Lecture 28: Selected Magnetic Density Problems

Monday, October 21, 2019

0.1 Magnetic Dipole Density Examples

Example. A magnetized ball: $\vec{M} = M_0 \hat{z}$ **1st Method**

$$\vec{J}_M = \boldsymbol{\nabla} \times \vec{M} = 0$$

$$\vec{K}_M = \vec{M} \times \hat{n} = M_0 \sin(\theta) \hat{\varphi}$$

This is exactly like the rotating sphere homework:

$$\vec{A} = \frac{\mu_0}{4\pi} \oint \frac{\vec{K}_M(x')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}a'$$

2nd Method

$$\vec{H} = -\boldsymbol{\nabla} \Phi_M$$

$$\vec{J}_{\text{free}} = 0$$

$$\nabla^2 \Phi_M = -[-\boldsymbol{\nabla} \cdot \vec{M}]$$

Recall that we derived the form of Φ_M:

$$\Phi_M = \frac{1}{4\pi} \int_\Omega \frac{-\boldsymbol{\nabla} \cdot \vec{M}}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x + \frac{1}{4\pi} \oint \frac{M_0 \cdot \hat{n}'}{|\vec{x} - \vec{x}'|} \, \mathrm{d}a'$$

By our definition of $\vec{M}$:

$$\boldsymbol{\nabla} \cdot \vec{M} = 0$$

However, there is a surface term:

$$\vec{M} \cdot \hat{n} = M_0 \cos(\theta)$$

Therefore:

$$\Phi_M = \frac{1}{4\pi} \oint_{S^2} \frac{M_0 \cos(\theta')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}\Omega' \, a^2 = \frac{M_0 a^2}{4\pi} \int \frac{\cos(\theta')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}\Omega' = \frac{M_0 a^2}{4\pi} \frac{4\pi}{3} \frac{r_<}{r_>^2} \underbrace{P_1(\cos(\theta))}_{\cos(\theta)}$$

Therefore,

$$\Phi_M = \begin{cases} \frac{M_0 a^2}{3} \frac{r}{a^2} \cos(\theta) = \frac{M_0}{3} z & r < a \\ \frac{M_0 a^2}{3} \frac{a}{r^2} \cos(\theta) = \frac{m \cos(\theta)}{4\pi r^2} & r > a \end{cases}$$

where $\vec{m} = \left(\frac{4\pi}{3} a^3\right) M_0 \hat{z}$.

$$\vec{H}_{\text{in}} = -\frac{M_0}{3} \hat{z}$$

$$\vec{H}_{\text{out}} \propto \text{dipole field}$$

$$\vec{B}_{\text{in}} = \mu_0 \left[-\frac{M_0}{3} \hat{z} + M_0 \hat{z}\right] = \frac{2}{3} \mu_0 M_0 \hat{z}$$

◇

Example. Let us consider putting such a sphere into an external field. We would then imagine, by superposition, that $\vec{B}_0+\frac{2}{3}\mu_0\vec{M}=\vec{B}_{\text{in}}$. Additionally, this means that $\vec{H}_{\text{in}}-\frac{1}{3}\vec{M}$. The solution must be self-consistent, such that

$$\vec{H}_{\text{in}}=\frac{1}{\mu}\vec{B}_{\text{in}}$$

This gives the relation

$$\frac{1}{\mu_0}\vec{B}_0-\frac{1}{3}\vec{M}=\frac{1}{\mu}\left[\vec{B}_0+\frac{2}{3}\mu_0\vec{M}\right]$$

so

$$\vec{M}=\frac{3}{\mu_0}\left[\frac{\mu-\mu_0}{\mu+2\mu_0}\right]\vec{B}_0$$

◇

Example. Magnetic Shielding We now have a shell with inner radius a and outer radius b with an external magnetic field. Again, let us assume $\vec{J}_{\text{free}}=0$ (the field is curl-free):

$$\vec{H}=-\boldsymbol{\nabla}\Phi_M$$

Because $\boldsymbol{\nabla}\cdot\vec{B}=0$,

$$\nabla^2\Phi_M=0$$

Using the azimuthal symmetry of this problem, we can write

$$\Phi_M=\begin{cases}\sum_{l=0}^{\infty}\alpha_l r^l P_l(\cos(\theta)) & r<a\\ \sum_{l=0}^{\infty}\left[\beta_l r^l+\frac{\gamma_l}{r^{l+1}}\right]P_l(\cos(\theta)) & a<r<b\\ -H_0 r\cos(\theta)+\sum_{l=0}^{\infty}\frac{\delta_l}{r^{l+1}}P_l(\cos(\theta)) & b<r\end{cases}$$

Our first boundary condition is that the magnetic field B is continuous normal to the boundaries at a and b. Additionally, the tangential component is H_θ, which must also be continuous at each boundary. This problem is left as an exercise for the reader. The solution is given in Jackson. If $\mu>>1$, there is strong magnetic shielding. ◇

0.2 Faraday's Law

For a surface Σ and loop Γ such that $\Gamma=\partial\Sigma$, the boundary of the surface, the energy gained by going around the loop once is

$$\mathscr{E}=-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\vec{B}\cdot\hat{n}\,\mathrm{d}a$$

where $\int_\Sigma\vec{B}\cdot\hat{n}\,\mathrm{d}a=$ flux so $\mathscr{E}=-\frac{\mathrm{dflux}}{\mathrm{d}t}$. This "electromotive force" or "emf" $\mathscr{E}$ corresponds to an electric field felt on the loop induced by the magnetic field in the rest frame of the loop. We can then say that

$$\mathscr{E}=\oint_{Gamma}\vec{E}'\cdot\mathrm{d}\vec{l}$$

The electric field can no longer be curl-free (it's in a loop, after all). Because the surface is fixed, we can write

$$\oint_\Gamma \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma \vec{B} \cdot \mathrm{d}\vec{a} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{a}$$

By Stokes' Theorem, this implies

$$\int \left(\boldsymbol{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) \cdot \mathrm{d}\vec{a} = 0$$

so we must now modify Maxwell's equations to include

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This is only in the rest frame!However, if $v << c$, $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$, so Faraday is consistent.