33-756 Homework 10

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1. Fresnel-Like Integrals

(a) $I = \int_{-\infty}^{\infty} e^{\imath\lambda x^2}\,\mathrm{d}x$.

We can define the following parameterization of a closed loop in complex space: $\Gamma = \gamma_1 + \gamma_2 + \gamma_3$ where

$$\begin{aligned}\gamma_1\colon &\quad x = t \quad \text{for} \quad t \in [0, R)\\ \gamma_2\colon &\quad x = Re^{\imath t} \quad \text{for} \quad t \in \left[0, \frac{\pi}{4}\right].\\ \gamma_3\colon &\quad x = te^{\imath\frac{\pi}{4}}\end{aligned}$$

The integral we want to calculate is

$$I = 2\int_{\gamma_1} e^{\imath\lambda x^2}\,\mathrm{d}x$$

as $R \to \infty$. In order to find this, we can show that the integral along γ_2 vanishes as $R \to \infty$, and because $e^{\imath\lambda x^2}$ is holomorphic, it will have no residues in this region so the integral around Γ will be equal to 0. Therefore, $\int_{\gamma_1} - \int_{\gamma_2} = \int_{\gamma_3}$ (here I have defined γ_3 in the anticlockwise direction so as to cancel the minus sign).
First, I will show that $\int_{\gamma_2} \to 0$ as $R \to 0$. The integral along this path is

$$I_{\gamma_2} = \int_{\gamma_2} e^{\imath\lambda x^2}\,\mathrm{d}x = \int_0^{\pi/4} e^{\imath\lambda R^2(\cos(2t)+\imath\sin(2t))}\imath Re^{\imath t}\,\mathrm{d}t$$

since $\mathrm{d}z = \imath Re^{\imath t}\,\mathrm{d}t$. We want to show that this integral vanishes, so we can equivalently show that its magnitude vanishes. by the triangle inequality,

$$\begin{aligned}\left|\int_0^{\pi/4} e^{\imath\lambda R^2(\cos(2t)+\imath\sin(2t))}\imath Re^{\imath t}\,\mathrm{d}t\right| &\le \int_0^{\pi/4} \left|e^{\imath\lambda R^2(\cos(2t)+\imath\sin(2t))}\right|\left|\imath Re^{\imath t}\right|\mathrm{d}t\\ &= R\int_0^{\pi/4} e^{-\lambda R^2\sin(2t)}\,\mathrm{d}t\end{aligned}$$

We can then use Jordan's inequality: $\frac{4t}{\pi} \le \sin(2t) \le 2t$ for $0 \le t \le \frac{\pi}{4}$:

$$|I_{\gamma_2}| \le R\int_0^{\pi/4} e^{-4\lambda R^2 t/\pi}\,\mathrm{d}t = \frac{\pi}{4\lambda R}\left(1 - e^{-\lambda R^2}\right)$$

It is obvious from here that this vanishes as $R \to \infty$, as long as $\lambda > 0$. Therefore, $I \equiv I_{\gamma_1} = I_{\gamma_3}$:

$$I = 2e^{\imath\frac{\pi}{4}}\int_0^{R\to\infty} e^{-\lambda x^2}\,\mathrm{d}x = e^{\imath\frac{\pi}{4}}\int_{-\infty}^{\infty} e^{-\lambda x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\lambda}}\left(e^{\imath\frac{\pi}{2}}\right)^{1/2} = \sqrt{\frac{i\pi}{\lambda}}$$

(b) $I = \int_{-\infty}^{\infty} e^{\imath(ax^2+bx+c)}\,\mathrm{d}x$.

In one dimension, the stationary phase approximation says that

$$\int_{-\infty}^{\infty} e^{\imath f(x)}\,\mathrm{d}x \approx \sum_{x_0\in\Sigma} e^{\imath f(x_0)+\mathrm{sign}(f''(x_0))\frac{\imath\pi}{4}}\sqrt{\frac{2\pi}{|f''(x_0)|}}$$

where Σ is the set of critical points $\partial_x f(x_0) = 0$. For this problem, there is only one critical point at $x_0 = -\frac{b}{2a}$, so we can plug in the proper values into the formula above to find that

$$I \approx e^{\imath\left(c-\frac{b^2}{4a}+\imath\frac{\pi}{4}\right)}\sqrt{\frac{2\pi}{2a}} = \sqrt{\frac{\imath\pi}{a}}e^{\imath\left(c-\frac{b^2}{4a}\right)}$$

To find the exact solution, we first must complete the square in the exponential:

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right)$$

Therefore, the integral will be

$$I = e^{\imath\left(c-\frac{b^2}{4a}\right)}\int_{-\infty}^{\infty} e^{\imath a\left(x+\frac{b}{2a}\right)^2}\,\mathrm{d}x$$

We can do a change of variables $x' = x+\frac{b}{2a}$, $\mathrm{d}x' = \mathrm{d}x$. The bounds of integration won't change under this transformation, and we have already done the integral of $e^{\imath a x'^2}$ above. We will therefore find

$$I = e^{\imath\left(c-\frac{b^2}{4a}\right)}\sqrt{\frac{\imath\pi}{a}}$$

which happens to be equal to the stationary phase approximation.

(c) $I = \int_{-\infty}^{\infty} e^{\imath(ax^2+Ax^4)}$.

Expanding in A we find

$$I \approx \int e^{\imath ax^2}\,\mathrm{d}x + \int \imath x^4 A e^{\imath ax^2}\,\mathrm{d}x + \mathcal{O}(A^2)$$

We already know the first integral, and the second one can be done by realizing that

$$\int \imath x^4 A e^{\imath ax^2}\,\mathrm{d}x = \int -\imath A\partial_a^2\left(e^{\imath ax^2}\right)\mathrm{d}x = -\imath A\partial_a^2\int e^{\imath ax^2}\,\mathrm{d}x$$

so

$$I = \sqrt{\frac{\imath\pi}{a}} - \imath A\partial_a^2\sqrt{\frac{\imath\pi}{a}} = \sqrt{\frac{\imath\pi}{a}} - \frac{3A}{4a^{5/2}}\imath\sqrt{\pi\imath}$$

I'd guess the condition for a convergent expansion is that $A << 1$ or maybe $A << a$. After some consideration, I don't think this was the correct way to do the problem. If we consider the stationary points to be $0, \pm\sqrt{-\frac{a}{2A}}$, we can see that expanding around 0 gives us

$$\begin{aligned} I &= e^{\imath 0+\imath\frac{\pi}{4}}\sqrt{\frac{2\pi}{|2a|}} + 2e^{-\imath\frac{a^2}{4A}-\imath\frac{\pi}{4}}\sqrt{\frac{2\pi}{6a}} \\ &= \sqrt{\frac{\imath\pi}{a}} + 2e^{-\imath\frac{a^2}{4A}}\sqrt{\frac{\imath\pi}{3a}} \end{aligned}$$

2. Frequency Space Propagator

The frequency space propagator for a particle moving in a potential V is given by

$$K(x_f, x_i, \omega) = \int_0^\infty K(x_f, x_i, t)e^{\imath\omega t}\,\mathrm{d}t = A\sum_n \frac{\sin(nrx_f)\sin(nrx_i)}{(E - \frac{\hbar^2 r^2}{2m}n^2)}$$

(a) Determine the potential V.

It's just a bit obvious that this is the propagator for a particle in a box. The sine functions in the numerator are eigenfunctions of that Hamiltonian and the energy in the denominator contains the eigenvalues with $r = \frac{\pi}{L}$. I will therefore derive the propagator for a particle in a box and show that it's Fourier transform is equal to the given equation. The particle in a box (in one dimension) can be described as

$$H = \frac{p^2}{2m} + V(x) \qquad V(x) = \begin{cases} 0 & 0 \le x \le L \\ \infty & \text{otherwise} \end{cases}$$

The eigensystem is

$$\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\Big(\frac{n\pi x}{L}\Big)$$
$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

The propagator is easily calculated:

$$\begin{aligned} K(x_f, x_i; t) &= \langle x| e^{-\imath Ht/\hbar} |x_0\rangle \\ &= \sum_n \langle x| e^{-\imath Ht/\hbar} |n\rangle \langle n|x_0\rangle \\ &= \sum_n e^{-E_n t/\hbar}\psi_n(x)\psi_0^*(x) \\ &= \frac{2}{L}\sum_n e^{\imath E_n t/\hbar}\sin\Big(\frac{n\pi x}{L}\Big)\sin\Big(\frac{n\pi x_0}{L}\Big) \end{aligned}$$

We can then Fourier transform this statement:

$$\begin{aligned} K(x_f, x_i; \omega) &= \frac{2}{L}\int_0^\infty \sum_n \sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{2\pi x_0}{L}\right)e^{-\imath Et/\hbar}e^{\imath\omega t}\,\mathrm{d}t \\ &= \frac{2}{L}\sum_n \sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{2\pi x_0}{L}\right)\int_0^\infty e^{\imath(\hbar\omega - E_n)t}\,\mathrm{d}t \\ &= \frac{2}{L}\sum_n \sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{2\pi x_0}{L}\right)\frac{\imath\hbar}{\hbar\omega - E_n} \\ &= A\sum_n \frac{\sin(nrx_f)\sin(nrx_i)}{(E - \frac{\hbar^2 r^2}{2m}n^2)} \end{aligned}$$

with $r = \frac{\pi}{L}$, $E = \hbar\omega$, and $A = \frac{2\imath\hbar}{L} = \frac{2\imath\hbar r}{\pi}$.

(b) Determine the constant A in terms of the other parameters in the problem.

See the end result of 2(a).

3. Sakurai 2.34

(a) Write down an expression for the classical action for a simple harmonic oscillator for a finite time interval.

The classical action for the interval $[t_a, t_b]$ for a simple harmonic oscillator is

$$S(t_a, t_b) \equiv \int_{t_a}^{t_b} \mathrm{d}t\, L(x(t), \dot{x}(t)) = \int_{t_a}^{t_b} \mathrm{d}t \left(\frac{1}{2} m \dot{x}(t)^2 - \frac{1}{2} m \omega^2 x(t)^2 \right)$$

(b) Construct $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$ for a simple harmonic oscillator using Feynman's prescription for $t_n - t_{n-1} = \Delta t$ small. Keeping only terms up to order $(\Delta t)^2$, show that it is in complete agreement with the $t - t_0 \to 0$ limit of the propagator given by (2.6.46).

We can write the action as

$$\begin{aligned} S(t_{n-1}, t_n) &= \int_{t_{n-1}}^{t_n} \mathrm{d}t \left[\frac{1}{2} m \dot{x}^2 - V(x) \right] \\ &= \Delta t \left[\frac{1}{2} m \left(\frac{x_n - x_{n-1}}{\Delta t} \right)^2 - V\left(\frac{x_n + x_{n-1}}{2} \right) \right] \\ &= \frac{1}{2\Delta t} m \left[x_n^2 - 2 x_n x_{n-1} + x_{n-1}^2 - \frac{\omega^2}{2} (x_n + x_{n-1})^2 \Delta t \right] \end{aligned}$$

Therefore, the propagator is

$$\begin{aligned} \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle &= \sqrt{\frac{m}{2\pi \imath \hbar \Delta t}} e^{\frac{\imath}{\hbar} \int \mathrm{d}t L} \\ &= \sqrt{\frac{m}{2\pi \imath \hbar \Delta t}} \exp\left[\frac{\imath \Delta t}{\hbar} \frac{m}{2} \left(\left[\frac{x_n - x_{n-1}}{\Delta t} \right]^2 - \omega^2 \left[\frac{x_n + x_{n-1}}{2} \right]^2 \right) \right] \\ &= \sqrt{\frac{m}{2\pi \imath \hbar \Delta t}} \exp\left[\frac{\imath}{2\Delta t} m \left[x_n^2 - 2 x_n x_{n-1} + x_{n-1}^2 - \frac{\omega^2}{2} (x_n + x_{n-1})^2 \Delta t \right] \right] \end{aligned}$$

As $\Delta t \to 0$, we can see that this will resemble the original formulation of the propagator, which was

$$\begin{aligned} K(x_n, t_n; x_{n-1}, t_{n-1}) = &\sqrt{\frac{m\omega}{2\pi \imath \hbar \sin(\omega \Delta t)}} \\ &\times \exp\left[\left(\frac{\imath m \omega}{2\hbar \sin(\omega \Delta t)} \right) \left((x_n^2 - x_{n-1}^2) \cos(\omega \Delta t) - 2 x_n x_{n-1} \right) \right] \end{aligned}$$

For small Δt, $\sin(\omega \Delta t) \approx \omega$ and $\cos(\omega \Delta t) \approx 1 - \frac{\omega^2}{2}$, which will make this equation resemble the derived version from the classical action.