Lecture 29: Potential Scattering

Wednesday, October 30, 2019

First, some review of the exam:

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle)$$

and

$$|F\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle)$$

$$Y^1 = [\psi_0] \odot [A] \odot [F]$$

so

$$\left|\psi^1\right\rangle = [F][A]\,|\psi_0\rangle = [F]\frac{1}{\sqrt{3}}\,|A\rangle = \frac{1}{3}\,|F\rangle$$

Next,

$$Y^2 = [\psi_0] \odot [A] \odot (I - [F])$$

so

$$\left|Y^2\right\rangle = (I - [F])[A]\,|\psi_0\rangle = (I - [F])\frac{1}{\sqrt{3}}\,|A\rangle$$

Instead of writing out $I - [F]$, we can just distribute:

$$\left|Y^2\right\rangle = \frac{1}{\sqrt{3}}\,|A\rangle - \frac{1}{3}\,|F\rangle$$

Let's look at one of the probability questions now:

$$\Pr([A]_1 \mid [\psi_0][F_1]) = \frac{\Pr(\psi_0, A_1, F_2)}{\Pr(\psi_0, F_1)} = \frac{\Pr\left(Y^1\right) = \frac{1}{9}}{\Pr(Y^1) + \Pr(Y^3) = \frac{1}{9} + 0} = 1$$

Now back to potential scattering:

0.0.1 Square Well Scattering

$$V(x) = \begin{cases} -V_0 & |x| > a/2 \\ 0 & |x| < a/2 \end{cases}$$

We know solutions are exponential, proportional to $e^{\pm\imath kx}$ with $k = \sqrt{E}$ outside the well and $e^{\pm\imath k'x}$ with $k' = \sqrt{V_0 + E} > k$ inside the well.

$$\phi = \begin{cases} Ae^{\imath kx} + Be^{-\imath kx} & x < -a/2 \\ Ce^{\imath k'x} + De^{-\imath k'x} & -a/2 < x < a/2 \\ Fe^{\imath kx} + Ge^{-\imath kx} & a/2 < x \end{cases}$$

Boundary conditions at $-a/2$ give us

$$Ae^{-\imath ka/2} + Be^{\imath ka/2} = Ce^{-\imath k'a/2} + De^{\imath k'a/2}$$

and

$$\underbrace{kAe^{-\imath ka/2} - kBe^{\imath ka/2}}_{P\begin{pmatrix}A\\B\end{pmatrix}} = \underbrace{k'Ce^{-\imath k'a/2} - k'De^{\imath k'a/2}}_{Q\begin{pmatrix}C\\D\end{pmatrix}}$$

so

$$\begin{pmatrix}A\\B\end{pmatrix} = P^{-1}Q\begin{pmatrix}C\\D\end{pmatrix}$$

where $P^{-1}Q = R$.

By time reversal symmetry (taking the complex conjugate of φ and noting that the Schrödinger equation is invariant under this operation), we can conclude that

$$\begin{pmatrix} A^* \\ B^* \end{pmatrix} = R \begin{pmatrix} C^* \\ D^* \end{pmatrix}$$

so $R_{11}^* = R_{22}$ and $R_{12}^* = R_{21}$

$$R = \sqrt{\frac{k'}{k}} \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$$

The incident current is

$$J_{\text{inc}} = k(|A|^2 - |B|^2) = J_{\text{transf}} = k'(|C|^2 - |D|^2)$$

therefore,

$$|\alpha|^2 - |\beta|^2 = 1$$

or

$$\det\left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \right\} = 1$$

Applying the same method of time-reversal and current conservation to the other boundary. We can reuse R evaluated at any other point in space (called $\tilde{R}$). Therefore

$$\begin{bmatrix} A \\ B \end{bmatrix} = \underbrace{R\tilde{R}}_{M} \begin{bmatrix} F \\ G \end{bmatrix}$$

$$M = \begin{bmatrix} \gamma & \delta \\ \delta^* & \gamma^* \end{bmatrix}$$

$$|\gamma|^2 - |\delta|^2 = 1$$

so

$$\gamma = \alpha^2 - \beta^2 = e^{\imath ka} \left[\cos(k'a) - \imath \frac{k^2 + k'^2}{2kk'} \sin(k'a) \right]$$

is the reflection coefficient.

For transmission, set $A = 1$, and we want $G = 0$. Our equation now becomes

$$\begin{bmatrix} 1 \\ B \end{bmatrix} = M \begin{bmatrix} F \\ 0 \end{bmatrix}$$

so

$$F = \frac{1}{M_{11}}$$

and

$$T = |F|^2 = \frac{1}{|\gamma|^2} = \frac{1}{1 + \frac{(k^2 - k'^2)^2}{2k^2k'^2} \sin^2(k'a)}$$

Recall that $k' = \sqrt{V_0 + k^2}$, so plotting the coefficients as a function of k, we see the following result:

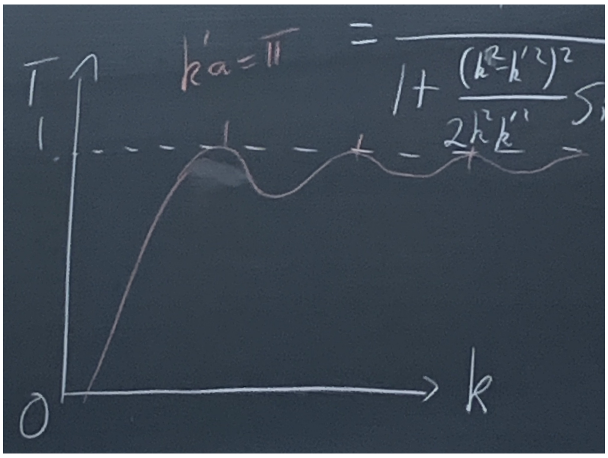

Figure 0.0.1: Graph of k vs T