Lecture 15:
Monday, October 05, 2020

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} = -\frac{\imath}{\hbar}\left(\langle f|\hat{H}'|i\rangle + \sum_{k\neq i}\frac{\langle f|\hat{H}'|k\rangle\ \langle k|\hat{H}'|i\rangle}{E_i - E_k}\right)e^{\imath(E_f - E_i)t}$$

From here,

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \sum_{k\neq i}\frac{\langle f|\hat{H}'|k\rangle\ \langle k|\hat{H}'|i\rangle}{E_i - E_k}$$

where

$$\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i) \qquad \text{(Fermi's Golden Rule)}$$

0.1 Decay Rates and Cross Sections

$$\rho(E_i) =$$

0.1.1 Phase Space and Normalization

In the Born approximation, we treat initial and final states as momentum eigenstates

$$\psi(x,t) = Ae^{\imath(\vec{\mathbf{p}}\cdot\vec{\mathbf{r}} - Et)}$$

To normalize within a cube of side a, $A = \frac{1}{a^{3/2}}$ with periodic boundary conditions: $\psi(x+a,y,z) = \psi(x,y,z)$ or rigid/open boundaries $\psi(a,y,z) = 0$. The allowed states that match the boundary conditions will be an array of $m_i \in \mathbb{Z}$: $\vec{\mathbf{p}} = (m_x, m_y, m_z)\frac{2\pi}{a}$.

The volume of phase space for $(2N_{\text{max}})^3$ states is $(2N_{\text{max}})^3\left(\frac{2\pi}{a}\right)^3$, so the volume of $\vec{\mathbf{p}}$-space per state is $\frac{(2\pi)^3}{V}$. Since the components can be positive or negative, to find all states with momentum between p and $p + \mathrm{d}p$, we can count states within a spherical shell of radius p and thickness $\mathrm{d}p$:

$$\mathrm{d}n = 4\pi p^2\frac{\mathrm{d}p}{\frac{(2\pi)^3}{V}}$$

so

$$\frac{\mathrm{d}n}{\mathrm{d}p} = \frac{4\pi p^2}{(2\pi)^3}V$$

Let's make the normalization volume $V = 1$ so $a = 1$. For each independent momentum,

$$\rho(E) = \mathrm{d}n\, E = \frac{4\pi p^2}{(2\pi)^3}\left|\frac{\mathrm{d}p}{\mathrm{d}E}\right|$$

For a decay to N particles, $\vec{\mathbf{p}} = \sum_{i=1}^{N}\vec{\mathbf{p}}_i$, so one $\vec{\mathbf{p}}_i$ is not independent:

$$\vec{\mathbf{p}}_N = \vec{\mathbf{p}}_a - \sum_{i=1}^{N-1}\vec{\mathbf{p}}_i$$

The total number of states for the N particles is

$$\mathrm{d}n = \prod_{i=1}^{N-1}\mathrm{d}n_i = \prod_{i=1}^{N-1}\frac{\mathrm{d}^3p_i}{(2\pi)^3}$$

or

$$\begin{aligned} \mathrm{d}n &= \prod_{i=1}^{N-1} \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \delta^3(\vec{\mathbf{p}}_a - \sum_{i=1}^{N} \vec{\mathbf{p}}_i)\, \mathrm{d}^3\vec{\mathbf{p}}_N \\ &= (2\pi)^3 \prod_{i=1}^{N} \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \delta^3(\vec{\mathbf{p}} - \sum_{i=1}^{N} \vec{\mathbf{p}}_i) \end{aligned}$$

0.1.2 Lorentz-Invariant Phase Space

Volume is not Lorentz invariant. The direction of motion is length contracted. To correct for this, we introduce ψ' with a different normalization. While $\langle\psi|\psi\rangle = 1$, we will have $\langle\psi'|\psi'\rangle = 2E$.