LECTURE 32: LIGHT

Wednesday, October 30, 2019

0.1 Light and Propagating Fields

Currently, Maxwell's equations look like this:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\vec{D} &= \rho \\
\boldsymbol{\nabla}\cdot\vec{B} &= 0 \\
\boldsymbol{\nabla}\times\vec{E} &= -\partial_t\vec{B} \\
(\boldsymbol{\nabla}\times\vec{H} &= \vec{J})^*
\end{aligned}$$

This last equation is incomplete!

By current conservation (taking the divergence of the last equation),

$$\partial_t\rho + \boldsymbol{\nabla}\cdot\vec{J} = 0$$

However, from the first equation, $partial_t\rho = \boldsymbol{\nabla}\cdot\partial_t\vec{D}$. Therefore, we must have

$$\boldsymbol{\nabla}\times\vec{H} = \vec{J} + \partial_t\vec{D}$$

In free space,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\vec{E} &= \rho/\epsilon_0 \\
\boldsymbol{\nabla}\cdot\vec{B} &= 0 \\
\boldsymbol{\nabla}\times\vec{E} &= -\partial_t\vec{B} \\
\boldsymbol{\nabla}\times\vec{B} &= \mu_0\vec{J} + \mu_0\epsilon_0\partial_t\vec{E}
\end{aligned}$$

Suppose there is no source in a region Ω. Now

$$\boldsymbol{\nabla}\times\vec{E} = -\partial_t\vec{B}$$

and

$$\boldsymbol{\nabla}\times\vec{B} = \mu_0\epsilon_0\partial_t\vec{E}$$

and $\boldsymbol{\nabla}\cdot\vec{E} = \boldsymbol{\nabla}\cdot\vec{B} = 0$. Seemingly by coincidence, $\mu_0\epsilon_0 = \frac{1}{c^2}$!

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\vec{E} = -\partial_t(\boldsymbol{\nabla}\times\vec{B}) = -\partial_t c^{-2}\partial_t\vec{E} = \boldsymbol{\nabla}\cancelto{0}{\boldsymbol{\nabla}\cdot\vec{E}} - \nabla^2\vec{E}$$

so

$$\nabla^2\vec{E} - \frac{1}{c^2}\partial_t^2\vec{E} = 0$$

and

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0$$

which are both wave equations, which have solutions $\varphi = f(x - ct) + g(x + ct)$. These are called plane waves because the strength of the field in a given plane is constant.

Let's look for solutions like

$$\vec{E} = \mathrm{Re}\left\{\vec{E}_0 e^{\imath\vec{k}\cdot\vec{r} - \imath\omega t}\right\}$$

Plugging this into our formula, we find

$$\nabla^2 \vec{E} = -k^2 \vec{E}$$

and

$$\partial_t^2 \vec{E} = (-\imath\omega)^2 \vec{E}$$

so as long as the solution is nonzero,

$$k = \frac{\omega}{c}$$

The curl acting on plane waves is just like $\imath\vec{k}\times$:

$$\imath\vec{k} \times \vec{E} = -\partial_t \vec{B}$$

so

$$\vec{B}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega}$$

so the electric and magnetic fields are always perpendicular.

This is the free wave solution, and adding a source will obviously complicate things. For sources, we have

$$\vec{E} = -\boldsymbol{\nabla}\Phi - \partial_t \vec{A}$$

and

$$\boldsymbol{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\partial_t \vec{E}$$

so

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \frac{1}{c^2}\left(-\boldsymbol{\nabla}\cdot\partial_t\Phi - \partial_t^2 \vec{A}\right)$$

If we write the left-hand side as

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{A}) - \nabla^2 \vec{A}$$

we get

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \boldsymbol{\nabla}\cdot\frac{1}{c^2}\partial_t\Phi - \frac{1}{c^2}\partial_t^2 \vec{A}$$

We want to keep the parts that look like a wave equation on one side:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{A} + \frac{1}{c^2}\partial_t\Phi) - \mu_0 \vec{J} = \nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A}$$

Recall we have some freedom in defining our potentials (Gauge freedom):

$$\boldsymbol{\nabla} \times \vec{A} + \boldsymbol{\nabla}\chi = \boldsymbol{\nabla} \times \vec{A}$$

and

$$-\boldsymbol{\nabla}\Phi - \partial_t\chi - \partial_t(\vec{A} + \boldsymbol{\nabla}\chi) = -\boldsymbol{\nabla}\Phi - \partial_t\vec{A}$$

so

$$\vec{A} \mapsto \vec{A} + \boldsymbol{\nabla}\chi$$

and

$$\Phi \mapsto \Phi + \partial_t\chi$$

We would like our final equation to be

$$\nabla^2\vec{A} - \frac{1}{c^2}\partial_t^2\vec{A} = -\mu_0\vec{J}$$

so let the gauge be

$$\boldsymbol{\nabla}\cdot\vec{A} + \frac{1}{c^2}\partial_t\Phi = 0$$

Doing the same process with the divergence of the electric field, we get that

$$-\nabla^2\Phi - \partial_t\boldsymbol{\nabla}\cdot\vec{A} = \rho/\epsilon_0$$

Using our gauge,

$$-\nabla^2\Phi - \frac{1}{c^2}\partial_t^2\Phi = \rho/\epsilon_0$$

What are the solutions for these equations? If we have

$$\nabla^2\Phi - \frac{1}{c^2}\partial_t^2\Phi = f(\vec{x}, t)$$

then we are looking for a Green's function with

$$\nabla^2 G(\vec{x}, t; \vec{x}', t') - \frac{1}{c^2}\partial_t^2 G = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t')$$

so

$$G = \int_{-\infty}^{\infty} G(\vec{x} - \vec{x}'; \omega)e^{\imath\omega(t-t')}\frac{1}{2\pi}\,\mathrm{d}\omega$$

so

$$\nabla^2 G(x - x'; \omega + \frac{\omega^2}{c^2}G = -4\pi\delta(x - x')$$

Therefore

$$(\nabla^2 + k^2)G(x - x'; \omega) = -4\pi\delta(x - x')$$

This is the operator for the Helmholtz equation.

This equation can be solved by

$$G(x - x'l\omega) = \frac{e^{\pm\imath k|x-x'|}}{|x - x'|}$$

Note that I stopped using vector arrows on x but they are vectors in general. Therefore

$$G(x - x', t - t')^{\pm} = \int e^{\pm\imath k|x-x'| - \imath\omega(t-t')}\frac{1}{|x - x'|}\frac{1}{2\pi}\,\mathrm{d}\omega = \frac{1}{2\pi}\frac{\delta(t - t' \pm \frac{|x-x'|}{c})}{|x - x'|}$$

These solutions must vanish at infinity, so they actually don't describe plane waves. The choice of $\pm$ concerns causality, and the $-$ case is the one where the past effects the future.