LECTURE 18:

Wednesday, February 26, 2020

Recap from last lecture: We can represent an object x in a Cartesian basis x_i, $i = 1, 2, 3$ or in a spherical basis x_a, $a = 1, 0, -1$ if if it is a representation of the rotation group (vectors, for example). We can transform both states and operators under the action of the group:

$$|\psi\rangle \to U |\psi\rangle$$

and

$$O \to UOU^{-1}$$

Let's try to relate the Cartesian and spherical bases. First, consider states. We know that the state $|i = 3\rangle = |m = 0\rangle$ by our definitions. They are both eigenstates of L_z with $L_z |m = 0\rangle = 0$. Under a rotation around the z-axis, the state $|i = 3\rangle$ transforms infinitesimally as

$$|i = 3\rangle \to U(\hat{\mathbf{n}} = \hat{\mathbf{e}}_3, \theta) |i = 3\rangle \approx (1 - \imath\theta L_z) |i = 3\rangle = |i = 3\rangle$$

by definition, so $L_z |i = 3\rangle = 0$. Next, let's rotate around the x-axis, first in Cartesian coordinates:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\theta \\ 0 \end{bmatrix}$$

since the rotation matrix is approximately (for small θ),

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta \\ 0 & \theta & 1 \end{bmatrix}$$

Therefore, the infinitesimal rotation of $|i = 3\rangle$ is $\delta_x(\theta) |i = 3\rangle = -\theta |i = 2\rangle$.

In the spherical basis, we have

$$\begin{aligned} e^{-\imath L_x \theta} |m = 0\rangle \approx |m = 0\rangle - \imath\theta L_x |m = 0\rangle &\approx \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \imath\theta \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= |m = 0\rangle - \frac{\imath\theta}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= |m = 0\rangle - \frac{\imath\theta}{\sqrt{2}} \left[|m = 1\rangle + |m = -1\rangle \right] \end{aligned}$$

All together,

$$\begin{aligned} \delta_x(\theta) |i = 3\rangle &= -theta |i = 2\rangle \\ \delta_x(\theta) |m = 0\rangle &= -\frac{\imath\theta}{\sqrt{2}} \left[|m = 1\rangle + |m = -1\rangle \right] \end{aligned}$$

so we can conclude that

$$|i = 2\rangle = \frac{\imath}{\sqrt{2}} \left[|m = 1\rangle + |m = -1\rangle \right]$$

We could do the same manipulation around the y-axis and find that

$$|i = 1\rangle = \frac{1}{\sqrt{2}} \left[|m = -1\rangle - |m = 1\rangle \right]$$

We can then use these results to show that

$$|m = \pm 1\rangle = \mp \frac{1}{\sqrt{2}}[\hat{\mathbf{x}} \pm \imath \hat{\mathbf{y}}]$$

We have just determined the relationship between the bases using states. Now let's do the same thing using operators:

$$O \to UOU^{-1} \approx (1 - \imath\theta\hat{\mathbf{n}} \cdot \vec{\mathbf{L}})O(1 + \imath\theta\hat{\mathbf{n}} \cdot \vec{\mathbf{L}}) = O - \imath\theta\Big[\vec{\mathbf{L}}, O\Big]$$

Therefore, the group action on an operator is the commutator. Again, we define $X_{m=0} \equiv X_{i=3}$ because both are invariant under rotations about the z-axis.

$$[J_\pm, X_{m=0}] = \sqrt{2}\hbar X_\pm$$

since this is the action of the raising and lowering operators. Now let's do the same action in the Cartesian basis:

$$\begin{aligned}
[J_x \pm \imath J_y, X_{i=3}] &= [(\vec{\mathbf{r}} \times \vec{\mathbf{p}})_x \pm \imath(\vec{\mathbf{r}} \times \vec{\mathbf{p}})_y, z] \\
&= [r_i p_j \epsilon_{ijx}, z] \pm \imath[r_i p_j \epsilon_{ijy}, z] \\
&= r_i[p_j, z]\epsilon_{ijx} \pm \imath r_i[p_j, z]\epsilon_{ijy}
\end{aligned}$$

$r_i p_j$ is zero unless $i = j$, but $\epsilon_{ijx} = 0$ when $i = j$, and r commutes with z, so we can pull it out of the commutator in the final step above.

Together, we have

$$\begin{aligned}
[J_x \pm \imath J_y, X_{i=3}] &= r_i\hbar(-\imath)\epsilon_{izx} + \imath r_i(-\imath\hbar)\epsilon_{izy} \\
&= y\hbar(-\imath)\epsilon_{yzx} \pm x\hbar\epsilon_{xzy} \\
&= (\mp x - \imath y)\hbar
\end{aligned}$$

We can then equate these results:

$$X_{m=\pm 1} = \frac{1}{\sqrt{2}}\left(\mp X_{X_{i=1}} - \imath X_{i=2}\right)$$

0.1 Recap: Wigner-Eckart Theorem

$$\langle lm, \alpha| O_L^s |l'm', \beta\rangle$$

is only nonzero if $m = s + m'$.

$$\langle lm, \alpha| O_L^s |l'm', \beta\rangle = \langle lm|Ls; l'm'\rangle \langle l, \alpha| |O_L| |l', \beta\rangle$$

Suppose we know

$$A = \left\langle \frac{1}{2}\frac{1}{2}\alpha \middle| z \middle| \frac{1}{2}\frac{1}{2}\beta \right\rangle$$

and we are interested in calculating

$$\left\langle \frac{1}{2}\frac{1}{2}\alpha \middle| x \middle| \frac{1}{2}, -\frac{1}{2}\beta \right\rangle = B$$

With Wigner-Eckart, we can write

$$A = \left\langle \frac{1}{2}\frac{1}{2} \middle| 1, 0; \frac{1}{2}\frac{1}{2} \right\rangle \left\langle \frac{1}{2}\alpha \middle| |z| \middle| \frac{1}{2}\beta \right\rangle$$

and

$$B = \left\langle \frac{1}{2}\frac{1}{2}\alpha \middle| \frac{1}{\sqrt{2}}[x_{m=1} + \imath x_{m=-1}] \middle| \frac{1}{2}, -\frac{1}{2}\beta \right\rangle$$

The $x_{m=-1}$ matrix element vanishes from our knowledge that $m = s + m'$ from above, and $-1 - \frac{1}{2}$ doesn't equal $\frac{1}{2}$ but $1 - \frac{1}{2}$ does:

$$B = \left\langle \frac{1}{2}\frac{1}{2}\alpha \right| \frac{1}{\sqrt{2}} x_{m=1} \left| \frac{1}{2}, -\frac{1}{2}\beta \right\rangle$$

Using Wigner-Eckart, we can now write

$$B = \left\langle \frac{1}{2}\frac{1}{2} \middle| 1, 1; \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}\alpha \right| |x| \left| \frac{1}{2}\beta \right\rangle$$

but x and z are both $l = 1$ states, so the reduced matrix elements must be the same thing, since it only depends on the total l of the operator. Therefore

$$\frac{A}{B} = \frac{\left\langle \frac{1}{2}\frac{1}{2} \left| 1, 0; \frac{1}{2}, \frac{1}{2} \right\rangle \right|}{-\frac{1}{\sqrt{2}} \left\langle \frac{1}{2}\frac{1}{2} \middle| 1, 1; \frac{1}{2}, -\frac{1}{2} \right\rangle} = 1 \implies A = B$$