33-756 Homework 2

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1. Total Time Derivative

Show that if you add a total time derivative to the Lagrangian ($\frac{\mathrm{d}}{\mathrm{d}t}F(x)$), it has no effect on the equations of motion.

Take a modified Lagrangian $L \to L' = L + \dot{F}(x)$. We can derive the modified Euler-Lagrange equations of motion as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial \dot{q}} = \frac{\partial L'}{\partial q}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}} + \frac{\partial \dot{F}}{\partial \dot{q}}\right] = \frac{\partial L}{\partial q} + \frac{\partial \dot{F}}{\partial q}$$

Next, we can remove the unmodified equations of motion $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial \dot{F}}{\partial q}$$

Next, because $\frac{\mathrm{d}t}{\mathrm{d}t} = 1$,

$$\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\partial F}{\partial q}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial q}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{F}}{\partial \dot{q}} \to \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial q} = \frac{\partial \dot{F}}{\partial q}$$

This is true because the derivatives commute. Therefore, the total time derivative has no effect on the equations of motion because it satisfies the Euler-Lagrange equations.

2. Transformations under Rotations

Show that $\vec{\mathbf{P}}$ transforms as a vector under an infinitesimal rotation using the representation of $\vec{\mathbf{P}}$ and $\vec{\mathbf{L}}$ as differential operators.

Consider a rotation of the operator $\vec{\mathbf{P}}' = U^\dagger \vec{\mathbf{P}} U$ where $U = e^{\frac{\imath}{\hbar}\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta}$. If we expand to first order, we find

$$\begin{aligned}\vec{\mathbf{P}}' &= \left(I - \frac{\imath}{\hbar}\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta\right)\vec{\mathbf{P}}\left(I + \frac{\imath}{\hbar}\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta\right)\\ &= \vec{\mathbf{P}} - \left(\frac{\imath}{\hbar}\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta\right)\vec{\mathbf{P}} + \vec{\mathbf{P}}\left(\frac{\imath}{\hbar}\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta\right)\end{aligned}$$

so

$$\delta\vec{\mathbf{P}} = \vec{\mathbf{P}}\left(\frac{\imath}{\hbar}\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta\right) - \left(\frac{\imath}{\hbar}\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta\right)\vec{\mathbf{P}}$$

or

$$\delta P_i = [P_i, L_j]\frac{\imath}{\hbar}n_j\theta$$

We can write these operators in differential form as

$$P_i = \frac{\hbar}{\imath}\partial_i$$

and

$$L_j = X_a P_b \epsilon_{abj} = \frac{\hbar}{\imath}X_a\partial_b\epsilon_{abj}$$

The commutator is therefore

$$\begin{aligned}[P_i, L_j] &= -\hbar^2\epsilon_{abj}[\partial_i, X_a\partial_b]\\ &= -\hbar^2\epsilon_{abj}\left(\underbrace{[\partial_i, X_a]}_{\delta_{ia}}\partial_b + X_a\underbrace{[\partial_i, \partial_b]}_{0}\right)\\ &= -\hbar^2\epsilon_{abj}\partial_b\delta_{ai} = -\hbar^2\epsilon_{ibj}\partial_b = -\frac{\hbar}{\imath}\epsilon_{ibj}P_b\end{aligned}$$

Therefore

$$\delta P_i = \epsilon_{ibj}P_b n_j\theta$$

or

$$\delta\vec{\mathbf{P}} = (\vec{\mathbf{P}}\times\hat{\mathbf{n}})\theta = -(\hat{\mathbf{n}}\times\vec{\mathbf{P}})\theta$$

Therefore, the infinitesimal rotation leads to a differential which acts moves the end of a vector perpendicular to itself, which is how vectors transform under rotations.

3. 3D Isotropic Harmonic Oscillator

Consider the isotropic harmonic oscillator in three dimensions. Its Hamiltonian can be written as

$$H = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + \frac{3}{2}\right)$$

Naively, one would think that the only symmetry is rotational. Let's assume that's the case—then the degeneracy would be $g = (2N+1)$. But this underestimates the degeneracy considerably. Show that for $n_x + n_y + n_z = N$, the degeneracy is $g = \frac{1}{2}(N+1)(N+2)$. To do this as a counting problem, you have 3 buckets and N marbles. How many ways are there to distribute the marbles?

Let's assume we know how many marbles are being put into the first bucket and call this number $n_1 \in [0, N]$. The other two bucket's contents must sum to $N - n_1$. How many ways are there to divide up $N - n_1$ marbles into two bins? We could lay out all the marbles in a row and place a separator to choose which marbles would go into which bin. Since we can also have the case where zero marbles go into a particular bin, there are $N - n_1 + 1$ spaces in which the separator can be placed.
Finally, to find the total number of ways we can separate N marbles, we have to sum over all the possible values of n_1:

$$\sum_{n_1=0}^{N} N - n_1 + 1 = \sum_{n_1=0}^{N} N + 1 - \sum_{n_1=0}^{N} n_1$$

The first sum is just the value $N+1$ summed over n_1+1 times (since n_1 goes from 0 to N). The second sum is the sum of all the numbers from 0 to N, which is well-known thanks to Gauss: $\frac{1}{2}N(N+1)$. Therefore, the degeneracy is

$$(N+1)(N+1) - \frac{1}{2}N(N+1) = \frac{1}{2}(n+1)(n+2)$$

So we see that there must be another symmetry that explains why the degeneracy grows more quickly with N. For the 3D SHO we have the symmetric tensor

$$T_{ij} = \frac{1}{\lambda}(p_i p_j + \lambda^2 r_i r_j)$$

Determine what λ must be in order for this tensor to be conserved. Show that the number of conserved quantities is equal to the number of generators in the group U(3). To determine the number of generators, determine how many independent parameters exist for the most general unitary 3-by-3 matrix. One can in fact show that this system has a U(3) symmetry.

If T_{ij} is conserved, $[T, H] = 0$. To calculate the commutator, I will first rewrite the Hamiltonian in terms of p and r:

$$H = \frac{p_k^2}{2m} + \frac{m\omega^2 r_k^2}{2}$$

so (factoring an $\frac{2}{m}$ to make some of this more symmetric)

$$\begin{aligned}
0 = [T, H] &= \frac{2}{\lambda m}\left\{(p_i p_j + \lambda^2 r_i r_j)(p_k p_k + m^2\omega^2 r_k r_k) - (p_k p_k + m^2\omega^2 r_k r_k)(p_i p_j + \lambda^2 r_i r_j)\right\} \\
&= (p_i p_j p_k p_k) + m^2\omega^2(p_i p_j r_k r_k) + \lambda^2(r_i r_j p_k p_k) + \lambda^2 m^2\omega^2(r_i r_j r_k r_k) \\
&\quad - (p_k p_k p_i p_j) - m^2\omega^2(r_k r_k p_i p_j) - \lambda^2(p_k p_k r_i r_j) - \lambda^2 m^2\omega^2(r_k r_k r_i r_j)
\end{aligned}$$

Since the position and momentum operators are all self-commuting, the red and blue terms cancel. The remaining terms can be written as:

$$0 = m^2\omega^2[p_i p_j, r_k r_k] - \lambda^2[p_k p_k, r_i r_j]$$

The first commutator is

$$\begin{aligned}
[p_i p_j, r_k r_k] &= p_i[p_j, r_k r_k] + [p_i, r_k r_k]p_j \\
&= p_i([p_j, r_k]r_k + r_k[p_j, r_k]) + ([p_i, r_k]r_k + r_k[p_i, r_k])p_j \\
&= p_i(-2\imath\hbar r_k \delta_{jk}) + (-2\imath\hbar r_k \delta_{ik})p_j \\
&= -2\imath\hbar(p_i r_j + r_i p_j) \\
&= -2\imath\hbar\{p_i, r_j\}
\end{aligned}$$

The second commutator is

$$\begin{aligned}
[p_k p_k, r_i r_j] &= [p_k p_k, r_i] r_j + r_i [p_k p_k, r_j] \\
&= (p_k [p_k, r_i] + [p_k, r_i] p_k) r_j + r_i (p_k [p_k, r_j] + [p_k, r_j] p_k) \\
&= (-2\imath\hbar p_k \delta_{ik}) r_j + r_i (-2\imath\hbar p_k \delta_{jk}) \\
&= -2\imath\hbar (p_i r_j + r_i p_j) \\
&= -2\imath\hbar \{p_i, r_j\}
\end{aligned}$$

As we can see, these commutators are the same, so we can cancel them, leaving

$$\lambda^2 = m^2\omega^2 \quad \text{or} \quad \lambda = \pm\sqrt{m\omega}$$

All unitary matrices are diagonalizable. Diagonalizing an N-by-N matrix will result in a diagonal matrix with N independent eigenvalue coefficients and a transformation matrix of N orthogonal basis vectors of dimension N. However, you only need to know $N-1$ orthogonal vectors to be able to exactly determine the last one, so in total a general unitary matrix has $N + N(N-1) = N^2$ independent coefficients (generators). Therefore, a 3-by-3 unitary matrix has 9 generators. Since T_{ij} is symmetric, it contains 6 independent parameters. The other 3 come from the 3D rotational symmetry group SO(3), which has 3 generators.