0.1 Tensor Operators

A tensor is something which transforms like a tensor. In the same vein, it is a representation of a group. A tensor is something which has some indices, and those indices label which representation it transforms under. For example, Cartesian tensors have indices which run over the three dimensions.

$$T_{ijk\dots}, \quad i, j, k, \dots \in \{0, 1, 2\}$$

whereas a spherical tensor can be represented as

$$T_{lm}$$

This uses an irreducible representation—there are no other ls or ms in this tensor. Under the group SU(2), such a tensor transforms as

$$O_l{}^m \to U O_l{}^m U^{-1}$$

in the same way that

$$|lm\rangle \to U\,|lm\rangle$$

$$O^s{}_l\,|l', m\rangle$$

transforms under the action of $l \otimes l'$:

$$O^s{}_l\,|l'm\rangle \to \left[U O^s{}_l U^{-1}\right]\left[U\,|l'm\rangle\right]$$

We can write this in terms of Wigner rotation matrices:

$$O^s{}_l\,|l'm\rangle \to \left(D^l_{s,s'}(\hat{\mathbf{n}}, \theta) O^{s'}{}_l\right)\left(D^{l'}_{m,\hat{m}}\,|l'\hat{m}\rangle\right)$$

To clarify, a rotation can be written as

$$\begin{aligned} e^{-\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta}\vec{\mathbf{x}}e^{\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta} &= \vec{\mathbf{x}} - \imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta\vec{\mathbf{x}} + \vec{\mathbf{x}}\cdot(\imath\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta) + \cdots \\ &= \vec{\mathbf{x}} - \imath[\vec{\mathbf{L}}\cdot\hat{\mathbf{n}}\theta]\vec{\mathbf{x}} + \dots \end{aligned}$$

but equivalently, we already know that the net action of this is a rotation of the coordinates: $R(\theta)x$.

From our previous analysis, the reducible representation $l \otimes l'$ breaks up into a tensor sum of irreducible representations:

$$l \otimes l' = |l + l'| \oplus \cdots \oplus |l - l'|$$

We can use our knowledge of addition of angular momentum to say something about this new state. Consider

$$J_z[O_l{}^s\,|l'm\rangle] = \hbar(s + m) O_l{}^s\,|l'm\rangle$$

This first term is a tensor product of the operator and the vector, so we know that the action will be an addition in terms of the irreducible representations labeled by s and m. The fact that O is an operator rather than a state is *almost* irrelevant. For the time being, it behaves in the exact same way as a state.

The maximum weight state for this particular tensor product is $O_l{}^l\,|l', l'\rangle$. This transforms like $|l, l; l', l'\rangle \equiv |l + l', l + l'\rangle$ so,

$$O_l{}^l\,|l', l'\rangle = K_{J=l+l'}\,|l + l', l + l'\rangle$$

The reason it is only proportional to this and not exactly equal can be shown by an example. Suppose we are in this maximum-weight state and there is an additional quantum number, α, which can be used to label the states:

$$O_l{}^l \left|l', l'; \alpha\right\rangle = K_{l+l'} \left|l + l', l + l'\right\rangle$$

Now K must depend on α, since we arrive at the states on the right purely through group theory which doesn't take α into account. The Hilbert space on the right doesn't have anything to do with α. We can write this in general as

$$O_l{}^s \left|j, m; \alpha\right\rangle = \sum_{J=|l-j|}^{|l+j|} K_J(\alpha) \left|J, M\right\rangle \quad \underbrace{\left\langle J, M | l, s; j, m\right\rangle}_{\text{Clebsch-Gordan Coefficients}}$$

Let's now project this onto a different state:

$$\left\langle j'm'; \beta\right| O_l{}^s \left|j, m; \alpha\right\rangle = K_j \left\langle j', m' | l, s; j, m\right\rangle$$

We require $m' = s + m$ since $M = s + m$ is required for the Clebsch-Gordan coefficients to be nonzero. Now what is this coefficient K_j?

$$K_J \left|J, l + m\right\rangle = \sum_\beta K_{\alpha\beta} \left|J, l + m; \beta\right\rangle$$

Therefore, taking the above matrix element means that K_j has to know about both α and β:

$$K_{\alpha\beta} = \left\langle J, \beta\right| |O| \left|J', \alpha\right\rangle$$

This is the standard notation for the "reduced matrix element". From here, we derive the Wigner-Eckart Theorem:

$$\left|J, M'; \beta\right\rangle O_l{}^s \left|j, m; \alpha\right\rangle = \delta_{M', s+m} \left\langle J, M' | j, m, l, s\right\rangle \left\langle J; \beta\right| |O_l| \left|j; \alpha\right\rangle \qquad \text{(Wigner-Eckart Theorem)}$$

Let's do an example that will hopefully clarify this.

Example. Suppose we go and measure the value of some matrix element in an experiment.

$$\left|\frac{1}{2}, \frac{1}{2}; \alpha\right\rangle Z \left|\frac{1}{2}, \frac{1}{2}; \beta\right\rangle = A$$

We want to then predict the value of

$$\left|\frac{1}{2}, \frac{1}{2}; \alpha\right\rangle X \left|\frac{1}{2}, -\frac{1}{2}; \beta\right\rangle = B$$

X transforms like $l = 1$ in Cartesian coordinates. Let's first write our operator in spherical coordinates, rotate it, and find the solution. The Z operator transforms like $Y_l^{m=0}$, whereas

$$X = \frac{1}{\sqrt{2}} \left[Y_{l=1}^{m=-1} - Y_{l=1}^{m=1}\right]$$

Only the second term will contribute:

$$\left\langle\frac{1}{2}, \frac{1}{2}; \alpha\right| X \left|\frac{1}{2}, -\frac{1}{2}; \beta\right\rangle = \left|\frac{1}{2}, \frac{1}{2}; \alpha\right\rangle - \frac{1}{\sqrt{2}} Y_{l=1}^{m=1} \left|\frac{1}{2}, -\frac{1}{2}; \beta\right\rangle$$

Technically there's also a radial component, but it's the same for Z so they will cancel. Let's now use Wigner-Eckart theorem:

$$\left\langle\frac{1}{2}, \frac{1}{2}; \alpha\right| X \left|\frac{1}{2}, -\frac{1}{2}; \beta\right\rangle = -\frac{1}{\sqrt{2}} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|1, \frac{1}{2}; 1, -\frac{1}{2}\right\rangle \left\langle\frac{1}{2}; \alpha\right| |Y_{l=1}^{m=1}| \left|\frac{1}{2}; \beta\right\rangle$$

We know that

$$A = \left\langle \frac{1}{2}, \frac{1}{2}; \alpha \right| Y_1^0 \left| \frac{1}{2}, \frac{1}{2}; \beta \right\rangle = \left\langle \frac{1}{2}, \frac{1}{2} \right| 1, 0; \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}; \alpha \right| |Y_l| \left| \frac{1}{2}; \beta \right\rangle$$

Therefore, the ratio of the two matrix elements is

$$\frac{\left\langle \frac{1}{2}, \frac{1}{2}; \alpha \right| Z \left| \frac{1}{2}, \frac{1}{2}; \beta \right\rangle}{\left\langle \frac{1}{2}, \frac{1}{2}; \alpha \right| X \left| \frac{1}{2}, -\frac{1}{2}; \beta \right\rangle} = \frac{\left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| 1, 0; \frac{1}{2}, \frac{1}{2} \right\rangle}{-\frac{1}{\sqrt{2}} \left\langle \frac{1}{2}, \frac{1}{2} \right| 1, \frac{1}{2}; 1, \frac{1}{2} \right\rangle} = 1$$

so $A = B$. $\diamond$