Lecture 8: Review

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The general solutions, again, are:

$$\nu \neq 0 \quad [a_\nu \rho^\nu + b_\nu \rho^{-\nu}] \sin(\nu\phi\alpha_\nu) \tag{1}$$

$$\nu = 0 \quad [a_0 + b_0 \ln \rho][A_0 + B_0 \rho] \tag{2}$$

In the case where the potential on both planes is V_0, $\alpha_\nu = 0$, and from the periodicity of the $\nu \neq 0$ condition, we can say $\nu = \frac{m\pi}{\beta}$. This discretizes ν:

$$\Phi = V_0 + \sum_{m=1}^{\infty} [a_m \rho^{\frac{m\pi}{\beta}} + b_m \rho^{-\frac{m\pi}{\beta}}] \sin\left(\frac{m\pi}{\beta}\phi\right) \tag{3}$$

There is another unspecified parameter which concerns what happens really far away and really up close. Let's assume we only want a solution which is finite at the vertex. Only the a_m terms will remain finite here:

$$\Phi = V_0 + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta}\phi\right) \tag{4}$$

What does the vector field look like?

$$\vec{E} = -\nabla\Phi = -\partial_\rho \Phi \hat{\rho} - \frac{1}{\rho}\partial_\rho \Phi \hat{\phi} = -a_m \rho^{\frac{m\pi}{\beta}-1} \sin\left(\frac{m\pi}{\beta}\phi\right)\hat{\rho} - \sum a_m \frac{m\pi}{\beta} \rho^{\frac{m\pi}{\beta}-1} \cos\left(\frac{m\pi}{\beta}\phi\right)\hat{\phi} \tag{5}$$

Suppose $\beta > \pi$. This implies $E \propto \rho^{\frac{\pi}{\beta}-1}$ as $\rho \to 0^+$, so the field diverges in the corner if the corner is a sharp edge.

0.1 Spherical Coordinates

$$\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2 \sin^2\theta}\partial_\phi^2 \tag{6}$$

Let us look at this from another perspective, an angular moment operator:

$$\vec{\mathbb{L}} = \vec{x} X(-\imath\vec{\nabla}) \tag{7}$$

$$\vec{x} \cdot \vec{\mathbb{L}} = 0 \tag{8}$$

$$\mathbb{L}_l = (-\imath)\varepsilon_{lmm} x_m \partial_n \tag{9}$$

$$-\mathbb{L}^2 = r^2\nabla^2 - \partial_r r^2 \partial_r \tag{10}$$

Now we can see that

$$\nabla^2 = \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{\mathbb{L}^2}{r^2} \tag{11}$$

If we are dealing with completely spherical boundaries, we need the full range of ϕ and θ.

If we say $\hbar = 1$, this is the same as the angular momentum operator from quantum:

$$[\mathbb{L}^2, f(r)] = 0 \tag{12}$$

$$[\mathbb{L}^2, \mathbb{L}_z] = 0 \tag{13}$$

$$[\mathbb{L}_z, f(r)] = 0 \tag{14}$$

$$\mathbb{L}^2|lm\rangle = l(l+1)|lm\rangle \tag{15}$$

$$\mathbb{L}_z|lm\rangle = m|lm\rangle \tag{16}$$

where

$$\langle\theta, \phi|lm\rangle = Y_{lm}(\theta, \phi) \tag{17}$$