Lecture 23: The Momentum Operator

Fri Oct 11 2019

0.1 The Momentum Operator

From last lecture, we said

$$\vec{P} = -\imath\hbar\nabla \tag{1}$$

We believe this is the momentum operator because

$$[\vec{R}_j, \vec{P}_k] = \imath\hbar I\delta_{jk} \tag{2}$$

Recall Ehrenfest's theorem from the previous lecture:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle_\varphi = \frac{1}{\imath\hbar}\langle\varphi|\,[A,H]\,|\varphi\rangle + \langle\varphi|\,\frac{\partial A}{\partial t}\,|\varphi\rangle \tag{3}$$

If

$$H = \frac{P^2}{2m} \tag{4}$$

Recall that we take the unitary operator for position to be

$$U(\vec{a}) = e^{-\imath\vec{a}\cdot\vec{P}/\hbar} \tag{5}$$

we can show that, for $A = X$, $U(\vec{a}) = e^{-\imath\vec{a}\cdot\vec{P}/\hbar}$

$$H' = UHU^\dagger = UU^\dagger H = H \tag{6}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X\rangle = \frac{1}{\imath\hbar}\langle[X,H]\rangle \tag{7}$$

Note that

$$[X,P^2] = XP^2 - P^2X + (PXP - PXP) = P[X,P] + [X,P]P = 2\imath\hbar P \tag{8}$$

therefore

$$[X,H] = \frac{2\imath\bar{P}}{2m} = \imath\hbar P_x/m \tag{9}$$

so

$$[\vec{R},H] = \imath\hbar\vec{P}/m \tag{10}$$

What we find from Ehrenfest's theorem is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\vec{R}\right\rangle = \vec{P}/m \tag{11}$$

which is the velocity. Therefore, it makes sense that this P operator is momentum because, for a massive Hamiltonian, it is mass times velocity. There's a third reason to call this operator momentum. Note that it commutes with the Hamiltonian and lacks any explicit time dependence. Therefore if we look at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\vec{P}\right\rangle = 0 \tag{12}$$

we discover this conservation law, particularly the conserved property whose conservation is associated with the invariance of the Hamiltonian under translation.

0.1.1 Position Basis

Recall that

$$e^{-\imath aP/\hbar} |x\rangle = |x+a\rangle \tag{13}$$

What does the momentum operator do to this basis?

$$e^{-\imath\delta P/\hbar} |x\rangle = (I - \imath\delta P/\hbar) |x\rangle = |x\rangle - \frac{\imath\delta}{\hbar} P |x\rangle = |x+\delta\rangle \tag{14}$$

where the last line comes from us knowing what this infinitesimal position operator must do. Therefore

$$P |x'\rangle = \lim \delta \to 0 \frac{\imath\hbar}{\delta} (|x'+\delta\rangle + |x'\rangle) \tag{15}$$

What does this look like in position space?

$$\begin{aligned}
\langle x| \left(P |x'\rangle\right. &= \lim_{\delta\to 0} \frac{\imath\hbar}{\delta} (|x'+\delta\rangle + |x'\rangle)\left.\right) && (16)\\
&= \lim_{\delta\to 0} \frac{\imath\hbar}{\delta} (\langle x| |x'+\delta\rangle + \langle x| |x'\rangle) && (17)\\
&= \imath\hbar\delta'(x-x') && (18)
\end{aligned}$$

What does this operator do to an arbitrary function on x? Recall that

$$\langle x| (X |\varphi\rangle) = x\varphi(x) \tag{19}$$

Now with momentum:

$$\langle x| (P |\varphi\rangle) = \langle x| PI |\varphi\rangle \tag{20}$$

we insert an identity:

$$I = \int \mathrm{d}x' \, |x'\rangle\langle x'| \tag{21}$$

so

$$\begin{aligned}
\langle x| P |\varphi\rangle &= \int \mathrm{d}x' \, \langle x| P |x'\rangle \langle x'| |\varphi\rangle && (22)\\
&= \int \mathrm{d}dx' \, \imath\hbar\delta'(x-x')\varphi(x') && (23)\\
&= \imath\hbar \left(\cancelto{0}{\delta(x-x')\varphi(x')\Big|_{x'=-\infty}^{\infty}} - \int \mathrm{d}x' \, \delta(x-x')\varphi'(x') \right) && (24)\\
&= -\imath\hbar\varphi'(x) && (25)
\end{aligned}$$

The cancellation occurs because φ is an element of the Hilbert space and is thus square integrable, so it vanishes at infinity. Therefore the evaluation of the $\varphi(x)$ at infinity will vanish.

0.1.2 Eigenstates of Momentum

$$P\,|p\rangle = p\,|p\rangle \tag{26}$$

what is $|p\rangle$?

$$\begin{aligned}
\langle x|\,|p\rangle &= \chi_p(x) && (27)\\
\langle x|\,P\,|p\rangle &= p\chi_p(x) && (28)\\
&= \langle p|\,P\,|x\rangle^* && (29)\\
&= \left[\int \mathrm{d}x'\,\langle p|\,|x'\rangle\,\langle x'|\,P\,|x\rangle\right]^* && (30)\\
&= -\imath\hbar\chi_p'(x) = p\chi_p(x) && (31)
\end{aligned}$$

This last line is a differential equation, which we can solve:

$$\chi_p(x) = \frac{1}{2\pi\hbar}e^{\imath px/\hbar} \tag{32}$$

In other words, the eigenstates of the momentum operators are plane waves. We can also use this as a basis:

$$I = \int \mathrm{d}p\,|p\rangle\langle p| \tag{33}$$

$$\int \mathrm{d}x\,\chi_p^*(x)\chi_{p'}(x) = \delta(p-p') \tag{34}$$