33-761 Homework 10

Nathaniel D. Hoffman

November 13, 2019

1 Jackson 6.3; (a), (b), and (c)

The homogeneous diffusion equation (5.160) for the vector potential for quasi-static fields in unbounded conducting media has a solution to the initial value problem of the form

$$\vec{\mathbf{A}}(\vec{\mathbf{x}},t) = \int \mathrm{d}^3x'\, G(\vec{\mathbf{x}}-\vec{\mathbf{x}}',t)\vec{\mathbf{A}}(\vec{\mathbf{x}}',0)$$

where $\vec{\mathbf{A}}(\vec{\mathbf{x}}',0)$ describes the initial field configuration and G is an appropriate kernel.

Equation 5.160:

$$\nabla^2\vec{\mathbf{A}} = \mu\sigma\partial_t\vec{\mathbf{A}}$$

(a) Solve the initial value problem by use of a three-dimensional Fourier transform in space for $\vec{\mathbf{A}}(\vec{\mathbf{x}},t)$. With the usual assumptions on interchange of orders of integration, show that the Green function has the Fourier representation,

$$G(\vec{\mathbf{x}}-\vec{\mathbf{x}}',t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, e^{-k^2t/\mu\sigma}e^{\imath\vec{\mathbf{k}}\cdot(\vec{\mathbf{x}}-\vec{\mathbf{x}}')}$$

and it is assumed that $t > 0$.

Let's first do a three-dimensional Fourier transform on $\vec{\mathbf{A}}(\vec{\mathbf{x}},t)$:

$$\vec{\mathbf{A}}(\vec{\mathbf{x}},t) = \frac{1}{(2\pi)^3}\int \vec{\mathbf{A}}(\vec{\mathbf{k}},t)e^{-\imath\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\,\mathrm{d}^3k$$

Looking back at the diffusion equation, we now move the Laplacian inside this integral, which pulls out two factors of $\imath\vec{\mathbf{k}}$, so the equation becomes

$$(\imath\vec{\mathbf{k}})^2\vec{\mathbf{A}} = \mu\sigma\partial_t\vec{\mathbf{A}}$$

or

$$\partial_t\vec{\mathbf{A}} = -\frac{k^2}{\mu\sigma}\vec{\mathbf{A}}$$

so

$$\vec{\mathbf{A}}(\vec{\mathbf{k}},t) = \vec{\mathbf{A}}(\vec{\mathbf{k}},0)e^{-\frac{k^2}{\mu\sigma}t}$$

This is the Fourier transform of the solution, so

$$\vec{\mathbf{A}}(\vec{\mathbf{x}},t) = \frac{1}{(2\pi)^3}\int \vec{\mathbf{A}}(\vec{\mathbf{k}},0)e^{-\frac{k^2}{\mu\sigma}t}e^{\imath\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\,\mathrm{d}^3k$$

and an inverse Fourier transform tells us that

$$\vec{\mathbf{A}}(\vec{\mathbf{k}}, 0) = \int \vec{\mathbf{A}}(\vec{\mathbf{x}}', 0) e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'} \, \mathrm{d}^3 x'$$

All together, we have

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) = \frac{1}{(2\pi)^3} \iint \vec{\mathbf{A}}(\vec{\mathbf{x}}', 0) e^{-\frac{k^2}{\mu\sigma} t} e^{\imath \vec{\mathbf{k}} \cdot (\vec{\mathbf{x}} - \vec{\mathbf{x}}')} \, \mathrm{d}^3 k \, \mathrm{d}^3 x'$$

which is equal to

$$= \int G(\vec{\mathbf{x}} - \vec{\mathbf{x}}', t) \vec{\mathbf{A}}(\vec{\mathbf{x}}', 0) \, \mathrm{d}^3 x'$$

so

$$G(\vec{\mathbf{x}} - \vec{\mathbf{x}}', t) = \frac{1}{(2\pi)^3} \int e^{-\frac{k^2}{\mu\sigma} t} e^{\imath \vec{\mathbf{k}} \cdot (\vec{\mathbf{x}} - \vec{\mathbf{x}}')} \, \mathrm{d}^3 k$$

(b) By introducing a Fourier decomposition in both space and time, and performing the frequency integral in the complex ω plane to recover the result of part a, show that $G(\vec{\mathbf{x}} - \vec{\mathbf{x}}', t)$ is the diffusion Green function that satisfies the inhomogeneous equation,

$$\frac{\partial G}{\partial t} - \frac{1}{\mu\sigma} \nabla^2 G = \delta^3(\vec{\mathbf{x}} - \vec{\mathbf{x}}') \delta(t)$$

and vanishes for $t < 0$.

The four-dimensional Fourier transform of a function of space and time is given by

$$G(\vec{\mathbf{x}}, t) = \frac{1}{(2\pi)^4} \int G(\vec{\mathbf{k}}, \omega) e^{\imath (\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} - \omega t)} \, \mathrm{d}^3 k \, \mathrm{d}\omega$$

Plugging this into the given inhomogeneous equation results in

$$\left[(-\imath\omega) - \frac{\imath \vec{\mathbf{k}}^2}{\mu\sigma} \right] G = e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'}$$

or

$$G(\vec{\mathbf{k}}, \omega) = \frac{e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'}}{\frac{k^2}{\mu\sigma} - \imath\omega}$$

Let us first invert the transform in time (using ω):

$$G(\vec{\mathbf{k}}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'}}{2\pi} \frac{e^{-\imath\omega t}}{-\imath\omega + \frac{k^2}{\mu\sigma}}$$

Our integrand is of the form $e^{\imath a\omega} g(\omega)$, where $a < 0$ for $t > 0$, so we want to perform this integral using a closed contour in the lower-half plane, by Jordan's lemma:

$$\lim_{R \to \infty} \int_{C_R} e^{\imath a \omega} g(\omega) \, \mathrm{d}\omega = 0$$

over a contour in the upper-half plane for *positive* a, however, we have negative a so the semicircle part will integrate to zero as we expand the radius to zero and thus take the integral from $-\infty$ to $+\infty$. In doing this, we will go around the pole in the lower-half plane at $\imath\omega = -\frac{k^2}{\mu\sigma}$, so the integral will be $-2\pi\imath$ times the residue:

$$G(\vec{\mathbf{k}}, t) = (-2\pi\imath) \frac{\imath e^{-\imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'}}{2\pi} e^{-\frac{k^2}{\mu\sigma} t} = e^{-\frac{k^2}{\mu\sigma} t - \imath \vec{\mathbf{k}} \cdot \vec{\mathbf{x}}'}$$

(I rearranged the equation to give me just $\omega +$ stuff in the denominator) In the upper-half plane, there are no poles, so the integral evaluates to 0, or $G = 0$, which is the evaluation for $t < 0$.

Finally, I have to transform back into position space to get the Green's function:

$$G(\vec{\mathbf{x}} - \vec{\mathbf{x}}', t) = \frac{\Theta(t)}{(2\pi)^3} e^{-\frac{k^2}{\mu\sigma}t} e^{\imath \vec{\mathbf{k}} \cdot (\vec{\mathbf{x}} - \vec{\mathbf{x}}')} \, \mathrm{d}^3 k$$

where $\Theta(t)$ is the Heaviside function, which gives us the vanishing function for $t < 0$ and evaluates to 1 for $t > 0$.

(c) Show that if σ is uniform throughout all space, the Green function is

$$G(\vec{\mathbf{x}}, t; \vec{\mathbf{x}}', 0) = \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} e^{\frac{-\mu\sigma|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2}{4t}}$$

If σ is uniform and doesn't depend on x, we can evaluate the integral by completing the square:

$$G(\vec{\mathbf{x}} - \vec{\mathbf{x}}', t) = \frac{\Theta(t)}{(2\pi)^3} \int \exp\left(-\frac{t}{\mu\sigma}\left|\vec{\mathbf{k}} - \imath\mu\sigma(\vec{\mathbf{x}} - \vec{\mathbf{x}}')/2t\right|^2 - \frac{\mu\sigma}{4t}|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2\right) \mathrm{d}^3 k$$

so

$$G(\vec{\mathbf{x}} - \vec{\mathbf{x}}', t) = \frac{\Theta(t)}{(2\pi)^3} \left(\frac{\pi\mu\sigma}{t}\right)^{3/2} e^{-\frac{\mu\sigma}{4t}|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2} = \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} e^{\frac{-\mu\sigma|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|^2}{4t}}$$

2 Jackson 6.10 (modification)

Assume we have free space, so only use ϵ_0 and μ_0. Maxwell stress tensor T_{ij} is the one given in Eq. (6.120) similarly $\vec{\mathbf{g}} = \epsilon_0 \vec{\mathbf{E}} \times \vec{\mathbf{B}}$ as in Eq. (6.118).

Equation 6.120:

$$T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\vec{\mathbf{E}} \cdot \vec{\mathbf{E}} + c^2 \vec{\mathbf{B}} \cdot \vec{\mathbf{B}}\right)\delta_{\alpha\beta}\right]$$

Equation 6.118:

$$\vec{\mathbf{g}} = \frac{1}{c^2}(\vec{\mathbf{E}} \times \vec{\mathbf{H}})$$

Discuss the conservation of angular momentum. Show that the differential and integral forms of the conservation law are

$$\frac{\partial}{\partial t}\left(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}\right) + \vec{\boldsymbol{\nabla}} \cdot \mathbf{M} = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}\right) \mathrm{d}^3 x + \int_S \vec{\mathbf{n}} \cdot \mathbf{M} \, \mathrm{d}a = 0$$

where the field angular-momentum density is

$$\mathscr{L}_{\text{field}} = \vec{\mathbf{x}} \times \vec{\mathbf{g}} = \mu\epsilon \times (\vec{\mathbf{E}} \times \vec{\mathbf{H}})$$

and the flux of angular momentum is described by the tensor

$$\mathbf{M} = \mathbf{T} \times \vec{\mathbf{x}}$$

Note that we define the dyadic notation as $[\vec{\mathbf{n}} \cdot \mathbf{M}]_j = \sum_i n_i M_{ij}$ and $M_{ijk} = T_{ij}x_k - T_{ik}x_j$. This tensor is antisymmetric in the j and k indices, so it has three independent elements.

I will begin by examining the force on a particle in free space under electric and magnetic fields.

We write this force according to the Lorentz force law:

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

However, since everything in this problem is in terms of a density, we should rewrite this in terms of the force density:

$$\vec{\mathbf{f}} = \rho\vec{\mathbf{E}} + \vec{\mathbf{J}} \times \vec{\mathbf{B}}$$

Next, we use Maxwell's equations to rewrite the charges in terms of the fields:

$$\vec{\mathbf{f}} = \epsilon_0(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}})\vec{\mathbf{E}} + \frac{1}{\mu_0}(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}) \times \vec{\mathbf{B}} - \epsilon_0 \partial_t \vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

Using the product rule, we can rewrite the last term:

$$\vec{\mathbf{f}} = \epsilon_0(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}})\vec{\mathbf{E}} + \frac{1}{\mu_0}(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}) \times \vec{\mathbf{B}} - \epsilon_0 \partial_t (\vec{\mathbf{E}} \times \vec{\mathbf{B}}) - \epsilon_0 \vec{\mathbf{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}})$$

By inserting a zero (from $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0$, we can make this equation look more symmetric:

$$\vec{\mathbf{f}} = \epsilon_0 \left[(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}})\vec{\mathbf{E}} - \vec{\mathbf{E}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}})\right] + \frac{1}{\mu_0}\left[(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}})\vec{\mathbf{B}} - \vec{\mathbf{B}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}})\right] - \epsilon_0 \partial_t (\vec{\mathbf{E}} \times \vec{\mathbf{B}})$$

Finally, by the identity

$$\frac{1}{2}\vec{\boldsymbol{\nabla}}\vec{\mathbf{A}} \cdot \vec{\mathbf{A}} = \vec{\mathbf{A}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}) + (\vec{\mathbf{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{A}}$$

we can rewrite this as

$$\vec{\mathbf{f}} = \epsilon_0 \left[(\vec{\boldsymbol{\nabla}}\vec{\mathbf{E}})\vec{\mathbf{E}} + (\vec{\mathbf{E}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{E}}\right] + \frac{1}{\mu_0}\left[(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}})\vec{\mathbf{B}} + (\vec{\mathbf{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}}\right] - \frac{1}{2}\vec{\boldsymbol{\nabla}}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \partial_t \vec{\mathbf{g}}$$

where $\vec{\mathbf{g}} = \epsilon_0(\vec{\mathbf{E}} \times \vec{\mathbf{B}})$ as defined in the problem.
In this form, we can see that

$$\vec{\mathbf{f}} = \vec{\boldsymbol{\nabla}} \times \mathbf{T} - \partial_t \vec{\mathbf{g}}$$

This force density can be related to the torque density in the system by

$$\vec{\mathbf{x}} \times \vec{\mathbf{f}} = \vec{\tau}$$

which in turn can be related to the mechanical angular momentum density:

$$\vec{\mathbf{x}} \times \vec{\mathbf{f}} = \vec{\tau} = \partial_t \mathscr{L}_{\text{mech}}$$

Therefore,

$$\begin{aligned}
\partial_t(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}) &= \vec{\mathbf{x}} \times \vec{\mathbf{f}} + \partial_t \vec{\mathbf{x}} \times \vec{\mathbf{g}} \\
&= -\vec{\mathbf{x}} \times \partial_t \vec{\mathbf{g}} + \vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}} \cdot \mathbf{T} + \vec{\mathbf{x}} \times \partial_t \vec{\mathbf{g}} \\
&= \vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}} \cdot \mathbf{T} = -\vec{\boldsymbol{\nabla}} \cdot (\mathbf{T} \times \vec{\mathbf{x}}) = -\vec{\boldsymbol{\nabla}} \cdot \mathbf{M}
\end{aligned}$$

so

$$\partial_t(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}) + \vec{\boldsymbol{\nabla}} \cdot \mathbf{M} = 0$$

If we integrate over a volume, the right-hand side will still be zero, but we can then use divergence theorem to convert the volume integral over $\vec{\boldsymbol{\nabla}} \cdot \mathbf{M}$ into a surface integral over $\vec{\mathbf{n}} \cdot \mathbf{M}$. This will yield the corresponding integral form of the conservation law.

3 Jackson 6.14; (a)

An ideal circular parallel plate capacitor of radius a and plate separation $d << a$ is connected to a current source by axial leads. The current in the wire is $I(t) = I_0 \cos(\omega t)$. Calculate the electric and magnetic fields between the plates to second order in powers of the frequency (or wave number), neglecting the effects of fringing fields.

If we neglect edge effects, we know that the electric field between two charged plates will be oriented in the $\hat{\mathbf{z}}$ direction, and varying the charge in time will give us a magnetic field in the $\hat{\varphi}$ direction, by the right-hand rule.
First, let's expand the fields in ω. It is important to note that, for the electric field, leading order will be proportional to the charge on the plates, but this is proportional to an integral over $I(t) \propto \frac{1}{\omega}$, so we need to insure the leading order term of the electric field is ω^{-1}. However, leading order in the electric field will be like an electrostatic solution, so there won't be a corresponding magnetic field. Additionally, we assume that the separation is small enough that the fields are uniform in z inside.

$$E_z = \sum_{n=-1}^{\infty} e_n(\rho)\omega^n$$

and

$$B_\varphi = \sum_{n=0}^{\infty} b_n(\rho)\omega^n$$

Since there are no currents between the plates, Ampere's law tells us

$$\frac{1}{\rho}\partial_\rho \rho B_\varphi = -\frac{\imath\omega}{c^2} E_z$$

and Faraday's law tells us

$$\partial_\rho E_z = -\imath\omega B_\varphi$$

We can substitute the second of these equations into the first (in two different ways) to give us second-order differential equations:

$$\frac{1}{\rho}\partial_\rho \rho \partial_\rho E_z = -\frac{\omega^2}{c^2} E_z$$

and

$$\partial_\rho \frac{1}{\rho}\partial_\rho \rho B_\varphi = -\frac{\omega^2}{c^2} B_\varphi$$

Expanding the derivatives on the left-hand sides of each of these equations, we find that they can be written as

$$\frac{\omega^2\rho^2}{c^2}\partial_\rho^2 E_z + \frac{\omega\rho}{c}\partial_\rho E_z + \frac{\omega^2\rho^2}{c^2} E_z = 0$$

and

$$\frac{\omega^2\rho^2}{c^2}\partial_\rho^2 B_\varphi + \frac{\omega\rho}{c}\partial B_\varphi + \left(\frac{\omega^2\rho^2}{c^2} - 1\right) B_\varphi = 0$$

These are solutions of the Bessel equation:

$$x^2\partial_x^2 y + x\partial_x y + (x^2 - n^2)y = 0$$

for $x = \frac{\omega\rho}{c}$ and $n = 0, 1$ for the E and B-fields, respectively. This means we can write our solutions in terms of Bessel functions of the first kind:

$$E_z(\rho) = E_z^0 J_0\left(\frac{\omega\rho}{c}\right)$$

and

$$B_\varphi(\rho) = B_\varphi^0 J_1\left(\frac{\omega\rho}{c}\right)$$

where E^0 and B^0 are found from boundary conditions. In our case, the charge on the plates at time t is

$$q = \int I_0 e^{-\imath\omega t} = \frac{\imath I_0}{\omega} e^{-im\omega t}$$

so the electric field will be $\frac{\sigma}{\epsilon_0} = -\frac{\imath I_0 e^{-\imath\omega t}}{\epsilon_0 \pi a^2 \omega}$. Using the fact that $J_0' = -J_1$, we find that

$$B_\varphi = \frac{\imath}{c} E_z^0 J_0' = -\frac{\imath}{c} E_z^0 J_1$$

we can conclude that

$$E_z = \frac{\imath I_0 e^{-\imath\omega t}}{\pi\epsilon_0 a^2 \omega} J_0\left(\frac{\omega\rho}{c}\right)$$

and

$$B_\varphi = -\frac{\mu_0 I_0 c e^{-\imath\omega t}}{\pi a^2 \omega} J_1\left(\frac{\omega\rho}{c}\right)$$

However, we only want the real parts of the fields, so we can change the exponential to a trig function. I will also expand the Bessel functions to second order:

$$E_z = -\frac{I_0 \sin(\omega t)}{\pi\epsilon_0 a^2 \omega}\left(1 - \frac{\rho^2\omega^2}{4c^2} + \mathcal{O}\big(\omega^4\big)\right)$$

and (factoring out a ρ and factoring in a c and ω to make this expansion look nicer):

$$B_\varphi = -\frac{\mu_0 I_0 \rho \cos(\omega t)}{2\pi a^2}\left(1 - \frac{\rho^2\omega^2}{8c^2} + \mathcal{O}\big(\omega^4\big)\right)$$

4 Propagation of Waves in Two Dimensions

Suppose we study propagations of waves in two dimensions (this could be sound propagation in a medium for example) with velocity v, and there are no boundaries, hence it is the full two-dimensional plane. The wave equation can be written as

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\psi(\vec{\mathbf{x}};t) - \frac{1}{v^2}\frac{\partial^2\psi(\vec{\mathbf{x}};t)}{\partial t^2} = 0.$$

Suppose that *initial conditions are specified*, that is, $\psi(\vec{\mathbf{x}};0)$ and $\frac{\partial\psi(\vec{\mathbf{x}};0)}{\partial t}$ are given functions. Using a Fourier transform, show that the solution is given by

$$\psi(\vec{\mathbf{x}};t) = \frac{1}{2\pi v}\frac{\partial}{\partial t}\int_{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\le vt} \frac{\mathrm{d}^2x'}{\sqrt{(vt)^2 - |\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2}}\psi(\vec{\mathbf{x}}';0) + \frac{1}{2\pi v}\int_{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\le vt} \frac{\mathrm{d}^2x'}{\sqrt{(vt)^2 - |\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2}}\frac{\partial\psi(\vec{\mathbf{x}}';0)}{\partial t}$$

First, by a Fourier transform, we can write

$$\psi(\vec{\mathbf{x}};t) = \frac{1}{(2\pi)^2}\int \psi(\vec{\mathbf{k}};t) e^{-\imath\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\,\mathrm{d}^2k$$

so the wave equation becomes

$$\left[-k^2 - \frac{1}{v^2}\partial_t\right]\psi(\vec{\mathbf{k}};t) = 0$$

We can solve this. It has solutions

$$\psi(\vec{\mathbf{k}};t) = A\cos(kvt) + B\sin(kvt)$$

At $t = 0$, we find that the initial conditions give this solution the following form:

$$\psi(\vec{\mathbf{k}}; t) = \psi(\vec{\mathbf{k}}; 0)\cos(kvt) + \partial_t\psi(\vec{\mathbf{k}}; 0)\sin(kvt)$$

where

$$\psi(\vec{\mathbf{k}}; 0) = \int \mathrm{d}^2x'\, e^{\imath\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}\psi(\vec{\mathbf{x}}; 0)$$

and similar for the time derivative.
Together, we have:

$$\begin{aligned}\psi(\vec{\mathbf{x}}; t) =& \frac{1}{(2\pi)^2}\iint \cos(kvt)e^{-\imath\vec{\mathbf{k}}(\vec{\mathbf{x}}-\vec{\mathbf{x}}')}\psi(\vec{\mathbf{x}}; 0)\,\mathrm{d}^2x'\,\mathrm{d}^2k \\ &+ \frac{1}{(2\pi)^2}\iint \sin(kvt)e^{-\imath\vec{\mathbf{k}}(\vec{\mathbf{x}}-\vec{\mathbf{x}}')}\partial_t\psi(\vec{\mathbf{x}}; 0)\,\mathrm{d}^2x'\,\mathrm{d}^2k\end{aligned}$$

If we rewrite $\vec{\mathbf{x}}-\vec{\mathbf{x}}' = |\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\cos(\theta)$ where θ is the angle between them, we see that $e^{-\imath\vec{\mathbf{k}}|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\cos(\theta)}$ gives Bessel functions of the first kind (in particular we get $\pi J_0(k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|)$ if we integrate over θ):

$$\int_0^\pi e^{\imath z\cos(\theta)}\,\mathrm{d}theta = \pi J_0(z)$$

so

$$\begin{aligned}\psi(\vec{\mathbf{x}}; t) =& \frac{1}{(2\pi)^2}\iint \cos(kvt)J_0(k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|)\psi(\vec{\mathbf{x}}; 0)\,\mathrm{d}^2k\,\mathrm{d}^2x' \\ &+ \frac{1}{(2\pi)^2}\iint \sin(kvt)J_0(k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|)\partial_t\psi(\vec{\mathbf{x}}; 0)\,\mathrm{d}^2k\,\mathrm{d}^2x'\end{aligned}$$

These integrals are only nonzero in the region where $|\vec{\mathbf{x}}-\vec{\mathbf{x}}'| \leq vt$, and they evaluate to $\frac{1}{\sqrt{(vt)^2-|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2}}$, so our answer becomes

$$\psi(\vec{\mathbf{x}}; t) = \frac{1}{(4\pi}\int_{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\leq vt}\frac{\mathrm{d}^2x'}{\sqrt{(vt)^2-|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2}}\psi(\vec{\mathbf{x}}'; 0) + \frac{1}{4\pi}\int_{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|\leq vt}\frac{\mathrm{d}^2x'}{\sqrt{(vt)^2-|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|^2}}\frac{\partial\psi(\vec{\mathbf{x}}'; 0)}{\partial t}$$

This isn't quite the correct answer, and I'm not sure where I missed the factor of $\frac{2}{v}$ or the time derivative in the first half.