes longer than the free-fall time for collapse.

If $t_s \ll t_{ff}$, then a sound wave can restore the equilibrium before the collapse time scale, leading to acoustic oscillations (sound waves). In summary, if $\frac{\pi c_s^2}{\lambda^2} > G\rho_0$, we get oscillations, otherwise we have Jeans instability and gravitational collapse. We can construct a length scale

$$L_J = \sqrt{\frac{\pi}{G\rho_0}} c_s \tag{Jeans Length}$$

If the wavelength of perturbation is larger than the Jeans length, then there is a Jeans instability and gravitational collapse. Small perturbations are unstable. Otherwise, we have oscillations. We can also construct a mass scale derived from the mass contained in a sphere of radius L_J:

$$M_J = \frac{4\pi}{3} \rho_0 L_J^3 \tag{Jeans Mass}$$

0.52.5 Consequences for Dark Matter

The important quantity is c_s. For a collisionless model, consider replacing c_s with $\sqrt{\langle v^2 \rangle}$, so $L_J \sim \sqrt{\langle v^2 \rangle} t_{ff}$.

For Cold Dark Matter (CDM), we consider small velocities and heavy, cold particles. $\langle v^2 \rangle \sim \frac{kT}{M} \implies \lambda_J$ is a few parsecs. WIMPS have $M \sim 100\text{GeV}$.

For Warm Dark Matter (WDM), $\lambda_J \sim 50-100\text{kpc}$ (scale of the galaxy). Sterile neutrinos are a proposed WDM particle, with $M \sim \text{keV}$.

For Hot Dark Matter, $c_s \sim c$ and $\lambda_J \sim 100\text{Mpc}$ (this would be like neutrino-mass particles). This is ruled out by the fact that galaxies form.

0.53 Ideal Self-Gravitating Fluids in an Expanding Cosmology

consider $\rho_0(t)$, $P_0(t)$, and $\vec{\mathbf{v}}(t) = H\vec{\mathbf{r}}$ where $\vec{\mathbf{r}}$ is the physical independent variable. For unperturbed fluids, ρ_0 and P_0 only depend on t and not $\vec{\mathbf{r}}$. Then

$$\left.\frac{\partial \rho_0}{\partial t}\right|_r + \rho_0 \vec{\boldsymbol{\nabla}}_r \cdot \vec{\mathbf{v}} = 0$$
$$\left.\frac{\partial v}{\partial t}\right|_r + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}}_r)\vec{\mathbf{v}} = -\vec{\boldsymbol{\nabla}}_r \phi$$
$$\nabla_r^2 \phi = 4\pi G \rho_0(t)$$

In an expanding cosmology, for unperturbed fluids, the Hubble flow is $\vec{\mathbf{r}}(t) = a(t)\vec{\mathbf{x}}$, where $\vec{\mathbf{x}}$ is comoving and time independent.

In a matter-dominated cosmology,

$$\left.\frac{\partial \vec{\mathbf{v}}}{\partial t}\right|_r = \dot{H}\vec{\mathbf{r}}$$

so from the Laplace equation,

$$\phi(\vec{\mathbf{r}}, t) = \frac{2\pi}{3} G \rho_0(t) \vec{\mathbf{r}}^2$$

implies that $\vec{\boldsymbol{\nabla}}\phi = \frac{4\pi}{3}\rho_0(t)\vec{\mathbf{r}}$. Using the second equation, we have

$$(\dot{H} + H^2)\vec{\mathbf{r}} = -\frac{4\pi}{3}\rho_0(t)\vec{\mathbf{r}} \implies \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_0(t)$$

This is the acceleration equation in the Newtonian limit ($P/\rho c^2 \ll 1$)!

Now if we go back to the first equation, $\vec{\boldsymbol{\nabla}}_r \cdot \vec{\mathbf{v}} = 3H$ implies

$$\dot{\rho}_0(t) + 3\frac{\dot{a}}{a}\rho_0 = 0$$

so

$$\rho_0(t) = \frac{\text{const}}{a^3(t)}$$

While you need Jeans swindle in the previous Minkowski derivation, with Hubble expansion there actually is a consistent solution of the unperturbed fluid equations.

Another problem with the Minkowski treatment applied to an expanding cosmology is that for long wavelength perturbations, $\lambda \gg L_J$, the time scale for collapse is on the same order as the time scale for instability, so expansion would severely modify the time evolution of Jeans unstable modes. Let's study these perturbations, starting with the velocity field:

$$\vec{\mathbf{v}} = \underbrace{\dot{a}(t)\vec{\mathbf{x}}}_{\frac{\dot{a}}{a}\vec{\mathbf{r}}} + \vec{\mathbf{v}}_1(\vec{\mathbf{r}}, t)$$

where $\vec{\mathbf{v}}_1$ is the peculiar velocity relative to the Hubble flow with $\vec{\mathbf{v}}_1 \ll H(t)\vec{\mathbf{r}}$. since the continuity and Euler equations have time derivatives at a constant $\vec{\mathbf{r}}$, we can write $\rho(\vec{\mathbf{r}}) \to \rho(\vec{\mathbf{x}})$ with $\vec{\mathbf{x}} = \vec{\mathbf{r}}/a$ so that

$$\left.\frac{\partial}{\partial t}\right|_r = \left.\frac{\partial}{\partial t}\right|_x - \underbrace{\frac{\dot{a}}{a^2}\vec{\mathbf{r}} \cdot \vec{\boldsymbol{\nabla}}_x}_{\frac{\dot{a}}{a}\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}_x}$$

The continuity equation reads

$$\left.\frac{\partial \rho}{\partial t}\right|_r + \vec{\boldsymbol{\nabla}}_r(\rho\vec{\mathbf{v}}) = 0$$

so the first term becomes

$$\left.\frac{\partial \rho}{\partial t}\right|_r = \left.\frac{\partial \rho}{\partial t}\right|_x - \frac{\dot{a}}{a}\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}_x\rho$$

The second term has $\vec{\boldsymbol{\nabla}}_r = \frac{1}{a}\vec{\boldsymbol{\nabla}}_x$, so

$$\frac{1}{a}\vec{\boldsymbol{\nabla}}_x(\rho \times (\dot{a}\vec{\mathbf{x}} + \vec{\mathbf{v}}_1)) = \frac{1}{a}\vec{\boldsymbol{\nabla}}_x(\rho\vec{\mathbf{v}}_1) + 3\frac{\dot{a}}{a}\rho + \frac{\dot{a}}{a}\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}_x\rho$$

Notice the final term here cancels the second term in the previous line. Therefore,

$$\left.\frac{\partial \rho}{\partial t}\right|_x + \frac{3\dot{a}}{a}\rho + \frac{1}{a}\vec{\boldsymbol{\nabla}}_x(\rho\vec{\mathbf{v}}_1) = 0$$

For perturbations, we write

$$\rho = \rho_0(t)[1 + \delta(\vec{\mathbf{x}}, t)]$$

where $\delta = \frac{\rho - \rho_0}{\rho_0}$.

$$\phi = \frac{2\pi}{3}G\rho_0 \underbrace{a^2\vec{\mathbf{x}}^2}_{\vec{\mathbf{r}}^2} + \phi_1(\vec{\mathbf{x}}, t)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_0$$

The first term is

$$-\frac{1}{2}\frac{\ddot{a}}{a}a^2\vec{\mathbf{x}}^2 = -\frac{1}{2}\ddot{a}a\vec{\mathbf{x}}^2$$

so

$$\phi = \underbrace{-\frac{1}{2}\ddot{a}a\vec{\mathbf{x}}^2}_{\phi_0(\vec{\mathbf{x}}, t)} + \phi_1(\vec{\mathbf{x}}, t)$$

Finally, if we look at the Euler equation with $P(\vec{\mathbf{x}}, t) = P_0(t) + P_1(\vec{\mathbf{x}}, t)$, we have

$$\left.\frac{\partial \vec{\mathbf{v}}}{\partial t}\right|_r + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}}_r)\vec{\mathbf{v}} = -\frac{\vec{\boldsymbol{\nabla}}P}{\rho} - \vec{\boldsymbol{\nabla}}\phi$$

Since $\vec{\mathbf{v}}_0 = \dot{a}\vec{\mathbf{x}}$, we have, to zeroth order

$$\left.\frac{\partial \vec{\mathbf{v}}_0}{\partial t}\right|_r + (\vec{\mathbf{v}}_0 \cdot \vec{\boldsymbol{\nabla}}_r)\vec{\mathbf{v}}_0 = -\vec{\boldsymbol{\nabla}}\phi_0$$

and

$$\underbrace{\left.\frac{\partial \vec{\mathbf{v}}_1}{\partial t}\right|_r}_{\left.\frac{\partial \vec{\mathbf{v}}_1}{\partial t}\right|_x - \frac{\dot{a}}{a}(\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}_x)\vec{\mathbf{v}}_1} + \underbrace{(\vec{\mathbf{v}}_0 \cdot \vec{\boldsymbol{\nabla}}_r)\vec{\mathbf{v}}_1}_{\frac{\dot{a}}{a}(\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}_x)\vec{\mathbf{v}}_1} + \underbrace{(\vec{\mathbf{v}}_1 \cdot \vec{\boldsymbol{\nabla}}_r)\vec{\mathbf{v}}_0}_{\frac{\dot{a}}{a}\vec{\mathbf{v}}_1} = -\frac{\vec{\boldsymbol{\nabla}}P_1}{\rho_0} - \vec{\boldsymbol{\nabla}}\phi_1 {}_{-\frac{1}{a\rho_0}} \vec{\boldsymbol{\nabla}}_x P_1 - \frac{1}{a}\vec{\boldsymbol{\nabla}}_x\phi_1$$

so

$$\frac{\partial \vec{\mathbf{v}}_1}{\partial t} + \frac{\dot{a}}{a}\vec{\mathbf{v}}_1 = -\frac{1}{a\rho_0}\vec{\boldsymbol{\nabla}}_x P_1 - \frac{1}{a}\vec{\boldsymbol{\nabla}}_x \phi_1$$

And with $\rho_1 = \rho_0 \delta$, we have

$$\frac{1}{a^2}\nabla_x^2 \phi_1 = 4\pi G \rho_0 \delta$$

For adiabatic expansion, we have $P_1 = c_s^2 \rho_1 = c_s^2 \rho_0 \delta$. In summary, we have

$$\begin{aligned}
\frac{\partial \delta}{\partial t} + \frac{1}{a}\left(\vec{\boldsymbol{\nabla}}_x \cdot \vec{\mathbf{v}}_1\right) = 0 \\
\frac{\partial \vec{\mathbf{v}}_1}{\partial t} + \frac{\dot{a}}{a}\vec{\mathbf{v}}_1 = -\frac{1}{a\rho_0}\vec{\boldsymbol{\nabla}}_x P_1 - \frac{1}{a}\vec{\boldsymbol{\nabla}}_x \phi_1 \\
\nabla_x^2 \phi_1 = 4\pi G a^2 \rho_0 \delta \\
P_1 = c_s^2 \rho_0 delta
\end{aligned}$$

If we take the time derivative of the first equation, we can use the other equations to get

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - \frac{c_s^2}{a^2}\nabla_x^2 \delta - 4\pi G \rho_0 \delta = 0$$

Taking the spatial Fourier transform with

$$\delta(x,t) = \tilde{\delta}_k(t) e^{\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}}$$

with $\vec{\mathbf{k}}$ comoving and $\frac{\vec{\mathbf{k}}}{a} = \vec{\mathbf{k}}_{ph}$, we get

$$\frac{\partial^2 \tilde{\delta}}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \tilde{\delta}}{\partial t} + [c_s^2 k_{ph}^2 - 4\pi G \rho_0]\tilde{\delta} = 0$$

If we took $\dot{a}/a = 0$ and $a = 1$, this is the same as Minkowski spacetime with $\left|\vec{\mathbf{k}}_{ph}\right| = \frac{2\pi}{\lambda_{ph}}$. The Jeans length in physical coordinates corresponds when the last term vanishes:

$$\lambda_{J,ph} = c_s \sqrt{\frac{\pi}{G\rho_0(t)}}$$

For matter domination, $\frac{\dot{a}}{a} = H = \frac{2}{3t}$ and $\rho_0 = \frac{3H^2}{8\pi G}$. The general solution is a combination of Bessel functions, but we can understand the growth of structure for $\lambda \gg \lambda_J \implies c_s^2 k^2 \to 0$ (vanishing pressure). This condition gives us

$$\frac{\partial^2 \tilde{\delta}}{\partial t^2} + \frac{4}{3t}\frac{\partial \tilde{\delta}}{\partial t} - \frac{2}{3t^2}\tilde{\delta} = 0$$

Solutions have the form

$$\tilde{\delta}(t) = At^{2/3} + \frac{B}{t}$$

where A and B are constants. Instead of exponential growth as in Minkowski, we now have a growing mode which is slower, $\tilde{\delta} \sim At^{2/3} \sim Aa(t)$. Expansion slows the growth of density perturbations. During radiation domination, the growth is only $\ln(t) \sim \ln(a(t))$, which is much slower because the expansion is faster. Structure formation via gravitational collapse of density perturbations grow only during (and after) the matter-dominated era.

0.54 Conclusion

This concludes the notes taken for the Particle Astrophysics class taught by Dr. Daniel Boyanovsky in the Spring Semester of 2021 at the University of Pittsburgh. These notes were transcribed by me, Nathaniel Dene Hoffman, following handwritten lecture notes distributed by the professor as well as notes taken concurrently in class. Any inaccuracies in the notes are likely my fault, and these notes have not been thoroughly proofread for accuracy or consistency. The final few lectures were rushed because we were running towards the end of the semester, and the professor chose not to have a final (or midterm), so the mathematical details were not critical to understanding the class. As a result, some of the less-important details in these lectures (and others) have been glossed over. I hope these notes will be useful to someone taking the class or at least as a reference in understanding the basics of cosmology and particle physics in an expanding universe. Any comments may be directed to dene@cmu.edu, although I cannot guarantee I will have time to update these notes with corrections. Thanks for reading!