Lecture 47: Review of Radiation

Monday, December 02, 2019

Suppose

$$\begin{pmatrix} \vec{\mathbf{H}} \\ \vec{\mathbf{E}} \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{H}}_\omega \\ \vec{\mathbf{E}}_\omega \end{pmatrix} e^{-\imath\omega t}$$

Then, Maxwell's equations become

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = -\mu_0 \frac{\partial}{\partial t} \vec{\mathbf{H}}$$

If we define $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$,

$$\vec{\mathbf{E}}_\omega = \frac{\imath Z_0}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}}_\omega$$

and

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{Z_0 k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}}_\omega$$

where $k = \frac{\omega}{c}$.

Recall that we want fields which satisfy the Helmholtz equation in spherical coordinates:

$$\left(\nabla^2 + k^2\right) \begin{pmatrix} \vec{\mathbf{E}}_\omega \\ \vec{\mathbf{H}}_\omega \end{pmatrix} = \vec{\mathbf{0}}$$

since both fields have no divergence in the radiation zone. We can solve this by realizing that the angular momentum operator, $\vec{\mathbb{L}} = \frac{1}{\imath} \vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}}$, commutes with the Laplacian, so if ψ is a solution to the Helmholtz equation, so is $\vec{\mathbb{L}}\psi$. Using this, we found that the general solutions to these fields is

$$\vec{\mathbf{E}}_\omega = \vec{\mathbb{L}}\psi + \frac{\imath Z_0}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\chi$$

and

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{Z_0 k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\psi + \vec{\mathbb{L}}\chi$$

Next, we found general solutions to the spherical Helmholtz equation using the spherical Bessel functions:

$$\psi = \sum_{l,m} \left[a_{lm} h_l^{(1)}(kr) + b_{lm} h_l^{(2)}(kr) \right] Y_{lm}(\Omega)$$

since the Hankel functions look like outgoing/incoming waves (1/2) in the $kr >> 1$ regime.

Now we want to act the angular momentum operator on this function. Since it is a spherical operator, it only acts on the Y_{lm} part, so

$$\vec{\mathbb{L}}\psi = \sum_{l,m} f_{lm}(kr) \frac{\vec{\mathbb{L}} Y_{lm}}{\sqrt{l(l+1)}}$$

where we are just scaling by $\sqrt{l(l+1)}$. We recognize these as the vector spherical harmonics:

$$\frac{\vec{\mathbb{L}}Y_{lm}}{\sqrt{l(l+1)}} = \mathbb{X}_{lm}$$

These make an orthonormal basis:

$$\int \vec{\mathbb{X}}^*_{l'm'} \cdot \vec{\mathbb{X}}_{lm}\,\mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}$$

Jackson scales the general fields by Z_0:

$$\vec{\mathbf{E}}_\omega = Z_0\left(\underbrace{\vec{\mathbb{L}}\psi}_{g_{lm}} + \frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\chi\right)$$

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\psi + \underbrace{\vec{\mathbb{L}}\chi}_{f_{lm}}$$

If we now expand in terms of the vector spherical harmonics, we find

$$\vec{\mathbf{E}}_\omega = Z_0 \sum_{l,m}\left(g_{lm}(kr)\vec{\mathbb{X}}_{lm} + \frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times \left(f_{lm}(kr)\vec{\mathbb{X}}_{lm}\right)\right)$$

$$\vec{\mathbf{H}}_\omega = Z_0 \sum_{l,m}\left(f_{lm}(kr)\vec{\mathbb{X}}_{lm} - \frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times \left(g_{lm}(kr)\vec{\mathbb{X}}_{lm}\right)\right)$$

We should emphasize that these are exact representations, we have not made any approximations yet. Also, f and g are switched with respect to the earlier lecture where this derivation was first done.

Notice that $\vec{\mathbf{x}} \cdot \vec{\mathbb{L}}\psi = 0$ for any ψ, so

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega = Z_0 \sum_{l,m} \frac{\imath}{k}\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} \times \left(f_{lm}(kr)\vec{\mathbb{X}}_{lm}\right) = -\frac{Z_0}{k}\sum_{l,m}\vec{\mathbb{L}}(f_{lm}(kr)\mathbb{X}_{lm})$$

The angular momentum operator commutes with the radial part of the dot product, so this becomes

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega = -\frac{Z_0}{k}\sum_{l,m} f_{lm}(kr)\vec{\mathbb{L}} \cdot \vec{\mathbb{X}}_{lm} = -\frac{Z_0}{k}\sum_{l,m}\sqrt{l(l+1)}Y_{lm}f_{lm}(kr)$$

If we want to find $f_{lm}(kr)$ we need to decompose $\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega$ into spherical harmonics:

$$f_{lm}(kr) = -\frac{k}{Z_0\sqrt{l(l+1)}}\int Y^*_{lm}\left(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega\right)\mathrm{d}\Omega$$

For radiation problems, we would typically have these functions be $a_E(l,m)h_l^{(1)}(kr) = f_{lm}(kr)$. We can do the same derivation to find the $\vec{\mathbf{H}}_\omega$ field:

$$g_{lm}(kr) = \frac{k}{\sqrt{l(l+1)}}\int Y^*_{lm}\left(\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega\right)\mathrm{d}\Omega$$

where, for radiation, $g_{lm}(kr) = a_M(l,m)h_l^{(1)}(kr)$. These a factors are the electric and magnetic multipoles.

In the radiation zone, using

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}} = \imath \left[\vec{\mathbf{x}}\nabla^2 - \vec{\boldsymbol{\nabla}}(1 + \vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}) \right]$$

we found that

$$\vec{\boldsymbol{\nabla}} \times (f_{lm}(kr)\vec{\mathbb{X}}_{lm})$$

only has a contribution from the factor of $e^{\imath kr}$ from the Hankel functions in the far-field limit, so we find terms like

$$(-\imath)^{l+1}\frac{e^{\imath kr}}{kr}\hat{\mathbf{n}} \times \vec{\mathbb{X}}_{lm}$$

In the radiation zone, the terms simplify to

$$\vec{\mathbf{H}}_\omega \to (-\imath)^{l+1}\frac{e^{\imath kr}}{kr}\sum_{l,m}\left[a_E(l,m)\vec{\mathbb{X}}_{lm} + a_M(l,m)\hat{\mathbf{n}} \times \vec{\mathbb{X}}_{lm} \right]$$

and

$$\vec{\mathbf{E}}_\omega = Z_0\vec{\mathbf{H}}_\omega \times \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}} = \frac{\vec{\mathbf{x}}}{r}$.