## Lecture 27: Normalization of Dirac Spinors

Monday, November 09, 2020

In the last lecture, we ended with

$$2E = u_1^\dagger u_1 = \frac{E^2 + M^2 + 2EM + p^2}{(E+M)^2}|A_1|^2 = \frac{2E(E+M)}{(E+M)^2}|A_1|^2$$

so

$$A_1 = \sqrt{E+M}$$

and similarly, $A_2 = A_1' = A_2' = \sqrt{E+M}$.

## 0.1 $\vec{\mathbf{S}}$ for an Anti-Proton Spinor

For an anti-proton state written in terms of physical $E$ and $\vec{\mathbf{p}}$ of anti-particles $\psi = N(E, \vec{\mathbf{p}})e^{-\imath(\vec{\mathbf{p}}\cdot\vec{\mathbf{r}} - Et)}$. Acting the Hamiltonian on this, we get

$$\hat{H}\psi = -E\psi$$

and

$$\hat{\vec{\mathbf{p}}}\psi = -\imath\vec{\boldsymbol{\nabla}}\psi = -\vec{\mathbf{p}}\psi$$

so we need to create some modified operators which give the proper $E$ and $\vec{\mathbf{p}}$ (not the negative of these):

$$\hat{H}^{(v)} = -\imath\partial_t \qquad \hat{\vec{\mathbf{p}}}^{(v)} = +\imath\vec{\boldsymbol{\nabla}}$$

so

$$\hat{\vec{\mathbf{L}}}^{(v)} = \vec{\mathbf{r}} \times \hat{\vec{\mathbf{p}}}^{(v)} = -\hat{\vec{\mathbf{L}}}$$

We want a spin operator $\hat{\vec{\mathbf{S}}}^{(v)}$ such that $\left[\hat{H}_D, \vec{\mathbf{L}}^{(v)} + \vec{\mathbf{S}}^{(v)}\right]$, so $\hat{\vec{\mathbf{S}}}^{(v)} = -\hat{\vec{\mathbf{S}}}$, so $v_1 \propto u_4$ is a spin-up antiparticle, whereas $u_4$ was a spin-down, negative-energy particle.

## 0.2 Charge Conjugation

The $C$-operator interchanges matter and antimatter particles. Using minimal substitution, $\hat{E} \to \hat{E} - q\Phi$ and $\vec{\mathbf{p}} \to \vec{\mathbf{p}} - q\vec{\mathbf{A}}$ causes $p_\mu \to p_\mu - qA_\mu$ and $\imath\partial_\mu \to \imath\partial_\mu - qA_\mu$.

The charge conjugation operator gives $Cu_1 \to v_1$ and $Cu_2 \to v_2$.

## 0.3 Spin and Helicity

For particles at rest, $u_1(E,0) = A_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $u_2(E,0) = A_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are eigenstates of

$$\hat{S}_z = \frac{1}{L}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1\end{pmatrix}$$

In general ($\vec{\mathbf{p}} \neq 0$), $u_1$ and $u_2$ are not eigenvectors of $S_z$. For particles moving in the $\pm\hat{\mathbf{z}}$ direction,

$$u_1 = A_1 \begin{pmatrix} 1 \\ 0 \\ \pm\frac{p}{E+m} \\ 0 \end{pmatrix}, \qquad u_2 = A_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \mp\frac{p}{E+m} \end{pmatrix}$$

and

$$v_1 = A_1' \begin{pmatrix} 0 \\ \mp\frac{p}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v_2 = A_2' \begin{pmatrix} \pm\frac{p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are eigenstates of $S_z$.

$$\hat{S}_z u_{1,2}(E,0,0,\pm p) = \pm_{(1,2)} \frac{1}{2} u_{1,2}(E,0,0,\pm p)$$

and

$$\hat{S}_z^v v_{1,2}(E,0,0,\pm p) = \pm_{(1,2)} \frac{1}{2} v_{1,2}(E,0,0,\pm p)$$

### 0.3.1 Helicity

Since $\left[\hat{H}_D, \hat{S}_z\right] \neq 0$, it is not generally possible to define a basis of simultaneous eigenstates of $\hat{H}_D$ and $\hat{S}_z$. We define helicity, $h$, as

$$\hat{h} = \frac{\hat{S} \cdot \hat{\vec{\mathbf{p}}}}{p} = \frac{1}{2p} \begin{pmatrix} \vec{\sigma} \cdot \vec{\mathbf{p}} & 0 \\ 0 & \vec{\sigma} \cdot \vec{\mathbf{p}} \end{pmatrix}$$

Then, $\left[\hat{H}_D, \hat{h}\right] = 0$ since $\hat{H}_D = \vec{\alpha} \cdot \vec{\mathbf{p}} + \beta m$. Let $\hat{h}u = \lambda u$ where $u = \begin{pmatrix} u_A // u_D \end{pmatrix}$ ($u_{A,D}$ are 2-spinors). Therefore we define

$$\vec{\sigma} \cdot \vec{\mathbf{p}} u_{A,B} = 2p\lambda u_{A,B}$$

so

$$\underbrace{(\vec{\sigma} \cdot \vec{\mathbf{p}})(\vec{\sigma} \cdot \vec{\mathbf{p}})}_{p^2} u_A = 4p^2\lambda^2 u_A$$

so $\lambda = \pm\frac{1}{2}$ are the eigenvalues of helicity. We define $\lambda = \frac{1}{2}$ to be "right-handed" helicity.

$$(\vec{\sigma} \cdot \vec{\mathbf{p}}) u_A = (E+m) u_B$$

so $u_B = 2\lambda \frac{p}{E+m} u_A$.

Let $\vec{\mathbf{p}} = (p\sin(\theta)\cos(\varphi), p\sin(\theta)\sin(\varphi), p\cos(\theta))$ so

$$\vec{\sigma} \cdot \vec{\mathbf{p}} = \begin{pmatrix} p\cos(\theta) & p\sin(\theta)e^{-\imath\varphi} \\ p\sin(\theta)e^{\imath\varphi} & -p\sin(\theta) \end{pmatrix}$$

With $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$, this becomes

$$p \begin{pmatrix} & \sin(\theta)e^{-\imath\varphi} \\ \sin(\theta)e^{\imath\varphi} & -\cos(\theta)) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2p\lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

so

$$a(\cos(\theta) - 2\lambda) + b\sin(\theta)e^{-\imath\varphi} = 0$$

or

$$\frac{b}{a} = \frac{2\lambda - \cos(\theta)}{\sin(\theta)} e^{\imath\varphi}$$

For right-handed helicity,

$$\frac{b}{a} = \frac{\sin(\theta/2)}{\cos(\theta/2)} e^{\imath\varphi}$$

From $u_D = +\frac{p}{E+m} u_A$, we have