
Lecture 39:

Monday, November 11, 2019

Recall we had the following expression for work in an electric system:

$$\int \vec{\mathbf{J}} \cdot \vec{\mathbf{E}} \, \mathrm{d}^3x$$

and we were trying to relate it to the change in time of some electromagnetic energy and some mechanical term

$$\mapsto \frac{\mathrm{d}}{\mathrm{d}t} U_{\mathrm{EM}} - \int \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{S}}$$

where the first term is

$$-\int \left(\vec{\mathbf{E}} \cdot \partial_t \vec{\mathbf{D}} + \vec{\mathbf{H}} \cdot \partial_t \vec{\mathbf{B}} \right) \mathrm{d}^3x$$

such that

$$\partial_t \left[u_{\mathrm{mech}} + u_{\mathrm{EM}} \right] = -\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{S}}$$

Suppose the electric field can be expanded as

$$\vec{\mathbf{E}}(x,t) = \int \vec{\mathbf{E}}(x,\omega) e^{-\imath\omega t} \, \mathrm{d}\omega \overbrace{\mapsto} * \int_{-\infty}^{\infty} \vec{\mathbf{E}}^*(\vec{\mathbf{x}},\omega) e^{\imath\omega t} \, \mathrm{d}\omega = \int_{-\infty}^{\infty} \vec{\mathbf{E}}^*(\vec{\mathbf{x}},-\omega) e^{-\imath\omega t}$$

Assuming we have a peak around ω_0, we can write $\vec{\mathbf{E}} \cdot \partial_t \vec{\mathbf{D}}$ as

$$\int E^*(\vec{\mathbf{x}},\omega') e^{\imath\omega' t} [-\imath\omega\epsilon(\omega)] \cdot \vec{\mathbf{E}}(\vec{\mathbf{x}},\omega) e^{-\imath\omega t} \, \mathrm{d}\omega \, \mathrm{d}\omega' \, \mathrm{d}^3x = \int \vec{\mathbf{E}}^*(\vec{\mathbf{x}},-\omega) e^{-\imath\omega t} [\imath\omega'\epsilon(-\omega')] e^{\imath\omega' t} \, \mathrm{d}\omega \, \mathrm{d}\omega' \, \mathrm{d}^3x$$

Brillouin, a student of Sommerfeld, used the equality of these expressions to rewrite it as

$$= \frac{1}{2} \vec{\mathbf{E}}^*(\vec{\mathbf{x}},\omega) [-\imath\omega\epsilon(\omega) + \imath\omega'\epsilon^*(\omega')] \vec{\mathbf{E}}(\vec{\mathbf{x}},\omega) e^{-\imath(\omega-\omega')t} \, \mathrm{d}\omega \, \mathrm{d}\omega' \, \mathrm{d}^3x$$

We then write the second term in the square brackets as an approximation

$$\imath\omega'\epsilon^*(\omega') \approx \imath\omega\epsilon^*(\omega) + \imath(\omega'-\omega) \frac{\mathrm{d}}{\mathrm{d}\omega} [\omega\epsilon(\omega)]$$

We can combine the first term of this expression with the first term in the square brackets to get $2\omega \, \mathrm{Im}[\epsilon(\omega)]$. The remaining term is approximately

$$\imath(\omega'-\omega) \frac{\mathrm{d}}{\mathrm{d}\omega} [\omega\epsilon(\omega)] \approx \frac{\mathrm{d}}{\mathrm{d}\omega} [\omega\epsilon(\omega)] + \frac{\mathrm{d}}{\mathrm{d}\omega'} [\omega'\epsilon^*(\omega')] \approx \left. \frac{\mathrm{d}}{\mathrm{d}\omega [\omega \, \mathrm{Re}[\epsilon(\omega)]]} \right|_{\omega_0}$$

All together, we have

$$-\int 2\omega_0 \, \mathrm{Im}[\epsilon(\omega_0)] \int \vec{\mathbf{E}}^*(\vec{\mathbf{x}},\omega) e^{-\imath(\omega-\omega')t} \, \mathrm{d}\omega \, \mathrm{d}\omega' \, \mathrm{d}^3x$$

We must also factor in the part with the time derivative

$$-\int 2\omega_0 \, \mathrm{Im}[\epsilon(\omega_0)] \left\langle \vec{\mathbf{E}}^2 \right\rangle_{\omega_0} \mathrm{d}^3x - \frac{\partial}{\partial t} int \frac{\mathrm{d}}{\mathrm{d}\omega} [\omega \, \mathrm{Re}[\epsilon\omega]] \left\langle \vec{\mathbf{E}}^2 \right\rangle_{\omega_0}$$

assuming the electric field is some oscillation with a slowly varying amplitude.

We can do the same for the momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{\mathrm{mech}} = -\frac{\partial}{\partial t}\int \left[\frac{\mathrm{d}}{\mathrm{d}\omega}[\omega \operatorname{Re}[\epsilon(\omega)]\Big|_{\omega_0} \left\langle \vec{\mathbf{E}}^2 \right\rangle_{\omega_0} + \frac{\mathrm{d}}{\mathrm{d}\omega}[\omega \operatorname{Re}[\mu(\epsilon)]]\Big|_{\omega_0} \left\langle \vec{\mathbf{H}}^2 \right\rangle_{\omega_0} \right] \mathrm{d}^3x$$

This evaluates to

$$-\int \left[2\omega_0 \operatorname{Im}[\epsilon(\omega_0)] \left\langle \vec{\mathbf{E}}^2 \right\rangle_{\omega_0} + 2\omega_0 \operatorname{Im}[\mu(\epsilon_0)] \left\langle \vec{\mathbf{H}}^2 \right\rangle_{\omega_0} \mathrm{d}^3x \right] - \int \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{S}}\, \mathrm{d}^3x = \frac{\mathrm{d}}{\mathrm{d}t}(U_{\mathrm{mech}} + U_{\mathrm{EM}})$$

The first term is dissipation and the second part is escaping energy.

0.1 Inhomogeneous Media

What if we had a medium for which $\lambda\left|\vec{\boldsymbol{\nabla}}\epsilon\right| << \epsilon$ and $\epsilon(\vec{\mathbf{x}})$ is a function of position. We now have

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \mu_0 \epsilon(x)(-\imath\omega)\vec{\mathbf{E}}(\vec{\mathbf{x}}, \omega)$$

and

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \imath\omega\vec{\mathbf{B}} = \imath\omega\vec{\mathbf{H}}$$

As we seem to always do with these sorts of equations, take the curl of the curl:

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \vec{\boldsymbol{\nabla}} \times \left(\epsilon(\imath\omega)\vec{\mathbf{E}} \right) = \epsilon(\vec{\mathbf{x}})(-\imath\omega)\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} - \imath\omega\mu_0 \left(\vec{\boldsymbol{\nabla}} \cdot \epsilon \times \vec{\mathbf{E}} \right)$$

The second term is small if we assume the permittivity changes slowly in space.

$$\cancelto{0}{\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{H}}} - \nabla^2\vec{\mathbf{H}} = \mu_0\omega^2\epsilon(\vec{\mathbf{x}})\vec{\mathbf{H}}$$

We can do the same thing with the electric field, except that the divergence of this field is not zero, but it's approximately zero, again because the permittivity changes slowly.

$$\nabla^2\vec{\mathbf{E}} + \omega^2\mu_0\epsilon(\vec{\mathbf{x}})\vec{\mathbf{E}}(\vec{\mathbf{x}}, \omega) = 0$$

and

$$\nabla^2\vec{\mathbf{H}} + \omega^2\mu_0\epsilon(\vec{\mathbf{x}})\vec{\mathbf{H}}(\vec{\mathbf{x}}, \omega) = 0$$

Alternatively we could write the factors in front of the field as $\omega^2\mu_0\epsilon(\vec{\mathbf{x}}) = \frac{n^2}{c^2}$.