33-761 Homework 11

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1. Jackson Problem 7.16 (a) and (b)

Plane waves propagate in a homogeneous, nonpermeable, but *anisotropic* dielectric. The dielectric is characterized by a tensor ϵ_{ij}, but if coordinate axes are chosen as the principle axis, the components of displacement along these axes are related to the electric-field components by $D_i = \epsilon_i E_i$ $(i = 1, 2, 3)$, where ϵ_i are the eigenvalues of the matrix ϵ_{ij}.

(a) Show that plane waves with frequency ω and wave vector $\vec{\mathbf{k}}$ must satisfy

$$\vec{\mathbf{k}} \times (\vec{\mathbf{k}} \times \vec{\mathbf{E}}) + \mu_0 \omega^2 \vec{\mathbf{D}} = 0$$

A plane wave will have

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 e^{\imath\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} - \omega t\right)}$$

and

$$\vec{\mathbf{H}} = \vec{\mathbf{H}}_0 e^{\imath\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} - \omega t\right)}$$

Using the Maxwell curl equations in free space, we find that

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \imath \vec{\mathbf{k}} \times \vec{\mathbf{E}} = \imath \omega \vec{\mathbf{B}} = \imath \omega \mu_0 \vec{\mathbf{H}}$$

and

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \imath \vec{\mathbf{k}} \times \vec{\mathbf{H}} = -\imath \omega \vec{\mathbf{D}}$$

Taking the cross product of $\imath\vec{\mathbf{k}}$ with the first equation, we find

$$\imath\vec{\mathbf{k}} \times (\imath\vec{\mathbf{k}} \times \vec{\mathbf{E}}) - \imath\omega\mu_0(\imath\vec{\mathbf{k}} \times \vec{\mathbf{H}}) = 0$$

Substituting in the second equation, we have

$$\imath\vec{\mathbf{k}} \times (\imath\vec{\mathbf{k}} \times \vec{\mathbf{E}}) - \imath\mu_0\omega(\imath\omega\vec{\mathbf{D}})$$

or

$$\vec{\mathbf{k}} \times (\vec{\mathbf{k}} \times \vec{\mathbf{E}}) + \mu_0 \omega^2 \vec{\mathbf{D}} = 0$$

once the $\imath$s are canceled.

(b) Show that for a given wave vector $\vec{\mathbf{k}} = k\vec{\mathbf{n}}$ there are two distinct modes of propagation with different phase velocities $v = \frac{\omega}{k}$ that satisfy the Fresnel equation

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = \frac{1}{\sqrt{\mu_0 \epsilon_i}}$ is called a principal velocity, and n_i is the component of $\vec{\mathbf{n}}$ along the ith principal axis.

Substituting $\vec{\mathbf{k}} = k\vec{\mathbf{n}}$ into the above equation and dividing out k, we find

$$\vec{\mathbf{n}} \times (\vec{\mathbf{n}} \times \vec{\mathbf{E}}) + \mu_0 v^2 \vec{\mathbf{D}} = 0$$

Using the double cross-product identity, this is equal to

$$\vec{\mathbf{n}}(\vec{\mathbf{n}} \cdot \vec{\mathbf{E}}) - \vec{\mathbf{E}} + \mu_0 v^2 \vec{\mathbf{D}} = 0$$

Taking the dot product with $\vec{\mathbf{n}}$:

$$(\vec{\mathbf{n}}^2 - 1)(\vec{\mathbf{n}} \cdot \vec{\mathbf{E}}) + \mu_0 v^2 \vec{\mathbf{n}} \cdot \vec{\mathbf{D}} = 0$$

Next, we can write this in index notation:

$$(n_i^2 - 1)(n_i E_i) + v^2 \mu_0 \epsilon_i E_i n_i = 0$$

We can recognize $\mu_0 \epsilon_i = \frac{1}{v_i^2}$ and divide out $E_i n_i$:

$$(n_i^2 - 1) + \frac{v^2}{v_i^2} = 0$$

or

$$\begin{aligned} v_i^2(n_i^2 - 1) + v^2 &= 0 \\ v_i^2 n_i^2 &= -(v^2 - v_i^2) \\ -v_i^2 \frac{n_i^2}{v^2 - v_i^2} &= 0 \\ \frac{n_i^2}{v^2 - v_i^2} &= 0 \end{aligned}$$

with the usual implied sum over i.

2. Radiation Power

For vacuum define $\vec{\mathbf{P}} = \int \mathrm{d}^3x\, \epsilon_0 \vec{\mathbf{E}} \times \vec{\mathbf{B}}$, and similarly define the total energy as $U = \frac{1}{2} \int \mathrm{d}^3x [\epsilon_0 \vec{\mathbf{E}}^2 + \frac{1}{\mu_0} \vec{\mathbf{B}}^2$ (assuming localized fields to make integrals convergent).

(a) Show that we always have,

$$c\left|\vec{\mathbf{P}}\right| \leq U$$

First note that by definition,

$$\left|\int f(x)\,\mathrm{d}x\right| \leq \int |f(x)|\,\mathrm{d}x$$

so

$$\left|\vec{\mathbf{P}}\right| \leq \int \mathrm{d}^3x \left|\epsilon_0 \vec{\mathbf{E}} \times \vec{\mathbf{B}}\right|$$

Note that

$$\epsilon_0 = \frac{1}{c}\sqrt{\frac{\epsilon_0}{\mu_0}},$$

so we can write this as

$$c\left|\vec{\mathbf{P}}\right| \leq \int \mathrm{d}^3x\,\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\times\vec{\mathbf{B}}\right|$$

By definition of the cross product,

$$\begin{aligned}
c\left|\vec{\mathbf{P}}\right| &\leq \int \mathrm{d}^3x\,\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\times\vec{\mathbf{B}}\right| \\
&= \int \mathrm{d}^3x\,\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\right|\left|\vec{\mathbf{B}}\right||\sin(\gamma)| \\
&\leq \int \left[\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\right|\left|\vec{\mathbf{B}}\right| + \frac{\epsilon_0}{2}\left(\vec{\mathbf{E}} - c\vec{\mathbf{B}}\right)^2\right]|\sin(\gamma)| \\
&= \int \left[\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\right|\left|\vec{\mathbf{B}}\right| + \frac{1}{2}\left(\sqrt{\epsilon_0}\vec{\mathbf{E}} - \frac{1}{\sqrt{\mu_0}}\vec{\mathbf{B}}\right)^2\right]|\sin(\gamma)| \\
&= \int \left[\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\right|\left|\vec{\mathbf{B}}\right| + \frac{\epsilon_0}{2}\vec{\mathbf{E}}^2 + \frac{1}{2\mu_0}\vec{\mathbf{B}}^2 - \frac{1}{2}2\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\vec{\mathbf{E}}\right|\left|\vec{\mathbf{B}}\right|\right]|\sin(\gamma)| \\
&= \int \frac{1}{2}\left[\epsilon_0\vec{\mathbf{E}}^2 + \frac{1}{\mu_0}\vec{\mathbf{B}}^2\right]|\sin(\gamma)| \leq U
\end{aligned}$$

where γ is the angle between the field vectors and $0 \leq |\sin(\gamma)| \leq 1$ for the final inequality.

(b) Show that if we demand the equality, this necessarily implies $c\left|\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{E}}\right|$ and $\vec{\mathbf{E}}\cdot\vec{\mathbf{B}} = 0$ as in plane wave solutions.

For the equality to hold, we demand that the quantity we added, $\frac{\epsilon_0}{2}\left(\sqrt{\epsilon}\vec{\mathbf{E}} - \frac{1}{\sqrt{\mu_0}}\vec{\mathbf{B}}\right)^2 = \frac{\epsilon_0}{2}\left(\vec{\mathbf{E}} - c\vec{\mathbf{B}}\right)^2 = 0$, which is true if $\left|\vec{\mathbf{E}}\right| = c\left|\vec{\mathbf{B}}\right|$, and $\sin(\gamma) = 1$, which is true if $\vec{\mathbf{E}}\cdot\vec{\mathbf{B}} = 0$.

3. Jackson Problem 9.3

Note that here the dominant mode is an electric dipole.

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos(\omega t)$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

First, we want to find an expansion for the potential. Due to the spherical and azimuthal symmetry, we can expand the potential as

$$\Phi = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos(\theta))$$

Outside the sphere, $A_l = 0$ and we can solve for B_l:

$$V(\theta) = \sum_l B_l R^{-(l+1)}P_l(\cos(\theta))$$

so

$$B_l = R^{l+1}\frac{2l+1}{2}\int_{-1}^{1} V(\cos(\theta))P_l(\cos(\theta))\,\mathrm{d}\cos(\theta)$$

Since the potentials are opposite in magnitude on either side of the sphere, this becomes

$$B_l = R^{l+1}\frac{2l+1}{2}V\left(\int_0^1 P_l(\cos(\theta))\,\mathrm{d}\cos(\theta) - \int_{-1}^0 P_l(\cos(\theta))\,\mathrm{d}\cos(\theta)\right)$$

For now, I am ignoring the time dependence in the potential and will just add it in at the end. Because the Legendre polynomials are even/odd if l is even/odd, the even l will cancel out:

$$B_l = R^{l+1}V(2l+1)\int_0^1 P_l(\cos(\theta))\,\mathrm{d}\cos(\theta) \quad \text{for} \quad l \quad \text{odd}$$

We assume this system looks approximately like a dipole in the radiation zone, and the potential with respect to the dipole moment is

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos(\theta) = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}P_1(\cos(\theta))$$

The $l = 1$ term in our first expansion is

$$V\frac{3}{2}\left(\frac{R}{r}\right)^2 P_1(\cos(\theta))$$

and setting these equal to each other, we find that

$$\vec{\mathbf{p}} = 4\pi\epsilon_0 V\frac{3}{2}R^2\hat{\mathbf{z}} = 6\pi\epsilon_0 R^2 V\hat{\mathbf{z}}$$

assuming the symmetry is about the $\hat{\mathbf{z}}$-axis. We can now calculate the fields in the radiation zone using the dipole moment, adding in the time dependence:

$$\vec{\mathbf{H}} = \frac{ck^2}{4\pi}\hat{\mathbf{n}}\times\vec{\mathbf{p}}\frac{e^{\imath kr}}{r} \mapsto -\frac{3V}{2Z_0}(kR)^2\sin(\theta)\frac{e^{\imath(kr-\omega t)}}{r}\hat{\varphi}$$

$$\vec{\mathbf{E}} = Z_0\vec{\mathbf{H}}\times\hat{\mathbf{n}} \mapsto -\frac{3V}{2}(kR)^2\sin(\theta)\frac{e^{\imath(kr-\omega t)}}{r}\hat{\theta}$$

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c^2 Z_0}{32\pi^2}|(\hat{\mathbf{n}}\times\vec{\mathbf{p}})\times\hat{\mathbf{n}}|^2 k^4 = \frac{9}{8}(kR)^4\frac{V^2}{Z_0}\sin^2(\theta)$$

and

$$P = \int\frac{\mathrm{d}P}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = 3\pi(kR)^4\frac{V^2}{Z_0}$$

4. Jackson Problem 9.8 (a) and (c)

(a) Show that a classical oscillating electric dipole $\vec{\mathbf{p}}$ with fields given by (9.18) radiates electromagnetic angular momentum to infinity at the rate

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\vec{\mathbf{p}}^*\times\vec{\mathbf{p}}]$$

Equation(s) 9.18:

$$\vec{\mathbf{H}} = \frac{ck^2}{4\pi}(\vec{\mathbf{n}}\times\vec{\mathbf{p}})\frac{e^{\imath kr}}{r}\left(1-\frac{1}{\imath kr}\right)$$

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\left\{k^2\,(\vec{\mathbf{n}}\times\vec{\mathbf{p}})\times\vec{\mathbf{n}}\frac{e^{\imath kr}}{r} + [3\vec{\mathbf{n}}(\vec{\mathbf{n}}\cdot\vec{\mathbf{p}})-\vec{\mathbf{p}}]\left(\frac{1}{r^3}-\frac{\imath k}{r^2}\right)e^{\imath kr}\right\}$$

The linear momentum density is defined as

$$\vec{\mathbf{g}} = \frac{1}{2c^2}\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*$$

Using this, we can define angular momentum density as

$$\vec{\ell} = \vec{\mathbf{r}} \times \vec{\mathbf{g}} = \vec{\mathbf{r}} \times (\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*) = \frac{1}{2c^2}\left[\vec{\mathbf{E}}(\vec{\mathbf{r}} \cdot \vec{\mathbf{H}}^*) - \vec{\mathbf{H}}^*(\vec{\mathbf{r}} \cdot \vec{\mathbf{E}})\right]$$

Note that

$$n_i \epsilon_{ijk} n_j p_k = \epsilon_{ijk}\delta_{ij} p_k = 0$$

so if $\vec{\mathbf{r}} = r\vec{\mathbf{n}}$, $\vec{\mathbf{r}} \cdot \vec{\mathbf{H}} = 0$ and $\vec{\mathbf{r}} \cdot \vec{\mathbf{E}}$ will only contain the second term in the large brackets:

$$\vec{\mathbf{r}} \cdot \vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}(2\vec{\mathbf{n}} \cdot \vec{\mathbf{p}})\left(\frac{1}{r^2} - \frac{\imath k}{r}\right)e^{\imath kr}$$

so

$$\vec{\ell} = \frac{1}{2c^2}\vec{\mathbf{H}}^*(\vec{\mathbf{r}} \cdot \vec{\mathbf{E}}) = \frac{\imath k^3}{16\pi^2\epsilon_0 c r^2}\left(1 + \frac{1}{k^2r^2}\right)(\vec{\mathbf{n}} \cdot \vec{\mathbf{p}})(\vec{\mathbf{n}} \times \vec{\mathbf{p}}^*)$$

We want to calculate the radiated angular momentum, so instead of integrating this over all space, we can just integrate it over a sphere of radius r and then take $r \to \infty$. In other words, $\mathrm{d}\vec{\ell} = \vec{\ell}\,\mathrm{d}a\,\mathrm{d}r = \vec{\ell}r^2\,\mathrm{d}r\,\mathrm{d}\Omega$ so

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \vec{\ell}r^2\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}\,\mathrm{d}\Omega = \vec{\ell}r^2 c\,\mathrm{d}\Omega$$

so

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \frac{\imath k^3}{16\pi^2\epsilon_0}\left(1 + \frac{1}{k^2r^2}\right)\int(\vec{\mathbf{n}} \cdot \vec{\mathbf{p}})(\vec{\mathbf{n}} \times \vec{\mathbf{p}}^*)\,\mathrm{d}\Omega$$

To perform this integral, note that the integrand, in index notation, is

$$n_i p_i \epsilon_{ijk} n_j p_k^* = n_i n_j \epsilon_{ijk} p_k^* p_i \mapsto -n_i n_j(\vec{\mathbf{p}}^* \times \vec{\mathbf{p}})$$

Finally, the integral over $\int n_i n_j\,\mathrm{d}\Omega = \frac{4\pi}{3}\delta_{ij}$, so, taking the limit as $r \to \infty$,

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = -\frac{\imath k^3}{16\pi^2\epsilon_0}\frac{4\pi}{3}(\vec{\mathbf{p}}^* \times \vec{\mathbf{p}}) = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\vec{\mathbf{p}}^* \times \vec{\mathbf{p}}]$$

(c) For a charge e rotating in the x-y plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $\frac{\mathrm{d}L_z}{\mathrm{d}t} = \frac{e^2k^3a^2}{6\pi\epsilon_0}$. What about a charge oscillating along the z axis?

The charge distribution for such a system is given by

$$\rho = e\delta(x - a\cos(\omega t))\delta(y - a\sin(\omega t))\delta(z)$$

The dipole moment is therefore

$$\vec{\mathbf{p}} = \int \vec{\mathbf{x}}\rho\,\mathrm{d}^3x = ea(\hat{\mathbf{x}}\cos(\omega t) + \hat{\mathbf{y}}\sin(\omega t)) = \mathrm{Re}[ea(\hat{\mathbf{x}} + \imath\hat{\mathbf{y}})e^{-\imath\omega t}]$$

If we just look at

$$\vec{\mathbf{p}}(\vec{\mathbf{x}}) = ea(\hat{\mathbf{x}} + \imath\hat{\mathbf{y}}),$$

we see that

$$\vec{\mathbf{p}}^* \times \vec{\mathbf{p}} = e^2a^2\imath(\hat{\mathbf{x}} \times \hat{\mathbf{y}} - \hat{\mathbf{y}} \times \hat{\mathbf{x}}) = 2e^2a^2\imath\hat{\mathbf{z}}$$

so

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \frac{k^3e^2a^2}{6\pi\epsilon_0}\hat{\mathbf{z}}$$

from the formula found in part (a).