Lecture 32: Muon-Electron Interactions

Friday, November 20, 2020

For the anti-muon, the right-handed helicity is

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}$$

and the left-handed spinor is

$$v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

where $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

This allows us to calculate j_e for each of the four initial-state helicity combinations (which we average) and j_μ for each of the four final-state helicity combinations (which we sum).

0.1 Muon and Electron Currents

The matrix element for a particular helicity combination of $e^- e^+ \to \mu^- \mu^+$ can be written $M_{fi} = -\frac{e^2}{s}(j_e \cdot j_\mu)$ where $j_e^\mu = \bar{v}(p_2)\gamma^\mu u(p_1)$ and $j_\mu^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$ are calculated with the combinations of right and left-handed helicity spinors found above.

In general, combinations of $\bar{\psi}\gamma^\mu\varphi$ can be evaluated explicitly using the Dirac-Pauli representation as

$$j^0 = \bar{\psi}\gamma^0\varphi = \psi^\dagger\gamma^0\gamma^0\varphi = \psi^\dagger\varphi$$

$$j^1 = \bar{\psi}\gamma^1 = \psi^\dagger\gamma^0\gamma^1\varphi = \psi_0^*\varphi_3 + \psi_1^*\varphi_2 + \psi_2^*\varphi_1 + \psi_3^*\varphi_0$$

and so on.

To calculate the matrix elements, we need to take into account all 16 helicity combinations between pairs of right or left-handed particles going to pairs of right/left-handed particles. We must find j_e and j_μ for each of the four possible combinations. For example, if the final state has $\mu_\uparrow^- \mu_\downarrow^+$,

$$\begin{aligned} j_\mu^0 &= \bar{u}_\uparrow(p_2)\gamma^0 v_\downarrow(p_4) \\ &= E\begin{pmatrix} c & s & c & s \end{pmatrix}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix} \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} j_\mu^1 &= \bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) \\ &= E(-c^2 + s^2 - c^2 + s^2) \\ &= 2E(s^2 - c^2) \\ &= -2E\cos(\theta) \end{aligned}$$

Doing all of these, we find that

$$j_{\mu,RL} = 2E\begin{pmatrix} 0 & -\cos(\theta) & \imath & \sin(\theta) \end{pmatrix}$$

You can also show that

$$j_{\mu,RR} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} = j_{\mu,LL}$$

These matrix elements only vanish in the relativistic limit, so they aren't necessarily zero at lower energies.

$$j_{\mu,LR} = 2E\begin{pmatrix} 0 & -\cos(\theta) & -\imath & \sin(\theta) \end{pmatrix}$$

We could similarly calculate j_e by brute force:

$$[\bar{u}\gamma^\mu v]^\dagger = v^\dagger(\gamma^\mu)^\dagger\gamma^0 u$$

If $\mu = 0$, $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$. If $\mu \neq 0$, we still get $\gamma^0\gamma^\mu$, so

$$[\bar{u}\gamma^\mu v]^\dagger = v^\dagger\gamma^0\gamma^\mu u = \bar{v}\gamma^\mu u$$

We can then find the electron current, $j_e^\nu = \bar{v}_e\gamma^\nu u_e$ by taking the Hermitian conjugate of j_μ^ν and setting $\theta = 0$:

$$j_{e,RL} = 2E\begin{pmatrix} 0 & -1 & -\imath & 0 \end{pmatrix}$$
$$j_{e,LR} = 2E\begin{pmatrix} 0 & -1 & \imath & 0 \end{pmatrix}$$

Now we have all the currents, so we can calculate the matrix element!

0.2 Electron-Muon Production Cross-Section

Of the 16 possible helicity combinations, only 4 have non-zero currents: $RL \to RL$, $RL \to LR$, $LR \to RL$, and $LR \to LR$. For the first,

$$M_{RL\to RL} = -\frac{e^2}{s(=(2E)^2)} j_e^\mu g_{\mu\nu} j_\mu^\nu = -e^2(-\cos(\theta) - 1) = e^2(1 + \cos(\theta))$$

$s = (2E)^2$ in the center of mass frame or for symmetric colliders. The same can be said about $M_{LR\to LR}$ by parity.

$$M_{RL\to LR} = M_{LR\to RL} = e^2(1 - \cos(\theta))$$

Since helicity states are orthogonal,

$$\begin{aligned} |M_{RL\to RL} + M_{LR\to LR} + M_{RL\to LR} + M_{LR\to RL}|^2 &= |M_{RL\to RL}|^2 + |M_{LR\to LR}|^2 + |M_{RL\to LR}|^2 + |M_{LR\to RL}|^2 \\ &= e^4\left[2(1+\cos(\theta))^2 + 2(1-\cos(\theta))^2\right] \\ &= 2e^4\left[2 + 2\cos^2(\theta)\right] \end{aligned}$$

We have to average over the four initial combinations, even though two of them are zero:

$$\left\langle |M_{fi}|^2 \right\rangle = e^4(1 + \cos^2(\theta))$$

where $e^2 = 4\pi\alpha$, so

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2 s}\cancelto{1}{\frac{p_f^*}{p_i^*}}\left\langle |M_{fi}|^2 \right\rangle \\ &= \frac{\alpha^2}{4s}\left(1 + \cos^2(thetai)\right) \end{aligned}$$

since $\frac{p_f^*}{p_i^*} \to 1$ in the relativistic limit.