Lecture 5: Symmetries and Conservation Laws

Friday, January 24, 2020

From last lecture, $[H, O] = 0$, then O is a constant of motion as long as O has no explicit time dependence. If U implements a symmetry group G, under the action of G, $H \to U^\dagger H U$. We can write this as

$$e^{-\imath\vec{\lambda}\cdot\vec{\mathbf{X}}} H e^{\imath\vec{\lambda}\cdot\vec{\mathbf{X}}}$$

As a consequence, if $\left[H, \vec{\mathbf{X}}\right] = 0$, then H is invariant under G, since we pull H through the exponentials and they will cancel out. Therefore, if H is invariant under G, X is conserved.

If L is invariant, then maybe (usually) H is invariant, so this is a similar result to Noether's theorem in classical mechanics. The simplest counterexample is boosts. Take

$$L = \frac{1}{2} m\dot{x}^2$$

A boost transformation is $x \to x + \delta v t$, so

$$L \to \frac{1}{2} m\dot{x}^2 + m x \delta v$$

The action is still invariant, since

$$\delta S = \int \mathrm{d}t\, \delta L = \int m\delta v x \,\mathrm{d}t = \left[\frac{\mathrm{d}}{\mathrm{d}t}(m\delta v x)\right] \mathrm{d}t = 0$$

Total derivatives have no effect on the equations of motion, since they don't change the Euler-Lagrange equations. A symmetry which takes $L \to L + \frac{\mathrm{d}}{\mathrm{d}t} f(x, \dot{x}, t)$ is still a symmetry. However, the Hamiltonian, which leads to quantum conservation laws, is not invariant under boosts, and there is no time integral to get rid of the consequences.

0.1 Degeneracy

Symmetries imply degeneracies. If G is a symmetry with generators $\vec{\mathbf{X}}$, then $\left[H, \vec{\mathbf{X}}\right] = 0$ implies

$$H\left|\lambda\right\rangle = E(\lambda)\left|\lambda\right\rangle \implies H\vec{\mathbf{X}}\left|\lambda\right\rangle = E(\lambda)\vec{\mathbf{X}}\left|\lambda\right\rangle$$

so if $\vec{\mathbf{X}}\left|\lambda\right\rangle \neq \left|\lambda\right\rangle$, there exists a degeneracy.

Let's first look at a case which is not degenerate: rotations. On the homework, we saw that the group defined by 3D rotations (SO(3)) has the same Lie algebra as SU(2). We are going to call the generators of SO(3) $J_i \in \mathfrak{so}(3)$ such that

$$[J_i, J_j] = \imath\hbar\epsilon_{ijk} J_k$$

In QM, there are two ways of forming a group representation. The first are matrices, and the second are differential operators acting on an infinite dimensional space of square integrable functions L^2 ($L^2 = \{f(x) \mid \int |f(x)|^2 \,\mathrm{d}x < \infty\}$). In other words, we can write

$$\vec{\mathbf{L}} = \imath\vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

but we can also write

$$\vec{\mathbf{p}} = -\imath\hbar\frac{\partial}{\partial\vec{\mathbf{x}}}$$

such that

$$\left[-\imath\hbar r_a \frac{\partial}{\partial r_b}\epsilon_{abi}, -\imath\hbar r_c \frac{\partial}{\partial r_d}\epsilon_{cdj}\right] = -\imath\hbar^2 \epsilon_{ijk} r_f \frac{\partial}{\partial r_g}\epsilon_{fgk}$$

Let's now find the matrix representations. First, find operators which commute with all elements of the Lie algebra:

$$\left[O, \vec{\mathbf{J}}\right] = 0$$

These are called Casimir operators. For rotations, these operators happen to be $\vec{\mathbf{J}}^2$ (for both SO(3) and SU(2)). As it turns out, this works for all vector operators:

$$U^\dagger(\hat{\mathbf{n}}, \theta)\{P_i, X_i, L_i\}U(\hat{\mathbf{n}}, \theta) = R(\hat{\mathbf{n}}, \theta)_{ij}\{P_j, X_j, L_j\}$$

$$\begin{aligned}
\left[\vec{\mathbf{J}}^2, J_i\right] &= \left[J^a J^a, J^i\right] \\
&= J^a \left[J^a, J^i\right] - \left[J^i, J^a\right] J^a \\
&= J^a \left(i\hbar\epsilon^{aik} J^k\right) - \left(\imath\hbar\epsilon^{iak} J^k\right) J^a \\
&= \imath\hbar \left[e^{aik} J^a J^k - \epsilon^{iak} J^k J^a\right] \\
&= \imath\hbar \left[\epsilon^{kia} J^k J^a - \epsilon^{iak} J^k J^a\right] \\
&= \left[\epsilon^{kia} - \epsilon^{iak}\right] J^k J^a = 0
\end{aligned}$$

Or you could just say $\vec{\mathbf{J}}^2$ is a scalar under rotations so it is invariant under rotations.

Every representation is labelled by eigenvalues of the Casimir operator.

Lemma 0.1.1 (Schur's Lemma)**.** *Any group element which commutes with all other group elements is proportional to I (the identity).*

The eigenvalues of $\vec{\mathbf{J}}^2$ are the total angular momentum:

$$\vec{\mathbf{J}}^2 |a\rangle = a |a\rangle$$

The next step is to choose a generator to diagonalize (in SO(3) you can only diagonalize one of them at a time since they don't commute with each other). We will arbitrarily choose J_z such that

$$J_z |a, b\rangle = b |a, b\rangle$$

We are working in a basis which are eigenvectors of J_z. There is nothing else we can diagonalize simultaneously, since the J's don't individually commute. These a's and b's label the states of the representation. a will not change if we operate on this state with J_x or J_y, but b will change. What are the possible values of b?

Define raising and lowering operators

$$J_\pm = \frac{(J_x \pm \imath J_y)}{\sqrt{2}}$$

such that

$$[J_z, J_\pm] = \pm\hbar J_\pm$$

$$J_z J_\pm |a, b\rangle = (b \pm \hbar) J_\pm |a, b\rangle$$

Call $b = \hbar\hat{b}$:

$$J_z J_\pm |a, b\rangle = \hbar\left(\hat{b} \pm 1\right) J_\pm |a, b\rangle$$

so the action of the raising and lowering operators is to raise and lower b.