Lecture 40: Angular Momentum

Monday, November 18, 2019

Chapter 1

Angular Momentum

In Classical mechanics, we know that the orbital angular momentum is $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$. We can examine components of this using the cross product and cyclic permutations ($L_x = yp_z - zp_y$).

In Quantum mechanics, we promote position and momentum to operators. L_x contains two terms and so does L_y so any commutator will contain four terms:

$$\left[\hat{\mathbf{L}}_x, \hat{\mathbf{L}}_y\right] = \left[\hat{\mathbf{Y}}\hat{\mathbf{P}}_z - \hat{\mathbf{Z}}\hat{\mathbf{P}}_y, \hat{\mathbf{Z}}\hat{\mathbf{P}}_x - \hat{\mathbf{X}}\hat{\mathbf{P}}_z\right] = \underbrace{\left[\hat{\mathbf{Y}}\hat{\mathbf{P}}_z, \hat{\mathbf{Z}}\hat{\mathbf{P}}_x\right]}_{\hat{\mathbf{Y}}\left[\hat{\mathbf{P}}_z, \hat{\mathbf{Z}}\right]\hat{\mathbf{P}}_x} + \underbrace{\left[\hat{\mathbf{Z}}\hat{\mathbf{P}}_y, \hat{\mathbf{X}}\hat{\mathbf{P}}_z\right]}_{\hat{\mathbf{X}}\left[\hat{\mathbf{Z}}, \hat{\mathbf{P}}_z\right]\hat{\mathbf{P}}uy} = -\imath\hbar\hat{\mathbf{Y}}\hat{\mathbf{P}}_x + \imath\hbar\hat{\mathbf{X}}\hat{\mathbf{P}}_y = \imath\hbar\hat{\mathbf{L}}_z$$

This is the orbital angular momentum for a single particle, but we might have many particles. Let's call total angular momentum $\vec{\mathbf{L}} = \sum_{i=1}^{N} \hat{\mathbf{L}}^{(i)}$. We can also have the total angular momentum, which includes the spin: $\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{S}}$.

Additionally, $\left[\hat{\mathbf{J}}_x, \hat{\mathbf{J}}_y\right] = \imath\hbar\hat{\mathbf{J}}_z$ (along with the other cyclic permutations).

There is another operator we want to look at:

$$\hat{\mathbf{J}}^2 = \vec{\mathbf{J}} \cdot \vec{\mathbf{J}} = \hat{\mathbf{J}}_x^2 + \hat{\mathbf{J}}_y^2 + \hat{\mathbf{J}}_z^2$$

This total angular momentum squared has interesting commutation relations:

$$\left[\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_x\right] = \left[\hat{\mathbf{J}}_y^2, \hat{\mathbf{J}}_x\right] + \left[\hat{\mathbf{J}}_z^2, \hat{\mathbf{J}}_x\right] = 0$$

We see that $\hat{\mathbf{J}}^2$ commutes with $\hat{\mathbf{J}}_x$, and it can be shown that it commutes with the other two components also. We are looking for a complete commuting set of observables (CCSO). Customarily, we choose this to be $\{\hat{\mathbf{J}}_z, \hat{\mathbf{J}}^2\}$ (because 20th century physicists love Jay-Z according to Dr. Widom).

Let's define

$$\hat{\mathbf{J}}_\pm = \hat{\mathbf{J}}_x \pm \imath\hat{\mathbf{J}}_y$$

such that we can redefine

$$\hat{\mathbf{J}}^2 = \frac{1}{2}(\hat{\mathbf{J}}_+\hat{\mathbf{J}}_- + \hat{\mathbf{J}}_-\hat{\mathbf{J}}_+) + \hat{\mathbf{J}}_z^2$$

We can show that

$$\left[\hat{\mathbf{J}}_z, \hat{\mathbf{J}}_\pm\right] = \pm\hbar\hat{\mathbf{J}}_\pm,$$

$$\left[\hat{\mathbf{J}}_+, \hat{\mathbf{J}}_-\right] = 2\hbar\hat{\mathbf{J}}_z,$$

and

$$\left[\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_\pm\right] = 0$$

Let's now talk about the eigenstates of angular momentum. Since $\langle\psi|\hat{\mathbf{J}}^2|\psi\rangle \geq 0$, we know that its eigenvalues must be non-negative. Note that $\hat{\mathbf{J}}$ is Hermitian, so the eigenvalues are real, therefore squaring them will result in non-negative numbers.

Let's call the eigenstates $|j\rangle$ and say that they have eigenvalue $j(j+1)\hbar^2$:

$$\hat{\mathbf{J}}^2 |j\rangle \equiv j(j+1)\hbar^2 |j\rangle$$

Similarly, we can define the eigenstates of $\hat{\mathbf{J}}_z$:

$$\hat{\mathbf{J}}_z |m\rangle = m\hbar |m\rangle$$

Note that j and m do not need to be (and rarely are) integers.

We will label shared eigenstates of $\{\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_z\}$ as $|kjm\rangle$, where we include k in case there's some extra degeneracy for which we need to distinguish states.

Some facts about these eigenstates:

- $-j \leq m \leq j$ Proof: $0 \leq \left|\hat{\mathbf{J}}_+ |jm\rangle\right|^2 = \langle jm| \hat{\mathbf{J}}_- \hat{\mathbf{J}}_+ |jm\rangle = \langle jm| \left(\hat{\mathbf{J}}^2 - \hat{\mathbf{J}}_z^2 - \hbar\hat{\mathbf{J}}_z\right) |jm\rangle$. Next, we can evaluate each of these operators acting on the state:

  $$0 \leq (j(j+1) - m^2 - m)\hbar^2$$

  so $m \leq j$. If we do the same trick starting with $\hat{\mathbf{J}}_-$ we will find the other half of the inequality.

- $j \geq 0$ ($\hat{\mathbf{J}}^2 \geq 0$)
- $m = \pm j$ iff $\hat{\mathbf{J}}_\pm |jm\rangle = 0$.

  $$\left|\hat{\mathbf{J}}_\pm |jm\rangle\right|^2 = \pm(j(j+1) - m(m+1))\hbar^2 = 0$$

- If $m \lesseqgtr -j$ then $\hat{\mathbf{J}}_z(\hat{\mathbf{J}}_\pm |jm\rangle) = (m \pm 1)\hbar(\hat{\mathbf{J}}_\pm |jm\rangle)$ and $\hat{\mathbf{J}}^2(\hat{\mathbf{J}}_\pm |jm\rangle) = j(j+1)\hbar^2(\hat{\mathbf{J}}_\pm |jm\rangle)$, so $\hat{\mathbf{J}}_\pm$ acts like a raising/lowering operator for m but leaves j unchanged.
- $j \in \mathbb{Z}/2$ (is a half-integer or integer) Proof: Consider lowering the state $|jm\rangle$ p times using $\hat{\mathbf{J}}_-$, we would find that $-j \leq m - p \leq -j + 1$ where p is an integer. Recall that if $m = -j$, lowering it will give us zero. $m - p - 1$ can't be less than $-j$. We could also end up exactly at $-j$, and if we tried to lower it again we would get 0. Therefore $\exists p \in \mathbb{Z}$ such that $m - p = -j$. We can also say that $\exists q \in \mathbb{Z}$ such that $m + q = +j$ using the raising process. We can take these two assertions and subtract them, which would give $p + q = 2j \in \mathbb{Z}$.