33-755 Homework 6

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Cohen-Tannoudji Exercise 2.9

Expectation Values in Energy Eigenstates

Let H be the Hamiltonian operator of a physical system. Denote by $|\varphi_n\rangle$ the eigenvectors of H, with eigenvalues E_n:

$$H|\varphi_n\rangle = E_n|\varphi_n\rangle$$

a. For an arbitrary operator A, prove the relation:

$$\langle\varphi_n|[A,H]|\varphi_n\rangle = 0.$$

Using the hermiticity of H and associativity of the inner product:

$$\begin{aligned}
\langle\varphi_n|[A,H]|\varphi_n\rangle &= \langle\varphi_n|AH - HA|\varphi_n\rangle \\
&= \langle\varphi_n|AH|\varphi_n\rangle - \langle\varphi_n|HA|\varphi_n\rangle \\
&= \langle\varphi_n|AE_n|\varphi_n\rangle - \langle\varphi_n|E_nA|\varphi_n\rangle \\
&= E_n\left(\langle\varphi_n|A|\varphi_n\rangle - \langle\varphi_n|A|\varphi_n\rangle\right) = 0
\end{aligned}$$

b. Consider a one-dimensional problem, where the physical system is a particle of mass m and of potential energy $V(X)$. In this case, H is written:

$$H = \frac{1}{2m}P^2 + V(X)$$

[α.] In terms of P, X, and $V(X)$, find the commutators: $[H,P]$, $[H,X]$, $[H,XP]$.

$$\begin{aligned}
[H,P] &= \frac{P^2}{2m}P + V(X)P - P\frac{P^2}{2m} - PV(X) \\
&= \frac{P^3}{2m} + V(X)P - \frac{P^3}{2m} - PV(X) \\
&= V(X)P - PV(X) = [V(X),P] = \imath\hbar\partial_X V(X) = \imath\hbar V'(X)
\end{aligned}$$

$$[H,X] = \frac{P^2}{2m}X + V(X)X - X\frac{P^2}{2m} - XV(X)$$

$$
\begin{aligned}
&= \frac{1}{2m}\left[P^2, X\right] + [V(X), X] \\
&= -\frac{1}{2m}\left[X, P^2\right] + 0 \\
&= -\frac{1}{2m} 2\imath\hbar P = -\frac{\imath\hbar P}{m}
\end{aligned}
$$

$$
\begin{aligned}
[H, XP] &= [H, X]P + X[H, P] \\
&= -\frac{\imath\hbar P^2}{m} + X[V(X), P] = -\frac{\imath\hbar P^2}{m} + \imath\hbar X V'(X)
\end{aligned}
$$

[β.] Show that the matrix element $\langle\varphi_n| P |\varphi_n\rangle$ (which we shall interpret in Chapter III as the mean value of the momentum in the state $|\varphi_n\rangle$) is zero.

From the previous problem (the commutator $[H, X] = -\frac{\imath\hbar P}{m}$):

$$
\begin{aligned}
P &= \frac{m}{\imath\hbar}[X, H] \implies \\
\langle\varphi_n| P |\varphi_n\rangle &= \frac{m}{\imath\hbar}\langle\varphi_n| [X, H] |\varphi_n\rangle = 0
\end{aligned}
$$

since, from the first problem, $\langle\varphi_n| [A, H] |\varphi_n\rangle = 0$.

[γ.] Establish a relation between $E_k = \langle\varphi_n| \frac{P^2}{2m} |\varphi_n\rangle$ (the mean value of the kinetic energy in the state $|\varphi_n\rangle$) and $\langle\varphi_n| X \frac{\mathrm{d}V}{\mathrm{d}X} |\varphi_n\rangle$. Since the mean value of the potential energy in the state $|\varphi_n\rangle$ is $\langle\varphi_n| V(X) |\varphi_n\rangle$, how is it related to the mean value of the kinetic energy when:

$$
V(X) = V_0 X^\lambda, \quad (\lambda = 2, 4, 6, \cdots; V_0 > 0)?
$$

First, $\partial_X V(X) = V'(X) = \frac{1}{\imath\hbar}[H, P]$ from problem α. Therefore,

$$
\begin{aligned}
X V'(X) &= \frac{1}{\imath\hbar} X[H, P] = \frac{1}{\imath\hbar}([H, XP] - [H, X]P) \\
\implies \langle\varphi_n|XV'(X)|\varphi_n\rangle &= \frac{1}{\imath\hbar}\cancelto{0}{\langle\varphi_n|[H, XP]|\varphi_n\rangle} + 2\langle\varphi_n|\frac{P^2}{2m}|\varphi_n\rangle \\
\langle\varphi_n|XV'(X)|\varphi_n\rangle &= 2E_k
\end{aligned}
$$

In the given case, this means that

$$
\begin{aligned}
E_k &= \frac{1}{2}\langle\varphi_n|XV'(X)|\varphi_n\rangle \\
&= \frac{\lambda}{2}\langle\varphi_n|V_0 X^\lambda|\varphi_n\rangle = \frac{\lambda}{2}\langle\varphi_n|V(X)|\varphi_n\rangle
\end{aligned}
$$