Lecture 10: Spherical Symmetry

September 23, 2019

For spherical solutions to the Laplace equation where we utilize the whole angular space,

$$\Phi(r,\theta,\phi) = \sum_l \sum_{-l\le m\le l} [A_l r^l + B_l r^{-(l+1)}] Y_{lm}(\theta,\phi) \tag{1}$$

. When there is an axis of symmetry, only the $m = 0$ term survives, and we can use $P_l(\cos\theta)$ rather than the Y_{lm} solutions, although they are not normalized.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{2}$$

is an expansion where we take $\vec{x}'$ to be an axis of symmetry and say $r' > r$. As $\vec{x} \to \vec{x}'$,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|r-r'|} = \frac{1}{r}\frac{1}{1-\frac{r'}{r}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum A_l(r')\frac{1}{r^{l+1}}P_l(\cos 0) \Rightarrow \sum \frac{1}{r^{l+1}}r'^l = \sum \frac{A_l(r')}{r^{l+1}} \tag{3}$$

.

Example. Take a circle of charge made from taking vectors of length c an angle α from the z-axis. Find the potential at a location $\vec{x}$.

If we imagine that the ring forms a sphere separating vectors which are smaller and larger than $\vec{c}$, $\Phi = \sum \frac{A_l}{r^{l+1}} P_l(\cos\theta)$ for $|\vec{x}| > c$. The axis of symmetry is the z-axis, so we can take a special vector along this axis to help us find the coefficients. On axis,

$$\Phi = \frac{2\pi(c\sin\alpha)\lambda}{4\pi\varepsilon_0\sqrt{r^2+c^2-2rc\cos\alpha}} = \sum \frac{A_l}{r^{l+1}} P_l(\cos\theta)\Big|_{\cos\theta\to 1} \tag{4}$$

. This also has an expansion on the right side, since

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{5}$$

so

$$\frac{\lambda}{2\varepsilon_0} \sum \frac{c^l}{r^{l+1}} P_l(\cos\alpha) = \sum \frac{A_l}{r^{l+1}} \tag{6}$$

so

$$\Phi(r,\theta) = \frac{\lambda c\sin\alpha}{2\varepsilon_0} \sum_l \frac{c^l P_l \cos\alpha}{r^{l+1}} P_l(\cos\theta) \tag{7}$$

for $r > c$. We can expand this further if we ignore the ϕ symmetry:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum \frac{g_{lm}(r',\theta',\phi')}{r^{l+1}} Y_{lm}(\theta,\phi) \tag{8}$$

.

Say $r > r'$. Then,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m}\sum_{l',m'} \frac{B_{lm;l'm'}r'^{l'}}{r^{l+1}} Y_{lm}(\theta,\phi)Y^*_{l'm'}(\theta',\phi') \tag{9}$$

.

Recall,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum \frac{r'^l}{r^{l+1}} P_l(\cos\gamma(\theta,\phi,\theta',\phi')) = \sum B_{l,m,m'}\left(\frac{r'^l}{r^{l+1}}\right) Y_{lm}(\theta,\phi)Y^*_{lm'}(\theta',\phi') \tag{10}$$

. If we let $\phi \to \phi'$, $\cos(\phi-\phi') \to 1$, so $m = m'$. Therefore

$$\sum \frac{r'^l}{r^{l+1}} P_l(\cos\gamma) = \sum B_{lm}\frac{r'^l}{r^{l+1}} Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \tag{11}$$

. Rotational symmetry reduces B_{lm} to B_l, so we can look at the special case where $\theta' \to 0$, $P_l(\cos\gamma) \to P_l(\cos\theta)$, and $Y_{lm} \to Y_{l0} = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$. Finally, we see that $B_l\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{2l+1}{4\pi}} = 1$ so $B_l = \frac{4\pi}{2l+1}$.

We have found

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m}\left(\frac{r^l_<}{r^{l+1}_>}\right)\frac{4\pi}{2l+1} Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \tag{12}$$

. ◇

0.0.1 Green's Functions in Spherical Coordinates

For Dirichlet Green's Functions, we must have that the function vanishes at the boundaries; the potential is specified and constant there. Suppose we have a problem of concentric spherical shells of radius $a < b$.

Recall if we know $G_D(x,x')$,

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int G_D(x,x')\rho(x')d^3x' - \frac{1}{4\pi}\oint_\Sigma \frac{\partial G_D}{\partial n'}\Phi(x')da' \tag{13}$$