Lecture 15: History Sample Space

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0.1 Consistent Histories

$$\begin{aligned}
Y^1 &= [z+] \odot [x+] \odot [z+] \\
Y^2 &= [z+] \odot [x+] \odot [z-] \\
Y^3 &= [z+] \odot [x-] \odot [z+] \\
Y^4 &= [z+] \odot [x-] \odot [z-] \\
Z &= [z-] \odot I \odot I.
\end{aligned}$$

Definition 1. The **History Sample Space** of a group of histories is $Y^{\vec{\gamma}} = Y^{\vec{\alpha}} + Y^{\vec{\beta}}$, where $Y^{\vec{\alpha}} Y^{\vec{\beta}} = 0$.

Because of this second condition, $|\vec{\gamma}\rangle = |\vec{\alpha}\rangle + \left|\vec{\beta}\right\rangle$, and $Pr(\vec{\gamma}) = Pr(\vec{\alpha}) + Pr(\vec{\beta}) = \langle\vec{\gamma}|\vec{\gamma}\rangle = \langle\vec{\alpha}|\vec{\alpha}\rangle + \left\langle\vec{\beta}\middle|\vec{\beta}\right\rangle + \left\langle\vec{\alpha}\middle|\vec{\beta}\right\rangle + \left\langle\vec{\beta}\middle|\vec{\alpha}\right\rangle$. It seems like our Generalized Born Rule has failed, due to these last two terms. However, if we require "consistency", the Rule still works.

Definition 2. A History Sample Space is **consistent** if $\left\langle\vec{\alpha}\middle|\vec{\beta}\right\rangle = 0, \forall \vec{\alpha} \neq \vec{\beta}$.

With these dynamics, the chainket for history 1 is $\left|Y^1\right\rangle = \frac{1}{2}|z+\rangle$:

$$\left|Y^1\right\rangle = |z+\rangle\langle z+|\, I\, |x+\rangle\langle x+|\, I\, |z+\rangle \tag{1}$$

$$\left\langle Y^1\middle|Y^3\right\rangle = \frac{1}{2}\langle z+|z+\rangle = \frac{1}{2} \neq 0 \tag{2}$$

Ergo, these histories are not consistent.

Let's imagine a system in a magnetic field $\vec{B} = B\hat{y}$:

$$T\colon\ |z+\rangle \to |x+\rangle \to |z-\rangle \to |x-\rangle \to -|z+\rangle \tag{3}$$

Under this dynamic(s)?, we find $\left|Y^1\right\rangle = \left|Y^3\right\rangle = \left|Y^4\right\rangle = 0$, $\left|Y^2\right\rangle = |z-\rangle$, so these histories are consistent in the dynamics of a constant magnetic field.

0.2 Beam Splitter

We will again use a discrete toy model space. Say we have three branches on a beam splitter, the branch a incoming, c outgoing perpendicular to a, and d outgoing parallel to a. We will call states in a $\{\ldots, -2a, -1a, 0a\}$ going toward the beam splitter from left to right. Similarly, $\{1c, 23c33, \ldots\}$ and $\{1d, 2d, 3d, \ldots\}$ go away from the beam splitter from left to right.

Our basis is $\mathcal{B} = \{|mz\rangle\, z \in a, c, d, m \in \mathbb{Z}\}$

$$T = S \Rightarrow S\,|mz\rangle = |(m+1)z\rangle \tag{4}$$

$$S\,|0a\rangle = \frac{1}{\sqrt{2}}\,(|1c\rangle + |1d\rangle) \tag{5}$$

For consistency, we also require a b branch with states labeled $\{\dots, -2b, -1b, 0b\}$ going parallel to c moving towards the beam splitter from left to right.

$$S\,|0b\rangle = \frac{1}{\sqrt{2}}\,(-\,|1c\rangle + |1d\rangle) \tag{6}$$

Our histories are then

$$[0a] \odot \{[1c], [1d]\} \odot \{[2c], [2d]\} \tag{7}$$

$$t = 0,\ \ |\psi_0\rangle = |0a\rangle \tag{8}$$

$$t = 1,\ \ |\psi_1\rangle = \frac{1}{\sqrt{2}}\,(|1c\rangle + |1d\rangle) \tag{9}$$

$$t = 2,\ \ |\psi_2\rangle = \frac{1}{\sqrt{2}}\,(|2c\rangle + |2d\rangle) \tag{10}$$

$$|(0a, 1c, 2c)\rangle = \frac{1}{\sqrt{2}}\,|2c\rangle \tag{11}$$

$$|(0a, 1d, 2d)\rangle = \frac{1}{\sqrt{2}}\,|2d\rangle \tag{12}$$

$$Pr([1c]_1, [2c]_2 \mid [0a]_0) = \frac{1}{2} = Pr([1d]_1, [2d]_2 \mid [0a]_0) \tag{13}$$

Additionally, we can calculate marginal probabilities from these:

$$Pr([1c]_1 \mid [0a]_0) = \frac{1}{2} = Pr([2c]_2 \mid [0a]_0) \tag{14}$$

$$Pr([1c]_1 \mid [2c]_2) = \frac{Pr([1c]_1, [2c]_2)}{Pr([2c]_2)} = 1 \tag{15}$$

$$Pr([2c]_2 \mid [2c]_2) = 0 \tag{16}$$

because that chainket would vanish:

$$|(0a, 1d, 2c)\rangle = [2c]_2 T_{21} [1d]_1 T_{10}\,|0a\rangle = 0 \tag{17}$$

In the coming lecture, we will introduce a measurement device on the c-branch, called $\hat{c}$. This device sits in the path and has two states, $0\hat{c}$ and $1\hat{c}$. Now our Hilbert space will have a basis $\{|mz\hat{c}\rangle\}$, so the whole space will be $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_d$, the product of the particle and detector spaces. Now our time evolution operator becomes $T = SR$, $R = I \otimes I$ except $R\,|2c, 0\hat{c}\rangle = |2c, 1\hat{c}\rangle$, switching the measurement device from the "ready" state to the "triggered" state. Acting R on a triggered state resets it to the ready state.