Lecture 43: The Helmholtz Equation in Spherical Coordinates

Monday, November 25, 2019

In the previous lecture, we were able to write the fields in the radiation zone in a form which utilized the magnetic dipole and multipole moments:

$$\vec{\mathbf{E}}_\omega^{\text{dipole}} = -\frac{Z_0 k^2}{4\pi}(\hat{\mathbf{n}} \times \vec{\mathbf{m}}_\omega)\frac{e^{\imath kr}}{r}$$

$$\vec{\mathbf{B}}_\omega^{\text{multipole}} = -\frac{\imath c k^3}{8\pi}\frac{1}{3}\left[\hat{\mathbf{n}} \times \vec{\mathbf{Q}}[\hat{\mathbf{n}}]\right]\frac{e^{\imath kr}}{r}$$

We can calculate the differential power as it relates to the solid angle by

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \left(\frac{1}{2\mu_0}\vec{\mathbf{E}}_\omega \times \vec{\mathbf{B}}_\omega\right) \cdot \hat{\mathbf{n}} r^2$$

so

$$\begin{aligned} P &\propto \int \left[(\hat{\mathbf{n}} \times \vec{\mathbf{m}}_\omega) \times (\hat{\mathbf{n}} \times \vec{\mathbf{Q}}^*)\right] \cdot \hat{\mathbf{n}} \,\mathrm{d}\Omega \\ &\propto \int \vec{\mathbf{m}}_\omega \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{Q}}^*) \,\mathrm{d}\Omega \\ &\propto \int [m_i \epsilon_{ijk} n_j Q_{kl} n_l] \,\mathrm{d}\Omega \\ &\propto m_i \epsilon_{ijk} Q_{kl}^* \delta_{jl} = 0 \end{aligned}$$

since

$$\int n_j n_l \,\mathrm{d}\Omega = \frac{4\pi}{3}\delta_{jl}$$

and δ is completely symmetric while ϵ is completely antisymmetric.

0.1 Helmholtz Equation in Spherical Coordinates

The Helmholtz equation

$$\left(\nabla^2 + k^2\right)\psi = 0$$

can be written in spherical coordinates as

$$\frac{1}{r^2}\partial_r r^2 \partial_r + \left(k^2 - \frac{l(l+1)}{r^2}\right) f_{lm} = 0$$

where

$$\psi = \sum_{lm} f_{lm}(r) Y_{lm}(\Omega)$$

Assuming spherical symmetry, $f_{lm} \to f_l$, and we can write $f_l = \frac{u_l}{\sqrt{r}}$ and solve for $u_l(r)$ to simplify this equation:

$$\left[\partial_r^2 + \frac{1}{r}\partial_r + \left(k^2 - \frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right)\right] u_l(r) = 0$$

This is very similar to the Bessel equation, and the solutions for u_l are known as the spherical Bessel functions:

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x)$$

which is regular at $x = 0$,

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x)$$

which is singular at $x = 0$, and

$$h_l^{(1,2)} = j_l(x) \pm \imath n_l(x)$$

These functions have the following recursion relations and expansions:

$$j_l(x) = (-x)^l \left[\frac{1}{x}\partial_x\right]^l \left(\frac{\sin(x)}{x}\right)$$

$$n_l(x) = -(-x)^l \left[\frac{1}{x}\partial_x\right]^l \left(\frac{\cos(x)}{x}\right)$$

As $x \to 0$ (or $x << 1$),

$$j_l(x) \mapsto \frac{x^l}{(2l+1)!!}\left[1 - \frac{x^2}{2(2l+3)} + \cdots\right]$$

$$n_l(x) \mapsto \frac{-(2l-1)!!}{x^{l+1}}\left[1 - \frac{x^2}{2(1-2l)} + \cdots\right]$$

As $x \to \infty$ (or $x >> 1$),

$$j_l(x) \mapsto \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right)$$

$$n_l(x) \mapsto -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right)$$

and

$$h_l^{(1)} \mapsto (-\imath)^{l+1}\frac{e^{\imath x}}{x}$$

This last equation is the kind of outgoing wave behavior which we want in a radiative solution.

Additionally, for all $j_l,\ n_l,\ h_l = z_l$,

$$\frac{2l+1}{x} z_l(x) = z_{l-1}(x) + z_{l+1}(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x z_l(x)\right] = x z_{l-1}(x) - l z_l(x)$$

Finally, the Wronskian for the spherical Bessel functions is

$$W\left[j_l, n_l\right] = \frac{1}{\imath} W\left[j_l, h_l^{(1)}\right] = \frac{1}{x^2}$$

Quote

"Almost everything you can imagine is a thing you cannot write"
- Turgut, on solutions to equations

Quote

"The world of functions is very wild and crazy"
- Turgut, also on solutions to equations

0.2 Green's Function for the Spherical Helmholtz Equation

$$(\nabla^2 + k^2) G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = -\delta(\vec{\mathbf{x}} - \vec{\mathbf{x}}') \mapsto \frac{e^{\imath k |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} = \frac{\delta(r - r')}{r^2} \underbrace{\delta(\Omega - \Omega')}_{\sum_{lm} Y^*_{lm}(\Omega') Y_{lm}(\Omega)}$$

Note the missing 4π in front of the δ-function. This is just a scaling factor and only slightly effects how the Green's function is applied.

We can therefore write the Green's function as

$$G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = \sum_{lm} g_l(r, r') Y_{lm}(\Omega) Y^*_{lm}(\Omega')$$

If we integrate the differential equation for g_l around r', we find that

$$\int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r \left[\frac{1}{r^2} \partial_r r^2 \partial_r g_l\right] = -\int_{r'-\epsilon}^{r'+\epsilon} \frac{\delta(r - r')}{r^2}\, \mathrm{d}r'$$

so

$$\left.\frac{\mathrm{d}g_l}{\mathrm{d}r}\right|_{r'+\epsilon} - \left.\frac{\mathrm{d}g_l}{\mathrm{d}r}\right|_{r'-\epsilon} = -\frac{1}{r'^2}$$

so

$$g_l(r, r') = A_l j_l(kr_<) h_l^{(1)}(kr_>)$$

since we want regular behavior at 0 and oscillatory behavior at ∞. We can use the Wronskian to determine the factor A_l:

$$G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = (\imath k) j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}(\Omega) Y_{lm}^*(\Omega')$$