Lecture 50: Relativity

Friday, December 06, 2019

0.1 Relativity

The theory of relativity is formulated on the fact/observation that light moves at a finite speed an that speed is the same for all observers. We define a four-position as the regular position with an additional component $x^0 = ct$. By this definition,

$$(x^0)sr - \vec{\mathbf{x}} \cdot \vec{\mathbf{x}} = 0 = \mathrm{d}s^2$$

for light.

In general, $\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x'^\nu} \mathrm{d}x'^\nu$. If we transform $\mathrm{d}s^2 \mapsto a(\vec{\mathbf{v}}) \mathrm{d}s^2$ under some shift to another inertial frame, we find that $\mathrm{d}s^2 \mapsto a(|\vec{\mathbf{v}}|) \mathrm{d}s^2$ and $a(|\vec{\mathbf{v}}_1|)a(|\vec{\mathbf{v}}_2|) \mathrm{d}s^2 = a(|\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2|) \mathrm{d}s^2$ so $a(|\vec{\mathbf{v}}| \to 0) = 1$, which implies $\mathrm{d}s^2$ is an invariant under Lorentz transformations.

We can write

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \, \mathrm{d}x^\nu \, \eta_{\mu\nu}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

is the Minkowski metric. This defines the Lorentz group $SO(3)$ since $\Lambda^\mu_\nu \Lambda^\sigma_\lambda \eta_{\mu\sigma} = \eta_{\nu\lambda}$.

We can also show that the most general linear transformation which preserves $\mathrm{d}s^2$ is

$$\begin{bmatrix} x'^0 \\ x'^1 \end{bmatrix} = \begin{bmatrix} \cosh(x) & -\sinh(x) \\ -\sinh(x) & \cosh(x) \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}$$

so that

$$-\frac{v}{c} = \frac{\mathrm{d}x'^1}{\mathrm{d}x'^0} = -\tanh(x)$$

which gives us the transformations

$$x'^0 = \gamma(x^0 - vx^1)$$

$$x'^1 = \gamma(x^1 - vx^0)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

How do we connect this to electrodynamics? Let's introduce a 4-vector source

$$J^\mu = (c\rho, \vec{\mathbf{J}})$$

4-vectors are geometric objects which transform like $\mathrm{d}x^\mu$ under Lorentz transforms:

$$a'^\mu = \Lambda^\mu_\lambda a^\lambda$$

We can write the 4-velocity as $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ where $c\,\mathrm{d}\tau = \mathrm{d}s$ is the proper time. In any other frame, $\mathrm{d}\tau = \sqrt{1-\frac{v^2}{c^2}}\,\mathrm{d}t$. We can also define the 4-momentum $p^\mu = mu^\mu$. As it turns out, we can write moving charges as

$$\rho = \sum q_i \delta(\vec{\mathbf{x}} - \vec{\mathbf{v}}_i(t))$$

and

$$\vec{\mathbf{J}} = \sum q\vec{\mathbf{v}}_i \delta(\vec{\mathbf{x}} - \vec{\mathbf{v}}_i(t))$$

Recall the charge conservation law

$$\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0$$

or

$$\partial_\mu J^\mu = 0$$

We can also write a 4-potential

$$A^\mu = \left(\frac{\Phi}{c}, \vec{\mathbf{A}}\right)$$

which implies that the Lorentz gauge which we used is actually just

$$\partial_\mu A^\mu = 0$$

Recall that using this, we found the wave equations

$$\nabla^2 \vec{\mathbf{A}} - \frac{1}{c^2}\partial_t \vec{\mathbf{A}} = -\mu_0 \vec{\mathbf{J}}$$

and

$$\nabla^2 \Phi - \frac{1}{c^2}\partial_t \Phi = -\frac{\rho}{\epsilon_0}$$

This wave operator is really

$$\nabla^2 - \frac{1}{c^2}\partial_t^2 = \partial_\mu \partial^\mu = \Box$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

so

$$\Box A^\nu = -\mu_0 J^\nu$$

If we define $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and recall that to raise and lower indices, we use

$$A_\mu \equiv \eta_{\mu\nu} A^\nu$$

$F_{\mu\nu}$ is a 2-tensor ($F'^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma F^{\lambda\sigma}$). We can show that

$$F^{0i} = E^i$$

and

$$\epsilon_{kij} F^{ij} = B_k$$

Using this 2-tensor, we can show that Maxwell's equations are simply

$$\partial_\mu F^{\mu\lambda} = \mu_0 J^\lambda$$

We can define the dual of this tensor as

$$* F^{\mu\lambda} = \frac{1}{2} \epsilon^{\mu\lambda\sigma\alpha} F_{\sigma\alpha}$$

then

$$\partial_\mu * F^{\mu\lambda} = 0$$

which describes the fact that the magnetic field has no sources.

If we write

$$A^\nu = -\mu_0 (\Box^{-1}) J^\nu$$

we can show that the inverse of the d'Alambertian is

$$\Box^{-1} = \frac{\delta\left(t - t' - \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} = \Theta(x^0 - x'^0)\delta((x - x')^2)$$

Finally, the Lorentz force is defined as

$$m \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = q F^{\mu\nu} u_\nu$$