33-755 Homework 12

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Cohen-Tannoudji 6.2: Observables in Angular Momentum Superposition States

Consider an arbitrary physical system whose four-dimensional state space is spanned by a basis of four eigenvectors $|j, m_z\rangle$ common to $\vec{\mathbf{J}}^2$ and J_z ($j = 0$ or 1; $-j \leq m_z \leq +j$), of eigenvalues $j(j+1)\hbar^2$ and $m_z\hbar$ such that:

$$J_\pm |j, m_z\rangle = \hbar\sqrt{j(j+1) - m_z(m_z \pm 1)}\, |j, m_z \pm 1\rangle$$
$$J_+ |j, j\rangle = J_- |j, -j\rangle = 0$$

a. Express in terms of the kets $|j, m_z\rangle$, the eigenstates common to $\vec{\mathbf{J}}^2$ and J_x, to be denoted by $|j, m_x\rangle$.

First, both bases share the ground state $|0, 0\rangle$ because this is the only eigenstate of $\vec{\mathbf{J}}^2$ with eigenvalue 0. Next, we can express J_x in terms of the raising and lowering operators: $J_x = \frac{1}{2}(J_+ + J_-)$.
If we act J_x on an arbitrary vector in the J_z basis, $|\psi\rangle = \alpha |1, -1\rangle + \beta |1, 0\rangle + \gamma |1, 1\rangle$, we find:

$$J_x |\psi\rangle = \frac{\sqrt{2}}{2} \left[(\alpha + \gamma) |1, 0\rangle + \beta(|1, -1\rangle + |1, 1\rangle)\right]$$

Therefore, the eigenstate of J_x with eigenvalue $+\hbar$,

$$J_x |1, m_x = +1\rangle = \hbar |1, m_x = +1\rangle$$

will be the state when

$$(\alpha + \gamma)\frac{\sqrt{2}}{2} = \beta \quad \text{and} \quad \beta\frac{\sqrt{2}}{2} = \alpha = \gamma$$

or

$$\beta = \sqrt{2}, \alpha = \gamma = 1$$

so

$$|1, m_x = +1\rangle = \frac{1}{2}\left[|1, -1\rangle + \sqrt{2}\, |1, 0\rangle + |1, -1\rangle\right]$$

accounting for normalization.
Similarly,

$$|1, m_x = -1\rangle = \frac{1}{2}\left[|1, 1\rangle - \sqrt{2}\, |1, 0\rangle + |1, -1\rangle\right]$$

and

$$|1, m_x = 0\rangle = \frac{1}{\sqrt{2}}\left[|1, 1\rangle - |1, -1\rangle\right]$$

b. Consider a system in the normalized state:

$$|\psi\rangle = \alpha\,|1,1\rangle + \beta\,|1,0\rangle + \gamma\,|1,-1\rangle + \delta\,|0,0\rangle$$

(i) What is the probability of finding $2\hbar^2$ and $\hbar$ if $\vec{\mathbf{J}}^2$ and J_x are measured simultaneously?

$$\Pr\left(\vec{\mathbf{J}}^2 = 2\hbar^2, J_z = \hbar\right) = |\langle 1,1|\psi\rangle|^2 = |\alpha|^2$$

(ii) Calculate the mean value of J_z when the system is in the state $|\psi\rangle$, and the probabilities of the various possible results of a measurement bearing only on this observable.

Only the $m_z \neq 0$ terms will contribute in the expectation value since the eigenvalue of J_z on a state with $m_z = 0$ is 0:

$$\langle J_z\rangle = (|\alpha|^2 - |\gamma|^2)\hbar$$

$$\Pr(J_z = \hbar) = |\alpha|^2$$

$$\Pr(J_z = -\hbar) = |\gamma|^2$$

$$\Pr(J_z = 0) = |\beta|^2 + |\delta|^2$$

(iii) Same questions for the observable $\vec{\mathbf{J}}^2$ and for J_x.

$$\begin{aligned}\Pr\left(J^2 = 2\hbar^2, J_x = \hbar\right) &= |\langle 1, m_x = 1|\psi\rangle|^2 \\ &= \frac{1}{4}\left|\langle 1,1|\psi\rangle + \sqrt{2}\,\langle 1,0|\psi\rangle + \langle 1,-1|\psi\rangle\right|^2 \\ &= \frac{1}{4}\left|\alpha + \sqrt{2}\beta + \gamma\right|^2\end{aligned}$$

$$\Pr(J_x = \hbar) = |\langle 1, m_x = 1|\psi\rangle|^2 = \frac{1}{4}\left|\alpha + \sqrt{2}\beta + \gamma\right|^2$$

$$\Pr(J_x = -\hbar) = |\langle 1, m_x = 1|\psi\rangle|^2 = \frac{1}{4}\left|\alpha - \sqrt{2}\beta + \gamma\right|^2$$

$$\Pr(J_x = 0) = |\langle 1, m_x = 0|\psi\rangle|^2 + |\langle 0, m_x = 0|\psi\rangle|^2 = |\alpha - \gamma|^2 + |\delta|^2$$

(iv) J_z^2 is now measured; What are the possible results, their probabilities, and their mean value?

The two possible outcomes are 0 and $\hbar^2$.

$$\Pr\left(J_z^2 = 0\right) = \Pr(J_z = 0) = |\beta|^2 + |\delta|^2$$

$$\Pr\left(J_z^2 = \hbar^2\right) = \Pr(J_z = \hbar) + \Pr(J_z = -\hbar) = |\alpha|^2 + |\gamma|^2$$

$$\left\langle J_z^2\right\rangle = \hbar\,\langle J_z\rangle = \hbar^2(|\alpha|^2 + |\gamma|^2)$$

Cohen-Tannoudji 6.6: Electric Quadrupole Hamiltonian

Consider a system of angular momentum $l = 1$. A basis of its state space is formed by the three eigenvectors of L_z: $|+1\rangle$, $|0\rangle$, $|-1\rangle$, whose eigenvalues are, respectively, $+\hbar$, 0, and $-\hbar$, and which satisfy:

$$L_\pm\,|m\rangle = \hbar\sqrt{2}\,|m \pm 1\rangle$$

$$L_+\,|1\rangle = L_-\,|-1\rangle = 0$$

This system, which possesses an electric quadrupole moment, is placed in an electric field gradient, so that its Hamiltonian can be written:

$$H = \frac{\omega_0}{\hbar}(L_u^2 - L_v^2)$$

where L_u and L_v are the components of $\vec{\mathbf{L}}$ along the two directions Ou and Ov of the xOz plane which form angles of 45° with Ox and Oz; ω_0 is a real constant.

a. Write the matrix which represents H in the $\{|+1\rangle, |0\rangle, |-1\rangle\}$ basis. What are the stationary states of the system and what are their energies? (These states are to be written $|E_1\rangle, |E_2\rangle, |E_3\rangle$, in order of decreasing energies.)

We can write

$$L_u = \frac{1}{\sqrt{2}}(L_x + L_z)$$

and

$$L_v = \frac{1}{\sqrt{2}}(L_x - L_z)$$

so that

$$H = \frac{\omega_0}{\hbar}(L_x L_z + L_z L_x)$$

With

$$L_z = \hbar \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

and

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

we can write

$$H = \frac{\omega_0 \hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The eigenstates of this matrix are

$$|E_1\rangle = \frac{1}{2}\left(-|+1\rangle - \sqrt{2}\,|0\rangle + |-1\rangle\right), \quad E_1 = \hbar\omega_0$$

$$|E_2\rangle = \frac{1}{2}\left(|+1\rangle + |-1\rangle\right), \quad E_2 = 0$$

$$|E_3\rangle = \frac{1}{2}\left(-|+1\rangle + \sqrt{2}\,|0\rangle + |-1\rangle\right), \quad E_1 = -\hbar\omega_0$$

b. At time $t = 0$, the system is in the state:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|+1\rangle - |-1\rangle\right]$$

What is the state vector $|\psi(t)\rangle$ at time t? At t, L_z is measured; What are the probabilities of the various possible results?

We can write the vector as a superposition of energy eigenstates:

$$|\psi(0)\rangle = -\frac{1}{\sqrt{2}}(|E_1\rangle + |E_3\rangle)$$

so

$$|\psi(t)\rangle = \frac{1}{2\sqrt{2}}\left[\left(e^{-\imath\omega t} + e^{\imath\omega t}\right)(|+1\rangle - |-1\rangle) + \left(e^{-\imath\omega t} - e^{\imath\omega t}\right)\sqrt{2}\,|0\rangle\right]$$

since

$$|E_1(t)\rangle = e^{-\imath\omega\hbar t/\hbar}$$

and

$$|E_3(t)\rangle = e^{\imath\omega\hbar t/\hbar}$$

If L_z is measured at time t, the probability to measure $\pm\hbar$ or 0 is given by

$$\Pr(L_z = +\hbar) = |\langle +1|\psi(t)\rangle|^2 = \left|\frac{1}{2\sqrt{2}}(e^{-\imath\omega t} + e^{\imath\omega t})\right|^2 = \frac{1}{2}\cos^2(\omega t)$$

Similarly,

$$\Pr(L_z = 0) = |\langle 0|\psi(t)\rangle|^2 = \sin^2(\omega t)$$

and

$$\Pr(L_z = -\hbar) = |\langle -1|\psi(t)\rangle|^2 = \frac{1}{2}\cos^2(\omega t)$$

c. Calculate the mean values $\langle L_x\rangle(t)$, $\langle L_y\rangle(t)$, and $\langle L_z\rangle(t)$ at t. What is the motion performed by the vector $\left\langle \vec{\mathbf{L}} \right\rangle$?

Using

$$L_y = \frac{\hbar}{\sqrt{2}\imath}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

I used Mathematica to quickly calculate

$$\langle L_x\rangle_t = \frac{1}{4}e^{-2\imath\omega t}(e^{2\imath\omega t} - 1)^2\hbar$$

$$\langle L_y\rangle_t = \frac{1}{4}\imath e^{-2\imath\omega t}(e^{4\imath\omega t} - 1)\hbar$$

$$\langle L_z\rangle_t = \frac{1}{4}e^{-2\imath\omega t}(1 + e^{4\imath\omega t})\hbar$$

I'm not sure what the second part of the question is asking.

d. At t, a measurement of L_z^2 is performed.

(i) Do times exist when only one result is possible?

As seen by the probabilities above, there are certainly times when only $\hbar^2$ is possible, and that happens when $\sin^2(\omega t) = 0$ or $t = n\pi$.

(ii) Assume that this measurement has yielded the result $\hbar^2$. What is the state of the system immediately after the measurement? Indicate, without calculation, its subsequent evolution.

I'm not quite sure, but I'd imagine that once it's measured, the system immediately after measurement is now in a superposition of $|+1\rangle$ and $|-1\rangle$ but I'm not sure how it evolves after that.

Cohen-Tannoudji 7.2: 3D Harmonic Oscillator in Magnetic Field

Consider a particle of mass μ, whose Hamiltonian is:

$$H_0 = \frac{\vec{\mathbf{P}}^2}{2\mu} + \frac{1}{2}\mu\omega_0^2\vec{\mathbf{R}}^2$$

(an isotropic three-dimensional harmonic oscillator), where ω_0 is a given positive constant.

a. Find the energy levels of the particle and their degrees of degeneracy. Is it possible to construct a basis of eigenstates common to H_0, $\vec{\mathbf{L}}^2$, L_z?

For an isotropic 3D harmonic oscillator, the energy eigenstates can be described as a tensor product of three harmonic oscillators:

$$\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z$$

where the energy levels of each particular Hilbert space are given by

$$E_i = \left(n_i + \frac{1}{2}\right)\hbar\omega_0$$

Therefore, the total energy of a given eigenstate will be

$$E = \sum_i E_i = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega_0$$

The ground state $|000\rangle$ is not degenerate. However, each excited state is increasingly degenerate. The first excited state has threefold degeneracy: $E_{100} = E_{010} = E_{001}$, the second excited state has sixfold degeneracy: $E_{200} = E_{020} = E_{002} = E_{110} = E_{011} = E_{101}$, and the third excited state has tenfold degeneracy.
Because the potential is spherically symmetric, it is possible to construct a basis of eigenstates common to H_0, $\vec{\mathbf{L}}^2$, and L_z because the angular momentum operators commute with the Hamiltonian.

b. Now assume that the particle, which has a charge q, is placed in a uniform magnetic field $\vec{\mathbf{B}}$ parallel to Oz. We set $\omega_L = -\frac{qB}{2\mu}$. The Hamiltonian H of the particle is then, if we chose the gauge $\vec{\mathbf{A}} = -\frac{1}{2}\vec{\mathbf{r}} \times \vec{\mathbf{B}}$:

$$H = H_0 + H_1(\omega_L)$$

where H_1 is the sum of an operator which is linearly dependent on ω_L (the paramagnetic term) and an operator which is quadratically dependent on ω_L (the diamagnetic term). Show that the new stationary states of the system and their degrees of degeneracy can be determined exactly.

From class, we showed that the contribution using the Coulomb gauge is

$$H_1 = \omega_L L_z + \frac{1}{2}\mu\omega_L^2\left(X^2 + Y^2\right)$$

Therefore, the total Hamiltonian can be written as

$$H = \frac{P_x^2 + P_y^2}{2\mu} + \omega_L L_z + \frac{1}{2}\mu(\omega_0^2 + \omega_L^2)\left(X^2 + Y^2\right) + \left(\frac{P_z^2}{2\mu} + \frac{1}{2}\omega_0^2 Z^2\right)$$

Note that the Hilbert space $\mathcal{H}_z$ is essentially unaffected by the introduction of a magnetic field in the $\hat{\mathbf{z}}$-direction. This is a reflection of how a particle traveling parallel to the magnetic field will not experience a force from it. From here, we can use the left and right circularized number operators to rewrite the Hamiltonian as

$$H = (N_d + N_g + 1)\hbar\sqrt{\omega_0^2 + \omega_L^2} + (N_d - N_g)\hbar\omega_L + \left(N_z + \frac{1}{2}\right)\hbar\omega_0$$

If we use the following basis of energy eigenstates:

$$\left|\psi_{n_d n_g n_z}\right\rangle = \frac{1}{\sqrt{n_d! n_g! n_z!}}(a_d^\dagger)^{n_d}(a_g^\dagger)^{n_g}(a_z^\dagger)^{n_z}\,|000\rangle$$

we can find the energy levels of any state to be

$$(n_d + n_g + 1)\hbar\sqrt{\omega_0^2 + \omega_L^2} + (n_d - n_g)\hbar\omega_L + \left(n_z + \frac{1}{2}\right)\hbar\omega_0$$

c. Show that if ω_L is much smaller than ω_0, the effect of the diamagnetic term is negligible compared to that of the paramagnetic term.

If $\omega_L << \omega_0$, then the term $\omega_0^2 + \omega_L^2 \sim \omega_0^2$, so the Hamiltonian will be approximately

$$H \approx H_0 + \omega_L L_z$$

d. We now consider the first excited state of the oscillator, that is, the states whose energies approach $\frac{5\hbar\omega_0}{2}$ when $\omega_L \to 0$. To first order in $\frac{\omega_L}{\omega_0}$, what are the energy levels in the presence of the field $\vec{\mathbf{B}}$ and their degrees of degeneracy (the Zeeman effect for a three-dimensional harmonic oscillator?) Repeat for the second excited state.

Expanding the Hamiltonian to first-order, we find

$$H = \frac{\hbar\omega_0}{2}\left[(N_d + N_g + 1) + \frac{\omega_L}{\omega_0}(N_d - N_g) + \left(N_z + \frac{1}{2}\right)\right]$$

We can rewrite this as

$$H = \frac{\hbar\omega_0}{2}\left[N_d\left(1 + \frac{\omega_L}{\omega_0}\right) + N_g\left(1 - \frac{\omega_L}{\omega_0}\right) + N_z + \frac{3}{2}\right]$$

Assuming $\omega_L < \omega_0$, the first excited states will be when $n_d = 1$, $n_g = 1$, or $n_z = 1$. These states will have energy

$$E_{100} = \frac{5}{4}\hbar\omega_0 + \frac{1}{2}\hbar\omega_L$$
$$E_{010} = \frac{5}{4}\hbar\omega_0 - \frac{1}{2}\hbar\omega_L$$
$$E_{001} = \frac{5}{4}\hbar\omega_0$$

using the shorthand $E_{n_d n_g n_z}$
The second excited states will have the following degeneracy:

$$E_{200} = \frac{7}{4}\hbar\omega_0 + \hbar\omega_L$$
$$E_{020} = \frac{7}{4}\hbar\omega_0 - \hbar\omega_L$$
$$E_{002} = \frac{7}{4}\hbar\omega_0$$
$$E_{110} = \frac{7}{4}\hbar\omega_0$$
$$E_{101} = \frac{7}{4}\hbar\omega_0 + \frac{1}{2}\hbar\omega_L$$
$$E_{011} = \frac{7}{4}\hbar\omega_0 - \frac{1}{2}\hbar\omega_L$$