Lecture 4: Conserved Charge of Rotational Invariance

Wednesday, January 22, 2020

Recall Noether's theorem from the previous lecture:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta L}{\delta \dot{\vec{\mathbf{x}}}}\delta\vec{\mathbf{x}}\right] = 0$$

If the action is rotationally invariant,

$$\vec{\mathbf{x}} \to R(\hat{\mathbf{n}},\theta)\vec{\mathbf{x}}$$

where

$$R(\hat{\mathbf{n}},\theta) = e^{\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta}$$

In the previous lecture, we found that

$$\imath(L^a)_{ij} = \epsilon^a_{ij} \equiv \epsilon_{aij}$$

If we expand the exponential to a few terms, we find

$$e^{\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta} \to 1 + \imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta + \mathcal{O}\big(\theta^2\big)$$

as $\theta \to 0$. We find $\delta\vec{\mathbf{x}}$ to be

$$\begin{aligned}\delta\vec{\mathbf{x}} &= \left[(\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}})_{ij}\theta\right]x_j\\ &= (\imath(L^a)_{ij}n^a x_j)\,\theta\\ &= (\epsilon_{aij}n_a x_j)\,\theta\end{aligned}$$

If our Lagrangian has the form

$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

we find that Noether's theorem gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{x}_i\delta x_i\right] = 0&\\ &= \frac{\mathrm{d}}{\mathrm{d}t}m\left[\dot{x}_i(n_a\theta\epsilon_{aij}x_j)\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{x}_i\epsilon_{aij}x_jn_a\theta\right]\end{aligned}$$

Because $\hat{\mathbf{n}}$ and θ are arbitrary and this equation must be true for all $\hat{\mathbf{n}}$ and θ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\underbrace{m\dot{x}_i}_{p_i}\epsilon_{aij}x_j\right] = 0$$

so

$$p_ix_j\epsilon_{aij} = \vec{\mathbf{x}}\times\vec{\mathbf{p}} = \vec{\mathbf{L}} \quad\longrightarrow\quad \text{invariant}$$

0.1 Conservation Laws in Quantum Mechanics

The fundamental time-evolution equation in QM is the Schrödinger equation:

$$\imath\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle$$

The Schrödinger picture is a formulation where we make the operators independent of time, but allow the wave functions to be time-dependent.

The Heisenberg picture is a formulation where all of the operators are time-dependent whereas the wave functions are time-independent.

There is a simple way to transform between the two using a time-evolution operator:

$$U(t', t) = e^{-\imath H(t'-t)/\hbar}$$

If we work in the Schrödinger picture, we know that $|\psi(t')\rangle = U(t', t) |\psi(t)\rangle$. If we consider the expectation value of some operator:

$$\langle\psi(t')|_S O_S |\psi(t)\rangle_S = \langle\psi(t')| U^\dagger(t', t) O U(t', t) |\psi(t)\rangle$$

We could equivalently define

$$O_H(t') = U^\dagger(t', t) O_S U(t', t)$$

such that

$$\langle\psi(t')|_S O_S |\psi(t)\rangle_S = \langle\psi|_H O_H(t') |\psi\rangle_H$$

where

$$|\psi\rangle_H \equiv U(t', t) |\psi(t)\rangle_S$$

We can use the Schrödinger equation on the Heisenberg picture operator:

$$\begin{aligned}
\imath\hbar\frac{\partial}{\partial t}O_H(t) = \\
\imath\hbar\frac{\partial}{\partial t}\langle\psi|_S O_S |\psi\rangle_S &= \left[\imath\hbar\frac{\partial}{\partial t}\langle\psi|_S\right] O |\psi\rangle_S + \langle\psi|_S O \left[\imath\hbar\frac{\partial}{\partial t}|\psi\rangle_S\right] \\
&= -\langle HO|\psi|HO\rangle + \langle OH|\psi|OH\rangle \\
&= \langle [O, H]|\psi|[O, H]\rangle \\
&= \imath\hbar \langle\psi|_H \frac{\mathrm{d}}{\mathrm{d}t}O_H(t) |\psi\rangle_H
\end{aligned}$$

so

$$\imath\hbar\frac{\mathrm{d}}{\mathrm{d}t}O_H(t) = [O, H]$$

What does this have to do with conserved quantities? If $[O, H] = 0$, O is time independent. In quantum mechanics, a symmetry is always expressible in terms of a unitary transformation

$$U = e^{\imath\vec{\mathbf{X}}\cdot\vec{\lambda}}$$

where $\vec{\mathbf{X}}^\dagger = \vec{\mathbf{X}}$ are the generators of the symmetry which obey a Lie algebra $[X_a, X_b] = \imath f_{abc}X_c$.

The difference between classical and quantum mechanics is that everything is an operator, and operators transform under symmetries:

$$O \to U^\dagger(\vec{\lambda}) O U(\vec{\lambda})$$

where $\vec{\lambda}$ is the set of parameters which determine the group element. Now consider some of the typical operators and how they transform. Under rotations, the position operator transforms as

$$\vec{\mathbf{x}} \to U(\hat{\mathbf{n}}, \theta)\vec{\mathbf{x}}U(\hat{\mathbf{n}}, \theta)$$

or

$$\vec{\mathbf{x}}' = e^{-\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta/\hbar}\vec{\mathbf{x}}e^{\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta/\hbar}$$

Consider an infinitesimal rotation ($\theta \to 0$):

$$\vec{\mathbf{x}}' = (1 - \imath\frac{\vec{\mathbf{L}}}{\hbar}\cdot\hat{\mathbf{n}}\theta)\vec{\mathbf{x}}(1 + \imath\frac{\vec{\mathbf{L}}}{\hbar}\cdot\hat{\mathbf{n}}\theta) + \mathcal{O}(\theta^2)$$

or

$$\vec{\mathbf{x}}' = \vec{\mathbf{x}} - (\imath \frac{\vec{\mathbf{L}}}{\hbar} \cdot \hat{\mathbf{n}}\theta)\vec{\mathbf{x}} + \vec{\mathbf{x}}(\imath \frac{\vec{\mathbf{L}}}{\hbar} \cdot \hat{\mathbf{n}}\theta)$$

so

$$\delta\vec{\mathbf{x}} = \vec{\mathbf{x}}(\imath \frac{\vec{\mathbf{L}}}{\hbar} \cdot \hat{\mathbf{n}}\theta) - (\imath \frac{\vec{\mathbf{L}}}{\hbar} \cdot \hat{\mathbf{n}}\theta)\vec{\mathbf{x}}$$

so

$$\delta x_a = [(x_a(\imath L_b)/\hbar) - (\imath L_b X_a)/\hbar]\, \hat{\mathbf{n}}_b\theta$$

We define the angular momentum operator as

$$\vec{\mathbf{L}} \equiv \vec{\mathbf{x}} \times \vec{\mathbf{p}} = \vec{\mathbf{x}} \times \left(\imath\hbar \frac{\partial}{\partial \vec{\mathbf{x}}}\right)$$

We can also write this in index notation:

$$L_b = -\imath\hbar x_i \partial_j \epsilon_{ijb}$$

Let's now apply this to our δx_a formula:

$$\delta x_a = (\imath \hat{\mathbf{n}}_b \theta)\left[x_a, \imath \frac{L_b}{\hbar}\right]$$

Now we just need to figure out what the commutator is.

$$\left[x_a, \imath \frac{L_b}{\hbar}\right] = [x_a, x_c p_d \epsilon_{cdb}]$$

ϵ_{cdb} is just a constant, we can take it out, and we are left with

$$[x_a, \imath x_c p_d] = \imath[x_a, x_c]p_d + x_c \imath[x_a, p_d]$$

since

$$[A, BC] = [A, B]C + B[A, C]$$

Position commutes with itself and $[x_a, p_d] = \imath\hbar\delta_{ad}$ so

$$[x_a, x_c p_d] = -x_c(\delta_{ad})$$

Finally

$$\begin{aligned}
\delta x_a &= \imath \hat{\mathbf{n}}_b \theta \epsilon_{cdb}(\imath\hbar x_c \delta_{ad}) \\
&= -\hat{\mathbf{n}}_b \theta x_c \epsilon_{cab} \\
&= -\hat{\mathbf{n}}_b \theta x_c \epsilon_{abc}
\end{aligned}$$

This is very similar to the classical case where

$$\delta x_a = \hat{\mathbf{n}}_b \theta x_c \epsilon_{abc}$$

Whereas in the quantum case we have

$$\delta x_a = -\hat{\mathbf{n}}_b \theta x_c \epsilon_{abc}$$

We have shown that the operator $\vec{\mathbf{x}}$ transforms just like a vector under rotation. If you did the same thing with $\vec{\mathbf{p}}$, you would find the exact same result (and a similar result with any vector operator).