33-761 Take-Home Final

Nathaniel D. Hoffman

December 8, 2019

1.

(a) Recall total angular momentum conservation we worked out in problem 6.10 of Jackson, show that the integral version can be recast into the following form,

$$\frac{\mathrm{d}\vec{\mathbf{L}}_{\text{total}}}{\mathrm{d}t} = \int_\Sigma \left[\mathrm{d}\vec{\mathbf{a}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{x}} \times \epsilon_0 \vec{\mathbf{E}}) + \mathrm{d}\vec{\mathbf{a}} \cdot \vec{\mathbf{B}}\left(\vec{\mathbf{x}} \times \frac{1}{\mu_0}\vec{\mathbf{B}}\right)\right] + \frac{1}{2}\int_\Sigma (\mathrm{d}\vec{\mathbf{a}} \times \vec{\mathbf{x}})\left[\epsilon \vec{\mathbf{E}}^2 + \frac{1}{\mu_0}\vec{\mathbf{B}}^2\right]$$

From problem 6.10, we found that

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = -\int_\Sigma \hat{\mathbf{n}} \cdot \bar{M}\,\mathrm{d}a$$

with the following definitions:

$$\bar{M} = \bar{T} \times \vec{\mathbf{x}}$$

and

$$\begin{aligned} T_{ij} &= \left[\epsilon E_i E_j + \mu H_i H_j - \frac{1}{2}\delta_{ij}\left(\epsilon E^2 + \mu H^2\right)\right] \\ &= \left[\epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right] \end{aligned}$$

since $\vec{\mathbf{H}} = \frac{1}{\mu_0}\vec{\mathbf{B}}$ and we assume we are working in a space with vacuum permittivity and permeability. Next, I will write this all in index notation, using

$$M_{il} = \epsilon_{ijk} T_{jl} x_k$$

to denote the dyadic cross product.

$$\begin{aligned} \frac{\mathrm{d}L_l}{\mathrm{d}t} &= -\int_\Sigma n_l M_{il}\,\mathrm{d}a \\ &= -\int_\Sigma n_l \epsilon_{ijk} T_{jl} x_k\,\mathrm{d}a \\ &= -\int_\Sigma n_l \epsilon_{ijk}\left[\epsilon_0 E_j E_l + \frac{1}{\mu_0}B_j B_l - \frac{1}{2}\delta_{jl}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right] x_k\,\mathrm{d}a \\ &= -\int_\Sigma \mathrm{d}a_l\, E_l(\epsilon_{ijk}\epsilon_0 E_j x_k) + \mathrm{d}a_l\, B_l\left(\epsilon_{ijk}\frac{1}{\mu_0}B_j x_k\right) - \frac{1}{2}\mathrm{d}a_l\,\epsilon_{ijk}\delta_{jl}x_k\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \\ &= \int_\Sigma \mathrm{d}a_l\, E_l\left(\epsilon_{ikj}x_k E_j\right) + \mathrm{d}a_l\, B_l\left(\epsilon_{ikj}\frac{1}{\mu_0}x_k B_j\right) + \frac{1}{2}\int_\Sigma \epsilon_{ijk}\,\mathrm{d}a_j\, x_k\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \end{aligned}$$

$$\boxed{\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \int_\Sigma \left[\mathrm{d}\vec{\mathbf{a}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{x}} \times \epsilon_0 \vec{\mathbf{E}}) + \mathrm{d}\vec{\mathbf{a}} \cdot \vec{\mathbf{B}}\left(\vec{\mathbf{x}} \times \frac{1}{\mu_0}\vec{\mathbf{B}}\right)\right] + \frac{1}{2}\int_\Sigma (\mathrm{d}\vec{\mathbf{a}} \times \vec{\mathbf{x}})\left[\epsilon \vec{\mathbf{E}}^2 + \frac{1}{\mu_0}\vec{\mathbf{B}}^2\right]}$$

(b) Consider a local current distribution which has *no electric dipole and electric quadrupole moments* but the current distribution generates to the leading order a magnetic dipole $\vec{\mathbf{m}}$ which oscillates in time with frequency ω. Find the radiated angular momentum for this case using the expression in part (a) and taking Σ as a sphere far away. NOTE that to find a nonzero result we should keep *next to leading order terms in* $\frac{1}{r}$, so the non-radiation part contributes to the radiated angular momentum. (It is common to average $\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}$ over a period, it does not vanish).

Such a current distribution will give rise to fields described by equations 9.35 and 9.36 from Jackson:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\left\{k^2(\hat{\mathbf{n}} \times \vec{\mathbf{m}}) \times \hat{\mathbf{n}}\frac{e^{\imath kr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}) - \vec{\mathbf{m}}]\left(\frac{1}{r^3} - \frac{\imath k}{r^2}\right)e^{\imath kr}\right\}$$

$$\vec{\mathbf{E}} = -\frac{Z_0}{4\pi}k^2(\hat{\mathbf{n}} \times \vec{\mathbf{m}})\frac{e^{\imath kr}}{r}\left(1 - \frac{1}{\imath kr}\right)$$

The final integral in the formula from (a) vanishes on the sphere, since $\mathrm{d}\vec{\mathbf{a}} \times \vec{\mathbf{x}}$ will be zero because those vectors will always be parallel. I will take the remaining terms one at a time. Note that we must use the complex conjugate of the field and divide by two in order to obtain the real part:

$$\begin{aligned}
\frac{1}{2}\,\mathrm{d}\vec{\mathbf{a}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{x}} \times \epsilon_0 \vec{\mathbf{E}}^*) &= \underbrace{\frac{Z^2\epsilon_0 k^4}{32\pi^2}\left(\frac{1}{r}\left(1 + \frac{1}{k^2r^2}\right)\right)}_{A}\mathrm{d}\vec{\mathbf{a}} \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{m}})(\vec{\mathbf{x}} \times (\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)) \\
&= A\,\mathrm{d}a\,\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{m}})(\vec{\mathbf{x}} \times (\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)) \\
&= 0
\end{aligned}$$

since $\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{m}}) = \hat{\mathbf{n}} \cdot (\vec{\mathbf{m}} \times \hat{\mathbf{n}}) - \vec{\mathbf{m}}(\hat{\mathbf{n}} \times \hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot (\vec{\mathbf{m}} \times \hat{\mathbf{n}}) = -\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{m}}) = 0$

Now let's examine the term which doesn't vanish. Note that on the sphere, $\vec{\mathbf{x}} = \vec{\mathbf{r}} = r\hat{\mathbf{n}}$:

$$\begin{aligned}
\frac{1}{2}\,\mathrm{d}\vec{\mathbf{a}} \cdot \vec{\mathbf{B}}\left(\vec{\mathbf{x}} \times \frac{1}{\mu_0}\vec{\mathbf{B}}^*\right) = \mathrm{d}a \Bigg\{ &\left(\frac{\mu_0 k^2}{8\pi}\frac{e^{\imath kr}}{r}\right)\overbrace{\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{m}}) \times \hat{\mathbf{n}}}^{*} \\
&+ \left(\frac{\mu_0}{8\pi}e^{\imath kr}\left(\frac{1}{r^3} - \frac{\imath k}{r^2}\right)\right)\overbrace{(3\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}) - \hat{\mathbf{n}} \cdot \vec{\mathbf{m}})}^{**}\Bigg\}\left(\vec{\mathbf{x}} \times \frac{1}{\mu_0}\vec{\mathbf{B}}^*\right)
\end{aligned}$$

$$\begin{aligned}
* &= \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{m}}) \times \hat{\mathbf{n}} \\
&= -\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{\mathbf{m}})) \\
&= -\hat{\mathbf{n}} \cdot ((\hat{\mathbf{n}} \cdot \vec{\mathbf{m}})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\vec{\mathbf{m}}) \\
&= -(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}) + (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
** &= 3(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}) - \hat{\mathbf{n}} \cdot \vec{\mathbf{m}} \\
&= 2(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}})
\end{aligned}$$

since $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$.

The final term is

$$\begin{aligned}
\vec{\mathbf{x}} \times \frac{1}{\mu_0}\vec{\mathbf{B}}^* = &\left(\frac{k^2}{4\pi}\frac{e^{-\imath kr}}{r}\right)\overbrace{(\vec{\mathbf{x}} \times ((\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*) \times \hat{\mathbf{n}}))}^{\dagger} \\
&+ \left(\frac{e^{-\imath kr}}{4\pi}\left(\frac{1}{r^3} + \frac{\imath k}{r^2}\right)\right)\overbrace{(\vec{\mathbf{x}} \times (3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}^*) - \vec{\mathbf{m}}^*))}^{\dagger\dagger}
\end{aligned}$$

$$
\begin{aligned}
\dagger &= r\left(\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*) \times \hat{\mathbf{n}})\right) \\
&= r\left(-\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*))\right) \\
&= -r\left(\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}^*)\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\vec{\mathbf{m}}^*)\right) \\
&= -r\left((\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}^*)(\hat{\mathbf{n}} \times \hat{\mathbf{n}}) - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)\right) \\
&= r(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)
\end{aligned}
$$

$$
\begin{aligned}
\dagger\dagger &= r\left(\hat{\mathbf{n}} \times (3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}^*)) - \hat{\mathbf{n}} \times \vec{\mathbf{m}}^*\right) \\
&= r\left(3(\hat{\mathbf{n}} \times \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}^*) - \hat{\mathbf{n}} \times \vec{\mathbf{m}}^*\right) \\
&= -r(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)
\end{aligned}
$$

All together, we now have

$$
\begin{aligned}
\mathscr{I} &= \frac{\mu_0}{8\pi} e^{\imath kr} \left(\frac{1}{r^3} - \frac{\imath k}{r^2}\right) 2(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}}) \left(\frac{e^{-\imath kr}}{4\pi} \left(k^2(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*) - \left(\frac{1}{r^2} + \frac{\imath k}{r}\right)(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)\right)\right) \\
&= -\left(\frac{\mu_0}{16\pi^2 r^5} + \frac{\mu_0 \imath k^3}{16\pi^2 r^2}\right)(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}})
\end{aligned}
$$

Now we must integrate this factor over a sphere of radius r and take the limit as $r \to \infty$:

$$
\begin{aligned}
\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} &= -\left(\frac{\mu_0}{16\pi^2 r^5} + \frac{\mu_0 \imath k^3}{16\pi^2 r^2}\right) \int_\Sigma (\hat{\mathbf{n}} \cdot \vec{\mathbf{m}})(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*)\, \mathrm{d}a \\
&= \left(\frac{\mu_0}{16\pi^2 r^5} + \frac{\mu_0 \imath k^3}{16\pi^2 r^2}\right)\left(\frac{4\pi}{3} r^2 (\vec{\mathbf{m}}^* \times \vec{\mathbf{m}})\right) \\
&= \left(\frac{\mu_0}{12\pi r^3} + \frac{\imath k^3 \mu_0}{12\pi}\right)(\vec{\mathbf{m}}^* \times \vec{\mathbf{m}})
\end{aligned}
$$

The integral is essentially the same one we had in the homework. The r^2 term comes from the spherical Jacobian, and we can use the fact that $(\hat{\mathbf{n}} \cdot \vec{\mathbf{m}})(\hat{\mathbf{n}} \times \vec{\mathbf{m}}^*) \mapsto n_i m_i \epsilon_{ijk} n_j m_k^* = n_i n_j \epsilon_{ijk} m_k^* m_i \mapsto -n_i n_j (\vec{\mathbf{m}}^* \times \vec{\mathbf{m}})$ and $\int n_i n_j \,\mathrm{d}\Omega = \frac{4\pi}{3}\delta_{ij}$.
Taking the limit as $r \to \infty$, we have

$$
\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \frac{\imath k^3 \mu_0}{12\pi}(\vec{\mathbf{m}}^* \times \vec{\mathbf{m}}) = \frac{k^3 \mu_0}{12\pi} \operatorname{Im}[\vec{\mathbf{m}} \times \vec{\mathbf{m}}^*]
$$

(note that I switched the cross product in the last step to get rid of the negative sign)

2.

Consider a very thin conductor of length d placed along the z-axis with its midpoint at the origin. Suppose that we have a current running in this conductor:

$$
I = I_0 \sin(kz) e^{-\imath\omega t}
$$

with $k = \frac{\omega}{c} = \frac{4\pi}{d}$. It is more convenient to express this current density in spherical coordinates for a multipole calculation.

(a) Obtain the *exact multipoles* for this current distribution. Note that we cannot use the approximation $kd << 1$ here (since $kd = 4\pi$).

To begin, we need to restate the current density in spherical coordinates. I will propose the following density and then justify each term:

$$\vec{\mathbf{J}}_\omega = \frac{I_0 \sin(kr\cos(\theta))}{r^2}\delta(\varphi)\left[\delta(\cos(\theta)-1)+\delta(\cos(\theta)+1)\right]\Theta\left(\frac{d}{2}-r\right)\left(\cos(\theta)\hat{\mathbf{r}}-\sin(\theta)\hat{\theta}\right)$$

where $\vec{\mathbf{J}} = \vec{\mathbf{J_0}} e^{-\imath\omega t}$.
First, the $I_0\sin(kr\cos(\theta))$ term comes directly from the formula, since $z = r\cos(\theta)$. Next, we have to confine this density to the thin conductor. Since it is running along the z-axis, we want the $\delta(\cos(\theta)\pm 1)$ terms, which set θ on the z-axis. The direction of this current must also be along the z-axis, which is where the $\cos(\theta)\hat{\mathbf{r}}-\sin(\theta)\hat{\theta} = \hat{\mathbf{z}}$ term comes from. The Heaviside function $\Theta\left(\frac{d}{2}-r\right)$ ensures the current density is only nonzero on the conductor (which is centered at 0 so $r = d/2$ at each end). Finally, for simplicity, we set $\varphi = 0$ ($\delta(\varphi)$) and normalize by $\frac{1}{r^2}$ so that when we integrate over the δ functions in spherical coordinates, the Jacobian in spherical coordinates cancels the $\frac{1}{r^2}$ term to give us the current we want in the problem.
We can further simplify this current density by noticing that the δ functions set $\cos(\theta) = \pm 1$ which will set $\sin(\theta) = 0$:

$$\vec{\mathbf{J}}_\omega = \frac{I_0 \sin(kr)}{r^2}\delta(\varphi)\left[\delta(\cos(\theta)-1)+\delta(\cos(\theta)+1)\right]\Theta\left(\frac{d}{2}-r\right)\hat{\mathbf{r}}$$

We can then find the associated charge density:

$$\rho_\omega = \frac{\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{J}}_\omega}{\imath\omega} = \frac{I_0}{\imath\omega}\left(\vec{\boldsymbol{\nabla}}\cdot\hat{r}\right)\left[\frac{\sin(kr)}{r^2}\left[\delta(\cos(\theta)-1)+\delta(\cos(\theta)+1)\right]\delta(\varphi)\Theta\left(\frac{d}{2}-r\right)\right]$$

The negative sign from $\sin(-kr) = -\sin(kr)$ when the $\delta(\cos(\theta)-1)$ is used cancels with the negative sign from the $\cos(\theta)\hat{\mathbf{r}}$. In spherical coordinates, $\vec{\boldsymbol{\nabla}}\cdot\hat{\mathbf{r}} = \frac{1}{r^2}\partial_r r^2$, so this becomes

$$\rho_\omega = \frac{I_0}{\imath\omega r^2}k\cos(kr)\delta(\varphi)\left[\delta(\cos(\theta)-1)+\delta(\cos(\theta)+1)\right]\Theta\left(\frac{d}{2}-r\right) + \cancelto{0}{\sin(kr)(\cdots)}$$

The second term will be zero here because the derivative of the Heaviside function is a δ function and $\delta\left(\frac{d}{2}-r\right)$ makes $kr = 2\pi$ and $\sin(2\pi) = 0$.
Now we can go about calculating the multipoles:

$$a_M = \frac{k^2}{\imath\sqrt{l(l+1)}}\int Y^*_{lm}\left(\vec{\boldsymbol{\nabla}}\cdot(\vec{\mathbf{r}}\times\vec{\mathbf{J}}_\omega)j_l(kr)\right)\mathrm{d}^3x = 0$$

since $\vec{\mathbf{r}}\times\vec{\mathbf{J}}_\omega = \vec{\mathbf{r}}\times J_\omega\hat{r} = 0$.
The other multipole is nonzero:

$$a_E = \frac{k^2}{\imath\sqrt{l(l+1)}}\int\left(\overbrace{Y^*_{lm}c\rho\partial_r\left[rj_l(kr)\right]}^{*} + \overbrace{Y^*_{lm}\imath k\left(\vec{\mathbf{r}}\cdot\vec{\mathbf{J}}\right)j_l(kr)}^{**}\right)\mathrm{d}^3x$$

$$\begin{aligned}
* &= \int \underbrace{[Y^*_{lm}(\pi,0)+Y^*_{lm}(0,0)]c\rho(r)\,\mathrm{d}^3x}_{u}\,\underbrace{\partial_r\left[rj_l(kr)\right]}_{\mathrm{d}v}\mathrm{d}r \\
&= \underbrace{[Y^*_{lm}(\pi,0)+Y^*_{lm}(0,0)]c\rho(r)rj_l(kr)r^2\Big|_{r=0}^{d/2}}_{uv} \\
&\quad - \int\underbrace{[Y^*_{lm}(\pi,0)+Y^*_{lm}(0,0)]c\partial_r[\rho r^2]rj_l(kr)\,\mathrm{d}r}_{v\mathrm{d}u} \\
&= uv - \int_0^{d/2}[Y^*_{lm}(\pi,0)+Y^*_{lm}(0,0)]c\partial_r\left[\frac{I_0 k}{\imath\omega}\cos(kr)\right]rj_l(kr)\,\mathrm{d}r
\end{aligned}$$

$$\begin{aligned}\partial_r\left[\rho r^2\right] &= -\frac{I_0 k}{\imath\omega} k\sin(kr)\\ &= -\frac{I_0 k^2}{\imath\omega}\sin(kr)\end{aligned}$$

so the second term becomes

$$-\int_0^{d/2}[Y_{lm}^*(\pi,0)+Y_{lm}^*(0,0)]\imath k I_0 \sin(kr) r j_l(kr)\,\mathrm{d}r$$

since $c=\frac{\omega}{k}$.
Next we will look at the other term:

$$\begin{aligned}** &= \int Y_{lm}^* \imath k\left(\vec{\mathbf{r}}\cdot\vec{\mathbf{J}}_\omega\right) j_l(kr)\,\mathrm{d}^3x\\ &= \int Y_{lm}^*\imath kr J_\omega j_l(kr)\,\mathrm{d}^3x\\ &= \int_0^{d/2}[Y_{lm}^*(\pi,0)+Y_{lm}^*(0,0)]\imath k I_0\sin(kr) r j_l(kr)\,\mathrm{d}r\end{aligned}$$

This exactly cancels the integral term we found above. Now we have only one term (the vu term from integration by parts):

$$\begin{aligned}a_E &= \frac{k^2}{\imath\sqrt{l(l+1)}}[Y_{lm}^*(\pi,0)+Y_{lm}^*(0,0)]c\rho(r)r^3 j_l(kr)\bigg|_{r=0}^{d/2}\\ &= \frac{I_0k^2}{\imath^2\sqrt{l(l+1)}}[Y_{lm}^*(\pi,0)+Y_{lm}^*(0,0)]\left[\cos(kr)rj_l(kr)\right]\bigg|_{r=0}^{d/2}\\ &= \frac{-I_0k^2}{\sqrt{l(l+1)}}[Y_{lm}^*(\pi,0)+Y_{lm}^*(0,0)]\frac{d}{2}j_l(2\pi)\end{aligned}$$

We can further simplify this by writing the spherical harmonics in terms of Legendre polynomials. Since there is azimuthal symmetry about the z-axis, we know that $m=0$, so

$$a_E = -\frac{dI_0k^2}{2\sqrt{l(l+1)}}\left[\sqrt{\frac{2l+1}{4\pi}}\left(P_l(-1)+P_l(+1)\right)\right]j_l(2\pi)$$

Additionally, $P_l(1)=(-1)^l P_l(-1)$, so

$$a_E = -\frac{dI_0k^2}{2\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}\left((-1)^l+1\right)P_l(1)j_l(2\pi)$$

We know that

$$(-1)^l+1=\begin{cases}2 & \text{if} \quad l \quad \text{even}\\ 0 & \text{if} \quad l \quad \text{odd}\end{cases}$$

and $P_l(1)=1$, so

$$a_E = -\frac{dI_0k^2}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}j_l(2\pi) = -I_0k^2 d\sqrt{\frac{(2l+1)}{4\pi l(l+1)}}j_l(2\pi), \quad l \quad \text{even}$$

and, lest we forget,

$$a_M = 0$$

(b) Find the angular distribution of radiated power as well as the total power radiated in terms of the multipoles.

Since $a_M = 0$, we can write the angular distribution of radiated power as

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{Z_0}{2k^2}\left|\sum_l \underbrace{(-\imath)^{l+1}}_{\imath(-1)^{l/2}}(a_E(l)\vec{\mathbb{X}}_{l0} \times \hat{\mathbf{n}})\right|^2, \quad l \quad \text{even}$$

The where $\vec{\mathbb{X}}_{l0}$ are the vector spherical harmonics:

$$\vec{\mathbb{X}}_{l0} = \frac{1}{\sqrt{l(l+1)}}\vec{\mathbb{L}}Y_{l0}$$

where

$$\vec{\mathbb{L}} = -\imath(\vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}})$$

Next, we want to take the cross product with $\hat{\mathbf{n}}$:

$$\begin{aligned}
\vec{\mathbb{L}}Y_{l0} \times \hat{\mathbf{n}} &= -\imath\left(\vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}}Y_{l0}\right) \times \hat{\mathbf{n}} \\
&= \imath\hat{\mathbf{n}} \times \left(\vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}}Y_{l0}\right) \\
&= \imath\left[(\hat{\mathbf{n}} \cdot \vec{\boldsymbol{\nabla}}Y_{l0})\vec{\mathbf{r}} - (\hat{\mathbf{n}} \cdot \vec{\mathbf{r}})\vec{\boldsymbol{\nabla}}Y_{l0}\right] \\
&= -\imath\left[r\vec{\boldsymbol{\nabla}}Y_{l0}\right]
\end{aligned}$$

since the first term would be derivatives of $Y_{lm}(\Omega)$ with respect to r, which are zero. Therefore, we are left with

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}\Omega} &= \frac{Z_0}{2k^2}\left|\sum_l \frac{\imath(-1)^{l/2}}{\sqrt{l(l+1)}}a_E(-\imath r\vec{\boldsymbol{\nabla}}Y_{l0})\right|^2, \quad l \quad \text{even} \\
&= \frac{Z_0}{2k^2}\left|\frac{(-1)^{l/2}}{\sqrt{l(l+1)}}a_E\sqrt{\frac{2l+1}{4\pi}}r\vec{\boldsymbol{\nabla}}P_l(\cos(\theta))\right|^2, \quad l \quad \text{even}
\end{aligned}$$

We can compute the gradient of the Legendre polynomials as:

$$\begin{aligned}
\vec{\boldsymbol{\nabla}}P_l(\cos(\theta)) &= \frac{1}{r}\partial_\theta P_l(\cos(\theta))\hat{\theta} \\
&= \sin(\theta)\frac{l}{\cos^2(\theta) - 1}\left(\cos(\theta)P_l(\cos(\theta)) - P_{l-1}(\cos(\theta))\right) \\
&= \frac{l}{\sin(\theta)}\left[\cos(\theta)P_l(\cos(\theta)) - P_{l-1}(\cos(\theta))\right]
\end{aligned}$$

so

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{Z_0}{2k^2}\left|\sum_l \frac{(-1)^{l/2}}{\sqrt{l(l+1)}}a_E\sqrt{\frac{2l+1}{4\pi}}\frac{lr}{\sin(\theta)}\left[\cos(\theta)P_l(\cos(\theta)) - P_{l-1}(\cos(\theta))\right]\right|^2, \quad l \quad \text{even}$$

Inserting the equation we had for a_E, we find:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{Z_0 d^2 k^2 I_0^2}{32\pi^2}\left|\sum_l (\imath^l)\frac{2l+1}{(l+1)}\frac{j_l(2\pi)}{\sin(\theta)}\left[\cos(\theta)P_l(\cos(\theta)) - P_{l-1}(\cos(\theta))\right]\right|^2, \quad l \quad \text{even}$$

Using the fact that $kd = 4\pi$, we can write this as

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{Z_0 I_0^2}{2}\left|\sum_l (\imath^l)\frac{2l+1}{l+1}\frac{j_l(2\pi)}{\sin(\theta)}\left[\cos(\theta)P_l(\cos(\theta)) - P_{l-1}(\cos(\theta))\right]\right|^2, \quad l \quad \text{even}$$

For the total power, we can use

$$
\begin{aligned}
P = \frac{Z_0}{2k^2} \sum_l |a_E(l)|^2 &= \frac{Z_0 I_0^2 k^2 d^2}{8\pi} \sum_l \left| \sqrt{\frac{2l+1}{l(l+1)}} j_l(2\pi) \right|^2, \quad l \quad \text{even} \\
&= \frac{Z_0 I_0^2 k^2 d^2}{8\pi} \sum_l \frac{2l+1}{l(l+1)} |j_l(2\pi)|^2, \quad l \quad \text{even} \\
&= 2 Z_0 I_0^2 \sum_l \frac{2l+1}{l(l+1)} |j_l(2\pi)|^2, \quad l \quad \text{even}
\end{aligned}
$$