The Turgut Lectures on Electromagnetism

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Contents

LECTURE 1: ELECTRODYNAMICS
Mon Aug 26 2019

0.1 Microscopic Theory

Basic objects: $\vec{E}$ and $\vec{B}$ fields with some source called Charge. Charge is a *locally conserved* quantity, meaning we will have some macroscopic current $\vec{J}$ and volume density ρ, and sometimes a surface density σ. Local conservation means that for a volume V with a current flowing out of it $\vec{J}$, and this volume has an orientable surface with an outward normal $\hat{n}$. The amount of charge leaking out is:

$$\oint \vec{J} \cdot \hat{n} da = -\frac{dQ}{dt}.$$

where $Q = \int_V \rho d^3x$. Therefore, conservation means

$$-\frac{d}{dt}\int_V \rho d^3x = \oint \vec{J} \cdot \hat{n} da = \int \nabla \cdot \vec{J} d^3x.$$

so

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} \right) dv = 0.$$

- $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$
- Charge cannot "magically" appear and disappear, it must be moved by a current.

These densities and currents are all macroscopic. We know that charge cannot be divided into arbitrarily small pieces, since electrons exist. SI units are the Coulomb, and even for macroscopic purposes, it's rather big. The electron charge in these units are $\|e\| \approx 1.6 \times 10^{-19}$C. Macroscopically, you don't have to think in terms of units of charge, since a single Coulomb will have 10^{19} particles in it.

In a vacuum, the Maxwell Equations relating these fields to their sources are:

1. $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$, ($\epsilon_0 = \frac{10^7}{4\pi c^2}$F/m)
2. $\nabla \cdot \vec{B} = 0$ (no monopoles/source for the magnetic field, *probably*)
3. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4. $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$, ($\mu_0 = 4\pi 10^{-7}$H/m)

Pro Tip

Discover magnetic charges for a free Nobel Prize

Additionally, this contains the connection between electrodynamics and light:

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

The Maxwell Equations imply local charge conservation. As long as there are no materials, the charges are like point charges.

0.2 "Idealized" Point Charges

We can understand a charge density to be

$$\rho(\vec{x}) = \sum_i q_i \delta(\vec{x} - \vec{r}_i(t))$$

Remark.

$$\delta(\vec{x} - \vec{x}' = \delta(x - x')\delta(y - y')\delta(z - z')$$

and

$$\int_{\vec{a} \in V} f(\vec{x})\delta(\vec{x} - \vec{a})d^3x = f(\vec{a})$$

and

$$\int_{\vec{a} \notin V} f(\vec{x})\delta(\vec{x} - \vec{a})d^3x = 0$$

Jackson notes that in this theory, we stay in the classical regime. Quantum is important if we deal with extremely strong fields, high energies, or short distances (on the order of a few Angstroms). If we go to large fields and short distances, we may need corrections. In this case, non-linearities appear which are not noticeable in ordinary electrodynamics.

In the ordinary macroscopic case for a large number of "photons", a 1mV/m field strength radio wave has a photon flux around 10^{12}photons/cm^{2}s. How do these fields interact with charges?

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

This cannot be obtained from Maxwell's Equations. The self-interaction problem is difficult to solve in classical electrodynamics, and it will generally be ignored in this class.

0.3 Materials (Continuous Media)

Microscopic dynamics of atoms of the medium are too complicated, so we find an approximate description. We take averages over a small volume element (and also short time scales).

$\vec{E} \mapsto \vec{P}$ (average dipole moment density of the medium) $\rho_b = -\nabla \cdot \vec{P}$ for surface charges $\sigma_b = \vec{P} \cdot \hat{n}$ Similarly, $\vec{B} \mapsto \vec{M}$ (average magnetic dipole density of the medium) $\vec{J} = \nabla \times \vec{M}$ for surface currents (and volume currents) $\vec{k} = \vec{M} \times \hat{n}$

It turns out, generally $\vec{P} = \vec{P}\left[\vec{E}, \vec{B}\right]$, where the brackets denote these are in general functionals; they could be nonlinear, could have time delays, etc. Similarly, $\vec{M} = \vec{M}\left[\vec{E}, \vec{B}\right]$. If you know these things, you can write down a set of consistent tensor equations. For linear media:

$$P_\alpha = \epsilon_{\alpha\beta} E_\beta$$

For isotropic, homogeneous media, this does not depend on position.

In media, Maxwell's Equations are modified to:

1. $\nabla \cdot \vec{E} = \frac{\rho_{\text{free}}}{\epsilon_0} + \frac{\rho_{\text{bound}}}{\epsilon_0} \implies \nabla \cdot \left[\epsilon_0 \vec{E} + \vec{P}\right] = \nabla \cdot \vec{D} = \rho_{\text{free}}$
2. $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$, so $\nabla \times \vec{H} = \vec{J}_{\text{free}}\left[\vec{E}, \vec{B}\right] + \frac{\partial \vec{D}}{\partial t}$
3. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4. $\nabla \cdot \vec{B} = 0$

Things like time-lag are responsible for wavelength dependence.

Lecture 2: Electrostatics

Mon Aug 26 2019

0.4 The Electric Field

Start with the first equation:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Assume there is no $\vec{B}$ field, or at least $\vec{B}$ is not changing in time (electrostatics). Also, $\vec{E}$ won't change with time, and there will be no currents, so the only equations left are the first one and the equation corresponding to the magnetic source ($\nabla \cdot \vec{B} = 0$), as well as $\nabla \times \vec{E} = 0$.

0.4.1 Integral Form

For a stationary surface Σ with charges inside, the divergence equation says that:

$$\int_V (\nabla \cdot \vec{E}) dv = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \oint_\Sigma \vec{E} \cdot d\vec{a}$$

Consider a stationary point charge q. Take a spherical shell around the charge (S^2 sphere) of radius r with outward normal vector $\hat{r}$. By symmetry, $\vec{E} = E(r)\hat{r}$ since the curl is zero. Gauss's law tells us

$$\oint \vec{E} \cdot d\vec{a} = E(r) \oint_{S^2} \hat{r} \cdot d\vec{a} = E(r) 4\pi r^2 = \frac{q}{\epsilon_0}.$$

Therefore, $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$ when q is at the center of the sphere.

$$\vec{E}(\vec{x}) = \frac{q}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|^2} \cdot \left[\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}\right] = \frac{q}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|^3} \cdot [\vec{x} - \vec{x}']$$

for an arbitrary position. Here, $\vec{x}'$ is the vector pointing from the origin to the charge and $\vec{x}$ is the vector pointing to the position of observation.

In general:

$$\vec{E} = \int_\Omega \frac{\rho(\vec{x}')dv'}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|^3} \cdot [\vec{x} - \vec{x}']$$

for some charge distribution in a volume Ω.

Let's look at $-\nabla\left[\frac{1}{|\vec{x}-\vec{x}'|}\right]$. If you were to expand out the denominator and take the gradient, you would get

$$-\frac{1}{2}\frac{1}{\text{something}^{3/2}} 2(x_i - x_i')\hat{e}_i$$

so

$$-\nabla\left[\frac{1}{|\vec{x} - \vec{x}'|}\right] = \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}.$$

Therefore, we can use

$$\vec{E}(\vec{x}) = -\nabla\left[\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')dv'}{|\vec{x} - \vec{x}'|}\right],$$

where the piece in the brackets is a scalar, called the scalar potential (only when there are no boundaries around). If $\vec{E} = -\nabla\Phi$ then $\nabla \times \vec{E} = \vec{0}$. Boundaries would mean some materials exist in the problem, so the properties of these materials will complicate the problem.

Remark.

$$\nabla \cdot \frac{\vec{r}}{r^3} \Rightarrow \partial_i \frac{x_i}{r^3} = \frac{3}{r^3} - 3\frac{x_i}{r^4}\frac{x_i}{r} = \frac{3}{r^3} - \frac{3r^2}{r^5} = 0.$$

However, this is not exactly true. It is true as long as $r \neq 0$. However, if it is, these derivatives are not justified.

$$\int_{S^3} \frac{\vec{r}}{r^3} dv = \oint_{S^2} \frac{\vec{r}}{r^3} \hat{r} d\vec{a} = 4\pi.$$

Therefore,

$$\nabla \cdot \frac{\vec{r}}{r^3} = 4\pi\delta(\vec{r}).$$

This is very useful, as we can show that,

$$\nabla \frac{q(\vec{x}-\vec{x}')}{4\pi\epsilon_0|\vec{x}-\vec{x}'|^3} = \frac{1}{\epsilon_0} q\delta(\vec{x}-\vec{x}').$$

Say we have a charge in an electric field moving from point A to point B. The change in kinetic energy is the integral of the work done, or

$$\Delta(\text{KE}) = -\int_A^B q\vec{E} \cdot d\vec{r} \Rightarrow \frac{1}{2}mv_B^2 + q\Phi(\vec{x}_B) = \frac{1}{2}mv_A^2 + q\Phi(\vec{x}_A)$$

0.4.2 Ideal Conductors

They are "ideal" meaning they have a sufficient number of charges such that in static equilibrium, $\vec{E} = \vec{0}$ inside an ideal conductor. This automatically means $\rho = 0$ inside—there is only surface charge σ on an ideal conductor. The electric field is zero inside, and $\vec{E} \parallel \vec{n}$ outside (perpendicular to the surface). $\vec{E} = -\nabla\Phi$ is perpendicular to Φ-constant surfaces. This implies Φ is constant on the surface of the conductor. $\vec{E} = 0$ on the inside implies conductors are equipotential regions.

On the surface, to calculate anything, we take a small Gaussian pillbox with a thickness $\delta \to 0$ across the boundary. The electric field will therefore be outwardly perpendicular to the surface (away from the conductor): $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}_+$.

Let us formulate a problem in an electrostatic system in the presence of a conductor. Either you put charges inside conductors or you put the conductors at certain potentials. Pretend we can keep the conductors at a certain constant potential with an "idealized" cable connected to a battery. We could also put charges on them, such that the total charge on a conductor is, say, Q. Maybe we'd have some ρ outside and ask what the potential is at a given point in space. Every vector field can be decomposed into a pure curl and pure gradient part. If we knew the surface charge distributions on all the conductors, we could write down the solution easily:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} + \sum_i \oint_{\Sigma_i} \frac{\sigma_i(\vec{x}')da'}{4\pi\epsilon_0|\vec{x}-\vec{x}'|}.$$

However, we don't know the σ_is. we can write down some equation $\epsilon_0[-\nabla\Phi\hat{n}] = \sigma$, or

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} + \sum_i \oint_{\Sigma_i} \frac{[-\nabla\Phi](\vec{x}')da'}{4\pi|\vec{x}-\vec{x}'|}.$$

This is not the most practical way to solve the problem. Typically, you turn this "integral" equation into a "differential" equation:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \nabla \cdot [-\nabla\Phi] = \nabla^2\Phi$$

Either Φ is given on the boundaries (Dirichlet Problem for the Poisson Equation), or $\partial_t\Phi$ is given (Neumann Problem).

Lecture 3: The Laplace Equation
Wed Aug 28 2019

0.5 Uniqueness of Solutions

Last time, we ended with the problem $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$(+ boundary conditions). Say we had a solution for ϕ; how do we know the solution is unique? Mathematically, we can prove such a solution always must exist, but the proof is beyond the scope of the course. From a physical perspective, it obviously has to exist (there must be a potential defined at every point in space). However, we can prove that there is only one unique solution using one of Green's identities:

$$\int_\Omega \nabla \cdot (U\nabla U)d^3x = \oint_\Sigma U\nabla U \cdot d\vec{a}$$

Suppose we have two solutions for the potential, called ϕ_1 and ϕ_2, and we will say the difference in the solutions is $U = \phi_1 - \phi_2$. As stated above, we will either be given $\phi\Big|_\Sigma$ or $\frac{\partial\phi}{\partial n}\Big|_\Sigma$, depending on the kind of problem. Our solution must differ on these boundaries, so either

$$\phi_1\Big|_\Sigma - \phi_2\Big|_\Sigma = 0 = U\Big|_\Sigma$$

or

$$\frac{\partial\phi_1}{\partial n}\Big|_\Sigma - \frac{\partial\phi_2}{\partial n}\Big|_\Sigma = 0 = \nabla U\Big|_\Sigma.$$

The second part of the identity then becomes 0, since one of the two terms must be zero. Now we can evaluate the first part (note that we can't just set the stuff inside the parentheses to zero, since we aren't evaluating on the boundary Σ, but rather on the surface Ω as a whole:

$$\int_\Omega \nabla \cdot (U\nabla U)d^3x = 0 = \int_\Omega [(\nabla U)^2 + (U\nabla^2 U)]d^3x$$

$\nabla^2 U = 0$ since U satisfies the Laplace equation subject to either of these boundary conditions. Therefore,

$$0 = \int_\Omega (\nabla U)^2 d^3x \Rightarrow \nabla U = 0.$$

Therefore, ϕ_1 and ϕ_2 can only differ by a constant value.

0.6 Solving the Laplace Equation

Note:

What follows is not the notes from class, but rather the same explanation from Jackson's textbook.

If our problems always involved localized discrete charges or completely continuous charge distributions with no boundaries, we could simply use

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')}{|x-x'|}d^3x'$$

as a convenient solution. However, when we have boundaries separating regions with and without charge, we almost always have to use Green's theorem (unless the problem is extremely simple). The Green's theorems/identities follow from the divergence theorem:

$$\int_V \nabla\cdot\vec{A}d^3x = \oint_S \vec{A}\cdot\vec{n}da$$

This applies to any "well-behaved" vector field $\vec{A}$ inside a volume V bounded by the closed surface S. Suppose $\vec{A} = \phi\nabla\psi$, where ϕ and ψ are two arbitrary scalar fields. From some vector math, we can show that:

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

and

$$\phi\nabla\psi\cdot\vec{n} = \phi\frac{\partial\psi}{\partial n}$$

where $\frac{\partial}{\partial n}$ is derivative at the surface directed outward normal from V. By substituting this into the divergence theorem, we obtain

Definition 0.6.1. Green's First Identity:

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}da$$

If we now switch ϕ with ψ, write the identity down again, and subtract it the identity that is not interchanged, the $\nabla\psi \cdot \nabla\phi$ terms cancel, and we get

Definition 0.6.2. Green's Second Identity (A.K.A. Green's Theorem):

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da$$

When we solve for different potentials, the main process will involve Green's Theorem. We will take some "magic" ψ, which we will name $G(x, x')$ (a Green's Function), and we will relabel $\phi = \Phi$ (our scalar potential). We will also use the fact that $\nabla^2\Phi = -\rho/\epsilon_0$.

Green's functions are a class of functions satisfying the equation

$$\nabla'^2 G(x, x') = -4\pi\delta(x - x'),$$

where the ∇' acts on the primed variable. One such function is $G = \frac{1}{|\vec{x}-\vec{x}'|}$, the potential of a unit point source. In general, Green's functions are of the form

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$$

where F satisfies the Laplace equation inside the volume: $\nabla'^2 F(x, x') = 0$.

If we were to solve Green's Second Identity for the potential using one of these Green's functions for ψ and $\phi = \Phi$, the potential, we would obtain:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')G(x, x')d^3x' + \frac{1}{4\pi}\oint_S \left[G(x, x')\frac{\partial\Phi}{\partial n'} - \Phi(x')\frac{\partial G(x, x')}{\partial n'}\right] da'$$

Examine the part with the surface integral. For **Dirichlet boundary conditions** (Φ is given on the surface), we can demand that $G_D(x, x') = 0$ for x' on the surface S. This will reduce the above equation to:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')G_D(x, x')d^3x' - \frac{1}{4\pi}\oint_S \Phi(x')\frac{\partial G_D(x, x')}{\partial n'}da'$$

Additionally, for Dirichlet boundary conditions, $G_D(x, x') = G_D(x', x)$. This is an observation of symmetry between observation and source points. The Green's function is a potential due to a unit point source, so switching which coordinate corresponds to the source and the observer does not change its value. For Neumann boundary conditions, symmetric functions can exist, although they aren't automatically symmetric and need a separate requirement to have this symmetry imposed. This is outside the scope of the class.

In the case of **Neumann boundary conditions**, we might think we can set $\frac{\partial G_N}{\partial n'}(x, x') = 0$ for x' on S, but if we apply Gauss's theorem, we can see that the surface integral does not vanish, in fact,

$$\oint_S \frac{\partial G}{\partial n'} da' = -4\pi$$

The simplest allowable boundary condition on G_N is

$$\frac{\partial G_N}{\partial n'}(x, x') = -\frac{4\pi}{\text{Area}(S)}$$

which reduces the solution to:

$$\Phi(x) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0} \int_V \rho(x') G_N(x, x') d^3x' + \frac{1}{4\pi} \oint_S G_N(x, x') \frac{\partial \Phi}{\partial n'} da'$$

where $\langle \Phi \rangle_S$ is the average value of the potential over the whole surface.

Note:

We have not solved any actual potentials yet, simply derived tools for solving them called Green's functions.

LECTURE 4: LAPLACE EQUATION

Fri Aug 30 2019

0.7 Review

Dirichlet Problem:

$$G_D(x, x') = 0$$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int G_D(x, x') \rho(x') + \frac{1}{4\pi} \oint_\Sigma \frac{\partial G_D}{\partial n'_+} da' \Phi(x')$$

$$G_D(x, x') = G_D(x', x)$$

Neumann Problem:

We can't impose $\left.\frac{\partial G_N}{\partial n_-}\right|_\Sigma = 0$, so we will impose $\left.\frac{\partial G_N}{\partial n_-}\right|_\Sigma = -\frac{4\pi}{\text{Area}(\Sigma)}$:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int G_N(x,x')\rho(x') + \langle\Phi\rangle_\Sigma + \oint_\Sigma G_N(x,x')\frac{\partial\Phi}{\partial n'_-}da'$$

$$G_N(x,x') = G_N(x',x)$$

If we only have conductors raised to potentials Φ_i (constants), then the charge in the jth conductor becomes:

$$\begin{aligned} Q_j &= -\frac{1}{4\pi}\oint_{\Sigma_j}\oint_{\Sigma_i}\frac{\partial^2 G}{\partial n_+ \partial n'_+}dada'\Phi_i \\ &= \sum_j C_{ji}\Phi_i \end{aligned}$$

Remark. For $\nabla^2\Phi = 0$, the potential satisfies this equation at charge free regions. In charge free regions, $\Phi(x)$ is given by an average over any sphere around x as long as the sphere is in the charge free region:

$$\Phi(x) = \frac{1}{4\pi b^2}\int_{S^2}\Phi(x + b\hat{\xi})da$$

where b is the radius of the sphere and $\hat{\xi}$ is the normal outwards. $da = b^2 d\Omega$ and

$$\frac{\partial}{\partial b}\langle\Phi\rangle_{S_b^2} = \frac{\partial}{\partial b}\frac{1}{4\pi}\oint\Phi(x + b\hat{\xi})d\Omega = \frac{1}{4\pi}\oint\nabla\Phi\cdot\hat{\xi}d\Omega = \frac{1}{4\pi}\int_V\nabla\cdot(\nabla\Phi)d^2x = 0$$

since $\nabla\cdot(\nabla\Phi) = 0$. This implies Φ has no max or min apart from the charged regions or boundaries. Suppose there was a maximum at x_*. Take a small sphere around x_* and average it, all the values on the sphere will be less than $\Phi(x_*)$, so the average will be less than the "true" value. Therefore, there are no true stable equilibrium points in electrostatics.

0.8 Energy Considerations

In free space, if we have point charges,

$$W = \frac{1}{2}\sum_{i\neq j}\frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|x_i - x_j|}$$

(Jackson uses "W" for energy). This is like the cost of bringing in charges from infinity. Alternatively, $W = \frac{1}{2}\epsilon\int_{\text{everywhere}} E^2 d^3x$ for continuous charge distributions (for point charges, you get infinities).

Let us derive $W = \frac{1}{2}\epsilon_0\int E^2 d^3x$:

The work to add an infinitesimal charge $\delta\rho(x)$ to a continuous distribution is

$$\delta W = \int_\Omega \Phi(x)\delta\rho(x)d^3x$$

$$\nabla \cdot \delta E = \delta\rho/\epsilon_0$$

$$\begin{aligned}
\delta W &= \epsilon_0 \int_\Omega \Phi(x)\nabla\cdot(\delta E)d^3x = \epsilon_0 \int \nabla\cdot[\Phi(x)\delta E]d^3x - \epsilon_0\int_\Omega \nabla\Phi\cdot\delta E d^3x \\
&= \epsilon_0 \oint_\Sigma \Phi(x)\delta E\cdot d\vec{a}_- + \epsilon\int_\Omega(-\nabla\Phi)\cdot\delta E d^3x \\
&= \epsilon_0 \sum_i \left(\oint_{\Sigma_i}\delta E\cdot da_-\right)\Phi_i + \epsilon_0\int E\cdot\delta E d^3x
\end{aligned}$$

$$\epsilon_0 \sum_i \left(\oint_{\Sigma_i}\delta E\cdot da_-\right)\Phi_i = 0,$$

so

$$\delta W = \epsilon\int_\Omega E\cdot\delta E d^3x = \delta\left(\frac{\epsilon_0}{2}\int_\Omega E^2 d^3x\right)$$

The $1/2$ here comes from pulling the δ out of the integral.

So $W = \frac{\epsilon_0}{2}\int_\Omega E^2d^3x + W_0$. $W\to 0$ as $|E|\to 0$ so $W_0\equiv 0$.

In the presence of conductors,

$$W = \frac{1}{2}\int_\Omega \Phi\rho d^3x + \frac{1}{2}\sum_{k=1}^N Q_k\Phi_k$$

Remark. $\delta W = \sum_i (C^{-1})_{ik}Q_k\delta Q_i$, therefore $\delta W = \delta\left(\frac{1}{2}\sum Q_i(C^{-1})_{ik}Q_k\right)$.

Lecture 5: Green's Functions for Special Geometries

Wed Sep 4 2019

The Method of Images

Suppose we have a grounded conductor along the xy-plane and a charge q at position (x', y', z') off of the plane. We claim the Green's function involves a charge at $-z'$ with opposite charge:

$$G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

$\nabla'^2 \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \right) = 0$ so we still have $-\nabla'^2 G_D = 4\pi\delta(\vec{x}-\vec{x}')$.

$$G_D(\vec{x},\vec{x}')\Big|_{\Sigma=\{z'=0|(x,y,0)\in\mathbb{R}^3\}} = 0$$

Solving for Potential

When we put an actual charge q at (x', y', z'),

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]$$

Surface charge:

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial z}\Big|_{z=0}$$

Note:

The method of images is a bit contrived, since the solution is assumed and the Green's function is then derived.

Grounded Sphere

Suppose we have a grounded sphere of radius a with a point charge q outside.

Suppose the distance from the center of the sphere to the charge is x'. Introduce a new point along the line joining the origin and the charge:

$$y' = \frac{a^2}{x'}\hat{n}'$$

(pointing away from the origin).

Geometrically, you take the tangent from the circle to the point and project the point on the sphere where it touches onto the radial axis. We can use this as a 1-to-1 map between the inside and outside of the sphere. The sphere itself is invariant under this transformation. Alternative mappings would only work if they obey the Laplace equation!

$$\Phi = \frac{q}{4\pi\epsilon_0|\vec{x}-\vec{x}'|} + \frac{q'}{4\pi\epsilon_0|\vec{x}-\frac{a^2}{x'}\hat{n}'|}.$$

The second part follows the Laplace equation for $\vec{x}$, but it is not trivial that it also is a solution for $\vec{x}'$ for particular choice of q.

$$\nabla^2\Phi = \frac{q}{\epsilon_0}$$

(outside, so $|\vec{x}| \geq a$)

$$\begin{aligned}\Phi\Big|_{\Sigma=S^2 \text{ with radius } a} &= \frac{q}{4\pi\epsilon_0|a\hat{n}-x'\hat{n}'|} + \frac{q'}{4\pi\epsilon_0|a\hat{n}-x'\hat{n}'|}\\ &= \frac{q}{4\pi\epsilon_0\sqrt{a^2+x'^2-2ax'\hat{n}\cdot\hat{n}'}} + \frac{q'}{4\pi\epsilon_0\sqrt{a^2+\frac{a^4}{x'^2}-2a\frac{a^2}{x'}\hat{n}\cdot\hat{n}''}}\\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2+x'^2-2ax'\hat{n}\cdot\hat{n}'}} + \frac{q'}{\frac{a}{x'}\sqrt{a^2+\frac{a^4}{x'^2}-2a\hat{n}\cdot\hat{n}'}}\right]\end{aligned}$$

On the surface,

$$\Phi\Big|_{\Sigma} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{\cdots}} + \frac{q'\frac{x'}{a}}{\sqrt{\cdots}}\right]$$

If $q' = -q\frac{a}{x'}$, $\Phi\Big|_{\Sigma} = 0$, so

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-\vec{x}'|} - \frac{q\frac{a}{x'}}{|\vec{x}-\frac{a^2}{x'^2}\vec{x}'|}\right]$$

$$\sigma\Big|_{\Sigma} = -\epsilon\frac{\partial\Phi}{\partial r}\Big|_{r=a}.$$

If we integrate this, $\oint_{S^2}\sigma da = q'$.

Remark. We actually construct the Green's function for this problem thanks to $-\nabla'^2 G_D = 4\pi\delta(\vec{x}-\vec{x}')$, where $G_D = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{\frac{a}{x'}}{\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}'\right|}$

Force of Attraction on Charges

What is the force between the sphere and the charge?

$$\oint_{S^2}\frac{1}{4\pi\epsilon_0}\frac{q\sigma(a^2 d\Omega)}{|\vec{x}-\vec{x}'|^2}\frac{(\vec{x}'-\vec{x})}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi\epsilon_0}\frac{q\cdot -q\frac{a}{x'}}{|\frac{a^2}{x'^2}\vec{x'}-\vec{x'}|^2}\hat{n}'$$

Basically the force between the surface charge σ and point charge q is the same as the force on q due to q', the image charge.

What about a charged sphere? Suppose we have a charge Q on the sphere. Find the grounded solution ($Q = 0$) and superimpose the effect. The grounded sphere has charge $q' = -\frac{q}{x'}a$, so the potential is now

$$\frac{1}{4\pi\epsilon_0}\frac{Q + \frac{q}{x'}a}{|\vec{x}|} + \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x} - \vec{x}'|} - \frac{q\frac{a}{x'}}{\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}\right]$$

Energy

What is the electrostatic energy of this configuration? Inside, $E = 0$, but so $W = 0$, but outside,

$$W = \frac{1}{2}\epsilon_0 \int E^2 dx = \frac{1}{2}\int \rho\Phi d^3x$$

We have to remove the infinity which comes from the E field from the point charge itself. The self-energy of this charge would work if it's a continuous distribution, but the discreteness of the charge messes with the integral.

N.B.

For a charged sphere, $W = \frac{2}{5}\frac{Q^2}{4\pi\epsilon_0 R}$. When $R \to 0$, we have an obvious problem. This is inherent in classical electrodynamics. It is "cured" in QM, but in a complicated way.

$$W = \frac{1}{2}\int q\delta(\vec{x} - \vec{x}')\Phi^{\text{reduced}}(\vec{x})d^3x$$

where

$$\Phi^{\text{reduced}} = \frac{1}{4\pi\epsilon_0}\frac{-\frac{q}{x'}a}{\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}.$$

LECTURE 6: GREEN'S FUNCTIONS ON SPECIAL GEOMETRIES, CONTINUED

Mon Sep 2 2019

0.9 Continued Sphere Example

$$G_D = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{\frac{a}{x'}}{\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}$$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int G_D(x,x')\rho(x')d^3x' - \frac{1}{4\pi}\oint \frac{\partial G_D(x,x')}{\partial n'}\Phi(x')da'$$

If $\rho(x') = 0$ and $\Phi(x)$ is given on the sphere, $\frac{\partial G_D}{\partial n'} = -\frac{\partial G_D}{\partial r'}$.

In spherical coordinates, you can compute

$$-\left.\frac{\partial G_D}{\partial r'}\right|_{r'=a} = \frac{(a^2 - r^2)}{a[r^2 + a^2 - 2ar\cos\gamma]^{3/2}}$$

$$\begin{aligned}\cos\gamma = \hat{n}\cdot\hat{n}' &= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - phi') \\ &= \cos\theta\cos\theta' + \sin\theta\sin\phi\sin\theta'\sin\phi' + \sin\theta\cos\phi\sin\theta'\cos\phi'.\end{aligned}$$

Now we can solve for the potential:

$$\Phi(x) = -\int \frac{(a^2 - r^2)}{a[r^2 + a^2 - 2ar\cos\gamma]^{3/2}}\Phi(a,\theta',\phi')a^2 d\Omega'.$$

$d\Omega' = \sin\theta' d\theta' d\phi'$ so

$$\Phi(r,\theta,\phi) = -\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \frac{a(a^2 - r^2)}{[r^2 + a^2 - 2ar\cos\gamma]^{3/2}}\Phi(a,\theta',\phi')\sin\theta'.$$

where $\Phi(a, ...)$ is the given potential on the surface of the sphere.

0.9.1 How to Solve the Laplace Equation Directly

We will compare this with the Green's function approach later. The Green's function is more painful to construct, but it applies to more general systems, whereas this solution will only apply to an individual problem.

Say we have a box joining the xy, xz, and yz-planes with planes at $x = a$, $y = b$, and $z = c$. Let's say all the surfaces are kept at $\Phi = 0$ except for the $z = c$ plane, which has some potential $V(x,y)$.

Separation of Variables

Assume $\Phi = X(x)Y(y)Z(z)$, so $\nabla^2\Phi = X''YZ + XY''Z + XYZ'' = 0$

Assuming Φ never vanishes inside, we can divide by XYZ:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Each of these is independent, so the only solutions involve these fractions being constants. Let us pick $\frac{X''}{X} = -\alpha^2$, $\frac{Y''}{Y} = -\beta^2$, and $\frac{Z''}{Z} = (\alpha^2 + \beta^2)$. The solutions for these are not difficult. $X = A\sin(\alpha x) + B\cos(\alpha x)$, $Y = C\sin(\beta y) + D\cos(\beta y)$.

We know $X(0) = 0$, so $B = 0$. Also, $X(a) = 0$ so $\sin(\alpha a) = 0$, so $\alpha = \frac{n\pi}{a},\ n = 1, 2, ...$ Additionally, $D = 0$ and $\beta = \frac{m\pi}{b},\ m = 1, 2, ...$

We can write out a solution for Z using hyperbolic functions:

$$Z(z) = E\sinh\left(\sqrt{\alpha^2+\beta^2}z\right) + F\cosh\left(\sqrt{\alpha^2+\beta^2}z\right).$$

$Z(0) = 0 \Rightarrow F = 0$. We have one more boundary condition, namely $Z(c) = V(x, y)$. By superposition,

$$\Phi = \sum_{m,n} A_{mn}\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}z\right)$$

Now we are given $\Phi(x, y, z = c) = V(x, y)$. If we set this up, we would find

$$V(x,y) = \sum_{m,n} A_{mn}\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}c\right)$$

If $V(x, y)$ is well-behaved,

$$A_{mn} = \frac{4}{ab\sinh\left(\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}c\right)}\int_0^a da\int_0^b dyV(x,y)\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{b}y\right)$$

We will not justify this solution, Dirichlet solved it a long time ago. Right after this, Sturm and Liouville showed that any differential operator $\mathcal{D} = \frac{d}{dx}(p(x)\frac{d}{dx}) + q(x)$ will have a complete basis on a finite interval.

Digression:

We have $\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ as a basis of the set of "nice" functions on $[0,a]$. This is a complete basis, so summing over all n of two basis elements will give $\delta(x-x')$ and integrating over x over two basis elements on the interval will yield δ_{nm}.

Sturm-Liouville problems require boundary conditions where either the function or its derivative (not both) vanish at the boundaries (you also need to specify which occurs at which boundary for both boundaries).

$$\int_0^a [f(x)(\mathcal{D}g(x)) - (\mathcal{D}f)g(x)]dx = [f(x)g'(x) - g(x)f'(x)]\Big|_0^a = 0$$

then $\mathcal{D}f_\lambda = \lambda f_\lambda$, so $|f_\lambda\rangle$ form an orthonormal basis.

LECTURE 7: STURM-LIOUVILLE PROBLEMS WITH PERIODIC FUNCTIONS
Fri Sep 6 2019

The other case where Sturm-Liouville still works is when the function is periodic.

Example. Periodic Functions: Suppose $\mathcal{D} = -\frac{d^2}{dx^2}$. The spectrum here is $\mathcal{D}\sin\frac{n\pi x}{a} = \frac{n\pi^2}{a^2}\sin\frac{n\pi x}{a}$. $f_n \equiv \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ for odd functions which are periodic on $[-a,a]$.

If you let the intervals become $(-\infty,+\infty)$, operators like $\imath\frac{d}{dx} \to \frac{e^{\imath kx}}{\sqrt{2\pi}}$, $k \in \mathbb{R}$. The eigenvalues are no longer discrete, but $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath kx}e^{\imath k'x}dx = \delta(k-k')$ still (a generalized orthonormality condition). Also, $\frac{1}{2\pi}\int dke^{\imath k(x-x')} = \delta(x-x')$ as a generalized completeness theorem. ◇

0.10 Cylindrical Symmetry

Take concentric cylinders with inner radius R_1, outer radius R_2, and given potentials on the edges of each cylinder. We will use cylindrical coordinates (ρ,ϕ,z). First, we will deal with the 2D version in which Φ has no z-dependence. In the example in Jackson, we have two large conducting sheets joined at an angle β at an insulated corner. To find the solution inside the wedge, given the potentials on each sheet, we can also use cylindrical coordinates.

What is the Laplacian for cylindrical coordinates?

$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$ or $d\vec{x} = d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}$.

$$\nabla\Phi \cdot d\vec{x} = d\Phi = \partial_\rho \Phi d\rho + \cdots$$

$$\nabla\Phi = [\frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} + \frac{\partial\Phi}{\partial z}\hat{z}] \cdot d\vec{x}$$

We can use this to find the Laplacian:

$\int(\nabla\Phi)^2 d^3x = -\int \Phi\nabla^2\Phi d^3x$. The volume element is $\rho d\rho d\phi dz$. We can solve this by integrating by parts. The Laplacian is therefore

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}.$$

If you have z-dependence, you have to use Bessel functions. Let's avoid that for now. With no z-dependence, we have,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi = 0$$

, assuming no inside charge.

Assume separation of variables, $\Phi = R(\rho)\Psi(\phi)$. Now we have

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}R + \frac{1}{\rho^2}\frac{1}{\Psi}\frac{d^2}{d\phi^2}\Psi = 0 \Rightarrow \frac{1}{R}\left(\rho\frac{d}{d\rho}\right)\left(\rho\frac{d}{d\rho}\right)R + \frac{1}{\Psi}\frac{d^2}{d\phi^2}\Psi = 0 = +\nu^2 - \nu^2$$

(they must be constants because we can vary ϕ and ρ independently but they still add up to 0).

$$\Psi_\nu = A_\nu \sin\nu\phi + B_\nu\cos\nu\phi$$

$$R_n u = a_\nu\rho^\nu + b_\nu\rho^{-\nu}$$

If $\nu = 0$, the logarithm also solves the original equation; $R_0 = a_0 + b_0\ln\rho$. We can't have a logarithm of a dimension-full thing, so the a_0 must also have some dimensional log term to divide it out.

The general solution by superposition can be written:

$$\Phi = (A_0 + B_0\phi)(a_0 + b_0\ln\rho) + \int d\nu(a_\nu\rho^\nu + b_\nu\rho^{-\nu})(\sin(\nu\phi + \alpha_\nu))$$

In the angle problem, we restrict $\Phi\Big|_{\phi=0} = V_0$, so $\alpha_\nu = 0$, $b_0 = 0$, and $b_\nu = 0$ (if we assume finite fields at $\rho = 0$). With $\Phi\Big|_{\phi=\beta} = V_0$, we can say $A_0 = V_0$, $B_0 = 0$. $\sin\nu\beta = 0$ would mean that killing the term at the boundary requires a discrete $\nu = \frac{m\pi}{\beta}$. Therefore

$$\Phi = V_0 + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta} + \phi\right).$$

However, the power term is infinite as $m \to \infty$.

Lecture 8: Review

Mon Sep 9 2019

The general solutions, again, are:

$$\nu \neq 0 \quad [a_\nu \rho^\nu + b_\nu \rho^{-\nu}] \sin(\nu\phi\alpha_\nu)$$

$$\nu = 0 \quad [a_0 + b_0 \ln \rho][A_0 + B_0 \rho]$$

In the case where the potential on both planes is V_0, $\alpha_\nu = 0$, and from the periodicity of the $\nu \neq 0$ condition, we can say $\nu = \frac{m\pi}{\beta}$. This discretizes ν:

$$\Phi = V_0 + \sum_{m=1}^{\infty} [a_m \rho^{\frac{m\pi}{\beta}} + b_m \rho^{-\frac{m\pi}{\beta}}] \sin\left(\frac{m\pi}{\beta}\phi\right)$$

There is another unspecified parameter which concerns what happens really far away and really up close. Let's assume we only want a solution which is finite at the vertex. Only the a_m terms will remain finite here:

$$\Phi = V_0 + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta}\phi\right)$$

What does the vector field look like?

$$\vec{E} = -\nabla\Phi = -\partial_\rho \Phi \hat{\rho} - \frac{1}{\rho}\partial_\rho \Phi \hat{\phi} = -a_m \rho^{\frac{m\pi}{\beta}-1} \sin\left(\frac{m\pi}{\beta}\phi\right)\hat{\rho} - \sum a_m \frac{m\pi}{\beta} \rho^{\frac{m\pi}{\beta}-1} \cos\left(\frac{m\pi}{\beta}\phi\right)\hat{\phi}$$

Suppose $\beta > \pi$. This implies $E \propto \rho^{\frac{\pi}{\beta}-1}$ as $\rho \to 0^+$, so the field diverges in the corner if the corner is a sharp edge.

0.11 Spherical Coordinates

$$\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2$$

Let us look at this from another perspective, an angular moment operator:

$$\vec{\mathbb{L}} = \vec{x} \times (-\imath\vec{\nabla})$$

$$\vec{x}\cdot\vec{\mathbb{L}} = 0$$

$$\mathbb{L}_l = (-\imath)\epsilon_{lmn}x_m\partial_n$$

$$-\mathbb{L}^2 = r^2\nabla^2 - \partial_r r^2\partial_r$$

Now we can see that

$$\nabla^2 = \frac{1}{r^2}\partial_r r^2\partial_r - \frac{\mathbb{L}^2}{r^2}$$

If we are dealing with completely spherical boundaries, we need the full range of ϕ and θ.

If we say $\hbar = 1$, this is the same as the angular momentum operator from quantum:

$$[\mathbb{L}^2, f(r)] = 0$$

$$[\mathbb{L}^2, \mathbb{L}_z] = 0$$

$$[\mathbb{L}_z, f(r)] = 0$$

$$\mathbb{L}^2|lm\rangle = l(l+1)|lm\rangle$$

$$\mathbb{L}_z|lm\rangle = m|lm\rangle$$

where

$$\langle\theta,\phi|lm\rangle = Y_{lm}(\theta,\phi)$$

Lecture 9: More on Y_{lm} Functions

Mon Sep 9 2019

N.B.

I will be using L for $\mathbb{L}$ from here onward.

If we don't have the full range of the spherical angles, we actually have to solve the original ∇^2 differential equations and can't use L and L^2 or the Y_{lm} functions.

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{\imath m\phi}$$

where

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

Orthogonality tells us:

$$\int Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) d\Omega = \delta_{ll'}\delta_{mm'}$$

The spectral decomposition tells us:

$$\sum_{lm} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

since $\delta(f(x)) = \frac{1}{f'(x_0)}\delta(x - x_0)$.

N.B.

In EM, we write Y_{lm} such that $Y_{lm}(\theta, \phi) = (-1)^m Y^*_{l,-m}(\theta, \phi)$.

For general spherical solutions,

$\Phi = \sum g_{lm}(r) Y_{lm}(\theta, \phi)$ or

$$\frac{1}{r^2}\partial_r r^2 \partial_r g - \frac{l(l+1)}{r^2} = 0$$

so $r^2\partial_r^2 g + 2r\partial_r g - l(l+1)g = 0$.

Suppose $g = r^\lambda$:

$$[\lambda(\lambda-1) + 2\lambda - l(l+1)]r^\lambda = 0$$

so $\lambda = l$ or $-(l+1)$

Therefore, the general solution in spherical systems (which use the periodicity in both ϕ and θ) is:

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A_l r^l + B_l r^{-(l+1)}] Y_{lm}(\theta, \phi).$$

0.11.1 Systems with ϕ-independence

If we have an axis of symmetry, set the z-axis as the axis of symmetry. Therefore solutions should be independent of the angle around the z-axis (ϕ).

This means $Y_{lm} \to Y_{l0}$ so $P_l^m \to P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l$, which are not normalized for "historical reasons". The differential equation then becomes:

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_l + l(l+1)P_l = 0,\ x \in [-1, 1].$$

Remark. There are other solutions $Q_l(x) = \frac{1}{2}P_l(x) \ln\left[\frac{1-x}{1+x}\right] + R_l(x)$ where R_l is a polynomial of degree $l-1$.

Additionally $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2l+1}{2}d_{ll'}$

Jackson notes some "easy" relations from Rodriguez's formula:

1. $\frac{d}{dx}P_{l+1} - \frac{d}{dx}P_{l-1} - (2l+1)P_l = 0$
2. $(l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0$
3. $P_{2k}(0) = \frac{(2k-1)!!}{2^k k!}(-1)^k$

It can be shown that:

$$\frac{1}{|\vec{x} - \vec{x}'|} = \begin{cases} \sum \frac{1}{r^{l+1}} A_l P_l(\cos\gamma) & r > r' \\ \sum r^l B_l P_l(\cos\gamma) & r < r' \end{cases}$$

where γ is the angle between the vectors.

Equivalently,

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

where $r_<$ and $r_>$ correspond to the smaller and larger vector.

LECTURE 10: SPHERICAL SYMMETRY

Wed Sep 11 2019

For spherical solutions to the Laplace equation where we utilize the whole angular space,

$$\Phi(r,\theta,\phi) = \sum_l \sum_{-l\le m\le l} [A_l r^l + B_l r^{-(l+1)}] Y_{lm}(\theta,\phi)$$

. When there is an axis of symmetry, only the $m = 0$ term survives, and we can use $P_l(\cos\theta)$ rather than the Y_{lm} solutions, although they are not normalized.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

is an expansion where we take $\vec{x}'$ to be an axis of symmetry and say $r' > r$. As $\vec{x} \to \vec{x}'$,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|r-r'|} = \frac{1}{r}\frac{1}{1-\frac{r'}{r}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum A_l(r')\frac{1}{r^{l+1}}P_l(\cos 0) \implies \sum \frac{1}{r^{l+1}} r'^l = \sum \frac{A_l(r')}{r^{l+1}}$$

.

Example. Take a circle of charge made from taking vectors of length c an angle α from the z-axis. Find the potential at a location $\vec{x}$.

If we imagine that the ring forms a sphere separating vectors which are smaller and larger than $\vec{c}$, $\Phi = \sum \frac{A_l}{r^{l+1}} P_l(\cos\theta)$ for $|\vec{x}| > c$. The axis of symmetry is the z-axis, so we can take a special vector along this axis to help us find the coefficients. On axis,

$$\Phi = \frac{2\pi(c\sin\alpha)\lambda}{4\pi\epsilon_0\sqrt{r^2+c^2-2rc\cos\alpha}} = \sum \frac{A_l}{r^{l+1}} P_l(\cos\theta)\bigg|_{\cos\theta\to 1}$$

. This also has an expansion on the right side, since

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

so

$$\frac{\lambda}{2\epsilon_0}\sum \frac{c^l}{r^{l+1}} P_l(\cos\alpha) = \sum \frac{A_l}{r^{l+1}}$$

so

$$\Phi(r,\theta) = \frac{\lambda c\sin\alpha}{2\epsilon_0}\sum_l \frac{c^l P_l(\cos\alpha)}{r^{l+1}} P_l(\cos\theta)$$

for $r > c$. We can expand this further if we ignore the ϕ symmetry:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum \frac{g_{lm}(r',\theta',\phi')}{r^{l+1}} Y_{lm}(\theta,\phi)$$

.

Say $r > r'$. Then,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m}\sum_{l',m'} \frac{B_{lm;l'm'} r'^{l'}}{r^{l+1}} Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta',\phi')$$

.

Recall,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum \frac{r'^l}{r^{l+1}} P_l(\cos\gamma(\theta,\phi,\theta',\phi')) = \sum B_{l,m,m'} \left(\frac{r'^l}{r^{l+1}}\right) Y_{lm}(\theta,\phi) Y^*_{lm'}(\theta',\phi')$$

. If we let $\phi \to \phi'$, $\cos(\phi-\phi') \to 1$, so $m = m'$. Therefore

$$\sum \frac{r'^l}{r^{l+1}} P_l(\cos\gamma) = \sum B_{lm} \frac{r'^l}{r^{l+1}} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi')$$

. Rotational symmetry reduces B_{lm} to B_l, so we can look at the special case where $\theta' \to 0$, $P_l(\cos\gamma) \to P_l(\cos\theta)$, and $Y_{lm} \to Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$. Finally, we see that $B_l \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{2l+1}{4\pi}} = 1$ so $B_l = \frac{4\pi}{2l+1}$.

We have found

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \left(\frac{r_<^l}{r_>^{l+1}}\right) \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi')$$

. ◇

0.11.2 Green's Functions in Spherical Coordinates

For Dirichlet Green's Functions, we must have that the function vanishes at the boundaries; the potential is specified and constant there. Suppose we have a problem of concentric spherical shells of radius $a < b$.

Recall if we know $G_D(x,x')$,

$$\Phi = \frac{1}{4\pi\epsilon_0} \int G_D(x,x')\rho(x') d^3x' - \frac{1}{4\pi} \oint_\Sigma \frac{\partial G_D}{\partial n'} \Phi(x') da'$$

LECTURE 11: SPHERICAL SYMMETRY, CONTINUED

Wed Sep 11 2019

We will restrict $a \le r \le b$. To form the Green's Function, we put an imaginary point charge somewhere with the normalization condition of $-4\pi \to (-4\pi)\delta(\vec{x}-\vec{x}')$:

$\nabla'^2 G = -4\pi\delta(\vec{x}-\vec{x}')$ which is equivalent to $\nabla^2 G$ in this case, since the Green's function is symmetric. Also, the Green's function must vanish on the boundaries.

$$\nabla^2 = \frac{1}{r}\partial_r^2 r - \frac{\mathbb{L}^2}{r^2}$$

Additionally, we can use

$$\sum_l \sum_{-l\leq m\leq l} Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

. We can use this to write

$$\delta(\vec{x}-\vec{x}') = \delta(\phi-\phi)\delta(\cos\theta-\cos\theta')\frac{\delta(r-r')}{r^2}$$

Let's suppose

$$G = \sum_{l,m} g_l(r,r')Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')$$

so

$$\sum_{l,m}\left[\frac{1}{r}\frac{d^2}{dr^2}r\frac{l(l+1)}{r^2}\right]g_l(r,r')Y_{lm}Y^*_{lm} = (-4\pi)\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')\frac{\delta(r-r')}{r^2}$$

comes from acting the Laplacian on G.

For $a \leq r < r' < b$ or $a < r' < r \leq b$, we have $\frac{1}{r}\frac{d^2}{dr^2}rg_l - \frac{l(l+1)}{r^2}g_l = 0$

Suppose $g_l = A_l r^l + B_l r^{-(l+1)}$

If $r = a$,

$$A_l a^l + B_l a^{-(l+1)} = 0 \implies B_l = -A_l a^{2l+1}$$

so for $r < r'$,

$$g_l = A_l\left[r^l - \frac{a^{2l+1}}{r^{l+1}}\right] = y^{(1)}$$

On the other boundary, $r = b$, we get that for $r' < r$,

$$g_l = E_l\left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right] = y^{(2)}$$

Because of the symmetric nature of the Green's function, our complete solution must be formed from these two solutions.

$$g_l(r,r') = C_l\left[r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right]\left[\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right]$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$. Apparently this is related to the product space.

What happens when $r = r'$?

$$\int_{r'-\epsilon}^{r'+\epsilon} \frac{1}{r}\frac{d^2}{dr^2} r g_l - \frac{l(l+1)}{r^2} g_l = \int_{r'-\epsilon}^{r'+\epsilon} -4\pi \frac{\delta(r-r')}{r^2} = -4\pi \frac{1}{r'}$$

On the right side, we assume $\frac{g_l}{r^2} \to 0$ so we are left with

$$\frac{d}{dr}(r g_l)\bigg|_{r'-\epsilon}^{r'+\epsilon} = -\frac{4\pi}{r'} \frac{d}{dr}[r g_l]\bigg|_{r'+\epsilon > r'} - \frac{d}{dr}[r g_l]\bigg|_{r'-\epsilon < r'}$$

so we are taking

$$C_l \frac{d}{dr}\left(r \left[r'^l - \frac{a^{2l+1}}{r'^{l+1}} \right] \left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right] \right)_{r \to r'}$$

and similar for the case where $r' > r$. Taking the derivatives and limits will tell us what C_l must be.

$$C_l = \frac{4\pi}{(2l+1)\left(1 - \left(\frac{a}{b}\right)^{2l+1}\right)}$$

Finally, we can write our general spherical Green's function:

$$G(r,\theta,\phi,r',\theta',\phi') = \sum_{l,m} \frac{4\pi}{(2l+1)\left(1 - \left(\frac{a}{b}\right)^{2l+1}\right)} \left[r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right] \left[\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right] Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi')$$

1. As $a \to 0$ and $b \to \infty$, we get back the original $G = \frac{1}{|\vec{x}-\vec{x}'|}$
2. As $a \neq 0$ and $b \to \infty$, $G = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a/x'}{|\vec{x}-\frac{a^2}{x'^2}\vec{x}'|}$ from our method of images (this will not look the same if you just write out these limits, but it can be found through some careful algebra).
3. As $a = 0$ and b is finite and say $\rho(x') = 0$, we have $G = \sum \frac{4\pi}{2l+1}[r_<^l] \left[\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right] Y_{lm} Y_{lm}^*$. As we approach the boundary, $r' > r$
$$\partial_{r'} G\bigg|_{r' \to b} = \sum \frac{4\pi}{2l+1} r^l \left[-\frac{(l+1)}{r'^{l+2}} - l \frac{r'^{l-1}}{b^{2l+1}} \right] Y_{lm} Y_{lm}^* \bigg|_{r' \to b}$$
. There's some more to do here but we basically get
$$\Phi(\vec{x}) = \sum \frac{r^l}{b^{l+2}} Y_{lm}(\theta,\phi) \int Y_{lm}^*(\theta',\phi') V(\theta',\phi') b^2 d\Omega'.$$

Lecture 12: Green's Functions in Cylindrical Coordinates

Fri Sep 20 2019

0.12 Review

$$\int_0^a J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu m}\frac{\rho}{a}\right)\rho d\rho = \delta_{nm}\frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2$$

$$\sum_{n=1}^{\infty} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho'}{a}\right) = \frac{\delta\rho-\rho'}{\rho}$$

If we remove the boundary, eliminating conditions on $k=\frac{x_m}{a}$ as $a\to\infty$ densely fills the interval $[0,\infty)$:

$$\int_0^\infty J_\nu(k\rho)\, J_\nu(k'\rho)\,\rho d\rho = \frac{\delta(k-k')}{k}$$

$$\int_0^\infty J_\nu(k\rho)\, J_\nu(k\rho')\,\rho d\rho = \frac{\delta(\rho-\rho')}{\rho}$$

These are the Hankel transforms.

0.13 Writing Green's Functions

Let's try to write our Green's function in terms of these integrals:

$$\frac{1}{|\vec{x}-\vec{x}'|} = G(\vec{x},\vec{x}')$$

$$\nabla^2 G = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z')$$

$$G(\vec{x},\vec{x}') = \sum_{m=0}^{\infty}\int_0^\infty kdkJ_m(k\rho)J_m(k\rho')\frac{e^{\imath m(\phi-\phi')}}{2\pi}g_{km}(z,z')$$

Recall that when we write these as separable equations, we select functions such that $R(\rho)\mapsto J_m(k\rho)$, $Q(\phi)\mapsto e^{\imath m\phi}$ and $Z(z)\mapsto Z''-k^2Z=0$. If we pick the radial part by hand and the angular part by a periodic equation, we are forced to use this representation of Z. We will later see a different solution where we don't select the radial portion first. To fix the last separable part, we will need functions g_{km} such that

$$\frac{d^2}{dz^2}g_{km}(z,z') - k^2 g_{km}(z,z') = -4\pi\delta(z-z')$$

We know that the solutions for these functions are

$$g_{km}\propto e^{\pm kz}$$

Let's try

$$g_{km}(z,z') = Ce^{-kz_>}e^{+kz_<}$$

where, as usual,

$$z_> = \max(z,z'),\ \ z_< = \min(z,z')$$

If we use the "jumping" condition, going through $z = z'$, we find

$$\int_{z'-\epsilon}^{z'+\epsilon} \frac{d^2}{dz^2} g_{km} - k^2 \cancelto{0}{\int_{z'-\epsilon}^{z'+\epsilon} g_{km}} = -4\pi \int_{z'-\epsilon}^{z'+\epsilon} \delta(z-z')dz$$

$$\left.\frac{dg}{dz}\right|_{z\to z'+} - \left.\frac{dg}{dz}\right|_{z\to z'-} = -4\pi$$

$$C\left\{e^{+kz'}\left(\left.\frac{d}{dz}e^{-kz}\right|_{z\to z'}\right) - \left(\left.\frac{d}{dz}e^{-kz'}\right|_{z\to z'}\right)\right\} = -4\pi$$

so

$$C = \frac{2\pi}{k},\ g_k = \frac{2\pi}{k}e^{-kz_>}e^{+kz_<}$$

Therefore,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{2\cancel{\pi}}{\cancel{2}\pi}\sum_{m=0}^{\infty}\int_0^{\infty}\frac{\cancel{k}}{\cancel{k}}dkJ_m(k\rho)J_m(k\rho')e^{\imath m(\phi-\phi')}e^{-k[z_>-z_<]}$$

Alternatively, we can use $Z'' + k^2 Z = 0$ so the Bessel equation becomes $\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(k^2 + \frac{m^2}{\rho^2}\right)R = 0$, so

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z')$$

and

$$\int_{-\infty}^{\infty} e^{\imath k(z-z')}\frac{dk}{2\pi} = \delta(z-z')$$

Now we must use the Macdonald functions for ρ: $I_m(k\rho)$ and $K_m(k\rho)$:

$$G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty}\int_0^{\infty}\cos(k(z-z'))\frac{dk}{\pi}e^{\imath m(\phi-\phi')}g_{km}(\rho,\rho')$$

We expect that the solution is representable by $I_m(k\rho)$ and $K_m(k\rho)$, but we know that $I_m(x)\to\infty$ as $x\to\infty$ and $K_m(x)\to\infty$ as $x\to 0^+$. Therefore we choose

$$C_k I_m(k\rho_<)K_m(k\rho_>)$$

Our original equation for the radial part was

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} \equiv \frac{d^2}{d\rho^2} + \frac{1}{rho}\frac{d}{d\rho}$$

Using the jump condition again,

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon}\rho d\rho\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}g\right) - \cancelto{0}{\int_{\rho'-\epsilon}^{\rho'+\epsilon} g} = -\int\frac{4\pi\delta(\rho-\rho')}{\rho}d\rho\rho$$

so

$$\left.\frac{dg}{d\rho}\ \rho'+\epsilon\right| - \left.\frac{dg}{d\rho}\ \rho'-\epsilon\right| = -\frac{4\pi}{\rho'}$$

which is

$$C[K_m'(k\rho')I_m(k\rho') - I_m'(k\rho')K_m(k\rho')] = -\frac{4\pi}{\rho'}$$

where $K' = \frac{dK}{d\rho}$ and $I' = \frac{dI}{d\rho}$

Theorem 0.13.1. *Wronskian Theorem:*

$$W(x) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$$\left(p\frac{d^2}{dx^2} + q\frac{d}{dx} + s\right) y = 0$$

then

$$W' = -\frac{q}{p} W$$

and

$$W(x) = W(x_0) e^{-\int \frac{q}{p} dx}$$

Therefore

$$W[K_m, I_m] = \frac{C}{x}$$

since

$$e^{-\int \frac{q}{p}} = e^{-\int \frac{dx}{x}} = \frac{1}{x}$$

$$\frac{dK_m}{dx} = -\frac{1}{2}(K_{m-1}(x) + K_{m+1}(x))$$

$$\frac{dI_m}{dx} = +\frac{1}{2}(I_{m-1}(x) + I_{m+1}(x))$$

$$\lim_{x\to 0^+} K_\nu(x) = \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu$$

and

$$\lim_{x\to 0^+} I_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

so

$$\lim_{\rho'\to 0^+} C\left(\frac{dK_m}{d\rho}\bigg|_{\rho'} I_m - \frac{dI_m}{d\rho}\bigg|_{\rho'} K_m\right) = -\frac{4\pi}{\rho'}$$

$$4\pi I_m(k\rho_<)K_m(k\rho_>)$$

Finally,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4}{\pi}\sum_{m=0}^{\infty}\int_0^\infty dk \cos(k(z-z'))e^{\imath m(\phi-\phi')} I_m(k\rho_<)K_m(k\rho_>)$$

Lecture 13: Direct Construction of Green's Functions

Mon Sep 23 2019

"YAMOCGF" - Yet Another Method of Constructing Green's Functions

Say we are looking at a region Ω with Ψ boundary$\Big| \equiv 0$ is a general function. We are looking for Green's functions for the operator $(\nabla^2 + f(\vec{x}) + \lambda)$ where λ is a given real number and f is a function. Suppose we have solutions to this eigenvalue equation for special values of λ_n:

$$(\nabla^2 + f)\Psi_n = -\lambda_n \Psi_n$$

or

$$(\nabla^2 + f + \lambda_n)\Psi_n = 0$$

Note that $G(\vec{v}, \vec{x}')$ is an inverse to our operator:

$$(\nabla^2 + f + \lambda) \cdot G = -4\pi\delta(\vec{x} - \vec{x}')$$

Let us construct the following object:

$$\sum_{n=1}^{\infty} \frac{\Psi_n^*(\vec{x}')\Psi_n(\vec{x})}{\lambda - \lambda_n}$$

Theorem 0.13.2. *If we have a finite domain, the eigenvalue equation has an infinite number of eigenvalues with finite degeneracy such that $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$ as $n \to \infty$. (If Ω becomes unbounded, this set becomes continuous and $\Psi_n(x)$'s are not normalizable in the usual sense.) Moreover, $\Psi_n(x)$ form a complete basis.*

Back to our object. Say $\lambda \neq \lambda_n$, $n = 1, 2, \ldots, \infty$. Let us try acting our operator on this sum:

$$\sum_{n=1}^{\infty} (\nabla^2 + f + \lambda) \frac{\Psi_n(\vec{x})\Psi_n^*(\vec{x}')}{\lambda - \lambda_n} = \sum_{n=1}^{\infty} \frac{(\lambda - \lambda_n)\Psi_n(\vec{x})\Psi_n^*(\vec{x}')}{\lambda - \lambda_n} = \sum_{n=1}^{\infty} \Psi_n(\vec{x})\Psi_n^*(\vec{x}') = \delta(\vec{x} - \vec{x}')$$

So we have found that our Green's function is, in general,

$$G = (-4\pi) \sum_{n=1}^{\infty} \frac{\Psi_n^*(\vec{x}')\Psi_n(\vec{x})}{\lambda - \lambda_n}$$

Example. Conducting Box: Our box has sides a, b, c corresponding to the lengths in $\vec{x}$x $\vec{y}$, and $\vec{z}$.

We know that the solutions to the Dirichlet conditions are

$$\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \sqrt{\frac{2}{b}} \sin\left(\frac{m\pi y}{b}\right) \sqrt{\frac{2}{c}} \sin\left(\frac{k\pi z}{c}\right)$$

so

$$(-\nabla^2)\sqrt{\frac{8}{abc}} \sin\left(n\pi\frac{x}{a}\right) \sin\left(m\pi\frac{y}{b}\right) \sin\left(k\pi\frac{z}{c}\right) = \lambda_{nmk}\Psi_{nmk}$$

Here we know that

$$\lambda_{nmk} = \left(\frac{n\pi}{a}\right)^2 \left(\frac{m\pi}{b}\right)^2 \left(\frac{k\pi}{c}\right)^2$$

So we can write out our general Green's function:

$$G = -4\pi \sum_{n,m,k=1}^{\infty} \left[\frac{8}{abc}\right] \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\left(\frac{n\pi}{a}\right)^2\left(\frac{m\pi}{b}\right)^2\left(\frac{k\pi}{c}\right)^2}$$

◇

Let us look at $-\nabla^2$ on $\mathbb{R}^3$: $\frac{1}{|\vec{x}-\vec{x}'|}$. We know that

$$\int_{-\infty}^{\infty} \frac{e^{-\imath\vec{k}\vec{x}}e^{\imath\vec{k}\vec{x}'}}{(2\pi)^3} d^3k = \delta(\vec{x}-\vec{x}')$$

$$= (4\pi)\int_{-\infty}^{\infty} d^3k \frac{1}{k^2}\Psi_{\vec{k}}(\vec{x}) * (\vec{x}')\Psi_{\vec{k}}(\vec{x})$$

Let's evaluate this in momentum space:

$$\int_0^{\infty} \cancel{k^2} dk d\Omega \frac{e^{\imath k|\vec{x}-\vec{x}'|\cos\theta}}{\cancel{k^2}(2\pi)^3} = \int_0^{\infty} \frac{dk 2\pi}{(2\pi)^3}\int_0^{2\pi} d\phi \int 0^{\pi} d\theta \sin\theta e^{\imath k|\vec{x}-\vec{x}'|\cos\theta}$$

Notice that $d\theta\sin\theta = -d(\cos\theta)$:

$$= \frac{1}{(2\pi)^2}\int_0^{\infty} dk \frac{e^{\imath k|\vec{x}-\vec{x}'|} - e^{-\imath k|\vec{x}-\vec{x}'|}}{\imath k|\vec{x}-\vec{x}'|}$$

Let us change our integration variable, on the second half, to $k \to -k$. This makes the integral $\int_{-\infty}^{0}$ of the same value.

$$= \frac{1}{(2\pi)^2}\left[\int_{-\infty}^{\infty} \frac{e^{\imath k|\vec{x}-\vec{x}'|}}{-k^2}\right]\frac{1}{|\vec{x}-\vec{x}'|}$$

We have this singularity at $k = 0$ which we must "jump" around, and then take the limit as the size of our jump approaches zero. If we suppose k is complex, we have an exponential decay term if the imaginary part of $k > 0$. Therefore, the contour can be closed at infinity, and we can evaluate the limit in the upper-half plane. Therefore, this integral can be solved through the residue theorem, where the residue is $\imath\pi$. The integral then is

$$\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}$$

where multiplying by 4π will give us the desired result.

In the next lecture, we will cover multipole expansions.

Lecture 14: The Multipole Expansion

Mon Sep 23 2019

Imagine we have a region of charge with density $\rho(\vec{x})$. The potential for this can be expanded as

$$\frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')d^3x \sum_{l=1}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma)$$

where $\cos\gamma = \hat{x}\cdot\hat{x}'$. This can be written as

$$\frac{1}{4\pi\epsilon_0}\overbrace{\left[\int \rho(\vec{x}')d^3x'\right]}^{Q}\frac{1}{r} + \frac{1}{4\pi\epsilon_0}\left[\int \rho(\vec{x}')d^3x'\hat{x}\cdot\hat{x}'\right]\frac{r'}{r^2} + \frac{1}{4\pi\epsilon_0}\left[\int \rho(\vec{x}')d^3x'\left(\frac{3}{2}(\hat{x}\cdot\hat{x}')^2 - \frac{1}{2}\right)\right]\frac{r'^2}{r^3} + \dots$$

This is the multipole expansion. We can further simplify the first term:

$$\frac{1}{4\pi\epsilon_0}\frac{q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\left(\int \rho(\vec{x}')\vec{x}' d^3x'\right)\cdot\hat{x}}{r^2} + \frac{\rho(\vec{x}')\left[\frac{3}{2}(\vec{x}'\cdot\hat{x})^2 - r'^2\hat{x}\cdot\hat{x}\right]}{4\pi\epsilon_0 r^3}$$

This last term is the quadrupole moment:

$$\frac{1}{4\pi\epsilon_0}\underbrace{\int d^3x'\rho(\vec{x}')\left[\frac{3}{2}x_i'x_j' - \frac{1}{2}r'^2\delta_{ij}\right]}_{Q_{ij}}\hat{x}_i\hat{x}_j\frac{1}{r^3}$$

Therefore, the potential can be written, in general as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{x}}{r^2} + \frac{1}{4\pi\epsilon_0}\frac{\sum_{i,j}Q_{ij}\hat{x}_i\hat{x}_j}{r^3} + \frac{1}{4\pi\epsilon_0}\frac{\sum_{i,j,k}Q_{ijk}\hat{x}_i\hat{x}_j\hat{x}_k}{r^4} + \dots$$

Remark.

$$\int \rho(\vec{x}')d^3x'[\text{homogeneous polynomial of degree } l \text{ in } x_1', x_2', x_3']$$

where the polynomial can be expanded in terms of $P_l(\hat{x}\cdot\hat{x}')$. This "Q" is traceless. Therefore there are $2l+1$ degrees of freedom per multipole moment.

Recall that

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m}\frac{4\pi}{2l+1}\left(\frac{r'^l}{r^{l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

so

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')d^3x'\sum\frac{4\pi}{2l+1}\left(\frac{r'^l}{r^{l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

which is equal to

$$\frac{4\pi}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\underbrace{\sum_{m=-l}^{l}}_{2l+1\text{ terms}}\frac{1}{2l+1}\frac{1}{r^{l+1}}\underbrace{\left[\int Y_{lm}^*(\theta',\phi')r'^l\rho(\vec{x}')d^3x'\right]}_{q_{lm}^*=q_{l,-m}(-1)^m}Y_{lm}(\theta,\phi)$$

We can construct Y_{lm}'s as homogeneous polynomials on $x-\imath y$, $x+\imath y$ and z.

$$q_{00} = \frac{1}{\sqrt{4\pi}}Q$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\int(x'-\imath y')\rho(\vec{x}')d^3x' = -\sqrt{\frac{3}{8\pi}}(p_1 - \imath p_2)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}}\int z'\rho(\vec{x}')d^3x' = \sqrt{\frac{3}{4\pi}}p_3$$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int(x'-\imath y')^2\rho(\vec{x}')d^3x' = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(Q_{11} - 2\imath Q_{12} - Q_{22})$$

0.13.1 Dipole Case

Suppose we have $Q = 0$ and a point dipole $\vec{p}$.

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{x}}{r^2}$$

Assuming $r \neq 0$, can write the electric field from

$$\vec{E} = -\nabla\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{x})\hat{x} - \vec{p}}{r^3}\right]$$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{n})\hat{x} - \vec{p}}{|\vec{x} - \vec{x}_0|^3}\right]$$

This is actually not correct. There are no point dipoles for electric fields. Atoms can have induced dipole moments, but there is no solution at x_0, the center of the dipole. To find this term, suppose we have a distribution $\rho(\vec{x}')$ which creates an electric field. Say we take a point $\vec{y}$ and a sphere around this point, and average the electric field in this region. Say $\vec{y} = \vec{0}$ for convenience.

$$\int_{\text{ball around }\vec{0}} \vec{E} d^3y = \frac{4\pi}{3}\vec{E}(\vec{0})$$

If the ball contains the charge,

$$\int_{\text{ball around }\rho(\vec{x})} \vec{E} d^3y = -\frac{1}{3\epsilon_0}\vec{p}$$

If you were to repeat this process for our point dipole, you would find that this integral evaluates to 0. It misses this $-\frac{1}{3\epsilon_0}\vec{p}$ term. For an ideal point dipole,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{n})\hat{x} - \vec{p}}{|\vec{x} - \vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\delta(\vec{x})\right]$$

LECTURE 15:
Wed Sep 25 2019

0.14 Finding Potentials for Continuous Charge Densities

Suppose we have a charge density $\rho(\vec{x}')$ and a ball about $\vec{0}$. We know that $\int_{\text{ball}} \vec{E}(\vec{x}) d^3x = -\oint_{\text{sphere}} \Phi d\vec{a}$.

We could also imagine that the charge density is inside the sphere.

$$\begin{aligned}
-\oint \Phi d\vec{a} &= -\frac{1}{4\pi\epsilon_0}\oint\int \frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|}R^2 d\Omega\hat{x} \\
&= -\frac{1}{4\pi\epsilon_0}\int d^3x'\rho(\vec{x}')R^2\int d\Omega\hat{x}\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}}\right)P_l(\cos\gamma) \\
&= -\frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')d^3x'R^2\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}}\right)\int d\Omega\hat{x}P_l(\cos\gamma)
\end{aligned}$$

We can perform this final integral. If we rotate so that our $\vec{x}$ is the new z-axis, we can see that, due to the orthogonal condition on P_l, the only nonzero term is $\hat{x} \to (\hat{x}')P_1(\cos\gamma)$, so the final answer is

$$\int_{\text{ball}} \vec{E}d^3x = -\frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')d^3x'R^2\hat{x}'\frac{4\pi}{3}\frac{r_<}{r_>^2}$$

where $r_< = \min(|\hat{x}'|, R)$ and $r_> = \max(|\hat{x}'|, R)$. So

$$\int_{\text{ball}} \vec{E}d^3x = \begin{cases} \frac{4\pi}{3}R^3\int \frac{\rho(\vec{x}')[-\hat{x}']d^3x'}{4\pi\epsilon_0|\vec{x}'|^2} = \frac{4\pi}{3}R^3\vec{E}(0) & \text{charge outside sphere} \\ -\int \frac{\rho(\vec{x}')\vec{x}'d^3x'}{3\epsilon_0} = -\frac{\vec{p}}{3\epsilon_0} & \text{charge inside sphere} \end{cases}$$

0.14.1 Ideal Point Dipoles

What is the immediate application? What is the ideal point dipole? Naïvely, we would think

$$\Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{x}}{r^2}$$

$$\vec{E} = -\nabla\Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{x})\hat{x}x\vec{p}}{|\vec{x}|^3}$$

but this implies that the average electric field in a small ball around the dipole is zero, which contradicts our previous result!We fix this by adding the term by hand:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{x})\hat{x}-\vec{p}}{|\vec{x}|^3} - \frac{\vec{p}}{3\epsilon_0}\delta(\vec{x})$$

0.14.2 Energy Calculations

Problem: Calculate energy for a charge distribution immersed into the field of an external charge distribution. We assume our distribution $\rho(\vec{x})$ is centered somewhere and we have some external charges generating some fields far away.

$$W = \int \rho(x)\Phi_{\text{ext}}(x)d^3x$$

Suppose the length scale of our distribution is L. If $\frac{|\nabla\Phi_{\text{ext}}|}{L} \ll 1$,

$$W = \int \rho(x)[\Phi_{\text{ext}}(0) + x^i\partial_{x^i}\Phi_{\text{ext}}\Big|_0 + \frac{1}{2}x^ix^j\partial_{x^i}\partial x^j\Phi_{\text{ext}}\Big|_0 + \ldots]d^3x$$

so

$$W = \left(\int \rho(x)d^3x\right)\Phi_{\text{ext}}(0) - \left[\int \rho(x)x^2d^3x\right]\overbrace{[-\partial_{x^i}\Phi_{\text{ext}}]}^{\vec{E}_{\text{ext}}(0)} + (-)\int \frac{d^3x\rho(x)}{3}\left[\frac{3}{2}x_ix_j - \frac{1}{2}r^2\partial_{ij}\right]\partial_{x^j}E_i\Big|_0$$

so

$$W \approx Q\Phi_{\text{ext}}(0) - \vec{p}\cdot\vec{E}_{\text{ext}}(0) - \frac{1}{3\cdot 2}Q_{ij}\partial_{[i}E^{\text{ext}}_{j]}(0) + \dots$$

0.14.3 Dipole-Dipole Interactions

Suppose we have two dipoles now, with $\hat{n}_{21}$ is the vector pointing from the first to the second.

$$W = -p_1\left[\frac{3(p_2\cdot\hat{n}_{12})\hat{n}_{12} - \vec{p_2}}{|\vec{x}_1 - \vec{x}_2|}\right] = \frac{\vec{p_1}\cdot\vec{p_2} - 3(\vec{p_2}\cdot\hat{n}_{12})(\vec{p_1}\cdot\hat{n}_{12})}{|\vec{x}_1 - \vec{x}_2|^3}$$

There is, of course, the dipole correction term, but it has a delta function in it. Our dipoles never overlap, so this term drops out.

LECTURE 16: ELECTROSTATICS OF DIELECTRICS

Friday Sep 27 2019

0.15 Microscopic vs. Macroscopic Structure

The micro scale is $\propto 10^{-9} \to 10^{-8}$ meters, while the macro scale is $\propto 10^{-6}$ meters. We can look in the range right between these to average out these microscopic fields. In this range, $\vec{B}_{\text{micro}} \approx \vec{0}$. Microscopic electric fields may be induced, and averaging over these can be modeled by a macroscopic dipole density $\vec{P}(\vec{x})$. This is our working, unjustified assumption to be discussed further by some models. If we believe this assumption, we can write down the potential as:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{\|\vec{x}-\vec{x}'\|}\,\mathrm{d}^3x + \frac{1}{4\pi\epsilon_0}\int \frac{\vec{p}(\vec{x}'\cdot(\vec{x}-\vec{x}')}{\|\vec{x}-\vec{x}'\|^3}\,\mathrm{d}^3x'$$

This is equivalent to

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{\|\vec{x}-\vec{x}'\|}\,\mathrm{d}^3x + \frac{1}{4\pi\epsilon_0}\int \frac{\vec{\nabla}\cdot'\vec{P}(x')}{\|\vec{x}-\vec{x}'\|}\,\mathrm{d}^3x' + \int_\Omega \frac{-\vec{\nabla}'\vec{P}}{\|\vec{x}-\vec{x}'\|}\,\mathrm{d}^3x'$$

The numerator of the second term here is the bound surface charge of the medium. We can think of the measured field as $\vec{E} = \vec{E}^{\text{ext}} + \overline{\vec{E}}_{\text{micro}}$. We have a working hypothesis that $\vec{\nabla}\times\vec{E} = 0$, because the microscopic magnetic field does not change in time, so $\vec{E} = -\vec{\nabla}\cdot\Phi$. $\vec{P}$ is a function of "local" $\vec{E}$ for the static case. We assume the linear term is the dominant contribution (it can be nonlinear, a simple model of permanent dipoles depends non-linearly on temperature, for example).

$$\vec{P} = \epsilon_0\chi$$

Isotropic materials have $\chi_{ij} = \chi\delta_{ij}$. Homogeneous materials have $\chi(\vec{x}) = \chi$. We can therefore show that

$$\rho_{\text{bound}} = -\vec{\boldsymbol{\nabla}} \cdot \vec{P}$$

so

$$\vec{\boldsymbol{\nabla}} \cdot \vec{E} = \frac{\rho_{\text{free}}}{\epsilon_0} - \frac{\vec{\boldsymbol{\nabla}}\vec{P}}{\epsilon_0}$$

or

$$\vec{\boldsymbol{\nabla}} \underbrace{(\epsilon_0 \vec{E} + \vec{P})}_{\vec{D}} = \rho_{\text{free}}$$

If we assume $\vec{P} = \epsilon_0 \chi \vec{E}$,

$$D = \epsilon_0 (1 + \chi)\vec{E}$$

where $\epsilon_0(1+\chi) \equiv \epsilon$. This brings us the familiar Poisson equation on the potential:

$$\epsilon \nabla^2 \Phi = -\rho$$

Charge free regions still satisfy $\nabla^2 \Phi = 0$, and we can use boundary conditions to determine solutions.

0.15.1 Boundary Conditions

If we take a Gaussian pillbox around a boundary, we know that $\vec{\boldsymbol{\nabla}} \cdot \vec{D} = \rho_{\text{free}}$, so

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{12} = 0$$

Also, the normal component of $\vec{D}$ is continuous in a linear material, since $\vec{D} = \epsilon \vec{E}$, so

$$\epsilon_1 (\vec{E}_1)_n = \epsilon_2 (\vec{E})_n$$

Additionally, the tangential components of $\vec{E}$ are continuous:

$$(\vec{E}_1 - \vec{E}_2)_{\text{tangent}} = \vec{0}$$

or

$$(\vec{E}_1 - \vec{E}_2) \times \hat{n}_{12} = \vec{0}$$

LECTURE 17: DIELECTRICS

Mon Sep. 30 2019

0.16 Dielectrics

In isotropic and homogeneous materials, we said that

$$\vec{P} = \epsilon_0 \chi \vec{E}$$

so

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

and

$$\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$$

Of course, we also must satisfy

$$\vec{\nabla} \times \vec{E} = \vec{0} \implies \vec{E} = -\vec{\nabla}\Phi$$

so

$$\vec{D} = \underbrace{(\epsilon_0 + \epsilon_0 \chi)}_{\epsilon} \vec{E}$$

so

$$\epsilon \nabla^2 \Phi = -\rho_{\text{free}}$$

In more general cases,

$$D_i = \epsilon_{ij}(x) E_j$$

so

$$\partial_i(\epsilon_{ij}(x)\partial_j \Phi) = -\rho_{\text{free}}$$

which is in general pretty hard to solve.

If we recall our boundary conditions:

$$\vec{E}_t \quad \text{is continuous across a boundary}$$

and

$$\vec{D}_n \quad \text{is continuous}$$

Example. For a dielectric ($\epsilon \neq 0$ sphere inserted into a uniform electric field),

$$\vec{E}_0 = E_0 \hat{z}$$

There are no free charges and symmetry around z

$$\Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))$$

The potential cannot go to zero at infinity, since there is an electric field there, so

$$\Phi_{\text{out}} = -E_0 z + B_0 + \sum_{l=0}^{\infty} C_l r^{-(l+1)} P_l(\cos(\theta))$$

We can set $B_0 = 0$ because the potential is invariant up to a constant. We know that Φ must be continuous across the $r = a$ boundary, so

$$\Phi(a)_{\text{out}} = -E_0 a \underbrace{P_1(\cos(\theta))}_{\cos(\theta)} + \sum_{l=0}^{\infty} C_l a^{-(l+1)} P_l(\cos(\theta))$$

so for $l = 1$

$$A_1 = -E_0 + \frac{C_1}{a^3}$$

and for $l \neq 1$

$$A_l = \frac{C_l}{a^{2l+1}}$$

We know that $D = \epsilon E$ and we know the inside and outside permittivity, so to maintain continuity,

$$(-\epsilon \vec{\boldsymbol{\nabla}}\Phi) \cdot \hat{n}\Big|_{r\to a^-} = (-\epsilon_0 \vec{\boldsymbol{\nabla}}\Phi) \cdot \hat{n}\Big|_{r\to a^+}$$

or

$$-\epsilon \frac{\partial \Phi_{\text{in}}}{\partial r}\Big|_{r=a} = -\epsilon_0 \frac{\partial \Phi_{\text{out}}}{\partial r}\Big|_{r-a}$$

so

$$\epsilon E_0 P_1(\cos(\theta)) - \epsilon \sum_{l=0}^{\infty} [-(l+1)a^{-(l+2)} C_l P_l(\cos(\theta)) = -\epsilon_0 \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos(\theta))$$

so when $l \neq 1$:

$$\epsilon(l+1)C_l a^{-(l+2)} = -\epsilon_0 l A_l a^{l-1}$$

and when $l = 1$:

$$\epsilon E_0 + \epsilon(1+l)a^{-3}C_1 = -\epsilon_0 A_1$$

From this and the previous boundary condition, we see that all of the $l \neq 1$ terms are 0, and solving between the remaining $l = 1$ terms, we see that

$$A_1 = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0$$

and

$$C_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a^3 E_0$$

so

$$\Phi_{\text{in}} = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 r \cos(\theta)$$

and

$$\Phi_{\text{out}} = -E_0 r \cos(\theta) + \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \frac{a^3 E_0}{r^2} \cos(\theta)$$

Taking the proper derivatives, we see that

$$\vec{E}_{\text{in}} = \frac{3\epsilon_0 E_0}{\epsilon + 2\epsilon_0} \hat{z}$$

and if we say that

$$\vec{p} = (4\pi a^3) \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right) \epsilon_0 E_0 \hat{z}$$

$$\vec{E}_{\text{out}} = E_0 \hat{z} + \frac{1}{4\pi\epsilon_0} \left[\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} \right]$$

We see here that the field inside is a reduction of the constant field outside, and the field outside has been amplified by the inclusion of the dielectric (unless the material is "active"), $\epsilon > \epsilon_0$. It is also clear here that $\vec{P} = \vec{D} - \epsilon_0 \vec{E}$. ◇

Quote

"What do we want to do with this example? Of course, we don't want to do anything with it - minimum action principle."

Example. Imagine two media which meet at a straight boundary. On the left side, we have ϵ_2 and on the right we have ϵ_1. Imagine placing a charge q a distance d from the boundary. We know that maintaining the boundary condition on the interface (the Green's function must vanish) must create some sort of image charge at $-d$. On the right side, $z > 0$,

$$\Phi = \frac{q}{4\pi\epsilon_1\sqrt{\rho^2 + (z-d)^2}} + \frac{q'}{4\pi\epsilon_1\sqrt{\rho^2 + (z+d)^2}}$$

and on the other side,

$$\Phi = \frac{1}{r\pi\epsilon_2}\frac{q''}{\sqrt{\rho^2 + (z-d)^2}}$$

where q'' is some "blurred" charge seen from the left side of the boundary. However, since Φ is continuous across the boundary, we know that

$$\frac{q}{4\pi\epsilon_1\cancel{\sqrt{\rho^2+d^2}}} + \frac{q'}{4\pi\epsilon_1\cancel{\sqrt{\rho^2+d^2}}} = \frac{q''}{4\pi\epsilon_2\cancel{\sqrt{\rho^2+d^2}}}$$

so

$$q + q' = \frac{\epsilon_1}{\epsilon_2}q''$$

By taking $\vec{D}$ having no jump across the boundary,

$$q - q' = q''$$

$\diamond$

Lecture 18: Image Method in Mediums, Continued

Mon Sep 30 2019

0.17 Image Method in Mediums, Continued

From last time, we had, from the continuity of the potential

$$q + q' = \frac{\epsilon_1}{\epsilon_2}q''$$

and from continuity of D:

$$-\epsilon_2\frac{\partial\Phi}{\partial z}\bigg|_{z\to 0^-} = -\epsilon_2\frac{\partial\Phi}{\partial z}\bigg|_{z\to 0^+} \implies q'' = q - q'$$

Therefore,

$$q' = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} q$$

and

$$q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} q$$

In this sense, conductors can be thought of as dielectrics with $\epsilon \to \infty$ limits. If we take the plane to be normal to $\hat{z}$ with region ϵ_1 in the positive direction, we find that

$$\begin{aligned} \vec{P}_2 \cdot \hat{n}_{12} + \vec{P}_2 \cdot \hat{n}_{21} &= \vec{P}_2 \cdot \hat{z} - \vec{P}_1 \cdot \hat{z} \\ &= \underbrace{\epsilon_2 \vec{E}_2 \cdot \hat{z} - \epsilon_1 \vec{E}_1 \cdot \hat{z}}_{=0 \quad \text{since} \quad \vec{D} \quad \text{is continuous}} \qquad + \epsilon_0 [\vec{E}_1 \cdot \hat{z} - \vec{E}_2 \cdot \hat{z}] \end{aligned}$$

so

$$\sigma_{\text{excess}} = \frac{1}{2\pi} \frac{\epsilon_0}{\epsilon_1} \left[\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \right] \frac{qd}{[\rho^2 + d^2]^{3/2}}$$

0.17.1 Energy Considerations in Dielectrics

For a number of dielectrics in a space,

$$\delta W = \int_\Omega \delta\rho_{\text{free}} \cdot \Phi \, \mathrm{d}^3x = \int_\Omega \vec{\nabla} \cdot (\delta \vec{D}\, \Phi \, \mathrm{d}^3x = \underbrace{\sum_{k=1}^{N} \oint_{\Sigma_k \Phi \delta \vec{D}} \cdot \, \mathrm{d}\vec{a}}_{=0} + \int_\Omega \vec{E} \cdot \delta \vec{D} \, \mathrm{d}^3x$$

so

$$\delta W = \int_\Omega E_i \epsilon_{ij}(x) \, \delta E_j \, \mathrm{d}^3x$$

or

$$W = \frac{1}{2} \int_\Omega \epsilon_{ij} E_i E_j \, \mathrm{d}^3x$$

In our special case for homogeneous dielectrics,

$$W = \frac{1}{2} \int \epsilon E^2 \, \mathrm{d}^3x$$

In the no dielectric case, we have $\frac{1}{2} \int \mathrm{d}^3x \, \vec{E}_0 \cdot \vec{D}_0$. When a dielectric is inserted, we can look at the change in energy, or

$$\begin{aligned} \Delta W &= \frac{1}{2} \int \vec{E} \cdot \vec{D} \, \mathrm{d}^3x - \frac{1}{2} \int \vec{E}_0 \cdot \vec{D}_0 \, \mathrm{d}^3x \\ &= \frac{1}{2} \left[\int \vec{E} \cdot \vec{D}_0 \, \mathrm{d}^3x - \int \vec{E}_0 \cdot \vec{D} \, \mathrm{d}^3x + \int (\vec{E} + \vec{E}_0) \cdot (\vec{D} - \vec{D}_0) \, \mathrm{d}^3x \right] \end{aligned}$$

The final term here is

$$\begin{aligned}\int (\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0)\,\mathrm{d}^3x &= -\int \vec{\nabla}(\Phi+\Phi_0)\cdot(\vec{D}-\vec{D}_0)\,\mathrm{d}^3x \\ &= \underbrace{-\int \vec{\nabla}\cdot[(\vec{D}-\vec{D}_0)(\Phi+\Phi_0)]\,\mathrm{d}^3x}_{-\left\{\sum_k \oint_{\Sigma_k}\vec{D}(\Phi+\Phi_0)\mathrm{d}\vec{a}-\oint_{\Sigma_k}\vec{D}_0(\Phi+\Phi_0)\mathrm{d}\vec{a}\right\}=0} \\ &\quad+\underbrace{\int [\overbrace{\vec{\nabla}\vec{D}}^{\rho_{\text{free}}}-\overbrace{\vec{\nabla}\vec{D}_0}^{\rho_{\text{free}}}](\Phi+\Phi_0)\,\mathrm{d}^3x}_{=0} \\ &= 0\end{aligned}$$

so

$$\Delta W = -\frac{1}{2}\int \vec{P}\cdot\vec{E}_0\,\mathrm{d}^3x$$

Again, the field will be $\vec{E}_0$ if there were no dielectrics.

We can find the force due to this dielectric:

$$F_i = -\frac{\partial W}{\partial \xi^i}\bigg|_{Q_k=\quad\text{fixed}}$$

where $\vec{\xi}$ is some displacement of the dielectric.

We also know that $W=\frac{1}{2}\int\rho\Phi$ so

$$\delta W = \frac{1}{2}\int(\delta\rho\,\Phi+\rho\,\delta\Phi)\,\mathrm{d}^3x = \int \delta\rho\,\Phi\,\mathrm{d}^3x$$

so if you have batteries keeping the dielectrics at constant potential, they will do some work on the system $\sum_k \delta Q_k\,\Phi_k$ which will have to be accounted for.

Lecture 19: Molecular Theory of Polarization

Wed Oct 2 2019

0.18 Molecular Theory of Polarization

Recall that for a permanent dipole in a field,

$$W = -\vec{p}\cdot\vec{E}$$

In dielectrics, we see that similarly

$$W = -\int \vec{P}\vec{E}_0\,\mathrm{d}^3x$$

The following is a list of interesting phenomena whose derivations are beyond the scope of this class:

- Pyroelectric materials have permanent dipole densities $\vec{P}_0$.
- Ferroelectric materials have a strong $\vec{P}$ after a field $\vec{E}$ has been applied. The polarization has a hysteresis (removing the field doesn't remove the polarization).
- Hook's (actual) law:
$$\sigma_{il} = \sum_{k,m} \lambda_{ilkm} u_{km}$$
where σ is the stress tensor and
$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^k} + \frac{\partial u_k}{\partial x^k}\right)$$
where u_i is the strain.
- Piezoelectricity and electrostriction are results of this. If you write down the free energy of a stressed/strained system, you see that
$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{2}\int \lambda_{iklm} u_{lm} u_{ik}\,\mathrm{d}^3x + \frac{1}{2}\int \gamma_{ikl} u_{ik} E_l\,\mathrm{d}^3x + \frac{1}{2}\int \epsilon_{ij} E_i E_j\,\mathrm{d}^3x$$
$$\sigma_{ik} = \frac{\delta\mathcal{F}}{\delta u^{ik}} = \lambda_{iklm} u_{lm} + \gamma_{ikl} E_l$$
and
$$\mathcal{D}_i = \frac{\delta\mathcal{F}}{\delta E_i} = \gamma_{kli} u_{kl} + \epsilon_{ij} E_j$$
where the ϵ terms are the dielectric constants of the material (not the Levi-Civita symbol). In these sorts of materials, stresses and strains can generate polarizations inside the material.

$$\vec{E} \mapsto \vec{E}_{\text{macro}}$$
is found by averaging over "small" regions. Take a region with a molecule in the center. We divide the electric field into an external (outside our averaging region) and an internal portion, which is
$$\vec{E}(0) = \frac{1}{\frac{4}{3}\pi R^3}\int_{\text{ball}(R)} \vec{E}^{\text{ext}}\,\mathrm{d}^3x$$
If we consider the nearby charges,
$$-\frac{\vec{P}_T}{3\epsilon_0} = \int_{\text{ball}} \vec{E}^{\text{interior}}\,\mathrm{d}^3x$$
where $\vec{P}_T$ is the total dipole moment $\int_{\text{ball}} \rho(\vec{x}')\vec{x}'\,\mathrm{d}^3x' \approx \frac{4\pi}{3}R^3\vec{P}$. Therefore, the macroscopic field of the center is $\vec{E}^{\text{ext}} -$ average $+ \vec{E}^{\text{int}}_{\text{average}}$. When there is a molecule at the center of the region, we should remove the internal field contribution and introduce some near-field contribution.
$$\vec{E}^{\text{near}} \approx \vec{0}$$

for most uniform crystals and random media. Essentially, for many things, the main contribution to the field is $\vec{E}^{\text{ext}}(0) = \vec{E}_{\text{macro}} + \frac{\vec{P}}{3\epsilon_0}$ which includes our dipole correction term. What does this mean?

$$\langle \vec{p}_{\text{mol}} \rangle = \epsilon_0 \gamma_{\text{mol}} \left(\vec{E}_{\text{macro}} + \frac{\vec{P}}{3\epsilon_0} \right)$$

so

$$\vec{P} = N \langle \vec{p}_{\text{mol}} \rangle$$

where N is the number density of the molecules. Therefore

$$\vec{P} = \overbrace{\frac{\gamma_{\text{mol}} N}{1 - \frac{\gamma N}{3}}}^{\kappa} \epsilon_0 \vec{E}$$

Mossotti and Classius discovered the relationship

$$\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} = 3N\gamma_{\text{mol}}$$

In all of these equations, $\gamma = \frac{\vec{p}}{\epsilon_0}$ We can write a Hamiltonian for a molecule

$$H = \sum \frac{P_i^2}{2m} - \sum_i \frac{(Ze)e}{\|\vec{x}_i\|} + \frac{1}{2} \sum_{i,j} \frac{e^2}{\|\vec{x}_i - \vec{x}_j\|}$$

We can perturb this Hamiltonian by applying an electric field

$$H \to H - eEz_i$$

where the second part is much smaller than the original energy levels. This can be approximated as a harmonic oscillator, which implies that

$$m\omega^2 \vec{x} = e\vec{E}$$

where we ignore any $m\vec{x}''$ in the static case, so

$$\vec{x} = \frac{e}{m\omega^2} \vec{E}$$

where $m\omega^2$ is sort of like a spring constant between the atoms in the molecule. Therefore

$$\vec{p} = e\vec{x} = \sum_{i=1}^{n} \frac{e^2}{m\omega_i^2} \vec{E} \propto \gamma_{\text{mol}} \vec{E}$$

In solids, $N \propto 10^{28,30}$ so $\epsilon \propto 10^{0,1}$.

0.18.1 Permanent Dipoles

Some molecules (like water) have permanent dipole moments. There will also be temperature contributions, since the probability of aligning to an external field is $\propto e^{-\vec{p}\cdot\vec{E}/(k_b T)}$.

Lecture 21: Magnetostatics, Continued

Mon Oct 7 2019

For a current moving in a circle of radius a,

$$A_y \mapsto A_\varphi = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \mathrm{d}\varphi' \, \frac{\cos(\varphi')}{\sqrt{a^2 + r^2 - 2ar\hat{x}\cdot\hat{x}'}}$$

where $\hat{x}\cdot\hat{x} = \cos(\gamma) = \sin(\theta)\cos(\varphi')$.

We expand

$$\frac{1}{\sqrt{a^2 + r^2 - 2ar\cos(\gamma)}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos(\gamma))$$

Recall (from Jackson) that

$$P_l(\hat{x}\cdot\hat{x}') = P_l^0(\cos(\theta))P_l^0(\cos(\theta)) + 2\sum_{m=1}^{\infty} \frac{(l-m)!}{(l+m)!} P_l^m(\cos(\theta))P_l^m(0)\cos(m[\varphi - \varphi'])$$

$$A_\varphi = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \mathrm{d}\varphi' \cos(\varphi')\left[\frac{r_<^l}{r_>^{l+1}} P_0(\cos(\theta))P_0(0) + \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} 2\sum_{m=0}^{l}(\dots)\right]$$

This removes all but the $m = 1$ term:

$$\int_0^{2\pi} \mathrm{d}\varphi' \cos(\varphi')\cos(m(\varphi - \varphi')) = \delta_{ml}\pi$$

$$A_\varphi = \frac{\mu_0 I a}{4\pi}(2\pi)\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} \underbrace{P_l^1(\cos(\theta))P_l^1(0)\underbrace{\frac{(l-1)!}{(l+1)!}}_{\frac{1}{l(l+1)}}}_{\frac{(-1)^{s+1}(2s-1)!!}{2^{s+1}s!(2s+2)}}$$

where $l = 2s + 1$.

$$A_\varphi = -\frac{\mu_0 I a}{2}\sum_{s=0}^{\infty}\frac{r_<^{2s+1}}{r_>^{2s+2}}\left[\frac{(-1)^2(2s-1)!!}{2^{s+1}s!(2s+2)} P_{2s+1}^1(\cos(\theta))\right]$$

Now we need to figure out what $\vec{B}$ is!We will once more rewrite this expression so that it matches what Jackson uses:

$$A_\varphi = -\frac{\mu_0 I a}{4}\sum_{s=0}^{\infty}\frac{r_<^{2s+1}}{r_>^{2s+2}}\frac{(-1)^s(2s+1)!!}{2^{s+1}s!(s+1)} P_l^1(\cos(\theta))$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$B_r = \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)A_\varphi)$$
$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)$$

(this is symmetric about φ so we don't need to calculate that component)

If you plug this vector potential into these formulas, you should use the trick that

$$\sin(\theta) = (1-x^2)^{1/2}$$

for small angles, which is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)^{1/2}P_l^1 = \frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)^{1/2}(-1)(1-x^2)^{1/2}P_l = \frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_l = l(l+1)P_l$$

Finally:

$$B_\theta = \frac{\mu_0 I a}{2r}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+1)!!}{2^s s!}\frac{r_<^{2s+1}}{r_>^{2s+2}}P_{2s+1}(\cos(\theta))$$

and

$$B_r = -\frac{\mu_0 I a}{4}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+1)!!}{2^s(s+1)!}\begin{cases} -\frac{2s+2)}{(2s+1)}\frac{1}{a^2}\left(\frac{r}{a}\right)^{2s} & r<a \\ \frac{1}{r^3}\left(\frac{a}{r}\right)^{2s} & r>a \end{cases} P_{2s+1}^1(\cos(\theta))$$

In retrospect, it might make more sense to write this in cylindrical coordinates. We call our current $\vec{J} = I_0\delta(\rho-a)\delta(z)\hat{\varphi}$.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

$$\begin{aligned}\vec{A} = \frac{\mu_0}{4\pi}&\int I_0\delta(\rho'-a)\delta(z')[-\sin(\varphi')\hat{\imath}+\cos(\varphi')\hat{\jmath}]\rho'\,\mathrm{d}\rho'\,\mathrm{d}\theta'\,\mathrm{d}z' \\ &\times\frac{4}{\pi}\int_0^\infty \cos(k(z-z'))\,\mathrm{d}k\left[I_0(k\rho_<)K_0(k\rho_>)+2\sum_{m=1}^{\infty}I_m(k\rho_<)K_m(k\rho_>)\cos(m(\varphi-\varphi'))\right]\end{aligned}$$

LECTURE 22: RING OF CURRENT IN CYLINDRICAL COORDINATES
Mon Oct 7 2019

0.19 Ring of Current in Cylindrical Coordinates

From last time

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4}{\pi}\int_0^\infty \mathrm{d}k\cos(k(z-z'))\left[\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>)+\sum_{m=1}^{\infty}I_m(k\rho_<)K_m(k\rho_>)\cos(m(\varphi-\varphi'))\right]$$

$$\begin{aligned}\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' \\ &= \frac{\mu_0}{4\pi}\int I_0\delta(\phi'-a)\delta(z')[-\sin(\varphi')\hat{\imath}+\cos(\varphi')\hat{\jmath}] \\ &\times \frac{1}{|\vec{x}-\vec{x}'|}\rho'\,\mathrm{d}\rho'\,\mathrm{d}z'\,\mathrm{d}\varphi'\end{aligned}$$

where $\hat{\varphi} = [-\sin(\varphi)\hat{\imath} + \cos(\varphi)\hat{\jmath}$. We can choose $\varphi = 0$ since we believe the system is symmetric about φ. By doing this, we can reduce the equation to

$$\vec{A}(\vec{x}) = \frac{\mu_0 I_0 a}{\pi}\int_0^\infty \mathrm{d}k\cos(kz)I_1(k\rho_<)K_1(k\rho_>)\hat{\jmath}$$

We can then use the previous formulation to write down the elements of the $\vec{B}$ field using the curl:

$$B_\rho = \frac{1}{\rho}\partial_z A_\varphi$$

and

$$B_z = \frac{1}{\rho}\partial_\rho(\rho A_\varphi)$$

or if we rewrite the potential with $\hat{\jmath} = \hat{\varphi}$

$$B_\rho = \frac{1}{\rho}\frac{\mu_0 I_0 a}{\pi}\int_0^\infty \mathrm{d}k\,(-k)\sin(kz)I_1(k\rho_<)K_1(k\rho_>)$$

and

$$B_z = \frac{\mu_0 I_0 a}{\pi}\int_0^\infty \mathrm{d}k\cos(kz)\begin{cases}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho I_1(ka)K_1(k\rho)\right] & \rho > a\\ \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho I_1(k\rho)K_1(ka)\right] & \rho < a\end{cases}$$

Lecture 23: More about Multipole Expansions

Wed Oct 9 2019

0.20 Multipole Expansion for Vector Potential

Recall that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

and

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + (\vec{x}'\cdot\nabla)\frac{1}{|\vec{x}|} + \frac{1}{2}(\vec{x}'\cdot\nabla)^2\frac{1}{|\vec{x}|} + \cdots$$

This is equal to

$$\frac{1}{|\vec{x}|} + x_i'x_i\frac{1}{|\vec{x}|^3} + \cdots$$

$$\partial_i(x_j J_i) = \delta_{ij} J_i + x_j \delta_i J_i = \delta_{ij} J_i = J_j$$

since $\vec{\boldsymbol{\nabla}} \cdot \vec{J} = 0$ in the static case.

Therefore

$$\int \partial_i(x_j J_i)\,\mathrm{d}^3x = 0 = \int J_j\,\mathrm{d}^3x$$

Now we say that

$$\vec{A} = \overbrace{\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x})}{|\vec{x}|}\,\mathrm{d}^3x}^{0} + \frac{\mu_0}{4\pi}\frac{1}{|x|^3}(\int \vec{J}(x')x_i'\,\mathrm{d}^3x')x_i + \cdots$$

Trick:

$$\int \partial_i(x_{j1}x_{j2}J_i) = \int (x_{j2}J_{j1} + x_{j1}J_{j2} + x_{j1}x_{j2}J_i = \int x_{(j1}J_{j2)} = 0$$

Using this notation,

$$A_j = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\; J_j(x')x_i')(x_i\frac{1}{|\vec{x}|^3} + \cdots$$

This integrand is $J_j(x')x_i' = x'_{[i}J_{j]} + x'_{(i}J_{j)} = \frac{1}{2}(x_i'J_j - x_j'J_i) + \frac{1}{2}(x_i'J - j + x_j'J_i)$. We know the integral over the symmetrized part is zero from our trick, so

$$A_j = \frac{\mu_0}{4\pi}\frac{1}{2}\int \mathrm{d}^3x'\,[x_i'J_j - x_j'J_i]\frac{x_i}{|x|^3} + \cdots$$

A useful identity:

$$\epsilon_{ijk}[\epsilon_{klm}x_l'J_m] = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})x_l'J_m = x_i'J_j - x_j'J_i$$

Therefore

$$A_j = \frac{\mu_0}{4\pi}\frac{1}{2}[\int \mathrm{d}^3x'\,\vec{x}' \times \vec{J}(x')] \times \vec{x}\frac{1}{|x|^3} + \cdots$$

And we will call

$$\vec{m} = \frac{1}{2}\int \mathrm{d}^3x'\,\vec{x}' \times \vec{J}(x')$$

All together

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|x|^3} + \cdots$$

Luckily, when we compute $\vec{B} = \vec{\boldsymbol{\nabla}} \times \vec{A}$, we find

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{x})\hat{x}}{|\vec{x}|^3} - \frac{\vec{m}}{|\vec{x}|^3}\right]$$

This expansion has a problem if we want to model point dipoles. If we take the average field over a ball, we can integrate in a ball which does not contain the dipole or in a ball that does contain it (similar to electric case):

$$\int_{\text{average over Ball(r)}} \vec{B}\,\mathrm{d}^3x = \int_{\text{ball}} \vec{\nabla}\times\vec{A}\,\mathrm{d}^3x = \oint_{S^2} (\hat{n}\times\vec{A})\,\mathrm{d}^2\Omega\,R^2$$

This surface integral is

$$\oint_{S^2} (\hat{n}\times\vec{A})\,\mathrm{d}^2\Omega\,R^2 = \frac{\mu_0}{4\pi}\oint \hat{n}times\int\frac{J(x')\,\mathrm{d}^3x'}{|\vec{x}-\vec{x}'|}R^2\,\mathrm{d}\Omega = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\,\vec{J}(\vec{x}')\times\oint\frac{\hat{n}}{|\vec{x}-\vec{x}'|}R^2\,\mathrm{d}\Omega$$

where the surface integral here is equivalent to

$$\frac{4\pi}{3}\hat{x}'\frac{r_<}{r_>^2}$$

(we did this same derivation for the electric dipole)

Therefore,

$$\int_{\text{ball}} \vec{B}\,\mathrm{d}^3x = \begin{cases} \frac{\mu_0}{4\pi}\frac{4\pi}{3}\int\frac{J(x')\times R^3}{|x'|^2}\hat{x}' = \frac{4\pi}{3}R^3\vec{B}(0) & \text{outside} \\ \frac{\mu_0}{4\pi}\frac{4\pi}{3}\int J(x')\times\hat{x}'r'\,\mathrm{d}^3x' = \frac{\mu_0}{4\pi}\frac{8\pi}{3}\vec{m} & \text{inside} \end{cases}$$

In conclusion,

$$\int_{\text{average}} \vec{B}\,\mathrm{d}^3x = \begin{cases} \frac{4\pi R^3\vec{B}(0)}{3} & \text{outside} \\ \frac{2\mu_0}{3}\vec{m} \end{cases}$$

This is following the notation Jackson, where some of the constants are absorbed into $\vec{m}$ to make it look similar to:

$$\int_{\text{average}} \vec{E}\,\mathrm{d}^3x = \begin{cases} \frac{4\pi R^3\vec{E}(0)}{3} & \text{outside} \\ -\frac{1}{3\epsilon_0}\vec{p} & \text{inside} \end{cases}$$

This is all important for dealing with materials. If we had some structure with some microcurrents $\vec{j}$, we need to model the effects of these things. One way to do this is to take small regions and average them out over small volume elements.

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{x})\hat{x}-\vec{m}}{|x|^3}\right] + \frac{2\mu_0}{3}\vec{m}\delta(\vec{x})$$

This last term has to be added to give us the correct average, just like in electric dipole.

Lecture 24: Force Acting on a Localized Current Distribution

Fri Oct 11 2019

From the previous lecture, we found that

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J}(x)\,\mathrm{d}^3x$$

We assume that

$$\vec{F} = q\vec{v} \times \vec{B}$$

or

$$\vec{F} = \int \vec{J} \times \vec{B}\,\mathrm{d}^3x$$

We will Taylor expand this in components (to deal with the cross product):

$$\begin{aligned}
F_i &= \int \epsilon_{ijk} J_j \left[B_k(0) + x_l \partial_l \Big|_0 B_k + \frac{1}{2}(x_l \partial_l)^2 \Big|_0 B_k + \cdots \right] \mathrm{d}x \\
&= \epsilon_{ijk} B_k(0) \cancelto{0}{\int J_j\,\mathrm{d}^3x} + \int \mathrm{d}^3x\, \epsilon_{ijk} J_j x_l \partial_l \Big|_0 B_k + \ldots
\end{aligned}$$

Recall the way we split up index notation in last lecture:

$$\int x_l J_j\,\mathrm{d}^3x = \int x_{[l} J_{j]} + \cancelto{0}{x_{(l} J_{j)}}\,\mathrm{d}^3x = \frac{1}{2}\int \underbrace{x_l J_j - x_j J_l}_{\epsilon_{ljm}(\vec{x}\times\vec{J})_m}\,\mathrm{d}^3x$$

so

$$\begin{aligned}
F_i &= \epsilon_{ijk}\epsilon_{ljm}\frac{1}{2}\int (\vec{x}\times\vec{J})_m\,\mathrm{d}^3x\,\partial_l \Big|_0 B_k \\
&= (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) m_m \partial_l \Big|_0 B_k = m_k \partial_i \Big|_0 B_k - \cancelto{0}{\underbrace{m_i \partial_k \Big|_0 B_k}_{\vec{\nabla}\cdot B=0}}
\end{aligned}$$

We can then say that

$$\vec{F} \approx \nabla \Big|_0 (\vec{m}\cdot\vec{B})$$

Recall that since $\vec{\nabla}\times\vec{B} = 0$ (we suppose this magnetic field is external), $\partial_i B_k - \partial_k B_i = 0$, so

$$m_k \partial_i B_k = m_k \partial_k B_i$$

so

$$\vec{F} \approx (\vec{m}\cdot\vec{\nabla})\Big|_0 \vec{B}$$

What is the torque on this system?

Notation

Jackson uses "n", but we will use $\mathcal{T}$

$$\mathcal{T} = \int \vec{x} \times (\vec{J} \times \vec{B}) \, \mathrm{d}^3 x$$

Again, let's look at the elements:

$$\begin{aligned} \mathcal{T}_i &= \int (J_i(x_k B_k) - B_i(x_k J_k)) \, \mathrm{d}^3 x \\ &= \int J_i(x_k B_k \, \mathrm{d}^3 x - \cancelto{0}{\int B_i x_k J_k \, \mathrm{d}^3 x} \end{aligned}$$

because we can expand B_i as

$$B_i(0) + \vec{x} \cdot \nabla \bigg|_0 \vec{B} + \cdots$$

and

$$B_i(0) \int x_k J_k \, \mathrm{d}^3 x = 0$$

because these are symmetrized indices.

We can expand the other side as

$$\begin{aligned} \mathcal{T}_i &= \int J_i x_k [B_k(0) + (\vec{x} \cdot \nabla) \bigg|_0 \vec{B} + \ldots] \, \mathrm{d}^3 x \\ &= \int \frac{1}{2} (x_k J_i - x_i J_k) B_k(0) \, \mathrm{d}^3 x + \ldots \\ &= \epsilon_{kil} m_l B_k(0) \\ &= \epsilon_{ilk} m_l B_k(0) \\ &= \vec{m} \times \vec{B}(0) \end{aligned}$$

Remark

$$\vec{\boldsymbol{\nabla}} \times \vec{B} = \mu_0 \vec{J}$$
$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} \times \vec{B} = \mu_0 \vec{\boldsymbol{\nabla}} \times J$$
$$\vec{\boldsymbol{\nabla}} \cancelto{0}{\vec{\boldsymbol{\nabla}} \cdot \vec{B}} - \nabla^2 \vec{B} = \mu_0 \vec{\boldsymbol{\nabla}} \times \vec{J}$$

so

$$\nabla^2 \vec{B} = -\mu_0 \vec{\boldsymbol{\nabla}} \times \vec{J}$$

From quantum mechanics, (circularly polarized light, for example), we know that these fields must carry some information about angular momentum. This can't be derived from our current expansions of $\vec{B}$ and $\vec{E}$. There is a more "transparent" expansion, but of course, it requires a "roundabout" way of doing the expansion. In the special case of

$\vec{J} = 0$, we find that $\vec{\nabla} \times \vec{B} = 0$ so $\vec{B} = -\vec{\nabla} \cdot \Phi_M$, where Φ_m is some scalar potential for the magnetic field. There is a problem with this. If we were to look at some path of current and integrate over a path overlapping it (passing through x_0),

$$\oint_\Gamma \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I$$

this would imply that

$$\int \vec{\nabla}\Phi_m \cdot \mathrm{d}\vec{l} = \Phi_m(\vec{x}_0) - \Phi_m(\vec{x}_0) = 0$$

unless we allow the potential to be multivalued (which we shouldn't).

If we look at $\vec{x} \cdot \vec{B}$ instead, we see that

$$\vec{x} \cdot \nabla^2 \vec{B} = \nabla^2(\vec{x} \cdot \vec{B}) - 2\cancelto{0}{\vec{\nabla} \cdot \vec{B}}$$

so

$$\nabla^2 \vec{B} = -\mu_0 \vec{\nabla} \times \vec{J} \implies \vec{x} \cdot \nabla^2 \vec{B} = -mu_0 \vec{x} \cdot \vec{\nabla} \times \vec{J} = \nabla^2(\vec{x} \cdot \vec{B})$$

We can now start playing with this expression:

$$\begin{aligned}
\nabla^2(\vec{x} \cdot \vec{B}) &= -\mu_0 \vec{x} \cdot \vec{\nabla} \times \vec{J} \\
&\to x_i \epsilon_{ijl} \partial_j J_l \\
&= \epsilon_{ijl} x_i \partial_j J_l \\
&= (x \times \nabla) \cdot \vec{J}
\end{aligned}$$

so

$$\begin{aligned}
\nabla^2(\vec{x} \cdot \vec{B}) &= -\mu_0 (\vec{x} \times \nabla) \cdot \vec{J} \\
&= -\imath\mu_0 (\underbrace{-\imath\vec{x} \times \nabla}_{\mathbb{L}}) \cdot \vec{J} \\
&= -\imath\mu_0 \vec{\mathbb{L}} \cdot \vec{J}
\end{aligned}$$

so

$$\vec{x} \cdot \vec{B} = \frac{\imath\mu_0}{4\pi} \int \frac{\vec{\mathbb{L}} \cdot \vec{J}(x')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

We now expand the denominator in terms of our spherical harmonics:

$$\begin{aligned}
\vec{x} \cdot \vec{B} &= \frac{\imath\mu_0}{4\pi} \int (\vec{\mathbb{L}} \cdot \vec{J})(x') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi) \frac{r^l_<}{r^{l+1}_>} \, \mathrm{d}^3 x' \\
&= \frac{\imath\mu_0}{4\pi} = \int \sum_{l,m} Y^*_{lm}(\Omega') \mathbb{L} \cdot \vec{J}(\Omega', r') \, \mathrm{d}\Omega' \, \mathrm{d}r' \, \frac{4\pi}{2l+1} \frac{r^l_<}{r^{l+1}_>} \, \mathrm{d}^3 x' \, Y_{lm}(\Omega)
\end{aligned}$$

It turns out that we can also express

$$\vec{x} \cdot \vec{B} = -r \frac{\partial \Phi_m}{\partial r}$$

so

$$
\begin{aligned}
\frac{\partial \Phi_m}{\partial r} &= \frac{-\imath\mu_0}{4\pi}\frac{1}{r}\sum_{l,m}\int \frac{4\pi}{2l+1}Y^*_{lm}(\Omega')(\vec{\mathbb{L}}\cdot\vec{J})(\Omega',r')r'^l\,\mathrm{d}\Omega'\,\mathrm{d}r'\,\frac{Y_{lm}(\Omega)}{r^{l+1}} \\
&= \left(\frac{-\imath\mu_0}{4\pi}\right)\sum_{l,m}\left(\frac{4\pi}{2l+1}\int Y^*_{lm}(\Omega)\vec{\mathbb{L}}\cdot\vec{J}\,\mathrm{d}\Omega'\,r'^l\,\mathrm{d}r'\right)\frac{Y_{lm}}{r^{l+1}} \\
&= \Phi_M = \frac{\imath\mu_0}{\sqrt{l+1}}\sqrt{l}\sum_{l,m}\left(\frac{4\pi}{2l+1}\right)\left\{\int \frac{\vec{\mathbb{L}}Y^*_{lm}}{\sqrt{l(l+1)}}\cdot\vec{J}\,\mathrm{d}\Omega'\,r'^l\,\mathrm{d}r'\right\}\frac{Y_{lm}}{r^{l+1}}
\end{aligned}
$$

where $\vec{\mathbb{L}}Y^*_{lm}$ are the vector spherical harmonics

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Lecture 25:

Monday, October 14, 2019

Recall

$$\vec{\boldsymbol{\nabla}}\times B = \mu_0\vec{J}$$

and we used this last lecture to show that

$$\nabla^2\vec{B} = -\mu_0\vec{\boldsymbol{\nabla}}\times\vec{J}$$

which we solved to find

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{(\vec{\boldsymbol{\nabla}}\times\vec{J}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

We also did the same formulation with $\vec{x}\cdot\vec{B}$:

$$\nabla^2\vec{x}\cdot\vec{B} = -\mu_0\vec{x}\cdot\vec{\boldsymbol{\nabla}}\times\vec{J}$$

$$\vec{x}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int\frac{(\vec{x}'\cdot\vec{\boldsymbol{\nabla}}\times\vec{J}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m}\frac{4\pi}{2l+1}Y^*_{lm}(\Omega')Y_{lm}(\Omega)$$

If we assume

$$\vec{B} = -\vec{\boldsymbol{\nabla}}\Phi_M$$

we found

$$\vec{x}\cdot\vec{\boldsymbol{\nabla}}\Phi_M = r\frac{\mathrm{d}}{\mathrm{d}r}\Phi_m$$

so

$$\Phi_m = \sum_{l,m}\frac{4\pi}{2l+1}\sqrt{\frac{l}{l+1}}\frac{1}{\sqrt{(l+1)l}}\left[\int r'^l(\vec{\mathbb{L}}Y^*_{lm})\cdot\vec{J}\,\mathrm{d}\Omega'\,r'^2\,\mathrm{d}r'\right]\frac{Y_{lm}(\Omega)}{r^{l+1}}$$

where

$$\frac{1}{\sqrt{l(l+1)}}\vec{\mathbb{L}}Y_{lm} = \vec{\mathbb{X}}_{lm}$$

are the vector spherical harmonics.

$$\int \vec{\mathbb{X}}^*_{lm} \cdot \vec{\mathbb{X}}_{l'm'}\, \mathrm{d}\Omega = \int Y^*_{lm} \vec{\mathbb{L}} \cdot \vec{\mathbb{L}} Y_{l'm'}\, \mathrm{d}\Omega \, \frac{1}{\sqrt{l(l+1)l'(l'+1)}}$$
$$= \frac{l(l+1)}{l(l+1)} \delta_{ll'}\delta_{mm'} = \delta_{ll'}\delta_{mm'}$$

so the vector spherical harmonics are an orthonormal basis. Our expansion is now

$$\Phi_M = \sum_{l,m} \frac{4\pi}{2l+!} \imath \sqrt{\frac{l}{l+1}} \left[\int r'^l r'^2\, \mathrm{d}r'\, \mathrm{d}\Omega'\, \vec{\mathbb{X}}^*_{lm} \cdot \vec{J}(\vec{x}') \right] \frac{Y_{lm}(\Omega)}{r^{l+1}}$$

The idea is, we want to turn this into an expansion for $\vec{A}$, the vector potential for the magnetic field. We want something like $\vec{\boldsymbol{\nabla}} \times \vec{A}$ since $\vec{B} = -\vec{\boldsymbol{\nabla}}\Phi_M = \vec{\boldsymbol{\nabla}} \times \vec{A}$. We use the following identity:

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}} = -\imath \vec{x} \nabla^2 + \imath \vec{\boldsymbol{\nabla}}(1 + \vec{x} \cdot \nabla)$$

In spherical coordinates, $\vec{x} \cdot \nabla = r\frac{\mathrm{d}}{\mathrm{d}r}$. Additionally recall that,

$$\nabla^2 \left(\frac{Y_{lm}}{r^{l+1}} \right) = 0$$

Let us then write

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}} \left(\frac{Y_{lm}}{r^{l+1}} \right) = \cancelto{0}{-\imath \vec{x} \nabla^2 \left(\frac{Y_{lm}}{r^{l+1}} \right)} + \underbrace{\imath \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \left[1 + r\frac{\mathrm{d}}{\mathrm{d}r} \right] \left(\frac{Y_{lm}}{r^{l+1}} \right)}_{(1-l-1)\frac{1}{r^{l+1}} Y_{lm}}$$

Therefore

$$\vec{\boldsymbol{\nabla}} \times \left[\frac{1}{l\imath} \right] \vec{\mathbb{L}} \left(\frac{Y_{lm}}{r^{l+1}} \right) = -\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \left(\frac{Y_{lm}}{r^{l+1}} \right)$$

Using this, we see that

$$-\vec{\boldsymbol{\nabla}}\Phi_M = -\nabla \sum_{l,m} B_{lm} \frac{Y_{lm}}{r^{l+1}} = \sum_{l,m} B_{lm} \left(-\nabla \frac{Y_{lm}}{r^{l+1}} \right) = \sum_{l,m} B_{lm} \left[\frac{1}{\imath l} \vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}} \frac{Y_{lm}}{r^{l+1}} \right]$$

Therefore, we see that

$$\vec{\boldsymbol{\nabla}} \times \sum_{l,m} \left[\frac{B_{lm}}{\imath l} \frac{\vec{\mathbb{L}} Y_{lm}}{r^{l+1}} \right] = \vec{\boldsymbol{\nabla}} \times \vec{A}$$

$$\vec{B} = \vec{\boldsymbol{\nabla}} \times \left[\sum_{l,m} \frac{4\pi}{2l+1} \frac{\imath}{\imath l} \sqrt{\frac{l}{l+1}} \left(\int \mathrm{d}^3x'\, r'^l \vec{\mathbb{X}}^* \cdot \vec{J} \right) \frac{\vec{\mathbb{L}} Y_{lm}}{r^{l+1}} \right]$$
$$= \vec{\boldsymbol{\nabla}} \times \left[\sum_{l,m} \frac{4\pi}{2l+1} \left(\int \mathrm{d}^3x'\, r'^l \vec{\mathbb{X}}^*_{lm} \cdot \vec{J} \right) \frac{\vec{\mathbb{X}}_{lm}}{r^{l+1}} \right]$$

Therefore, the true multipole expansion for the vector potential is

$$\vec{A} = \left[\sum_{l,m} \frac{4\pi}{2l+1} \left(\int \mathrm{d}^3x'\, r'^l \vec{\mathbb{X}}^*_{lm} \cdot \vec{J} \right) \frac{\vec{\mathbb{X}}_{lm}}{r^{l+1}} \right]$$

Example. Let us have an example of using Φ_M. It can be useful in some situations and is not just an operational trick. This is the homework problem for a rotating sphere. We have a charged sphere rotating with angular velocity ω with a surface current density $\vec{J} = \sigma\omega a \sin(\theta)\hat{\varphi}\delta(r-a)$. We want to find the B field. The homework is to solve for $\vec{A}$. However, there are two regions that are free from currents, the inside of the sphere and the outside. Φ_M works when there are no currents, so we could just glue these regions together using Φ_M. Recall that on the surface, the normal component of B in the two regions should be equal and continuous because $\vec{\nabla} \cdot \vec{B} = 0$. If we had a surface current, the tangential components must jump by the surface current (not a volume current) across the boundary.

In the outside region,

$$\vec{\nabla} \cdot \vec{B} = 0 \implies \nabla^2 \Phi_M = 0$$

$$\Phi_M = \begin{cases} \sum A_l r^l P_l(\cos(\theta)) & r < a \\ \sum \frac{B_l}{r^{l+1}} P_l(\cos(\theta)) & r > a \end{cases}$$

additionally, the continuity of the field across the boundary implies

$$-\frac{\partial \Phi_M}{\partial r}\bigg|_{r\to a^- = r \to a^+} \implies A_l = -\frac{l+1}{l}\frac{B_l}{a^{2l+1}}$$

Our other boundary condition tells us

$$B_\theta^{\text{outside}} - B_\theta^{\text{inside}} = k_\varphi = \sigma a \omega \sin(\theta)$$

so

$$-\frac{1}{r}\frac{\partial \Phi_M}{\partial \theta}\bigg|_{r\to a^+} + \frac{1}{r}\frac{\partial \Phi_M}{\partial r}\bigg|_{r\to a^-} = \sigma a \omega \sin(\theta)$$

◇

Lecture 26: Spinning Charged Sphere, continued

Monday, October 14, 2019

Example. From the last lecture, $\vec{J} = \sigma\omega a \sin(\theta)\delta(r-a)\hat{\varphi}$ and we are using $\vec{B} = -\vec{\nabla}\Phi_M$ with the boundary conditions $B_n^{(I)} - B_n^{(II)} = 0$ and $B_\theta^{(II)} - B_{\theta=\mu_0 K_\varphi}^{(I)} = \mu_0 \sigma a \omega \sin(\theta)$.

$$A_l = -\frac{l+1}{l}\frac{B_l}{a^{2l+1}}$$

Paramagnetic	Diamagnetic	Ferromagnetic
$-\vec{m}\cdot\vec{B} \iff k_B T$	Langevin Model	Heisenberg Model: $\mathbb{H} = -J\sum_{\langle ij\rangle}\vec{\delta}_i\vec{\delta}_j \to$ Ising Model in the Classical limit

Table 0.21.1: Three Different Types of Materials

This was from the first boundary condition. Next,

$$\sum_l \left[-\frac{1}{a}\frac{\partial}{\partial\theta}\frac{B_l}{a^{l+1}}P_l(\cos(\theta)) + \frac{1}{a}\frac{\partial}{\partial\theta}a^l A_l P_l(\cos(\theta))\right] = \mu_0\sigma\omega a\sin(\theta)$$

$$\sum_l \left[-\frac{B_l}{a^{l+1}}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}P_l(\cos(\theta)) + a^l A_l\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}P_l(\cos(\theta))\right] = \mu_0\sigma\omega a^2$$

$$\sum_l \left[+\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}(\cos(\theta))}P_l(\cos(\theta)) - a^l A_l\frac{\mathrm{d}}{\mathrm{d}(\cos(\theta))}P_l(\cos(\theta))\right] = \mu_0\sigma\omega a^2$$

$$\sum_l \left[\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}x}P_l + \frac{l+1}{l}\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}x}P_l\right] = \mu_0\sigma\omega a^2$$

$$\sum_l \left[\frac{2l+1}{l}\frac{B_l}{a^{l+1}}\frac{\mathrm{d}}{\mathrm{d}x}P_l\right] = \mu_0\sigma\omega a^2$$

There is no x dependence on the right side, and the only way to make that true on the left side is for $l = 1$. Therefore

$$B_1 = \frac{\mu_0\sigma\omega a^4}{3}$$

so

$$\Phi_M = \begin{cases} \frac{\mu_0\sigma\omega a^4}{3}\frac{\cos(\theta)}{r^2} & r > a \\ \frac{2\mu_0\sigma\omega a^4}{3}r\cos(\theta) & r < a \end{cases}$$

so

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{8\pi}{3}a^3\sigma\omega\right]\hat{z} \quad \text{if} \quad r < a$$

is constant inside the sphere. Outside the sphere the field, the field looks like a dipole field. ◇

0.21 Materials with Magnetic Properties

In the macroscopic limit, we average out the microscopic distribution:

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}_{\text{free}(\vec{x}')}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x + \frac{\mu_0}{4\pi}\int\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,\mathrm{d}^3x$$

The second term here is basically

$$\vec{M}(\vec{x}')\times\vec{\nabla}\cdot\frac{1}{|\vec{x}-\vec{x}'|} \to -\vec{\nabla}\times\left[\vec{M}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\right]$$

so the integration gives us

$$-\int \vec{\nabla} \times \vec{M}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' + \int \frac{\vec{\nabla}\times\vec{M}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

so

$$\vec{A}_{\text{matter}} = \frac{\mu_0}{4\pi}\left\{\oint \frac{\hat{n}\times\vec{M}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}a' + \int \frac{\vec{\nabla}\times\vec{M}}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'\right\}$$

From this we can find that the material description is reduced to

$$\vec{J}_{\text{matter}} = \vec{\nabla}\times\vec{M},$$

an effective medium current, and

$$\vec{K}_{\text{matter}} = \hat{n}\times\vec{M},$$

an effective surface current. From here, we modify Maxwell's equations to

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_{\text{free}} + \mu_0\vec{\nabla}\times\vec{M}$$

just like we did with the electric field (where $P_{\text{bound}} = -\vec{\nabla}\cdot\vec{P}$ and $\sigma_{\text{bound}} = \vec{P}\cdot\hat{n}$).

Lecture 27: Magnetic Materials

Monday, October 21, 2019

From the electrons in a magnetic material, we get magnetic dipoles $\vec{m} = g\mu_B\vec{S}$. Naively, we can think of this as some electron rotating at radius r, such that the current is $\frac{e}{\frac{2\pi r}{v}}\pi r^2 = m = \frac{e}{2}rv = \frac{e}{2m_e}\overbrace{m_e v r}^{L}$. Therefore, $\vec{m} = \frac{e}{2m_e}\vec{S}$. The spin is $S = \frac{1}{2}$, but $g_e \approx 2$, so the dipole moment is actually not just a trivial electron orbiting, and the actual corrections to the gyromagnetic constant come from QFT.

Let us introduce a macroscopic volume density of magnetic dipoles $\vec{M}$. Recall that we typically find higher order terms in magnetic dipoles are negligible.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\,\frac{\vec{J}_{\text{free}}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\,\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

Let us rewrite this second term using our usual trick:

$$= \frac{\mu_0}{4\pi}\left\{\int -\vec{\nabla}\times\left[\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right]\mathrm{d}^3x' + \int \frac{\vec{\nabla}\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'\right\}$$

Using the divergence theorem relations, this first term is equal to

$$\oint \frac{\vec{M}(\vec{x}')\times\hat{n}'}{|\vec{x}-\vec{x}'|}\,\mathrm{d}a'$$

so *effectively*:

$$\vec{J}_M = \vec{\nabla}\times\vec{M}$$

and

$$\vec{K}_M = \vec{M} \times \hat{n}$$

Now let us rewrite these in a differential form:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{\text{free}} + \mu_0 \vec{\nabla} \times \vec{M}$$

and of course

$$\vec{\nabla} \cdot \vec{B} = 0$$

Pro Tip

Again, if you find one of these "magnetic monopoles" you will be eternally famous.

This first equation is usually rewritten as

$$\vec{\nabla} \times \underbrace{\left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right)}_{\vec{H}} = \vec{J}_{\text{free}}$$

What is $\vec{M}$? How do we link $\vec{M}$ to $\vec{B}$. The usual approach is to write $\vec{H}$ as a function of $\vec{B}$ or vice-versa. For ferromagnets, there is a "saturation point" where an externally applied $\vec{H}$ will no longer create more $\vec{B}$. Reducing the applied magnetic field will result in hysteresis. Physically, this is because we are aligning all of the spin moments of the ferromagnet, and when we then reduce the applied field, they are in a lower energy state than the disordered state they started in, so they won't return to the same ground state, but rather to a lower energy state. If we include temperature, this gets more complicated, but one result is that magnetization cannot exist above the Curie temperature because the dipoles are moving around more with increasing temperature, and at some point, there is no way for the material to maintain the orderly orientation of the magnetic dipoles.

In linear materials,

$$\frac{1}{\mu_0} \vec{B} - \vec{M}[\vec{B}] = \frac{1}{\mu} \vec{B} = \vec{H}$$

since $\vec{M}[\vec{B}]$ is generated linearly by $\vec{B}$. Here, we define μ as the magnetic permeability. It can be inhomogeneous, and/or non-isotropic, but these solutions get very complicated very fast.

$$\vec{H} = \frac{1}{\mu} \vec{B} \quad \text{and} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A} \implies \vec{\nabla} \times \vec{H} = \vec{J}_{\text{free}} - \vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right)$$

Remember that A is not uniquely defined, and is invariant up to the gradient of some constant field χ:

$$\vec{A}' = \vec{A} + \vec{\nabla} \cdot \chi$$

$$\vec{B} = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \cancelto{0}{\vec{\nabla} \times \vec{\nabla} \cdot \chi}$$

We will choose a gauge where $\vec{\boldsymbol{\nabla}} \cdot \vec{A} = 0$. If we have constant μ,

$$\frac{1}{\mu}\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{A}) = \vec{J}_{\text{free}} = \frac{1}{\mu}\left[\cancelto{0}{\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{\nabla}} \cdot \vec{A}} - \nabla^2 \vec{A}\right]$$

so

$$\nabla^2 \vec{A} = -\mu \vec{J}_{\text{free}}$$

Now let us examine our boundary conditions. Using the Gaussian pillbox,

$$B_n^{(I)} - B_n^{(II)} = 0$$

If we have a free surface current,

$$H_t^{(I)} - H_t^{(II)} = \vec{K} \cdot (\hat{t} \times \hat{n})$$

where $\hat{t}$ is orthogonal to the current and the normal. This is sometimes written

$$(H_t^{(I)} - H_t^{(II)}) \times \hat{n} = \vec{K}$$

Let's examine a special case: $J_{\text{free}} = 0$. Now $\vec{\boldsymbol{\nabla}} \times \vec{H} = 0$. We could now claim

$$\vec{H} = -\vec{\boldsymbol{\nabla}}\Phi_M$$

We must be careful here, since Φ_M is usually not single-valued over all space.

$$\vec{B} = \vec{H} + \mu_0 \vec{M} \implies \vec{\boldsymbol{\nabla}} \cdot \vec{B} = 0 = \vec{\boldsymbol{\nabla}} \cdot (-\vec{\boldsymbol{\nabla}}\Phi_M) + \mu_0 \vec{\boldsymbol{\nabla}} \cdot \vec{M}$$

or

$$0 = -\nabla^2 \Phi_M + \mu_0 \vec{\boldsymbol{\nabla}} \cdot \vec{M}$$

so

$$\nabla^2 \Phi_M = -\mu_0 \underbrace{(-\vec{\boldsymbol{\nabla}} \cdot \vec{M})}_{\text{source}}$$

We would now think that

$$\Phi_M = \frac{\mu_0}{4\pi} \int_\Omega \frac{(-\vec{\boldsymbol{\nabla}} \cdot \vec{M}}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

However, we must also have a surface correction:

$$\Phi_M = \frac{\mu_0}{4\pi} \int_\Omega \frac{(-\vec{\boldsymbol{\nabla}} \cdot \vec{M}}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x' + \frac{\mu_0}{4\pi} \oint_\Sigma \frac{\vec{M}(\vec{x}') \cdot \hat{n}'}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

The justification for this is that the volume integral over $-\vec{\boldsymbol{\nabla}} \cdot \vec{M}$ must equal the surface integral of $\vec{M} \cdot \hat{n}$, but increasing the volume of our integrating surface will not change the integral, since we only integrate over nonzero $\vec{M}$, but it will increase the surface integral. Without this correction, the field would appear as a monopole from far away.

For linear materials, $\vec{H} = \frac{1}{\mu}\vec{B}$ and $\vec{\boldsymbol{\nabla}} \times \vec{H} = 0$ ($\vec{J}_{\text{free}} = 0$), so

$$\vec{H} = -\vec{\boldsymbol{\nabla}}\Phi_M \implies \vec{B} = \mu \vec{H}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{B} = -\vec{\boldsymbol{\nabla}} \cdot \mu \vec{\boldsymbol{\nabla}} \Phi_m = 0$$

The philosophy is, if we have no free current, we can make our field scalar. Despite the similarity to the electric dipole correction, these are not connected. We can still solve this field as a scalar potential if there is an applied magnetic field. What kinds of problems will we try to solve? This afternoon, we will look at the following:

- $\vec{M} = M_0 \hat{z}$ on a ball of radius a
- Shell of inner radius a and outer radius b made of a linear material inserted into a uniform $\vec{B}$ field. We will show that the magnetic field nearly vanishes inside a.

LECTURE 28: SELECTED MAGNETIC DENSITY PROBLEMS
Monday, October 21, 2019

0.22 Magnetic Dipole Density Examples

Example. A magnetized ball: $\vec{M} = M_0 \hat{z}$ **1st Method**

$$\vec{J}_M = \vec{\boldsymbol{\nabla}} \times \vec{M} = 0$$

$$\vec{K}_M = \vec{M} \times \hat{n} = M_0 \sin(\theta) \hat{\varphi}$$

This is exactly like the rotating sphere homework:

$$\vec{A} = \frac{\mu_0}{4\pi} \oint \frac{\vec{K}_M(x')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}a'$$

2nd Method

$$\vec{H} = -\vec{\boldsymbol{\nabla}} \Phi_M$$

$$\vec{J}_{\text{free}} = 0$$

$$\nabla^2 \Phi_M = -[-\vec{\boldsymbol{\nabla}} \cdot \vec{M}]$$

Recall that we derived the form of Φ_M:

$$\Phi_M = \frac{1}{4\pi} \int_\Omega \frac{-\vec{\boldsymbol{\nabla}} \cdot \vec{M}}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x + \frac{1}{4\pi} \oint \frac{M_0 \cdot \hat{n}'}{|\vec{x} - \vec{x}'|} \, \mathrm{d}a'$$

By our definition of $\vec{M}$:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{M} = 0$$

However, there is a surface term:

$$\vec{M} \cdot \hat{n} = M_0 \cos(\theta)$$

Therefore:

$$\Phi_M = \frac{1}{4\pi} \oint_{S^2} \frac{M_0 \cos(\theta')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}\Omega' \, a^2 = \frac{M_0 a^2}{4\pi} \int \frac{\cos(\theta')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}\Omega' = \frac{M_0 a^2}{4\pi} \frac{4\pi}{3} \frac{r_<}{r_>^2} \underbrace{P_1(\cos(\theta))}_{\cos(\theta)}$$

Therefore,

$$\Phi_M = \begin{cases} \frac{M_0 a^2}{3} \frac{r}{a^2} \cos(\theta) = \frac{M_0}{3} z & r < a \\ \frac{M_0 a^2}{3} \frac{a}{r^2} \cos(\theta) = \frac{m \cos(\theta)}{4\pi r^2} & r > a \end{cases}$$

where $\vec{m} = \left(\frac{4\pi}{3} a^3\right) M_0 \hat{z}$.

$$\vec{H}_{\text{in}} = -\frac{M_0}{3} \hat{z}$$

$$\vec{H}_{\text{out}} \propto \text{dipole field}$$

$$\vec{B}_{\text{in}} = \mu_0 \left[-\frac{M_0}{3} \hat{z} + M_0 \hat{z} \right] = \frac{2}{3} \mu_0 M_0 \hat{z}$$

◇

Example. Let us consider putting such a sphere into an external field. We would then imagine, by superposition, that $\vec{B}_0 + \frac{2}{3}\mu_0 \vec{M} = \vec{B}_{\text{in}}$. Additionally, this means that $\vec{H}_{\text{in}} - \frac{1}{3}\vec{M}$. The solution must be self-consistent, such that

$$\vec{H}_{\text{in}} = \frac{1}{\mu} \vec{B}_{\text{in}}$$

This gives the relation

$$\frac{1}{\mu_0} \vec{B}_0 - \frac{1}{3} \vec{M} = \frac{1}{\mu} \left[\vec{B}_0 + \frac{2}{3} \mu_0 \vec{M} \right]$$

so

$$\vec{M} = \frac{3}{\mu_0} \left[\frac{\mu - \mu_0}{\mu + 2\mu_0} \right] \vec{B}_0$$

◇

Example. Magnetic Shielding We now have a shell with inner radius a and outer radius b with an external magnetic field. Again, let us assume $\vec{J}_{\text{free}} = 0$ (the field is curl-free):

$$\vec{H} = -\vec{\nabla} \Phi_M$$

Because $\vec{\nabla} \cdot \vec{B} = 0$,

$$\nabla^2 \Phi_M = 0$$

Using the azimuthal symmetry of this problem, we can write

$$\Phi_M = \begin{cases} \sum_{l=0}^{\infty} \alpha_l r^l P_l(\cos(\theta)) & r < a \\ \sum_{l=0}^{\infty} \left[\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right] P_l(\cos(\theta)) & a < r < b \\ -H_0 r \cos(\theta) + \sum_{l=0}^{\infty} \frac{\delta_l}{r^{l+1}} P_l(\cos(\theta)) & b < r \end{cases}$$

Our first boundary condition is that the magnetic field B is continuous normal to the boundaries at a and b. Additionally, the tangential component is H_θ, which must also be continuous at each boundary. This problem is left as an exercise for the reader. The solution is given in Jackson. If $\mu >> 1$, there is strong magnetic shielding. ◇

0.23 Faraday's Law

For a surface Σ and loop Γ such that $\Gamma = \partial\Sigma$, the boundary of the surface, the energy gained by going around the loop once is

$$\mathscr{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \vec{B}\cdot\hat{n}\,\mathrm{d}a$$

where $\int_\Sigma \vec{B}\cdot\hat{n}\,\mathrm{d}a = \text{flux}$ so $\mathscr{E} = -\frac{\mathrm{dflux}}{\mathrm{d}t}$. This "electromotive force" or "emf" $\mathscr{E}$ corresponds to an electric field felt on the loop induced by the magnetic field in the rest frame of the loop. We can then say that

$$\mathscr{E} = \oint_{Gamma} \vec{E}'\cdot\mathrm{d}\vec{l}$$

The electric field can no longer be curl-free (it's in a loop, after all). Because the surface is fixed, we can write

$$\oint_\Gamma \vec{E}\cdot\mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \vec{B}\cdot\mathrm{d}\vec{a} = -\int\frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{a}$$

By Stokes' Theorem, this implies

$$\int\left(\vec{\boldsymbol{\nabla}}\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right)\cdot\mathrm{d}\vec{a} = 0$$

so we must now modify Maxwell's equations to include

$$\vec{\boldsymbol{\nabla}}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

This is only in the rest frame!However, if $v << c$, $\vec{E}' = \vec{E} + \vec{v}\times\vec{B}$, so Faraday is consistent.

LECTURE 29: FARADAY'S LAW
Wednesday, October 23, 2019

Again, for a surface Σ with boundary Γ,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \vec{B}\cdot\hat{n}\,\mathrm{d}a = \mathscr{E}$$

where $\mathscr{E}$ is the electromotive force, the net energy gain after a unit charge moves around the loop. In the rest frame of the loop Γ, the $\vec{E}$-field does the work, so

$$\mathscr{E} = \oint_\Gamma \vec{E}'\cdot\mathrm{d}\vec{l}$$

If we fix the loop, we can bring the time derivative inside the integral, so

$$\oint_\Gamma \vec{E}\cdot\mathrm{d}\vec{l} = -\int_\Sigma\frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{a}$$

where $\vec{E}' = \vec{E}$ for a fixed loop. By Stokes theorem,

$$\int_\Sigma \vec{\nabla} \times \vec{E} \cdot \mathrm{d}\vec{a} + \int_\Sigma \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{a} = 0$$

so

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

> **Digression**
>
> $$\vec{B} = \vec{\nabla} \times \vec{A}$$
>
> now implies that
>
> $$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$
>
> or
>
> $$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

What happens if the loop does move? Let's assume rigid motion (no deformation of the loop itself, just translation in space). Suppose the loop moves with velocity $\vec{v}$ and $\vec{B}$ is constant in time but could vary in space. If we imagine connecting a surface Σ_0 and Σ_{dt}, we can find the flux:

$$\int_{\Sigma_0} \vec{B} \cdot \mathrm{d}\vec{a} + \int_{-\Sigma_{dt}} \vec{B} \cdot \mathrm{d}\vec{a} + \int_{\text{sides}} \vec{B} \cdot \mathrm{d}\vec{a} = 0$$

since there is no divergence of the $\vec{B}$ field. The flux through the opposite orientation can be found by just negating the middle integral. $\vec{v} \times \mathrm{d}\vec{l} = \mathrm{d}\vec{a}$ on the sides, so

$$\int_{\Sigma_{dt}} \vec{B} - \int_{\Sigma_0} \vec{B} = \oint_\Gamma (\vec{v} \times \mathrm{d}\vec{l}) \cdot \vec{B} dt$$

so

$$\mathscr{E} = -\oint_\Gamma (\vec{v} \times \mathrm{d}\vec{l}) \cdot \vec{B} = \oint_\Gamma (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{l}$$

Notice how similar this is to the magnetic force on the charges. Technically, the motion of the charges includes a drift velocity along the loop in addition to the motion of the loop itself, but because that velocity is parallel to $\mathrm{d}\vec{l}$, its contribution is zero. By relating this equation to our loop frame, we find that, non-relativistically,

$$\vec{E}' = \vec{v} \times \vec{B}$$

so

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

This is the general formulation for changing reference frames in electromagnetism (for $v << c$). This is how we "make" magnetic fields do work. If the charges are constrained, like on a loop, we can move them and use a magnetic field to do work on them.

0.24 Energy Stored in Magnetic Fields

Let's look at a loop Γ upon which we are establishing a current. As we increase the flux, there will be a back-emf generated due to the loop's own magnetic field.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -I \cdot \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t}$$

where $\mathcal{F}$ is the flux. Therefore

$$\mathrm{d}W = I\mathrm{d}\mathcal{F}$$

We lose the minus sign because this is the back-emf, so the current for it is going in the opposite direction. If we suppose the flux contains some geometric factor L (self-inductance) $\mathcal{F} = L \cdot I$,

$$\mathrm{d}W = IL\mathrm{d}I$$

so

$$W = \frac{1}{2}LI^2$$

Now let's generalize to some current density $\vec{J}$ with $\vec{\nabla} \cdot \vec{J} = 0$. Imagine we perform this adiabatically, such that $\frac{\partial \rho}{\partial t} \approx 0$. If we look at a small cross-section $\mathrm{d}\vec{\sigma}$, we have $\underbrace{\vec{J} \cdot \mathrm{d}\vec{\sigma}}_{\mathrm{d}I} \underbrace{\int_S \delta\vec{B} \cdot \mathrm{d}\vec{a}}_{\delta\mathcal{F}}$ where S is the area of the cross-section, so

$$\delta(\mathrm{d}W) = \mathrm{d}I \int_S \delta\vec{B} \cdot \mathrm{d}\vec{a} = \mathrm{d}I \int_S \vec{\nabla} \times \delta\vec{A} \cdot \mathrm{d}\vec{a} = \mathrm{d}I \oint_\Gamma \delta\vec{A} \cdot \mathrm{d}\vec{l}$$

so

$$\delta(\mathrm{d}W) = \oint \delta\vec{A} \cdot \mathrm{d}\vec{I}\,\mathrm{d}l = \oint \delta\vec{A} \cdot \vec{J}\,\mathrm{d}\sigma\,\mathrm{d}l$$

If we sum over all of these segmented loops (all $\mathrm{d}\sigma\,\mathrm{d}l$), we say that this becomes a volume integral over the region.

$$\delta W = \int_\Omega \vec{J} \cdot \delta\vec{A}\,\mathrm{d}^3x$$

Lecture 30: Energy Calculations for Current Densities, Continued

Monday, October 28, 2019

Recall from last lecture we claimed that

$$\begin{aligned}
\delta W &= \int \vec{J}\cdot\delta\vec{A}\,\mathrm{d}l\,\mathrm{d}\sigma \\
&= \int_\Omega \vec{J}\cdot\delta\vec{A}\,\mathrm{d}^3x \\
&= \int_\Omega (\vec{\nabla}\times\vec{H})\cdot\delta\vec{A}\,\mathrm{d}^3x \\
&= \underbrace{\int_\omega \vec{\nabla}\cdot\vec{H}\times\delta\vec{A}\,\mathrm{d}^3x}_{\int_\Sigma \cancelto{0}{\vec{H}}\times\delta\vec{A}\cdot\mathrm{d}\vec{a}} + \int_\Omega \vec{H}\cdot\underbrace{(\vec{\nabla}\times\delta\vec{A})}_{\delta\vec{B}}\,\mathrm{d}^3x \\
&= \int_\Omega \vec{H}\cdot\delta\vec{B}\,\mathrm{d}^3x
\end{aligned}$$

If $\vec{H}=\frac{1}{\mu}\vec{B}$, this simplifies (if the material is linear):

$$\delta W = \frac{1}{2}\delta\int_\Omega \vec{H}\cdot\vec{B}\,\mathrm{d}^3x$$

so

$$W = \cancelto{0}{W_0} + \frac{1}{2\mu}\int_\Omega B^2\,\mathrm{d}^3x$$

An application of this is, if you insert a linear material into an existing magnetic field,

$$\Delta W = \frac{1}{2}\int_\Omega \vec{H}\cdot\vec{B}\,\mathrm{d}^3x - \frac{1}{2}\int_\Omega \vec{H}_0\cdot\vec{B}_0\,\mathrm{d}^3x$$

where the second term is the existing energy in the field. Recall that for the similar problem with dielectrics, we have

$$\Delta W = -\frac{1}{2}\int_\Omega \vec{P}\cdot\vec{E}_0\,\mathrm{d}^3x$$

By the same kinds of tricks we used to find this, we see that for magnetic fields,

$$\Delta W = \frac{1}{2}\int_\Omega \vec{M}\cdot\vec{B}_0\,\mathrm{d}^3x$$

Recall that to find the calculation for the electric work, we had to keep the charges fixed. For this scenario, we have to keep the currents fixed. If the currents were allowed to run, introducing the magnetic field would reduce the currents. This means that there is an extra work done by the source of the magnetic field to maintain $\vec{B}_0$. You could also do this calculation by finding the forces using the induced currents $\vec{J}_M=\vec{\nabla}\times\vec{M}$ and $\vec{K}_M=\vec{M}\times\hat{n}$, and you will get the same answer, but sometimes these currents are hard to calculate.

0.25 Self-Inductance

Recall for a loop in a magnetic field,

$$\mathscr{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

for flux

$$\Phi = \int_{\Sigma} \vec{B} \cdot \mathrm{d}\vec{a}$$

If the current in the wire is I, the self-inductance L of the wire is

$$\mathscr{E} = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

Now suppose we have multiple loops. We can now find the mutual inductance. The influence of the ith loop on the jth loop is

$$\mathscr{E}_i = -M_{ij}\frac{\mathrm{d}I_j}{\mathrm{d}t}$$

If we are building current I_i in the presence of nothing,

$$\underbrace{\mathscr{E}_i^{\text{ext}} \underbrace{I_i \, \mathrm{d}t}_{\mathrm{d}Q_i}}_{\mathrm{d}W_i^{\text{ext}}} = L_i \frac{\mathrm{d}I_i}{\mathrm{d}t} I_i \, \mathrm{d}t$$

so

$$W = \frac{1}{2} L_i I_i^2$$

However, with another loop, there is an additional factor on the right side:

$$\mathrm{d}W_i = L_i I_i \, \mathrm{d}I_i + M_{ji} \frac{\mathrm{d}I_i}{\mathrm{d}t} \underbrace{I_j dt}_{\mathrm{d}Q_j}$$

As we increase the current in loop i, we have to do work to ensure the loop j maintains its current. After you sum over all such contributions, you find that

$$W = \sum_{i=1}^{N} \frac{1}{2} L_i I_i^2 + \sum_{i<j} M_{ij} I_i I_j$$

Additionally, we could switch the labels on the loops, which should give the same formula. This gives us, non-trivially, that $M_{ij} = M_{ji}$. This is how transformers work, the ratios of the loops can be used to change the ratios of the voltages on either side. We send power over high-voltage lines because there is lower current, which means less power is dissipated over the wire ($P = I^2 R$). The power (VI) is fixed, so increasing V decreases I.

0.26 Effect of Magnetic Fields in Conductors

In a static case, there is no effect. However, if you have time dependent magnetic fields, you will create eddy currents. Suppose we have no free currents, only the response to the magnetic field: $\vec{J} = \sigma \vec{E}$. The magnetic field changing will create an electric field. Recall

$$\vec{\boldsymbol{\nabla}} \times \vec{E} = -\partial_t \vec{B}$$

and

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}_C = \mu\sigma\vec{E}$$

The curl of the second equation is

$$\underbrace{\vec{\nabla} \times \vec{\nabla} \times Bve}_{\cancelto{0}{\vec{\nabla}\vec{\nabla}\cdot\vec{B}} - \nabla^2\vec{B}} = \mu\vec{\nabla} \times \vec{J}_C = \mu\sigma(-\partial_t\vec{B})$$

so

$$\nabla^2\vec{B} = \mu\sigma\partial_t\vec{B}$$

Let's look at the boundary of the material. Recall that $B_\perp$ is continuous across a boundary and $H_\parallel$ is continuous if there are no surface currents. We suppose there are only volume currents induced by the magnetic field. Say we have a $\vec{B} = B_3(z,t)$ and our material exists in the area $z > 0$. Suppose at $z = 0$, $B_0(t) = \mathrm{Re}[B_0 e^{\imath\omega t}]$. The time dependence of these equations must be the same, so we assume $B_3(z,t) = \mathrm{Re}[B_3(z)e^{\imath\omega t}]$

$$\partial_z^2 B_3(z,t) = \imath\mu\omega\sigma B_3$$

Lecture 31: Magnetic Fields Inside Conductors, Continued

Monday, October 28, 2019

From last lecture, we had

$$\vec{J}_C = \sigma\vec{E}$$

and for linear media,

$$\vec{H} = \frac{1}{\mu}\vec{B}$$

Recall that

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

so

$$\nabla^2\vec{B} = \sigma\mu\partial_t\vec{B}$$

In last lecture, there was a problem with the proposed boundary conditions. The following is a correction.

Imagine with have a material above the z-axis and free space below it. In the space below, suppose we have a magnetic field $B_x(t)$ oriented in the $+\hat{x}$ direction. The magnetic field must be continuous on both sides of the z-axis boundary (it is). Now say the field is $B_x(t) = B_0\cos(\omega t)$, so $\vec{B} = B_x(z,t)\hat{x}$. We want to solve our Laplace equation using this field.

$$\vec{B} = \mathrm{Re}[B_x(z)e^{-\imath\omega t}]\hat{x}$$

Remember that the parallel H-field is continuous, so

$$\frac{1}{\mu_0}B_0 e^{-\imath\omega t}\bigg|_{z\to 0^-} = \frac{1}{\mu}B_x(z)\bigg|_{z\to 0^+} e^{-\imath\omega t}$$

so

$$\frac{1}{\mu}B_x(0^+) = \frac{1}{\mu_0}B_x(0^-) = \frac{1}{\mu_0}B_0$$

so

$$B_x(0^+) = \frac{\mu}{\mu_0}B_0$$

Our Laplace equation is now

$$\partial_z^2 B_x e^{-\imath\omega t} = \sigma\mu(-\imath\omega)B_x(z)e^{-\imath\omega t}$$

or

$$\partial_z^2 B_x + \sigma\mu\omega\imath B_x = 0$$

Solving this, we assume $B_x = e^{\imath kz}A$, so $k^2 = \sigma\mu\omega\imath = \sigma\mu\omega e^{\frac{\pi}{2}\imath}$. If we define $\delta = \sqrt{\frac{2}{\sigma\mu\omega}}$ as the "skin depth," we find that

$$B_x(z,t) = \mathrm{Re}\left\{\frac{\mu}{\mu_0}B_0 e^{\imath\left[\sqrt{\frac{\sigma\mu\omega}{2}}(1+\imath)\right]z}e^{-\imath\omega t}\right\} = e^{-\frac{2}{\delta}}\cos(\omega t - \delta z)$$

Therefore, δ can be thought of as the characteristic length that the field goes into the conductor.

We can use this field to find the electric field inside the conductor:

$$E_y = \frac{\omega\delta}{\sqrt{2}}\frac{\mu}{\mu_0}B_0\cos\left(\frac{2}{\delta} - \omega t + \frac{3\pi}{4}\right)$$

and eddy currents:

$$J_y = \sigma E_y = \frac{\sigma\omega\delta}{\sqrt{2}}\frac{\mu}{\mu_0}B_0\cos\left(\frac{2}{\delta} - \omega t + \frac{3\pi}{4}\right)$$

This next example will probably be in the homework:

Example. Suppose we have a sheet of thickness a with current running in opposite directions on opposite sides. This will induce a perpendicular current inside the material, and using Ampere's law, we can solve to find the magnetic field inside. Suppose we turn off the current suddenly. All of the energy is inside the region, an now now the H field, which was nonzero inside the material, will begin to decay (but it takes time, it won't decay immediately). ◇

0.27 Superconductors

A result of $\vec{J} = \sigma\vec{E}$ is that $m_e\partial_t\vec{v} = e^-\vec{E} - \underbrace{\frac{m_e\vec{v}}{\tau}}_{\text{scattering}}$. The drift velocity can be thought of as $\vec{v}_d = \frac{e^-\tau\vec{E}}{m_e}$. This leads to currents $\vec{J} = n_e e^-\vec{v}_d = \frac{n_e(e^-)^2\tau}{m_e}\vec{E}$.

In superconductors, we imagine there are supercurrents $\vec{J}_s$ which have no such scattering effect:

$$\partial_t \vec{J}_s = \frac{n_s (e^-)^2}{m_e} \vec{E}$$

where $n_s + n_N = n_e$ where n_N are "normal" electrons. This scenario is not what really happens, but it is a good approximation (Drude model). Define $\Lambda = \frac{n_s(e^-)^2}{m_e}$ and take the curl of the previous equation:

$$\vec{\nabla} \times \partial_t \vec{J}_s = \Lambda \left(-\partial_t \vec{B} \right)$$

so

$$\partial_t \left[\vec{\nabla} \times \vec{J}_s + \Lambda \vec{B} \right] = \vec{0}$$

London's assumption was that $\vec{B} = \frac{1}{\Lambda} \vec{\nabla} \times \vec{J} s$, and $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_s$. Taking the curl of this, we find that

$$-\nabla^2 \vec{B} = \mu_0 \Lambda \vec{B}$$

Quote:

"Of course this theory is not right."

However, it very close to a complete theory of superconductivity, as it predicts that magnetic fields will not penetrate the superconducting medium. The current theory is BCS theory.

LECTURE 32: LIGHT

Wednesday, October 30, 2019

0.28 Light and Propagating Fields

Currently, Maxwell's equations look like this:

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\partial_t \vec{B} \\ (\vec{\nabla} \times \vec{H} &= \vec{J})^* \end{aligned}$$

This last equation is incomplete!

By current conservation (taking the divergence of the last equation),

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$$

However, from the first equation, $\partial_t \rho = \vec{\boldsymbol{\nabla}} \cdot \partial_t \vec{D}$. Therefore, we must have

$$\vec{\boldsymbol{\nabla}} \times \vec{H} = \vec{J} + \partial_t \vec{D}$$

In free space,

$$\begin{aligned}
\vec{\boldsymbol{\nabla}} \cdot \vec{E} &= \rho/\epsilon_0 \\
\vec{\boldsymbol{\nabla}} \cdot \vec{B} &= 0 \\
\vec{\boldsymbol{\nabla}} \times \vec{E} &= -\partial_t \vec{B} \\
\vec{\boldsymbol{\nabla}} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \partial_t \vec{E}
\end{aligned}$$

Suppose there is no source in a region Ω. Now

$$\vec{\boldsymbol{\nabla}} \times \vec{E} = -\partial_t \vec{B}$$

and

$$\vec{\boldsymbol{\nabla}} \times \vec{B} = \mu_0 \epsilon_0 \partial_t \vec{E}$$

and $\vec{\boldsymbol{\nabla}} \cdot \vec{E} = \vec{\boldsymbol{\nabla}} \cdot \vec{B} = 0$. Seemingly by coincidence, $\mu_0 \epsilon_0 = \frac{1}{c^2}$!

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} \times \vec{E} = -\partial_t (\vec{\boldsymbol{\nabla}} \times \vec{B}) = -\partial_t c^{-2} \partial_t \vec{E} = \vec{\boldsymbol{\nabla}} \cancelto{0}{\vec{\boldsymbol{\nabla}} \cdot \vec{E}} - \nabla^2 \vec{E}$$

so

$$\nabla^2 \vec{E} - \frac{1}{c^2} \partial_t^2 \vec{E} = 0$$

and

$$\nabla^2 \vec{B} - \frac{1}{c^2} \partial_t^2 \vec{B} = 0$$

which are both wave equations, which have solutions $\varphi = f(x - ct) + g(x + ct)$. These are called plane waves because the strength of the field in a given plane is constant.

Let's look for solutions like

$$\vec{E} = \mathrm{Re}\left\{ \vec{E}_0 e^{\imath \vec{k} \cdot \vec{r} - \imath \omega t} \right\}$$

Plugging this into our formula, we find

$$\nabla^2 \vec{E} = -k^2 \vec{E}$$

and

$$\partial_t^2 \vec{E} = (-\imath \omega)^2 \vec{E}$$

so as long as the solution is nonzero,

$$k = \frac{\omega}{c}$$

The curl acting on plane waves is just like $\imath \vec{k} \times$:

$$\imath \vec{k} \times \vec{E} = -\partial_t \vec{B}$$

so

$$\vec{B}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega}$$

so the electric and magnetic fields are always perpendicular.

This is the free wave solution, and adding a source will obviously complicate things. For sources, we have

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}$$

and

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\partial_t \vec{E}$$

so

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \frac{1}{c^2}\left(-\vec{\nabla} \cdot \partial_t \Phi - \partial_t^2 \vec{A}\right)$$

If we write the left-hand side as

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

we get

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \vec{\nabla} \cdot \frac{1}{c^2}\partial_t \Phi - \frac{1}{c^2}\partial_t^2 \vec{A}$$

We want to keep the parts that look like a wave equation on one side:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t \Phi) - \mu_0 \vec{J} = \nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A}$$

Recall we have some freedom in defining our potentials (Gauge freedom):

$$\vec{\nabla} \times \vec{A} + \vec{\nabla}\chi = \vec{\nabla} \times \vec{A}$$

and

$$-\vec{\nabla}\Phi - \partial_t \chi - \partial_t(\vec{A} + \vec{\nabla}\chi) = -\vec{\nabla}\Phi - \partial_t \vec{A}$$

so

$$\vec{A} \mapsto \vec{A} + \vec{\nabla}\chi$$

and

$$\Phi \mapsto \Phi + \partial_t \chi$$

We would like our final equation to be

$$\nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A} = -\mu_0 \vec{J}$$

so let the gauge be

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t \Phi = 0$$

Doing the same process with the divergence of the electric field, we get that

$$-\nabla^2 \Phi - \partial_t \vec{\nabla} \cdot \vec{A} = \rho/\epsilon_0$$

Using our gauge,

$$-\nabla^2 \Phi - \frac{1}{c^2}\partial_t^2 \Phi = \rho/\epsilon_0$$

What are the solutions for these equations? If we have

$$\nabla^2\Phi - \frac{1}{c^2}\partial_t^2\Phi = f(\vec{x}, t)$$

then we are looking for a Green's function with

$$\nabla^2 G(\vec{x}, t; \vec{x}', t') - \frac{1}{c^2}\partial_t^2 G = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t')$$

so

$$G = \int_{-\infty}^{\infty} G(\vec{x} - \vec{x}'; \omega) e^{\imath\omega(t-t')} \frac{1}{2\pi} \,\mathrm{d}\omega$$

so

$$\nabla^2 G(x - x'; \omega + \frac{\omega^2}{c^2} G = -4\pi\delta(x - x')$$

Therefore

$$(\nabla^2 + k^2) G(x - x'; \omega) = -4\pi\delta(x - x')$$

This is the operator for the Helmholtz equation.

This equation can be solved by

$$G(x - x' l\omega) = \frac{e^{\pm\imath k|x - x'|}}{|x - x'|}$$

Note that I stopped using vector arrows on x but they are vectors in general. Therefore

$$G(x - x', t - t')^{\pm} = \int e^{\pm\imath k|x-x'| - \imath\omega(t-t')} \frac{1}{|x - x'|} \frac{1}{2\pi} \,\mathrm{d}\omega = \frac{1}{2\pi} \frac{\delta(t - t' \pm \frac{|x-x'|}{c})}{|x - x'|}$$

These solutions must vanish at infinity, so they actually don't describe plane waves. The choice of $\pm$ concerns causality, and the $-$ case is the one where the past effects the future.

Lecture 33: Green's Functions for the Helmholtz Equation

Friday, November 01, 2019

From last lecture,

$$\nabla^2 G - \frac{1}{c^2}\partial_t^2 G = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t')$$

leads to

$$\nabla^2 G(\vec{x} - \vec{x}'; \omega + k^2 G = -4\pi\delta(\vec{x} - \vec{x}')$$

and

$$G^{\pm} = \frac{e^{\pm\imath k|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|}$$

Then,

$$G = \int_{-\infty}^{\infty} \frac{e^{\pm\imath\frac{\omega}{c}|\vec{x}-\vec{x}'| - \imath\omega(t-t')}}{|\vec{x} - \vec{x}'|} \frac{1}{2\pi} \,\mathrm{d}\omega = \frac{\delta(t - t' \pm \frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x} - \vec{x}'|}$$

Our solutions for waves with sources look like

$$\Psi(x,t) = \Psi^{\text{inc}}(x,t) + \int G(\vec{x}-\vec{x}';t-t') f(\vec{x}',t')\, \mathrm{d}^3x'\, \mathrm{d}t'$$

In practical terms, only the light-cone intersection contributes to this integral. The only sources that matter are sources which the observer can see, which are sources which happened long enough ago to propagate across a distance to reach the observation point. This is called the Huygen's principle. In two dimensions, an interesting thing occurs, and you actually need to include the interior of the light-cone in your calculation, because the information can travel slower than c. For example, lightning is approximately two-dimensional, and the shock-wave it creates is approximately cylindrical, and the reason we don't hear a sharp impact is because the sound can travel at slower speeds. Apparently we are going to solve this as a homework problem.

Let us write our potential using the Green's function:

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x'\, \mathrm{d}t\, \frac{\delta\left[t' - \left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right]}{|\vec{x}-\vec{x}'|} \rho(\vec{x}',t')$$

or

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x'\, \frac{[\rho(\vec{x},t')]_{\text{ret}}}{|\vec{x}-\vec{x}'|}$$

where we use the retarded time $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$. For the vector potential, we can write the same thing:

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3x'\, \frac{[\vec{J}(\vec{x},t')]_{\text{ret}}}{|\vec{x}-\vec{x}'|}$$

Recall that

$$\vec{E} = -\partial_t \vec{A} - \vec{\boldsymbol{\nabla}}\Phi$$

and

$$\vec{B} = \vec{\boldsymbol{\nabla}} \times \vec{A}$$

If we wanted to find the advanced solutions, we would use $t' = t + \frac{|\vec{x}-\vec{x}'|}{c}$.

An alternative to this was popularized by Jefimenko. Suppose we are in a vacuum, so

$$\begin{aligned}
\vec{\boldsymbol{\nabla}} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\
\vec{\boldsymbol{\nabla}} \cdot \vec{B} &= 0 \\
\vec{\boldsymbol{\nabla}} \times \vec{B} &= \mu_0 \vec{J} + \frac{1}{c^2}\partial_t \vec{E} \\
\vec{\boldsymbol{\nabla}} \times \vec{E} &= -\partial_t \vec{B}
\end{aligned}$$

We know that

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{E}) = \vec{\boldsymbol{\nabla}}\frac{\rho}{\epsilon_0} - \nabla^2 \vec{E} = -\partial_t(\vec{\boldsymbol{\nabla}} \times \vec{B}) = -\partial_t(\mu_0 \vec{J}) - \frac{1}{c^2}\partial_t^2 \vec{E}$$

so

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = \frac{1}{\epsilon_0}\vec{\boldsymbol{\nabla}}\rho + \underbrace{\mu_0}_{\frac{1}{\epsilon_0}\frac{1}{c^2}} \partial_t \vec{J}$$

Similarly, for $\vec{B}$,

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = -\mu_0 \vec{\boldsymbol{\nabla}} \times \vec{J}$$

These are inhomogeneous wave equations, so we can solve them with the Green's functions. We can directly write

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\left[-\vec{\boldsymbol{\nabla}}'\rho - \frac{1}{c^2}\partial_{t'}\vec{J}\right]_{\text{ret}}}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

and

$$\vec{B}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\left[\vec{\boldsymbol{\nabla}} \times' \vec{J}\right]_{\text{ret}}}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

Note that

$$[\vec{\boldsymbol{\nabla}}'\rho]_{\text{ret}} \neq \vec{\boldsymbol{\nabla}}'[\rho]_{\text{ret}}$$

since the derivative will either act on $\vec{x}'$ or $\vec{x}'$ and $|\vec{x} - \vec{x}'|$, respectively.

$$\vec{\boldsymbol{\nabla}}'[\rho]_{\text{ret}} = \nabla' \bigg|_{t'} \rho + \partial_{t'}\rho \bigg|_{\vec{x}'} \left[-\nabla' \frac{|\vec{x} - \vec{x}'|}{c}\right] = \vec{\boldsymbol{\nabla}}'\rho + \partial_{t'}\rho \bigg|_{\vec{x}'} \frac{1}{c} \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$$

Note the extra term at the end. Let's call $\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|} \equiv \hat{R}$, so

$$[\vec{\boldsymbol{\nabla}}'\rho]_{\text{ret}} = \vec{\boldsymbol{\nabla}}'[\rho]_{\text{ret}} - \frac{1}{c}\left[\partial_{t'}\rho\right]_{\text{ret}} \hat{R}$$

We can apply this correction everywhere, integrating by parts:

$$\int \frac{1}{|\vec{x} - \vec{x}'|} \vec{\boldsymbol{\nabla}}'\rho \, \mathrm{d}^3 x' = \cancelto{0}{\int \vec{\boldsymbol{\nabla}}'(\cdots)} - \int \rho \vec{\boldsymbol{\nabla}}' \frac{1}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x'$$

so

$$\vec{E}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 x' \left\{ \frac{[\rho]_{\text{ret}}}{|\vec{x} - \vec{x}'|^2} \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|} - (-1)\frac{1}{c}\left[\partial_{t'}\rho\right]_{\text{ret}} \frac{1}{|\vec{x} - \vec{x}'|}\hat{R} - \frac{1}{c^2}\partial_t \left[\vec{J}\right]_{\text{ret}} \frac{1}{|\vec{x} - \vec{x}'|} \right\}$$

The first term is like a retarded Coulomb potential. For the magnetic field we do the same

$$\vec{B} = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 x' \left\{ \frac{[\vec{J}]_{\text{ret}} \times (\vec{x} - \vec{x}')}{R^3} + \frac{[\partial_{t'}]_{\text{ret}} \times \hat{R}}{cR} \right\}$$

These were originally found by Jefimenko (in a form that was generalized). Heaviside applied it to one of the fields and Feynman applied it to the other, the professor doesn't remember who did which part, but Feynman used this formulation in his lectures and didn't know that anyone had done it before. This is called the Heaviside-Feynman formulation.

We want to apply this to a point charge. The particle can only go through a backward light-cone one time (twice would require it going faster that c). Therefore there is a unique solution to $t' = t - \frac{|\vec{x}-\vec{r}(t')|}{c}$, so $t' \to t'(x,t)$. We will solve this next week (this is also done in Jackson).

Lecture 34: More on Jefimenko's Formulation

Monday, November 04, 2019

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left\{ \frac{\hat{R}}{R^2}[\rho(\vec{x}',t')]_{\text{ret}} + \frac{\hat{R}}{cR}\partial_t[\rho(\vec{x}',t')]_{\text{ret}} - \frac{1}{c^2}\frac{1}{R}\partial_t[\vec{J}(\vec{x}',t')]_{\text{ret}} \right\}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \left\{ [\vec{J}(\vec{x}',t')]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \partial_t[\vec{J}(\vec{x}',t')]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\}$$

For a point source, $\rho(\vec{x},t) = q\delta(\vec{x}-\vec{r}(t))$ and $\vec{J}(\vec{x},t) = q\vec{v}\delta(\vec{x}-\vec{r}(t))$. Here, we define t' by

$$t' = t - \frac{|\vec{x}-\vec{r}(t')|}{c}$$

which is intrinsically hard to solve. Therefore, the δ-functions will look like:

$$\delta(\vec{x}'-\vec{r}(t')) = \delta\left(\vec{x}' - \vec{r}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$$

If we make a local orthogonal transformation such that $\hat{y}_1' \parallel \vec{v}$. We need to factor in the Jacobian in these integrals, which should be of unit magnitude.

$$\text{Jacobian} = \left[\frac{\partial\tilde{x}}{\partial x}\right]^{-1}$$

$$\frac{\partial}{\partial x_i'}\left[x_i' - r_i\left(t - \frac{|\vec{x}-\vec{x}|}{c}\right)\right] = \delta_{ij} - \frac{\partial r_j}{\partial t'}\left[\frac{\partial}{\partial x'^i}\left(t - \frac{|\vec{x}-\vec{x}|}{c}\right)\right] = \delta_{ij} - \left.\frac{\partial r_j}{\partial t'}\right|_{\text{ret}}\frac{x_i - x_i'}{|\vec{x}-\vec{x}|}$$

Essentially this is like the identity matrix times a tensor. We can show that $\det(I + a\otimes b) = 1 + \vec{a}\cdot\vec{b}$.

$$\text{Jacobian} \sim \left(\left|\begin{bmatrix} 1 - \left.\frac{\partial\vec{r}}{\partial t'}\right|_{\text{ret}}\cdot\frac{\hat{R}}{R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|\right)^{-1}$$

I believe there are factors alongside the identity, but he chose not to write them out. Apparently you find that the Jacobian is

$$|J| = \frac{1}{\left[1 - \vec{v}\cdot\frac{\hat{R}}{R}\right]_{\text{ret}}} = \frac{1}{\kappa(v)}$$

so we have

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left\{ \left.\frac{\hat{R}}{R^2}\right|_{\text{ret}} \frac{1}{\kappa(v)\big|_{\text{ret}}} + \frac{1}{c}\partial_{t'}\left.\frac{\hat{R}}{\kappa(v)R}\right|_{\text{ret}} - \frac{1}{c^2}\partial_t\left[\frac{\vec{v}(t')}{R}\right]_{\text{ret}} \right\}$$

and

$$\vec{B} = \frac{q\mu_0}{4\pi}\left\{ \frac{\vec{v}\times\hat{R}}{\kappa(v)R^2}\bigg|_{\text{ret}} + \frac{1}{c}\partial_t\left[\frac{\vec{v}\times\hat{R}}{\kappa(v)R}\right]_{\text{ret}}\right\}$$

Feynmann wrote it in a simplified way:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\left[\frac{\hat{R}}{R^2}\right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c}\partial_t\left[\frac{\hat{R}}{R^2}\right]_{\text{ret}} + \frac{1}{c^2}\partial_t^2[\hat{R}]_{\text{ret}}\right)$$

Heaviside wrote out a nice form for the $\vec{B}$-field:

$$\vec{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\vec{v}\times\hat{R}}{\kappa(v)R^2}\right]_{\text{ret}} + \frac{1}{c[R]_{\text{ret}}}\partial_t\left[\frac{\vec{v}\times\hat{R}}{\kappa(v)}\right]_{\text{ret}}\right\}$$

0.29 Energy Transfer

$$\frac{\mathrm{d}E_{\text{mech}}}{\mathrm{d}t} = \int \vec{J}\cdot\vec{E}\,\mathrm{d}^3x$$

Note that $\vec{\boldsymbol{\nabla}}\times\vec{H} = \vec{J} + \partial_t\vec{D}$

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{E}\times\vec{H}) \implies \partial_i[\epsilon_{ijk}E_jH_k] = \epsilon_{ijk}\partial_iE_jH_k + E_j\epsilon_{ijk}\partial_iH_k$$

so

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int(\vec{\boldsymbol{\nabla}}\times\vec{H})\cdot\vec{E} + \partial_t\vec{D}\cdot\vec{E}\,\mathrm{d}^3x = -\int\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\vec{E}\times\vec{H}) + \vec{H}(\vec{\boldsymbol{\nabla}}\times\vec{E}) - \partial_t\vec{D}\cdot\vec{E}\,\mathrm{d}^3x \\ &= -\int\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{E}\times\vec{H}\,\mathrm{d}^3x - \int\left(\partial_t\vec{B}\cdot\vec{H} + \partial_t\vec{B}\cdot\vec{H}\right)\mathrm{d}^3x\end{aligned}$$

Now we have to start making assumptions and approximations, since in general, media are not nice and linear. If we say that the medium is linear, has no dispersion or loss, and assuming $\partial_tE \sim 0$ and ∂_tB implies static expressions are recovered, then $\vec{H} \sim \frac{1}{\mu}\vec{B}$ and $\vec{D} \sim \epsilon\vec{E}$. Now we can say that the energy density is found from

$$\frac{\mathrm{d}E_{\text{mech}}}{\mathrm{d}t} + \partial_t\frac{1}{2}\int(\epsilon E^2) + \frac{1}{\mu}B^2\,\mathrm{d}^3x = -\oint\vec{S}\cdot\hat{n}\,\mathrm{d}a$$

where

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$$

is the Poynting Vector. We will see that even with small amounts of dispersion and loss, we get a completely different set of equations. To eliminate the assumptions entirely, a complete thermodynamic evaluation must be made.

Lecture 35: Dispersion Relations in Media

Monday, November 04, 2019

Let's now look at momentum instead of energy. We can treat $\rho\vec{E} + \vec{J} \times \vec{B}$ as the force density, so

$$
\begin{aligned}
\frac{\mathrm{d}\vec{P}_{\text{mech}}}{\mathrm{d}t} &= \int (\rho\vec{E} + \vec{J} \times \vec{B})\,\mathrm{d}^3x = \int \left[\epsilon_0 \vec{\nabla} \cdot \vec{E}\vec{E} + \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} - \epsilon_0 \partial_t \vec{E} \times \vec{B}\right] \mathrm{d}^3x \\
&= \int \left[\epsilon_0(\vec{\nabla} \cdot \vec{E})\vec{E} - \epsilon_0 \partial_t(\vec{E} \times \vec{B}) + \epsilon_0 \vec{E} \times \partial_t \vec{B} - \mu_0 \vec{B} \times (\vec{\nabla} \times \vec{B})\right] \mathrm{d}^3x \\
&= \int \left[\epsilon_0(\vec{\nabla} \cdot \vec{E})\vec{E} + \epsilon_0 c^2 \underbrace{(\vec{\nabla} \cdot \vec{B})}_{0} \vec{B} + \epsilon_0 \vec{E} \times (-\vec{\nabla} \times \vec{E}) - \mu_0 \vec{B} \times (\vec{\nabla} \times \vec{B})\right] \mathrm{d}^3x
\end{aligned}
$$

Note that

$$
\begin{aligned}
\epsilon_{ijk} E_j \epsilon_{klm} \partial_l E_m &= \epsilon_{ijk} \epsilon_{klm} E_j \partial_l E_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_l E_m \\
&= E_j \partial_i E_j - E_j \partial_j E_i
\end{aligned}
$$

so we get that

$$
\begin{aligned}
\frac{\mathrm{d}P^j_{\text{mech}}}{\mathrm{d}t} &= \int [\epsilon_0 \partial_j E_j E_i + \epsilon_0 c^2 \partial_j B_j B_i] - E_j \partial_j E_i - E_j \partial_i E_j + c^2 \epsilon_0 B_j \partial_j B_i - \epsilon_0 c^2 B_j \partial_i B_j \,\mathrm{d}^3x \\
&= \int \partial_j T_{ji}\,\mathrm{d}^3x = \oint_\Sigma T_{ji}\,\mathrm{d}a_j
\end{aligned}
$$

Here, Σ is the surface of the volume. If we factor in the momentum from electromagnetism, $-\epsilon_0 \partial_t \int (\vec{E} \times \vec{B})\,\mathrm{d}^3x$, we find

$$
\partial_t \left[\vec{P}_{\text{mech}} + \vec{P}_{\text{EM}}\right] = \oint_\Sigma T_{ji}\,\mathrm{d}a_j = \oint_\Sigma \left[(\epsilon_0 E_i E_j + \epsilon_0 c^2 B_i B_j) - \frac{1}{2}(\epsilon_0 E^2 + \epsilon_0 c^2 B^2)\delta_{ij}\right] \mathrm{d}a_j
$$

This $T_{ij} = T_{ji}$ is the Maxwell Stress Tensor. It describes the ith component of the momentum escaping in the jth direction. In typical gasses and materials which don't have shearing forces, this tensor is diagonal, but in many solids, the off-diagonals can be nonzero. For example, T_{xx} is the amount of momentum in the x-direction (P_x) which escapes in the x-direction, while T_{xy} is the amount of momentum in the x-direction which escapes in the y-direction (and by symmetry, the y-momentum in the x-direction).

If we look at the electromagnetic momentum $\vec{P} = \epsilon\vec{E} \times \vec{B}$, we expect the Pointing vector $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$ would give us the energy flux.

0.29.1 Angular Momentum Conservation

We can define an angular momentum tensor as

$$
M_{kji} = x_k T_{ji} - x_j T_{ki}
$$

0.30 Dispersive Media

Suppose we have a linear medium which is dispersive. This means we can have a frequency dependency in our expansion of $\vec{E}$:

$$\partial_t \vec{E} = \int \partial_t \vec{E}(\vec{x}, \omega) e^{-\imath\omega t} \frac{1}{2\pi} \, \mathrm{d}\omega = \int (-\imath\omega) \vec{E}(\vec{x}, \omega) e^{-\imath\omega t} \frac{1}{2\pi} \, \mathrm{d}\omega$$

so

$$\vec{D} = (\epsilon + \epsilon\chi)\vec{E} \mapsto \vec{D}(\vec{x}, \omega) = (\epsilon_0 + \epsilon_0 \chi(\omega))\vec{E}(\vec{x}, \omega)$$

Similarly, we get ω dependence in Maxwell's equations:

$$\vec{\nabla} \times \vec{E}(\vec{x}, \omega) = \imath\omega \vec{B}(\vec{x}, \omega)$$

and

$$\vec{\nabla} \times \vec{B} = \mu(\omega)\epsilon(\omega)(-\imath\omega)\vec{E}(\vec{x}, \omega)$$

A consequence of this is that waves of different frequencies no longer travel at the same speed, which is where diffraction comes from.

Remark

In general, ϵ and μ are complex, and the index of refraction is therefore complex: $n(\omega) = n_R(\omega) + \imath n_I(\omega)$.

Howe can we model this? Recall that we model electrons as harmonic oscillators. We know that they are not, they are in definite orbitals, but if we disturb them, we are disturbing an equilibrium position. Therefore, to first order, we can think of them like harmonic oscillators with a linear restoring force bringing the system back to equilibrium. That is why we think of motion as $m\ddot{x} + m\gamma\dot{x} + \omega_j^2 \vec{x} = -e\vec{E}(\omega)e^{-\imath\omega t}$ where ω is the natural frequency of a stable energy level.

To first order, we find solutions to this equation of motion are of the form

$$\vec{x}(t) = \vec{x}_0 e^{-\imath\omega t}$$

where

$$\vec{x}_0 = \frac{-e}{m\left[\omega_j^2 - \omega^2 - \imath\gamma\right]}\vec{E}$$

so we can model this as a dipole $-e\vec{x}(t) = \vec{p}(t)$:

$$\vec{p}(\omega) = \frac{e^2}{m} \frac{1}{\left[\omega_j^2 - \omega^2 - \imath\gamma\omega\right]} \vec{E}(\omega) = \chi(\omega)\vec{E}(\omega)$$

In general, this means we can describe multiple (N) electrons in the form

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - \imath\gamma\omega}$$

where f_j is the oscillator strength and ω_j are the possible frequencies of the electrons with those oscillation strengths. These will depend on all sorts of things, like what atom we are looking at, what orbital the electron is in, and the molecular configuration. In general, we normalize the strengths by $\sum_j f_j = Z$ where Z is the number of electrons per molecule.

An interesting consequence of this is that our visible spectrum correlates with the low-refraction wavelengths of light in water, since this is an evolutionarily beneficial span of wavelengths.

Lecture 36: Diffraction Continued

Wednesday, November 06, 2019

Recall from last lecture we can model optical diffraction by

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - \imath\gamma_j\omega}$$

When we add an imaginary part, we find

$$\frac{\epsilon_R(\omega) + \epsilon_I(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2} + \imath \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j\gamma_j\omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2} \sim \frac{1}{\gamma_j\omega_j}$$

In conductors, f_0 gives us a band. Note that

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \sigma\vec{\mathbf{E}} + \epsilon(\omega)\partial_t\vec{\mathbf{E}}$$

In the first part, the conductance is a function of ω, since the free electrons must have some delayed response, and in the second part, we get $(-\imath\omega)\epsilon(\omega)\vec{\mathbf{E}}$.

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = (-\imath\omega)\left[\frac{\imath\sigma(\omega)}{\omega} + \epsilon_b(\omega)\right]\vec{\mathbf{E}}$$

If $\omega_j = 0$,

$$\frac{Ne^2}{m\epsilon_0}\frac{f_0}{-\omega^2 - \imath\gamma_0\omega} = \frac{1}{\omega\left[\frac{\imath Ne^2 f_0}{m\epsilon_0(\gamma_0 - \imath\omega)}\right]}$$

We see that

$$\sigma(\omega) = \frac{Ne^2 f_0}{m\epsilon_0[\gamma_j - \imath\omega]}$$

so there is a pole around $\omega = 0$.

0.30.1 Light Traveling Through a Medium

Let's look at the electric field:

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = -\partial_t\vec{\mathbf{B}} = \imath\omega\vec{\mathbf{B}}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu(\omega)\epsilon(\omega)(-\imath\omega)\vec{\mathbf{E}}$$

$$\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}\right) = \mu(\omega)\epsilon(\omega)[-\imath\omega][\imath\omega]\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}}\right) - \nabla^2\vec{\mathbf{B}}$$

These products in frequency space are convolutions in time space, thanks to the Fourier transform, so

$$\vec{\mathbf{D}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{E}} + \int_{-\infty}^{t} \epsilon(t - t')\vec{\mathbf{E}}(\vec{\mathbf{x}}, t')\,\mathrm{d}t'$$

so

$$\nabla^2\vec{\mathbf{B}} + \mu(\omega)\epsilon(\omega)\omega^2\vec{\mathbf{B}} = 0 = \nabla^2\vec{\mathbf{B}} + \frac{n^2(\omega)\omega^2}{c^2}\vec{\mathbf{B}}$$

so

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 e^{\imath\vec{\mathbf{k}} \cdot \vec{\mathbf{r}}}$$

where

$$k^2 = \frac{n^2(\omega)\omega^2}{c^2}$$

It is generally nice to think of ω as a function of k ($\omega(k)$): $k^2 = \frac{\omega}{\left[\frac{c}{n(\omega)}\right]^2}$. This shows that the speed of light is modified in a medium.

The electric field must now be

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 e^{\imath\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \imath\omega t} \implies \vec{\mathbf{B}}_0 = \frac{\vec{\mathbf{k}} \times \vec{\mathbf{E}}_0}{\omega}$$

However, there is still the anomalous relationship between ω and k. It seems like the speed of the wave could exceed c. To understand the meaning of $\frac{n(\omega)\omega}{c}$, we assume a narrow band of signal $A(k)e^{\imath\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} - \imath\omega(k)t}$ heavily concentrated around some k_0 and there is some ω_0 for which $k_0 = \frac{n(\omega_0)\omega_0}{c}$. It's close to a plane wave, but it isn't exactly a plane wave since there is some spread in the frequency. Let's imagine this is a one-dimensional wave for simplicity.

$$\begin{aligned}\int_{-\infty}^{\infty} A(k)e^{\imath kx - \imath\omega(k)t}\,\mathrm{d}k &= \int_{-\infty}^{\infty} A(k)e^{\imath kx - \imath\left[\omega(k_0)t + \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0}(k-k_0)t\right]}\,\mathrm{d}k \\ &= e^{\imath k_0 x - \imath\omega(k_0)t}\int_{-\infty}^{\infty} A(k_0 + \bar{k})e^{\imath\left[x - \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0} t\right]\bar{k}}\,\mathrm{d}\bar{k}\end{aligned}$$

Quote

"These are like the songs of Neil Diamond. You listen to Neil Diamond and it's like the world is gone."

$\frac{\mathrm{d}\omega}{\mathrm{d}k}$ is the group velocity, and while we have an underlying signal with frequency $\omega(k_0)$, we have an envelope which travels like $A\left(x - \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0} t\right)$. The envelope travels at the group velocity, so even if the wave itself travels faster than c (the phase velocity), we won't actually be able to see the signal until we observe changes in amplitude or frequency, which must move at the group velocity. This can be proven more generally using causality (we might do this later in class). We are also really only looking in the "transparency region" where $n_R(\omega) \gg n_I(\omega)$. In general, you would find that the fields are proportional to

$$e^{\imath(k_R+\imath k_I)\hat{\mathbf{k}}\cdot\vec{\mathbf{r}}-\imath\omega t}$$

so you would get a term like $e^{-k_I\hat{\mathbf{k}}\cdot\vec{\mathbf{r}}}$ which attenuates the signal (loss). If you take the second order terms in the expansion we just did, using the method of steepest descent, you would find that the shape of the signal can change in time (to first order it doesn't). This is because some of the components move faster than others (think spin dispersion and spin echo in NMR).

Lecture 37:

Friday, November 08, 2019

Recall that last lecture we said that, for a (very) Gaussian wave packet sharply peaking around k_0, The fields are both proportional to

$$e^{\imath\mathbf{k}\cdot\mathbf{x}-\imath\frac{\omega n(\omega)}{c}t}$$

In reality, $\epsilon(\omega)\mu(\omega) \equiv \frac{n^2(\omega)}{c^2}$ is complex. Last time, we just assumed it wasn't (or the imaginary part was small), which allowed us to invert the relation around k_0.

Let's now expand this for complex k:

$$\int_{-\infty}^{\infty} e^{\ln A(k)} + \imath(k-k_0)x + \imath k_0 x - \imath\omega(k_0)t - \imath\left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0}(k-k_0)t - \imath\left.\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}\right|_{k_0}(k-k_0^2)t + \cdots$$

We can also expand

$$\ln A(k) = \ln A(k_0) + \cancelto{0}{\frac{A'(k_0)}{A(k_0)}}(k-k_0) + \frac{1}{2}\left[\frac{A''(k_0)}{A(k_0)} - \cancelto{0}{\frac{A'^2(k_0)}{A^2(k_0)}}\right](k-k_0)^2 + \cdots$$

so

$$\psi \approx \int_{-\infty}^{\infty} e^{\ln A(k_0)} - \frac{\alpha^2}{2}(k-k_0)^2 + \imath(k-k_0)x - \imath\left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0}(k-k_0)t - \imath\frac{\omega''(k_0)}{2}(k-k_0)^2 t\,\mathrm{d}k$$

where

$$-\alpha^2 = \frac{A''(k_0)}{A(k_0)} < 0$$

Therefore

$$\psi \approx A(k_0)e^{\imath(k_0x-\omega(k_0)t)}\cdot\int_{-\infty}^{\infty}\mathrm{d}\bar{k}\,e^{-\frac{\alpha^2}{2}\bar{k}^2-\imath\frac{\omega''(k_0)}{2}t\bar{k}^2-\imath\bar{k}\left(x-\frac{\mathrm{d}\omega}{\mathrm{d}k}\big|_{k_0}\right)}$$

so

$$\frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{n(\omega)}{c} + \frac{\omega}{c}\frac{\mathrm{d}n}{\mathrm{d}\omega}$$

so

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c}{n(\omega)+\omega\frac{\mathrm{d}n}{\mathrm{d}\omega}} < c$$

This does not work near the anomalous region where the derivative in the denominator is not positive, but in this region, the imaginary part takes over. In this region, the group velocity can become negative or greater than c, which is just evidence that all the approximations we made are not stable in this region. To be fair, we even approximated the atoms as harmonic oscillators and dipoles.

Some general, model independent ideas:

Causality must exist, this is a big thing!

$$\mathbf{D}(\mathbf{x},t) = \epsilon_0\mathbf{E}(\mathbf{x},t) + \int_0^\infty G(\tau)\mathbf{E}(\mathbf{x},t-\tau)\,\mathrm{d}\tau$$

Note that the Fourier transform of this convolution is just $G(\omega)\mathbf{E}(\mathbf{x},\omega)$. We are restricting the integral to go from 0 to ∞:

$$G(\omega) = \int_0^\infty G(\tau)e^{\imath\omega\tau}\,\mathrm{d}\tau$$

where we assume ω is complex.

Dielectrics have no poles on the real axis, so $\omega \to 0$ does not introduce any problems.

Conductors have a simple pole $\frac{\sigma(\omega)}{\omega}$ at $\omega = 0$.

If we take ω to be complex, we can see that $\epsilon(\omega)$ is meromorphic and can be analytically continued at least in the upper-half plane since $e^{-\omega_I\tau}$ is okay as long as $\omega_I > 0$. Therefore, we have that $\epsilon^*(\omega) = \int G(\tau)e^{-\imath\omega^*\tau}\,\mathrm{d}\tau = \epsilon(-\omega^*)$. If we do this integration on a contour in the upper-half plane, we will take the contour to be a circle at ∞, and there are no poles, so

$$\left[\frac{\epsilon(z)}{\epsilon_0} - 1\right] = \frac{1}{2\pi\imath}\oint \frac{\left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right]}{\omega' - z}\,\mathrm{d}\omega'$$

Remember that

$$\left[\frac{\epsilon(\omega)}{\epsilon} - 1\right] = \int_0^\infty G(\tau)e^{\imath\omega\tau}\,\mathrm{d}\tau = G(\tau)\left.\frac{e^{\imath\omega\tau}}{\imath\omega}\right|_0^\infty - \int_0^\infty G'(\tau)\frac{e^{\imath\omega\tau}}{\imath\omega}\,\mathrm{d}\tau = \frac{G(0)}{\imath\omega} - \frac{G'(0)}{(\imath\omega)^2} + \cdots$$

For a dielectric, $G(0) = 0$. This tells us that

$$\frac{\epsilon_R(\omega)}{\epsilon_0} \sim \mathcal{O}\left(\frac{1}{\omega^2}\right)$$

and

$$\frac{\epsilon_I(\omega)}{\epsilon_0} \sim \mathcal{O}\left(\frac{1}{\omega^3}\right)$$

so there is no problem with closing the contour at positive infinity. Now we need to approach the real axis. When we do this, we need to jump slightly around z when we are on the real axis, cutting symmetrically around z with a semicircle of radius δ. We can do this by finding the principle value of the integral:

$$\mathcal{P}\int \equiv \lim_{\delta\to 0^+}\left\{\int_{-\infty}^{\omega-\delta} + \int_{\omega+\delta}^{\infty}\right\}$$

so

$$\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]=\frac{1}{2\pi\imath}\pi\imath\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]+\frac{1}{2\pi\imath}\mathcal{P}\int_{\infty}^{\infty}\frac{\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]}{\omega'-\omega}\,\mathrm{d}\omega'=\frac{1}{\pi\imath}\mathcal{P}\frac{\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]}{\omega'-\omega}\,\mathrm{d}\omega'$$

This is the derivation of the Kramers-Krönig relations:

$$\frac{\epsilon_R(\omega)}{\epsilon_0}-1=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\frac{\epsilon_I(\omega')}{\epsilon_0}}{\omega'-\omega}\,\mathrm{d}\omega'$$

$$\frac{\epsilon_I(\omega)}{\epsilon_0}=\frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\left[\frac{\epsilon_R(\omega')}{\epsilon_0}-1\right]}{\omega'-\omega}\,\mathrm{d}\omega'$$

Therefore, on $\mathbb{R}$, $\epsilon_R(\omega)-\imath\epsilon_I(\omega)=\epsilon_R(-\omega)+\imath\epsilon_I(-\omega) \implies \epsilon_R(\omega)$ is even and $\epsilon_I(\omega)$ is odd.

Lecture 38: The Kramers-Krönig Relations

Monday, November 11, 2019

Recall from last lecture that, since the real part of $\epsilon(\omega)$ is an even function and the imaginary part is odd, we find the Kramers-Krönig relations:

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}]-1=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}[\frac{\epsilon(\omega')}{\epsilon_0}]}{\omega'-\omega}\,\mathrm{d}\omega'$$

$$\mathrm{Im}[\frac{\epsilon(\omega)}{\epsilon_0}]=-\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}[\frac{\epsilon(\omega')}{\epsilon_0}-1]}{\omega'-\omega}\,\mathrm{d}\omega'$$

By splitting this into separate integrals at 0, we can show that these are equivalent to

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}]-1=\frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\omega'\,\mathrm{Im}[\epsilon(\omega')]}{\omega'^2-\omega^2}\,\mathrm{d}\omega'$$

and

$$\mathrm{Im}[\frac{\epsilon(\omega)}{\epsilon_0}]=-\frac{2\omega}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\mathrm{Re}[\frac{\epsilon(\omega')}{\epsilon_0}-1]}{\omega'^2-\omega^2}\,\mathrm{d}\omega'$$

0.30.2 Region of Transparency

If $\mathrm{Im}[\epsilon(\omega)]\approx 0$ over a range $[\omega_1,\omega_2]$, such that $n(\omega)\sim\sqrt{\epsilon_R(\omega)}$ and $n_I(\omega)\approx 0$ in the region of transparency. What does this imply? In the Kramers-Krönig relations, we know that

$$\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}-1]\simeq\frac{2}{\pi\epsilon_0}\mathcal{P}\int_{0}^{\omega_1}\frac{\omega'\,\mathrm{Im}[\epsilon(\omega')]}{\omega'^2-\omega^2}\,\mathrm{d}\omega'\frac{2}{\pi\epsilon_s n}\mathcal{P}\int_{\omega_2}^{\mathrm{inf}}\frac{\omega'\,\mathrm{Im}[\epsilon(\omega')]}{\omega'^2-\omega^2}\,\mathrm{d}\omega'$$

These are convergent integrals, and we therefore don't actually need the principle values because ω' never comes near ω. This allows us to take the derivatives of these expressions.

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\,\mathrm{Re}[\frac{\epsilon(\omega)}{\epsilon_0}-1] = \frac{2}{\pi\epsilon_0}\int_0^{\omega_1}\frac{\omega\omega'\,\mathrm{Im}[\epsilon(\omega')]}{(\omega'^2-\omega^2)^2}\,\mathrm{d}\omega' + \frac{2}{\pi\epsilon_0}\int_{\omega_2}^{\infty}\frac{\omega\omega'\,\mathrm{Im}[\epsilon(\omega')]}{(\omega'^2-\omega^2)^2}\,\mathrm{d}\omega' > 0$$

We know that $n^2(\omega) \simeq \mathrm{Re}[\epsilon(\omega)]$ so

$$2n(\omega)\frac{\mathrm{d}}{\mathrm{d}\omega}n \simeq \frac{\mathrm{d}}{\mathrm{d}\omega}\,\mathrm{Re}[\epsilon(\omega)] > 0$$

so

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} > 0$$

Therefore, the sky is blue because of causality (neat).

0.31 Transmission of Waves and Propagation in an Arbitrary Region of Frequency

Suppose we have an $x = 0$ boundary and a material to the right with $n(\omega)$ and vacuum to the left. We send a signal to the left, which hits the boundary at $t = 0$. We want to describe what happens after this.

$$u(x,t) = \int_{-\infty}^{\infty}\left[A(\omega)e^{\imath kx-\imath\omega t} + B(\omega)e^{-\imath kx-\imath\omega t}\right]\frac{\mathrm{d}\omega}{2\pi}$$

in the $x \leq 0$ region and

$$u(x,t) = \int_{-\infty}^{\infty}F(\omega)e^{\imath k(\omega)x-\imath\omega t}\frac{\mathrm{d}\omega}{2\pi}$$

inside the material. These functions are real, and we can use time reversal symmetry (taking the complex conjugate) to find a relation between A and B:

$$A^*(-\omega) = B(\omega)$$
$$B^*(-\omega) = A(\omega)$$

Suppose we know what the incoming wave looks like, so we therefore know what $u(0,t)$ and $\frac{\partial u}{\partial x}\big|_{x=0}(t)$, so

$$\left\{\begin{cases}A(\omega)\\ B(\omega)\end{cases}\right\} = \frac{1}{2}\int_{-\infty}^{\infty}\left[u(0,t) \pm \frac{c}{\imath\omega}\,\frac{\partial u}{\partial x}\bigg|_{x=0}\right]e^{\imath\omega t}$$

The wave function and its time derivative must be continuous across the boundary, so

$$F(\omega) = \frac{2}{1+n(\omega)}A(\omega)$$

Note

We won't prove the following, but it turns out that $|n(\omega)| \to 1$ as $|\omega| \to \infty$ in the upper-half-plane. Also, $\epsilon_R(\omega)$ is never negative or zero if we assume $\epsilon_I(\omega) \geq 0$.

This implies the following interesting thing. $n^2(\omega) = \epsilon(\omega)\mu_0$, but if ϵ cuts the negative axis somewhere, the square root will not be uniquely defined (the square root has a branch cut along the negative real line). Therefore, if ϵ is always well-defined and never cuts this region, $n(\omega)$ becomes an analytic function when ω is in the upper-half-plane. From this, we know that $F(\omega)$ is analytic since $A(\omega)$ is analytic in the upper-half-plane because $u(x,t)$ is real. Therefore, the integral which defines $u(x,t)$ in the material can be written as a contour integral which evaluates to 0 minus the half-circle at infinity, so

$$\int_{-\infty}^{\infty} F(\omega) \mapsto \oint \frac{2}{1+n(\omega)} A(\omega) e^{\imath\left(\frac{\omega x}{c} - \omega t\right)\left(\frac{x}{c} - t\right)} > 0$$

so $x \leq ct$. Even though we can't use the group velocity here, we still see that the speed of propagation doesn't exceed c.

Lecture 39: The Eikonal Approximation

Wednesday, November 13, 2019

From last lecture, we supposed that $\epsilon \to \epsilon(\vec{\mathbf{x}})$ and $\left|\lambda\vec{\boldsymbol{\nabla}}\epsilon\right| << \epsilon$, so

$$\nabla^2\vec{\mathbf{H}} + \mu_0\omega^2\epsilon(\vec{\mathbf{x}})\vec{\mathbf{H}} = 0$$

and

$$\nabla^2\vec{\mathbf{E}} + \mu_0\omega\epsilon(\vec{\mathbf{x}})\vec{\mathbf{E}} = 0$$

We can rewrite $\omega\mu_0\epsilon_0\omega\frac{\epsilon(\vec{\mathbf{x}})}{\epsilon_0} = \frac{\omega^2}{c^2}n^2(\vec{\mathbf{x}})$. If $n(\vec{\mathbf{x}})$ were a constant, we would imagine that $\vec{\mathbf{E}} \sim e^{\imath\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}$ where $k^2 = \frac{\omega^2 n^2}{c^2}$ and $\hat{\mathbf{k}}\cdot\vec{\mathbf{x}} \sim S(\vec{\mathbf{x}}) \implies \vec{\mathbf{E}} \sim e^{\imath\frac{\omega}{c}S(\vec{\mathbf{x}})}$. Therefore,

$$\vec{\boldsymbol{\nabla}}\cdot\left[(\vec{\boldsymbol{\nabla}}S)\imath\frac{\omega}{c}e^{\imath\frac{\omega}{c}S}\right] + \frac{n^2(x)\omega^2}{c^2}e^{\imath\frac{\omega}{c}S} = 0$$

so

$$\left[\nabla^2 S\imath\frac{\omega}{c} + (\vec{\boldsymbol{\nabla}}S)^2\left(\imath\frac{\omega}{c}\right)^2 + \frac{n^2(x)\omega}{c^2}\right]e^{\imath\frac{\omega}{c}S} = 0$$

(where we take $\nabla^2 S << (\vec{\boldsymbol{\nabla}}S)^2\frac{\omega}{c}$). This implies

$$(\vec{\boldsymbol{\nabla}}S)^2 = n^2(\vec{\mathbf{x}})$$

or

$$\vec{\boldsymbol{\nabla}}S = n(\vec{\mathbf{x}})\hat{k}(x)$$

$S(\vec{\mathbf{x}})$ are basically constant-phase surfaces, so locally it looks like we have plane waves. This kind of makes sense, because if we are slowly changing the index of refraction, locally it is constant, which means there are plane wave solutions.

Imagine a ray going through one of these constant surfaces. It starts at $\vec{\mathbf{r}}_0$, and we can parameterize it by the length of the ray.

$$\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}s} = \hat{\mathbf{k}}(\vec{\mathbf{r}}(s))$$

where s is the length of the ray, so

$$n(\vec{\mathbf{r}}(s))\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}s} = \hat{\mathbf{k}}n(\vec{\mathbf{r}}) = \vec{\boldsymbol{\nabla}}S$$

Next, take the derivative of both sides

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left[n(\vec{\mathbf{r}}(s))\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}s}\right] &= \vec{\boldsymbol{\nabla}}\frac{\mathrm{d}S}{\mathrm{d}s}\\ &= \vec{\boldsymbol{\nabla}}\left[\vec{\boldsymbol{\nabla}}S\cdot\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}s}\right]\end{aligned}$$

and

$$n(\vec{\mathbf{x}})\hat{\mathbf{k}}(x)\cdot\hat{\mathbf{k}}(x) = n(\vec{\mathbf{r}}(s))$$

From this, we get a differential equation for the ray:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[n(\vec{\mathbf{r}}(s))\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}s}\right] = \vec{\boldsymbol{\nabla}}\Big|_{\text{ray}} n(\vec{\mathbf{r}}(s))$$

Example. Suppose we have a medium whose index of refraction varies in the $\hat{\mathbf{x}}$ direction. Moreover, suppose it decreases as $|x|$ increases. Suppose we have a ray which starts at $x = 0$ and has some angle $\theta(s)$ from the $\hat{\mathbf{z}}$-axis at a point s along its length. Equivalently we could use $\theta(x)$. Our equation of motion tells us

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[n(x)\frac{\mathrm{d}x}{\mathrm{d}s}\right] = \frac{\mathrm{d}n}{\mathrm{d}s}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[n(x)\underbrace{\frac{\mathrm{d}z}{\mathrm{d}s}}_{\cos(\theta(x))}\right] = \frac{\mathrm{d}n}{\mathrm{d}z} = 0$$

since $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$.

This tells us that there is a conserved quantity: $n(x)\cos(\theta(x)) = n(x_0)\cos(\theta(x_0))$. This is pretty neat (it's kind of the continuous version of Snell's Law. We could have used sine functions if we had formulated the problem differently). It means if $n(x)$ decreases, $\cos(\theta(x))$ must increase to conserve the quantity, but there is an upper bound on the cosine. By having this slowly varying $n(x)$, we can basically confine the angle of the wave such that it will never leave a certain region. This is not quite the same thing as total internal reflection, since there is no discontinuous boundary, and therefore no evanescent waves or losses, which makes it ideal for optical fibers. Then again, it is difficult and expensive to make materials like this, and this is only an approximation, so there will technically be small losses no matter what. If the initial angle is larger than a particular value, the ray will not turn around in time (given a finitely large radius of the fiber) and the ray will it a hard boundary. This is the acceptance angle of the fiber.

We can write x as a function of z. Previously, the right-hand side of the differential equation was the gradient along the ray. If we use this parameterization, $\frac{\mathrm{d}}{\mathrm{d}z}n(x(z))$ no longer vanishes.

$$n(x)\frac{\mathrm{d}z}{\mathrm{d}s} = n_0 = n(x)\frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s}$$

so

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{n_0}{n(x)}\frac{\mathrm{d}x}{\mathrm{d}z}$$

Therefore, plugging this into our differential equation gives us

$$n_0^2\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = n(x)\frac{\mathrm{d}n(x)}{\mathrm{d}x} = n(x)\frac{\mathrm{d}n(x)}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

so

$$n_0\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}n^2(x(z))$$

or

$$n_0^2\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}n^2(x(z))$$

Solving this, we have

$$z(x) = n_0\int_0^x \frac{\mathrm{d}x}{\sqrt{n^2(x) - n_0^2}}$$

where $n(x_0) = n_0$.

Note that this is also in Jackson. $\diamond$

0.32 Wave Guides

Suppose we have some perfect metal tube with constant cross-section in the xy-plane. By perfect metal, we mean $\sigma \to \infty$. Inside, $\vec{\mathbf{B}} = 0$ and $\vec{\mathbf{E}} = 0$. Across the boundary, B_n and $\vec{\mathbf{E}}_t$ are continuous.

Let's suppose the electric field is a function in the xy-plane and propagates in the z-direction:

$$\begin{cases}\vec{\mathbf{E}} \\ \vec{\mathbf{B}}\end{cases} = \begin{cases}\vec{\mathbf{E}}(x,y) \\ \vec{\mathbf{B}}(x,y)\end{cases} e^{\imath(kz-\omega t)}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}} = -\partial_t\vec{\mathbf{B}} \quad \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}} = 0 \quad \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{B}} = 0$$

In non-dispersive homogeneous materials,

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{H}} = \partial_t\vec{\mathbf{D}}$$

so

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}} = \mu\epsilon\partial_t\vec{\mathbf{E}}$$

so

$$\vec{\nabla} \times \vec{\mathbf{E}} = \imath\omega\vec{\mathbf{B}}$$

and

$$\vec{\nabla} \times \vec{\mathbf{B}} = -\imath\omega\mu\epsilon\vec{\mathbf{E}}$$

In order to satisfy the boundary conditions, we cant have fields which are only in the x and y-directions, and we will see in the next lecture that this means waves are guided along this material.

Lecture 40: Wave Guides

Friday, November 15, 2019

Recall our discussion last lecture on perfect conductors with constant cross-section along the $\hat{\mathbf{z}}$-axis. By "perfect" we mean $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ and $\vec{\mathbf{B}} = \vec{\mathbf{0}}$ inside the material. In reality, even highly-conductive materials can have some fields breach the skin depth of the material, but we will ignore this for the present discussion. Recall that $\vec{\mathbf{E}}_{tangent}$ and $\vec{\mathbf{B}}_{\text{normal}}$ are both continuous at the boundaries of the conductor. With these boundary conditions, we can essentially say that

$$\vec{\mathbf{E}}_{\parallel}\Big|_{\text{surface}} = \vec{\mathbf{B}}_n\Big|_{\text{surface}} = \vec{\mathbf{0}}$$

If the conductor is straight along the $\hat{\mathbf{z}}$-axis, the propagation along this axis is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(x, y)e^{\pm\imath kz - \imath\omega t}$$

We will choose $+$, which represents waves going in the positive direction, so

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}(x, y)e^{\imath kz - \imath\omega t}$$

Inside the waveguide, $\vec{\nabla} \cdot \vec{\mathbf{E}} = \vec{\nabla} \cdot \vec{\mathbf{B}} = \vec{\mathbf{0}}$.

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} = +\imath\omega\vec{\mathbf{B}}$$

so

$$\begin{aligned} \partial_y E_z - \imath k E_y &= \imath\omega B_x \\ \imath k E_x - \partial_x E_z &= \imath\omega B_y \\ \partial_x E_y - \partial_y E_x &= \imath\omega B_z \end{aligned}$$

Similarly, $\vec{\nabla} \times \vec{\mathbf{B}} = -\epsilon\mu\frac{\partial\vec{\mathbf{E}}}{\partial t}$:

$$\begin{aligned} \partial_y B_z - \imath k B_y &= -\imath\omega\epsilon\mu E_x \\ \imath k B_x - \partial_x B_z &= -\imath\omega\epsilon\mu E_y \\ \partial_x B_y - \partial_y B_x &= -\imath\omega\epsilon\mu E_z \end{aligned}$$

With six unknowns and six equations, we can probably solve this in terms of derivatives of the fields. If we solve this, we find (assuming $\omega^2\epsilon\mu \neq k^2$)

$$\begin{aligned} E_x &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_x E_z + \omega\partial_y B_z] \\ E_y &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_y E_z - \omega\partial_x B_z] \\ B_x &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_x B_z - \omega\epsilon\mu\partial\mu x E_z] \\ B_y &= \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_y B_z + \omega\epsilon\mu\partial_y E_z] \end{aligned}$$

If we find E_z and B_z, we get the other components. If $E_z = B_z = 0$ then this reduces to the case where $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \vec{\mathbf{0}}$ so $\vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\psi$ where the boundary conditions dictate that ψ is a constant, so there is no propagation.

If $E_z = 0$ we call these modes "TE" or "transverse-electric" modes, and if $B_z = 0$, we call these "TM" or "transverse-magnetic" modes.

By taking the curl of $\vec{\mathbf{E}}$ twice, we find that in general

$$(\nabla^2 + \omega^2\epsilon\mu)\begin{cases}\vec{\mathbf{E}} \\ \vec{\mathbf{B}}\end{cases} = \vec{\mathbf{0}}$$

However, with our boundary conditions applied, we can say

$$(\nabla_\perp^2 - k^2 + \omega^2\epsilon\mu)\begin{cases}\vec{\mathbf{E}} \\ \vec{\mathbf{B}}\end{cases} = \vec{\mathbf{0}}$$

where the perpendicular Laplacian refers to derivatives in only the x and y coordinates. Using the relations we found between the components, we can reduce our equations to

$$(\nabla_\perp^2 - k^2 + \omega^2\epsilon\mu)\begin{cases}\vec{\mathbf{E}}_z \\ \vec{\mathbf{B}}_z\end{cases} = \vec{\mathbf{0}}$$

From this, we see that $E_z\big|_{\text{surface}} = 0$ and $\vec{\mathbf{B}}_n\Big|_{\text{surface}} = \vec{\mathbf{0}}$.

Example. Let's look at a rectangular wave guide. We must impose boundary conditions on all four surfaces (not the ones parallel to the x/y-plane, just think of this as an infinite structure). Let's look for TE modes, where $E_z = 0$. We can write $B_z = X(x)Y(y)$ and set the boundaries at $x = 0, a$ and $y = 0, b$. Plugging in our definition of B_z,

$$\frac{X''}{X} + \frac{Y''}{Y} + (\omega^2\epsilon\mu - k^2) = 0$$

so we can say that $\frac{X''}{X} = -k_x^2$ and $\frac{Y''}{Y} = 0k_y^2$. Solving these, we find that

$$B_z = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)]$$

From our component relations, we have

$$B_x = \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_x B_z]$$

and

$$B_y = \frac{\imath}{\omega^2\epsilon\mu - k^2}[k\partial_y B_z]$$

By the boundary condition on the normal of B_z, we find that the derivatives in the equations above must be zero at the boundary, so we can show that $k_x a = n\pi$. Using B_y, we find that $k_y b = m\pi$:

$$B_z = A_{mn}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{\imath kz - \imath\omega t}$$

Recall that we have to satisfy the condition $-k_x^2 - k_y^2 - k^2 + \omega^2\epsilon\mu$, or

$$k^2 = \omega^2\epsilon\mu - \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)$$

What this means is that there is a cutoff frequency below which no waves will propagate, if we choose m and n. If you check the velocities, you find that $v_p = \frac{\omega}{k} > \frac{1}{\sqrt{\epsilon\mu}}$ but $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} < \frac{1}{\sqrt{\epsilon\mu}}$, and in fact $v_p v_g = \frac{1}{\epsilon\mu}$.

Note that the condition that we cant have $B_z = E_z = 0$ implies that we can't propagate waves straight into the wave guide. We actually have to bounce around along the walls to maintain a propagating wave. ◇

How can we generalize this? We can rewrite our previous equations as

$$\vec{\mathbf{E}}_\perp = \frac{\imath}{\mu\epsilon\omega^2 - k^2}[k\vec{\boldsymbol{\nabla}}_\perp E_z - \omega\hat{z}\times\vec{\boldsymbol{\nabla}}_\perp B_z]$$

$$\vec{\mathbf{B}}_\perp = \frac{\imath}{\mu\epsilon\omega^2 - k^2}[k\vec{\boldsymbol{\nabla}}_\perp B_z - \omega\hat{z}\times\vec{\boldsymbol{\nabla}}_\perp E_z]$$

We can see here that if we look only at TE or TM waves, we can reduce these further. For $E_z = 0$,

$$\vec{\mathbf{B}}_\perp = \frac{\imath k}{\mu\epsilon\omega^2 - k^2}\vec{\boldsymbol{\nabla}}_\perp B_z$$

and

$$\vec{\mathbf{E}}_\perp = \frac{-\imath\omega}{\mu\epsilon\omega^2 - k^2}\hat{z}\times\vec{\boldsymbol{\nabla}}_\perp B_z$$

and for $B_z - 0$,

$$\vec{\mathbf{E}}_\perp = \frac{\imath k}{\mu\epsilon\omega^2 - k^2}\vec{\boldsymbol{\nabla}}_\perp E_z$$

and

$$\vec{\mathbf{B}}_\perp = \frac{\imath\omega}{\mu\epsilon\omega^2 - k^2}\hat{z}\times\vec{\boldsymbol{\nabla}}_\perp E_z$$

We are looking for the solutions of

$$\left[\nabla_\perp^2 + (\omega^2\epsilon\mu - k^2)\right]\psi$$

for either $\psi\big|_S = 0$ or $\frac{\partial\psi}{\partial n}\big|_S = 0$, which we recognize as the Dirichlet and Neumann boundary conditions.

Lecture 41: Waveguides, Continued
Monday, November 18, 2019

Recall from last lecture that if we have a TE mode, $E_z = 0$ so

$$\vec{\mathbf{B}}_\perp = \frac{\imath}{\omega^2\mu\epsilon - k^2}[k\vec{\boldsymbol{\nabla}}_\perp B_z]$$

and for TM modes, $B_z = 0$:

$$\vec{\mathbf{E}}_\perp = \frac{\imath}{\omega^2\mu\epsilon - k^2}[k\vec{\boldsymbol{\nabla}}_\perp E_z]$$

so we find

$$\{\vec{\mathbf{E}}_\perp, \vec{\mathbf{B}}_\perp\} = \pm\frac{k}{\omega}\hat{\mathbf{z}} \times \{\vec{\mathbf{B}}_\perp, \vec{\mathbf{E}}_\perp\}$$

so in both cases,

$$\left[\nabla_\perp^2 + (\mu\epsilon\omega^2 - k^2)\right]\{B_z, E_z\} = 0$$

Digression

$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ are real vector fields. By the usual convention, we write

$$\frac{1}{2}\left[\vec{\mathbf{E}}(\vec{\mathbf{x}},\omega)e^{-\imath\omega t} + \vec{\mathbf{E}}^*(\vec{\mathbf{x}},\omega)e^{+\imath\omega t}\right] \times \frac{1}{2}\left[\vec{\mathbf{B}}(\vec{\mathbf{x}},\omega)e^{-\imath\omega t} + \vec{\mathbf{B}}^*(\vec{\mathbf{x}},\omega)e^{\imath\omega t}\right]$$

This is equal to

$$\frac{1}{4}\left[\vec{\mathbf{E}} \times \vec{\mathbf{B}}e^{-2\imath\omega t} + \vec{\mathbf{E}}^* \times \vec{\mathbf{B}} + \vec{\mathbf{E}} \times \vec{\mathbf{B}}^* + \vec{\mathbf{E}}^* \times \vec{\mathbf{B}}^* e^{2\imath\omega t}\right]$$

Taking the time average, we see that the first and last terms will give us zero, so

$$\left\langle \vec{\mathbf{S}} \right\rangle_t = \frac{1}{2\mu_0}\,\mathrm{Re}[\vec{\mathbf{E}} \times \vec{\mathbf{B}}^*]$$

Often the averaging is implied and the symbolism is left off.

Therefore, if we want to calculate the flow of energy $\vec{\mathbf{S}}$, we will use this time-averaged formula. If you actually compute the Poynting vector for these fields, (writing ψ in place of E_z or B_z, depending on what kind of problem we are looking at), you find

$$\vec{\mathbf{S}} = \frac{1}{2}\frac{\omega k}{[\mu\epsilon\omega^2 - k^2]^2}\begin{cases}\epsilon(\hat{\mathbf{z}}\left|\vec{\boldsymbol{\nabla}}_\perp\psi\right|^2 + \imath\frac{(\mu\epsilon\omega^2-k^2)}{k}\psi\vec{\boldsymbol{\nabla}}_\perp\psi^*) \\ \mu(\hat{\mathbf{z}}\left|\vec{\boldsymbol{\nabla}}_\perp\psi\right|^2 + \imath\frac{(\mu\epsilon\omega^2-k^2)}{k}\psi\vec{\boldsymbol{\nabla}}_\perp\psi^*)\end{cases} \qquad \begin{cases}\text{TM} \\ \text{TE}\end{cases}$$

and the power transmitted will be

$$P = \int_{\mathscr{A}} \vec{\mathbf{S}} \cdot \hat{\mathbf{z}}\,\mathrm{d}a = \frac{1}{2}\frac{\omega k}{[\omega^2\mu\epsilon - k^2]^2}\begin{cases}\epsilon \\ \mu\end{cases} \int \vec{\boldsymbol{\nabla}}_\perp\psi^* \cdot \vec{\boldsymbol{\nabla}}_\perp\psi\,\mathrm{d}a$$

The integral can be calculated as follows:

$$= \int \vec{\boldsymbol{\nabla}} \boldsymbol{\cdot}_{\perp} (\psi^* \vec{\boldsymbol{\nabla}}_{\perp} \psi) - \int \psi^* \nabla_{\perp}^2 \psi = \oint \cancelto{0}{\psi^* \frac{\partial \psi}{\partial n}} \, \mathrm{d}l - \int \psi^* (\nabla_{\perp}^2 \psi)$$

Finally, we already had the equation $\nabla_{\perp}^2 \psi + \left[\mu\epsilon\omega^2 - k^2\right] \psi = 0$, so

$$\vec{\mathbf{S}} = \frac{1}{2} \frac{\omega k}{\mu\epsilon\omega^2 - k} \begin{cases} \epsilon \\ \mu \end{cases} (\mu\epsilon\omega^2 - k) \int |\psi|^2 \, \mathrm{d}a$$

so the power transmitted will be

$$P = \frac{1}{2\sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{\frac{1}{2}} \begin{cases} \epsilon \\ \mu \end{cases} \int \psi_\lambda^* \psi_\lambda \, \mathrm{d}a$$

Therefore the energy transfer will be

$$\frac{1}{2} \left(\frac{\omega}{\omega_\lambda}\right)^2 \begin{cases} \epsilon \\ \mu \end{cases} \int \psi_\lambda^* \psi_\lambda \, \mathrm{d}a$$

We find that the group velocity is

$$v_g = \mathrm{d}\omega \, k = \frac{1}{\sqrt{\mu\epsilon}} \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{\frac{1}{2}}$$

so

$$P = v_g U$$

as expected. Interestingly, $v_g v_p = \frac{1}{\mu\epsilon}$ exactly, and there is a cut-off frequency below which there is no propagation of power.

0.33 Radiation

We found general solutions in the Lorenz gauge ($\vec{\boldsymbol{\nabla}} \boldsymbol{\cdot} \vec{\mathbf{A}} + \frac{1}{c^2} \partial_t \Phi = 0$) which were

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{J}}(\vec{\mathbf{x}}', t')}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} \delta \left(t - \left[t' + \frac{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}{c} \right] \right) \mathrm{d}^3 x' \, \mathrm{d}t'$$

Now let's assume we have a single frequency, so

$$\vec{\mathbf{J}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}) e^{-\imath\omega t}$$

Therefore,

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}, t) e^{-\imath\omega t} = \left[\frac{\mu_0}{4\pi} \int \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}') \frac{e^{\imath \frac{\omega}{c} |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} \, \mathrm{d}^3 x' \right] e^{-\imath\omega t}$$

$\frac{\omega}{c} = k$ in free space with no materials and no boundaries. We imagine there are some charges and currents somewhere and we want to see what they look like very far away.

We have three length scales: the dimension of the source, d, the emitted wavelength, λ, and the distance of observation, $\vec{\mathbf{r}}$. Typically, $d << \lambda$. We define the near field as $d << r << \lambda$, the intermediate zone as $d << \lambda \sim r$ and the far field as $d << \lambda << r$, where the last one is where we will look for radiation effects.

When we are far enough away from the source, $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}_\omega = \mu_0\epsilon_0(-\imath\omega)\vec{\mathbf{E}}_\omega$ or $\vec{\mathbf{E}}_\omega = \frac{\imath c^2}{\omega}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}_\omega = \frac{\imath c}{k}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}_\omega$ and $\vec{\mathbf{B}}_\omega = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}_\omega$.

The near zone is sort of uninteresting. In this regime, $e^{\frac{2\pi\imath}{\lambda}|r-d|} \approx 1$, so to zeroth-order, we find that the potential in the near zone is static, so you could solve it as a static system and add perturbations and corrections from higher-order terms. In the next lecture, we will study the far zone (and see a glimpse of the intermediate zone), which we can no longer approximate as static.

Lecture 42: Radiation in the Far Field

Monday, November 18, 2019

Wait, that's the only kind of radiation.

Recall our three regimes:

- Far Field: $d << \lambda << r$
- Intermediate $d << \lambda \sim r$
- Near Field $d << r << \lambda$ (static limit)

If we expand our solutions in the near field,

$$e^{\imath\frac{2\pi}{\lambda}|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} \sim 1 + \imath\frac{2\pi}{\lambda}|\vec{\mathbf{x}}-\vec{\mathbf{x}}'| + \cdots$$

where 1 represents the static point.

In the radiation zone, let's expand the exponential in the vector potential:

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')e^{\imath kr\left(1-\frac{2\vec{\mathbf{x}}\cdot\vec{\mathbf{x}}'}{r^2}+\frac{\vec{\mathbf{x}}'^2}{r^2}\right)^{\frac{1}{2}}}}{r\left[1-\frac{2\vec{\mathbf{x}}\cdot\vec{\mathbf{x}}'}{r^2}+\frac{\vec{\mathbf{x}}'^2}{r^2}\right]^{\frac{1}{2}}}\,\mathrm{d}^3x'$$

However, $r\frac{\vec{\mathbf{x}}\cdot\vec{\mathbf{x}}'}{r^2} \to k\hat{\mathbf{n}}\cdot\vec{\mathbf{x}}'$ is on the order of $\mathcal{O}\left(\frac{d}{\lambda}\right)$. In the radiation zone, $\frac{d}{r} << \frac{d}{\lambda}$. The next term also has a vanishing order.

Let's try ignoring both of these terms. We find, to zeroth order, that

$$\vec{\mathbf{A}}_\omega \simeq \frac{\mu_0}{4\pi}\int \frac{\mathrm{d}^3x'\,\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')e^{\imath kr}}{r}$$

In the radiation zone, we find

$$\vec{\mathbf{A}}_\omega \simeq \frac{\mu_0}{4\pi}\left[\int \mathrm{d}^3x'\,\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')e^{\imath k\hat{\mathbf{n}}\cdot\vec{\mathbf{x}}'}\right]\frac{e^{\imath kr}}{r}$$

We can expand this last term in the intermediate range as

$$\frac{e^{\imath k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = \imath k \sum_{l,m} j_l(kr')h_l^{(1)}(kr)Y_{lm}(\Omega)Y_{lm}^*(\Omega')$$

where

$$j_l(kr) = \frac{J_{l+1/2}(kr)}{\sqrt{r}}$$

and $h_l^{(1)}$ is a Hankel function of the first kind. We can expand this in the far field to get the radiation effects, since the Hankel function will look like an exponential for large r. We'll derive all of this later.

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi}\frac{e^{\imath kr}}{r}\int \vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')\,\mathrm{d}^3x'$$

Now let's look at the divergence of $\vec{\mathbf{A}}_\omega$:

$$\int_\Omega \partial_j(x_iJ_j)\,\mathrm{d}^3x = \int_\Omega \delta_{ij}J_j + x_i\partial_jJ_j$$

so

$$\int_\Omega J_i = -\int x_i\partial_jJ_j$$

Recall that $\partial_t\rho + \vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{J}} = 0$, so

$$\int_\Omega J_i = -\int \imath\omega\rho_\omega(\vec{\mathbf{x}}')x'\,\mathrm{d}^3x'$$

This term is actually the dipole moment of the charge distribution!

$$\vec{\mathbf{A}}_\omega = -\frac{\imath\mu}{4\pi}\omega\left[\int \underbrace{\mathrm{d}^3x'\,\rho_\omega(\vec{\mathbf{x}}')\vec{\mathbf{x}}'}_{\vec{\mathbf{p}}_\omega}\right]\frac{e^{\imath kr}}{r}$$

Therefore,

$$\vec{\mathbf{B}}_\omega = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}_\omega = +\frac{\imath\mu_0\omega}{4\pi}\vec{\mathbf{p}}_\omega\times\vec{\boldsymbol{\nabla}}\cdot\left(\frac{e^{\imath kr}}{r}\right)$$

and

$$\vec{\boldsymbol{\nabla}}e^{\imath kr} = \imath k\left(\frac{\vec{\mathbf{x}}}{r}\right)e^{\imath kr}$$

so

$$\vec{\mathbf{B}}_\omega = \left(\frac{\imath(\imath k)\mu_0\omega}{4\pi}\vec{\mathbf{p}}(\omega)\times\hat{\mathbf{n}}\right)\frac{e^{\imath kr}}{r}+\cdots$$

and

$$\vec{\mathbf{E}}_\omega = \frac{\imath c}{k}\vec{\boldsymbol{\nabla}}\times\left[-\frac{k^2\mu_0c}{4\pi}\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}}\right]\frac{e^{\imath kr}}{r} = \frac{\imath c^2k\mu_0}{4\pi}\left[(\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\times\imath k\hat{\mathbf{n}}\right]\frac{e^{\imath kr}}{r}$$

so

$$\vec{\mathbf{E}}_\omega = \frac{-c^2k^2}{4\pi}\mu_0\left[(\vec{\mathbf{p}}_\omega\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right]\frac{e^{\imath kr}}{r}$$

Now lets look at the power. We will calculate $\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \mathrm{Re}[\left\langle \vec{\mathbf{S}} \right\rangle \cdot r^2\hat{\mathbf{n}}]$, the change in power as a function of solid angle. This morning we found that the time average of the Poynting vector was something like $\frac{1}{2}(\vec{\mathbf{E}}_\omega \times \vec{\mathbf{B}}^*_\omega)$, which will get rid of the $e^{\imath kr}$ terms, so we will get something like

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} \sim \left[\frac{c^2k^2}{4\pi}\mu_0\frac{ck^2\mu_0}{4\pi}\right]\hat{\mathbf{n}}\cdot\left([(\vec{\mathbf{p}}_\omega \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}] \times (\vec{\mathbf{p}}_\omega \times \hat{\mathbf{n}})\right)$$

We can rewrite it using some vector identities, so its similar to

$$\sim k^4\left((\vec{\mathbf{p}}_\omega \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}\right) \cdot \left((\vec{\mathbf{p}}_\omega \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}\right)^*$$

Lecture 43: The Helmholtz Equation in Spherical Coordinates

Monday, November 25, 2019

In the previous lecture, we were able to write the fields in the radiation zone in a form which utilized the magnetic dipole and multipole moments:

$$\vec{\mathbf{E}}^{\text{dipole}}_\omega = -\frac{Z_0k^2}{4\pi}(\hat{\mathbf{n}} \times \vec{\mathbf{m}}_\omega)\frac{e^{\imath kr}}{r}$$

$$\vec{\mathbf{B}}^{\text{multipole}}_\omega = -\frac{\imath ck^3}{8\pi}\frac{1}{3}\left[\hat{\mathbf{n}} \times \vec{\mathbf{Q}}[\hat{\mathbf{n}}]\right]\frac{e^{\imath kr}}{r}$$

We can calculate the differential power as it relates to the solid angle by

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \left(\frac{1}{2\mu_0}\vec{\mathbf{E}}_\omega \times \vec{\mathbf{B}}_\omega\right) \cdot \hat{\mathbf{n}}r^2$$

so

$$\begin{aligned}
P &\propto \int \left[(\hat{\mathbf{n}} \times \vec{\mathbf{m}}_\omega) \times (\hat{\mathbf{n}} \times \vec{\mathbf{Q}}^*)\right] \cdot \hat{\mathbf{n}}\,\mathrm{d}\Omega \\
&\propto \int \vec{\mathbf{m}}_\omega \cdot (\hat{\mathbf{n}} \times \vec{\mathbf{Q}}^*)\,\mathrm{d}\Omega \\
&\propto \int [m_i\epsilon_{ijk}n_jQ_{kl}n_l]\,\mathrm{d}\Omega \\
&\propto m_i\epsilon_{ijk}Q^*_{kl}\delta_{jl} = 0
\end{aligned}$$

since

$$\int n_jn_l\,\mathrm{d}\Omega = \frac{4\pi}{3}\delta_{jl}$$

and δ is completely symmetric while ϵ is completely antisymmetric.

0.34 Helmholtz Equation in Spherical Coordinates

The Helmholtz equation

$$\left(\nabla^2 + k^2\right)\psi = 0$$

can be written in spherical coordinates as

$$\frac{1}{r^2}\partial_r r^2 \partial_r + \left(k^2 - \frac{l(l+1)}{r^2}\right) f_{lm} = 0$$

where

$$\psi = \sum_{lm} f_{lm}(r) Y_{lm}(\Omega)$$

Assuming spherical symmetry, $f_{lm} \to f_l$, and we can write $f_l = \frac{u_l}{\sqrt{r}}$ and solve for $u_l(r)$ to simplify this equation:

$$\left[\partial_r^2 + \frac{1}{r}\partial_r + \left(k^2 - \frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right)\right] u_l(r) = 0$$

This is very similar to the Bessel equation, and the solutions for u_l are known as the spherical Bessel functions:

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x)$$

which is regular at $x = 0$,

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x)$$

which is singular at $x = 0$, and

$$h_l^{(1,2)} = j_l(x) \pm \imath n_l(x)$$

These functions have the following recursion relations and expansions:

$$j_l(x) = (-x)^l \left[\frac{1}{x}\partial_x\right]^l \left(\frac{\sin(x)}{x}\right)$$

$$n_l(x) = -(-x)^l \left[\frac{1}{x}\partial_x\right]^l \left(\frac{\cos(x)}{x}\right)$$

As $x \to 0$ (or $x << 1$),

$$j_l(x) \mapsto \frac{x^l}{(2l+1)!!}\left[1 - \frac{x^2}{2(2l+3)} + \cdots\right]$$

$$n_l(x) \mapsto \frac{-(2l-1)!!}{x^{l+1}}\left[1 - \frac{x^2}{2(1-2l)} + \cdots\right]$$

As $x \to \infty$ (or $x >> 1$),

$$j_l(x) \mapsto \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right)$$

$$n_l(x) \mapsto -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right)$$

and

$$h_l^{(1)} \mapsto (-\imath)^{l+1}\frac{e^{\imath x}}{x}$$

This last equation is the kind of outgoing wave behavior which we want in a radiative solution.

Additionally, for all $j_l,\ n_l,\ h_l = z_l$,

$$\frac{2l+1}{x}z_l(x) = z_{l-1}(x) + z_{l+1}(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xz_l(x)\right] = xz_{l-1}(x) - lz_l(x)$$

Finally, the Wronskian for the spherical Bessel functions is

$$W\left[j_l, n_l\right] = \frac{1}{\imath}W\left[j_l, h_l^{(1)}\right] = \frac{1}{x^2}$$

Quote

"Almost everything you can imagine is a thing you cannot write"
- Turgut, on solutions to equations

Quote

"The world of functions is very wild and crazy"
- Turgut, also on solutions to equations

0.35 Green's Function for the Spherical Helmholtz Equation

$$(\nabla^2 + k^2)G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = -\delta(\vec{\mathbf{x}} - \vec{\mathbf{x}}') \mapsto \frac{e^{\imath k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = \frac{\delta(r-r')}{r^2}\underbrace{\delta(\Omega-\Omega')}_{\sum_{lm}Y_{lm}^*(\Omega')Y_{lm}(\Omega)}$$

Note the missing 4π in front of the δ-function. This is just a scaling factor and only slightly effects how the Green's function is applied.

We can therefore write the Green's function as

$$G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = \sum_{lm} g_l(r, r') Y_{lm}(\Omega) Y^*_{lm}(\Omega')$$

If we integrate the differential equation for g_l around r', we find that

$$\int_{r'-\epsilon}^{r'+\epsilon} \mathrm{d}r \left[\frac{1}{r^2}\partial_r r^2 \partial_r g_l\right] = -\int_{r'-\epsilon}^{r'+\epsilon} \frac{\delta(r-r')}{r^2}\,\mathrm{d}r'$$

so

$$\left.\frac{\mathrm{d}g_l}{\mathrm{d}r}\right|_{r'+\epsilon} - \left.\frac{\mathrm{d}g_l}{\mathrm{d}r}\right|_{r'-\epsilon} = -\frac{1}{r'^2}$$

so

$$g_l(r, r') = A_l j_l(kr_<) h_l^{(1)}(kr_>)$$

since we want regular behavior at 0 and oscillatory behavior at ∞. We can use the Wronskian to determine the factor A_l:

$$G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = (\imath k) j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}(\Omega) Y^*_{lm}(\Omega')$$

Lecture 44:
Friday, November 22, 2019

The total radiation energy/second is the famous Rayleigh scattering result, the leading order dipole term:

$$P = \frac{c^2 Z_0 k^4}{12\pi} |\vec{\mathbf{p}}_\omega|^2$$

Recall that

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

and

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{k^4 Z_0}{32\pi^2} |\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{\mathbf{m}}_\omega)|^2$$

The next order term of the power is

$$P = \frac{Z_0 k^4}{12\pi} |\vec{\mathbf{m}}_\omega|^2$$

Note that $\frac{|\vec{\mathbf{m}}_\omega|}{|\vec{\mathbf{p}}_\omega|} \sim \omega d$ so $\frac{1}{c^2}\frac{|m_\omega^2|}{|p_\omega|^2} \sim \frac{\omega^2 d^2}{c^2} \sim (kd)^2$.

Recall $\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi}\frac{e^{\imath kr}}{r} \int \mathrm{d}^3 x' \, \imath k(\text{sym} + \text{antisym})$ where we are concerned with the symmetric part of $x' \cdot J$,

$$\frac{1}{2}\left[x_i' J_j(x') + x_j' J_i(x')\right]$$

We found previously that the antisymmetric part was proportional to $\vec{\mathbf{m}}_\omega$. We will see later that there is a better expansion, and our method here is not that sophisticated.

$$\begin{aligned} 0 = \int_\Omega \partial_j \left[x_i J_j \right] \mathrm{d}^3 x &= \int \left(\delta_{jk} x_i J_i + x_k \delta_{ij} J_j + x_k x_i \partial_j J_j \right) \\ &= \int \left(x_i J_k + x_k J_i \right) + x_k x_i \partial_j J_j \end{aligned}$$

$$\frac{1}{2} \int_\Omega \left(x_i J_k + x_k J_i \right) \mathrm{d}^3 x = -\imath\omega \int x_k x_i \rho \rho_\omega \, \mathrm{d}^3 x$$

since

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t} = -(-\imath\omega)\rho_\omega$$

Therefore

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi} \frac{e^{\imath k r}}{r} \int \mathrm{d}^3 x' \, (\imath k)(-\imath\omega) \frac{1}{2} (\hat{\mathbf{n}} \cdot \vec{\mathbf{x}}') \vec{\mathbf{x}}' \rho_\omega$$

Recall $\vec{\mathbf{B}}_\omega = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}_\omega$ and $\vec{\mathbf{E}}_\omega = \frac{\imath c}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}_\omega$. We are only operating on the exponential over r term in the front of $\vec{\mathbf{A}}_\omega$, so $\vec{\mathbf{B}}_\omega = -\frac{\mu_0}{8\pi} \left[\int \mathrm{d}^3 x' \cdots \right] \times \left(\vec{\boldsymbol{\nabla}} e^{\imath k r} \right) \frac{1}{r}$. To leading order, this gives us

$$\vec{\mathbf{B}}_\omega \approx -\frac{\mu_0 k \omega}{8\pi} \left\{ \left[\int \mathrm{d}^3 x' \, (\hat{\mathbf{n}} \cdot \vec{\mathbf{x}}') \vec{\mathbf{x}}' \rho_\omega \right] \times (\imath k) \hat{\mathbf{n}} \right\} \frac{e^{\imath k r}}{r}$$

Inside the integral, we can write

$$n_i \left[x_i' x_j' - \frac{1}{3} \delta_{ij} r'^2 \right] = (n_i x_i') x_j' - n_j r'^2$$

and

$$\hat{\mathbf{n}} \cdot [\mathbf{Q}] = (\hat{\mathbf{n}} \cdot \vec{\mathbf{x}}') \vec{\mathbf{x}}' - \frac{1}{3} r'^2 \hat{\mathbf{n}}$$

so

$$\hat{\mathbf{n}} \cdot [\mathbf{Q}] \times \hat{\mathbf{n}} = (\hat{\mathbf{n}} \cdot \vec{\mathbf{x}}') \vec{\mathbf{x}}' \times \hat{\mathbf{n}}$$

where

$$Q_{ij} = \int \left[3 x_i x_j - \delta_{ij} x^2 \right] \mathrm{d}^3 x$$

which is symmetric in i and j. Therefore, if we define $\vec{\mathbf{Q}}[\hat{\mathbf{n}}]_i \equiv Q_{ij} \hat{\mathbf{n}}_j$,

$$\frac{1}{\mu_0} \vec{\mathbf{B}} = \frac{\omega k^2 \imath}{8\pi} \frac{1}{3} \frac{e^{\imath k r}}{r} \vec{\mathbf{Q}}[\hat{\mathbf{n}}] \times \hat{\mathbf{n}} = 0\imath \frac{c k^3}{24\pi} \left[\hat{\mathbf{n}} \times \vec{\mathbf{Q}}[\hat{\mathbf{n}}] \right] \frac{e^{\imath k r}}{r}$$

where we recognize $\vec{\mathbf{Q}}$ as a contraction of the electric multipole tensor. Now we can take the curl to find the electric field, where (rewriting using $\frac{\vec{\mathbf{B}}_\omega}{\mu_0}$),

$$\vec{\mathbf{E}}_\omega = \frac{\imath Z_0}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}}_\omega$$

so

$$\vec{\mathbf{E}}_\omega = \frac{\imath Z_0}{k}(-1)\left(-\imath\frac{ck^3}{24\pi}\right)\left[\hat{\mathbf{n}}\times\vec{\mathbf{Q}}(\hat{\mathbf{n}})\right]\times\left(\vec{\boldsymbol{\nabla}}e^{\imath kr}\right)\frac{1}{r} = -\frac{\imath c Z_0 k^3}{24\pi}\left(\left[\hat{\mathbf{n}}\times\vec{\mathbf{Q}}(\hat{\mathbf{n}})\right]\times\hat{\mathbf{n}}\right)\frac{e^{\imath kr}}{r}$$

Now we can write down $\frac{\mathrm{d}P}{\mathrm{d}\Omega}$:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{2\mu}\vec{\mathbf{E}}_\omega\times\vec{\mathbf{B}}^*_\omega\cdot\hat{\mathbf{n}}r^2$$

since we are averaging over time. The final dot product is because we want to see the actual scaled power far away in the solid angle. If you write this down, you find

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c^2 Z_0 k^6}{24\times 24\times 2\pi^2}\hat{\mathbf{n}}\cdot\left[\left(\left[\hat{\mathbf{n}}\times\vec{\mathbf{Q}}(\hat{\mathbf{n}})\right]\times\hat{\mathbf{n}}\right)\times\left(\hat{\mathbf{n}}\times\vec{\mathbf{Q}}^*(\hat{\mathbf{n}})\right)\right]$$

we can use the triple product identity to rewrite this as

$$\frac{c^2 Z_0 k^6}{1152\pi^2}\left|\left(\hat{\mathbf{n}}\times\vec{\mathbf{Q}}(\hat{\mathbf{n}})\right)\times\hat{\mathbf{n}}\right|^2$$

Clearly this must be a smaller term than the leading terms. To find the full power, we must now integrate over $\mathrm{d}\Omega$.

$$P^{(quad)} = \frac{c^2 Z_0 k^6}{1152\pi^2}\int_{S^2}\left[\left(\hat{\mathbf{n}}\times\vec{\mathbf{Q}}(\hat{\mathbf{n}})\right)\times\hat{\mathbf{n}}\right]\cdot\left(\left[\vec{\mathbf{Q}}(\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right]\times\hat{\mathbf{n}}\right)^*\mathrm{d}^2\Omega$$

We expand $\left[\left(\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\vec{\mathbf{Q}})\right)-\vec{\mathbf{Q}}\hat{\mathbf{n}}\hat{\mathbf{n}}\right]\cdot\left[\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\vec{\mathbf{Q}}^*)-\vec{\mathbf{Q}}^*\hat{\mathbf{n}}\hat{\mathbf{n}}\right] = (\hat{\mathbf{n}}\cdot\vec{\mathbf{Q}})(\hat{\mathbf{n}}\cdot\vec{\mathbf{Q}}^*)-\vec{\mathbf{Q}}\cdot\vec{\mathbf{Q}}^*$. This is

$$\int n_\alpha Q_{\alpha\beta}n_\beta n_\gamma Q^*_{\gamma\delta}n_\delta\,\mathrm{d}^2\Omega = \int\mathrm{d}^2\Omega\left[Q_{\alpha beta}\hat{\mathbf{n}}_\beta Q_{\alpha\lambda}\hat{\mathbf{n}}_\lambda\right]$$

These are rotationally invariant integrals, by symmetry, so

$$\int\mathrm{d}^2\Omega\,\hat{\mathbf{n}}_\beta\hat{\mathbf{n}}_\lambda = \lambda^{(1)}\delta_{\alpha\beta}$$

To find the constant, we contract over $\beta=\lambda$, which means that $\lambda^{(1)}=\frac{4\pi}{3}$. The other integral is

$$\int\mathrm{d}^2\Omega\,\hat{\mathbf{n}}_\alpha\hat{\mathbf{n}}_\beta\hat{\mathbf{n}}_\gamma\hat{\mathbf{n}}_\delta = \lambda^{(2)}\left[\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta}\right]$$

Contracting over $\alpha=\beta$, we get

$$\int\mathrm{d}^2\Omega\,\hat{\mathbf{n}}_\gamma\hat{\mathbf{n}}_\delta = \lambda^{(2)}\left[5\delta_{\gamma\delta}\right]$$

so $\lambda^{(2)}=\frac{4\pi}{15}$. Therefore, using both integrals, we can write down our answer. Remember that the $\mathbf{Q}$ tensor is traceless:

$$P = \frac{c^2 Z_0 k^6}{1440\pi}\left[Q_{\alpha\beta}Q^*_{\alpha\beta}\right]$$

Lecture 46: The Helmholtz Equation in Spherical Coordinates

Monday, November 25, 2019

From last lecture, we showed that

$$\frac{e^{\imath k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = \sum_{l,m}(\imath k) j_l(kr_<)h_l^{(1)}(kr_>)Y_{lm}(\Omega)Y_{lm}^*(\Omega')$$

Recall that our vector potential was

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0}{4\pi}\int \frac{e^{\imath k|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')\,\mathrm{d}^3x'$$

so in the far field, we can expand this as

$$\vec{\mathbf{A}}_\omega = \frac{\mu_0 \imath k}{4\pi}\left[\int j_l(kr')Y_{lm}^*(\Omega')\vec{\mathbf{J}}_\omega(\vec{\mathbf{x}}')\,\mathrm{d}^3x'\right] h_l^{(1)}(kr)Y_{lm}(\Omega)$$

Of course, we can expand the spherical Bessel functions, but in general it won't decouple the equation nicely. We can expand $h^{(1)}$ in the radiation zone ($\frac{r}{\lambda} >> 1$), but this doesn't solve any problems on the inside of the integral, because of the vector components of $\vec{\mathbf{J}}_\omega$.

Instead, we have to find another (not so obvious) expansion. If we are away from the source region, the equations which we are solving are technically source-less:

$$\left(\nabla^2 + k^2\right)\begin{pmatrix}\vec{\mathbf{E}}_\omega \\ \vec{\mathbf{H}}_\omega\end{pmatrix} = 0$$

Let's take a step back and solve the Helmholtz equation in this region, away from the source: $(\nabla^2 + k^2)\psi = 0$. We want a vector solution, not a scalar. Note that $\vec{\mathbb{L}} = \frac{1}{\imath}\vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}}$ commutes with the Laplacian because the Laplacian is a scalar operator. This tells you that if you had a scalar solution, that solution would also satisfy

$$(\nabla^2 + k^2)\vec{\mathbb{L}}\psi = \vec{\mathbf{0}}$$

and $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbb{L}}\psi = 0$.

Also note that, through a few substitutions, $\vec{\mathbf{H}}_\omega = -\frac{\imath}{kZ_0}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}}_\omega$, so if

$$\vec{\mathbf{E}}_\omega = \vec{\mathbb{L}}\psi$$

then

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{kZ_0}\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\psi$$

We could also do this the other way around, where

$$\vec{\mathbf{H}}_\omega = \vec{\mathbb{L}}\chi$$

so

$$\vec{\nabla} \times \vec{\mathbf{H}}_\omega = \epsilon_0(-\imath\omega)\vec{\mathbf{E}}_\omega$$

so

$$\vec{\mathbf{E}}_\omega = \frac{\imath}{k} Z_0 \vec{\nabla} \times \vec{\mathbf{H}}_\omega$$

or

$$\vec{\mathbf{E}}_\omega = \frac{\imath}{k} Z_0 \vec{\nabla} \times \vec{\mathbb{L}}\chi$$

The addition of these solutions is indeed the general solution:

$$\vec{\mathbf{E}}_\omega = \frac{\imath}{k} Z_0 \vec{\nabla} \times \vec{\mathbb{L}}\chi + \vec{\mathbb{L}}\psi$$

and

$$\vec{\mathbf{E}}_\omega = \vec{\mathbb{L}}\chi - \frac{\imath}{kZ_0} \vec{\nabla} \times \vec{\mathbb{L}}\psi$$

Solutions to the source-less Helmholtz equation can be expanded as

$$\psi = \sum \underbrace{\left[A_{lm}^{(1)} h_l^{(1)}(kr) + A_{lm}^{(2)} h_l^{(2)}(kr)\right]}_{f_{lm}} Y_{lm}(\Omega)$$

and

$$\chi = \sum \underbrace{\left[B_{lm}^{(1)} h_l^{(1)}(kr) + B_{lm}^{(2)} h_l^{(2)}(kr)\right]}_{g_{lm}} Y_{lm}(\Omega)$$

so

$$\vec{\mathbf{E}}_\omega = \sum_{lm} \left[f_{lm}(kr) \underbrace{\vec{\mathbb{L}} Y_{lm}}_{\sim \vec{\mathbb{L}}_{lm}} + \frac{\imath Z_0}{k} \vec{\nabla} \times (g_{lm}(kr) \vec{\mathbb{X}}_{lm}) \right]$$

where $\vec{\mathbb{X}}_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{\mathbb{L}} Y_{lm}$ are the vector spherical harmonics, and

$$\vec{\mathbf{H}}_\omega = \sum_{lm} \left[-\frac{\imath}{kZ_0} \vec{\nabla} \times (f_{lm}(kr) \vec{\mathbb{X}}_{lm}) + g_{lm}(kr) \vec{\mathbb{X}}_{lm} \right]$$

If we only want outgoing solutions, we can just look at the $h^{(1)}$ terms and expand them as $(-\imath)^{l+1} \frac{e^{\imath kr}}{kr}$. In practice, we only do this to the $\vec{\mathbf{H}}$ field and then use Maxwell's equations to get the $\vec{\mathbf{E}}$ field. Suppose we absorb Z_0 into f_{lm} in the equation for $\vec{\mathbf{H}}_\omega$:

$$\vec{\mathbf{E}}_\omega = \sum_{lm} \left[f_{lm}(kr) \vec{\mathbb{X}}_{lm} + \frac{\imath}{k} \vec{\nabla} \times (g_{lm} \vec{\mathbb{X}}_{lm}) \right]$$

and

$$\vec{\mathbf{H}}_\omega = \sum_{lm} \left[-\frac{\imath}{k} \vec{\nabla} \times (f_{lm}(kr) \vec{\mathbb{X}}_{lm}) + g_{lm} \vec{\mathbb{X}}_{lm} \right]$$

Recall that we showed (on a homework) that

$$\imath \vec{\nabla} \times \vec{\mathbf{L}} = \vec{\mathbf{x}} \nabla^2 - \vec{\nabla} \times [1 + \vec{\mathbf{x}} \cdot \vec{\nabla}]$$

Lecture 47: Review of Radiation

Monday, December 02, 2019

Suppose

$$\begin{pmatrix} \vec{\mathbf{H}} \\ \vec{\mathbf{E}} \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{H}}_\omega \\ \vec{\mathbf{E}}_\omega \end{pmatrix} e^{-\imath\omega t}$$

Then, Maxwell's equations become

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = -\mu_0 \frac{\partial}{\partial t} \vec{\mathbf{H}}$$

If we define $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$,

$$\vec{\mathbf{E}}_\omega = \frac{\imath Z_0}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}}_\omega$$

and

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{Z_0 k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}}_\omega$$

where $k = \frac{\omega}{c}$.

Recall that we want fields which satisfy the Helmholtz equation in spherical coordinates:

$$\left(\nabla^2 + k^2\right) \begin{pmatrix} \vec{\mathbf{E}}_\omega \\ \vec{\mathbf{H}}_\omega \end{pmatrix} = \vec{\mathbf{0}}$$

since both fields have no divergence in the radiation zone. We can solve this by realizing that the angular momentum operator, $\vec{\mathbb{L}} = \frac{1}{\imath} \vec{\mathbf{x}} \times \vec{\boldsymbol{\nabla}}$, commutes with the Laplacian, so if ψ is a solution to the Helmholtz equation, so is $\vec{\mathbb{L}}\psi$. Using this, we found that the general solutions to these fields is

$$\vec{\mathbf{E}}_\omega = \vec{\mathbb{L}}\psi + \frac{\imath Z_0}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\chi$$

and

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{Z_0 k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\psi + \vec{\mathbb{L}}\chi$$

Next, we found general solutions to the spherical Helmholtz equation using the spherical Bessel functions:

$$\psi = \sum_{l,m} \left[a_{lm} h_l^{(1)}(kr) + b_{lm} h_l^{(2)}(kr)\right] Y_{lm}(\Omega)$$

since the Hankel functions look like outgoing/incoming waves (1/2) in the $kr >> 1$ regime.

Now we want to act the angular momentum operator on this function. Since it is a spherical operator, it only acts on the Y_{lm} part, so

$$\vec{\mathbb{L}}\psi = \sum_{l,m} f_{lm}(kr) \frac{\vec{\mathbb{L}} Y_{lm}}{\sqrt{l(l+1)}}$$

where we are just scaling by $\sqrt{l(l+1)}$. We recognize these as the vector spherical harmonics:

$$\frac{\vec{\mathbb{L}}Y_{lm}}{\sqrt{l(l+1)}} = \mathbb{X}_{lm}$$

These make an orthonormal basis:

$$\int \vec{\mathbb{X}}^*_{l'm'} \cdot \vec{\mathbb{X}}_{lm}\, \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}$$

Jackson scales the general fields by Z_0:

$$\vec{\mathbf{E}}_\omega = Z_0 \left(\underbrace{\vec{\mathbb{L}}\psi}_{g_{lm}} + \frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\chi \right)$$

$$\vec{\mathbf{H}}_\omega = -\frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}}\psi + \underbrace{\vec{\mathbb{L}}\chi}_{f_{lm}}$$

If we now expand in terms of the vector spherical harmonics, we find

$$\vec{\mathbf{E}}_\omega = Z_0 \sum_{l,m} \left(g_{lm}(kr)\vec{\mathbb{X}}_{lm} + \frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times (f_{lm}(kr)\vec{\mathbb{X}}_{lm}) \right)$$

$$\vec{\mathbf{H}}_\omega = Z_0 \sum_{l,m} \left(f_{lm}(kr)\vec{\mathbb{X}}_{lm} - \frac{\imath}{k}\vec{\boldsymbol{\nabla}} \times (g_{lm}(kr)\vec{\mathbb{X}}_{lm}) \right)$$

We should emphasize that these are exact representations, we have not made any approximations yet. Also, f and g are switched with respect to the earlier lecture where this derivation was first done.

Notice that $\vec{\mathbf{x}} \cdot \vec{\mathbb{L}}\psi = 0$ for any ψ, so

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega = Z_0 \sum_{l,m} \frac{\imath}{k}\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} \times (f_{lm}(kr)\vec{\mathbb{X}}_{lm}) = -\frac{Z_0}{k}\sum_{l,m} \vec{\mathbb{L}}(f_{lm}(kr)\mathbb{X}_{lm})$$

The angular momentum operator commutes with the radial part of the dot product, so this becomes

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega = -\frac{Z_0}{k}\sum_{l,m} f_{lm}(kr)\vec{\mathbb{L}} \cdot \vec{\mathbb{X}}_{lm} = -\frac{Z_0}{k}\sum_{l,m} \sqrt{l(l+1)}Y_{lm} f_{lm}(kr)$$

If we want to find $f_{lm}(kr)$ we need to decompose $\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega$ into spherical harmonics:

$$f_{lm}(kr) = -\frac{k}{Z_0\sqrt{l(l+1)}} \int Y^*_{lm} \left(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega \right) \mathrm{d}\Omega$$

For radiation problems, we would typically have these functions be $a_E(l,m)h_l^{(1)}(kr) = f_{lm}(kr)$. We can do the same derivation to find the $\vec{\mathbf{H}}_\omega$ field:

$$g_{lm}(kr) = \frac{k}{\sqrt{l(l+1)}} \int Y^*_{lm} \left(\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega \right) \mathrm{d}\Omega$$

where, for radiation, $g_{lm}(kr) = a_M(l,m)h_l^{(1)}(kr)$. These a factors are the electric and magnetic multipoles.

In the radiation zone, using

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}} = \imath\left[\vec{\mathbf{x}}\nabla^2 - \vec{\boldsymbol{\nabla}}(1 + \vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}})\right]$$

we found that

$$\vec{\boldsymbol{\nabla}} \times (f_{lm}(kr)\vec{\mathbb{X}}_{lm})$$

only has a contribution from the factor of $e^{\imath kr}$ from the Hankel functions in the far-field limit, so we find terms like

$$(-\imath)^{l+1}\frac{e^{\imath kr}}{kr}\hat{\mathbf{n}} \times \vec{\mathbb{X}}_{lm}$$

In the radiation zone, the terms simplify to

$$\vec{\mathbf{H}}_\omega \to (-\imath)^{l+1}\frac{e^{\imath kr}}{kr}\sum_{l,m}\left[a_E(l,m)\vec{\mathbb{X}}_{lm} + a_M(l,m)\hat{\mathbf{n}} \times \vec{\mathbb{X}}_{lm}\right]$$

and

$$\vec{\mathbf{E}}_\omega = Z_0\vec{\mathbf{H}}_\omega \times \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}} = \frac{\vec{\mathbf{x}}}{r}$.

Lecture 48: Radiation Review, Continued

Monday, December 02, 2019

Putting everything from the last few lectures together, we can write the frequency decomposed elements of the fields in the radiation zone ($kr >> 1$) as

$$\vec{\mathbf{H}}_\omega \mapsto (-\imath)^{l+1}\frac{e^{\imath kr}}{kr}\sum_{l,m}\left[a_E(l,m)\vec{\mathbb{X}}_{lm} + a_M(l,m)\hat{\mathbf{n}} \times \vec{\mathbb{X}}_{lm}\right]$$

and

$$\vec{\mathbf{E}}_\omega \mapsto Z_0\vec{\mathbf{H}}_\omega \times \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}} = \frac{\vec{\mathbf{x}}}{r}$.

If we now have this expansion in the radiation zone, how do we find the radiated power in some solid angle far away?

$$\frac{\mathrm{d}P_\omega}{\mathrm{d}\Omega} = r^2\hat{\mathbf{n}}\frac{1}{2}\left(\vec{\mathbf{E}}_\omega \times \vec{\mathbf{H}}^*_\omega\right)$$

where it is implied that we are taking the real part of this expression (which often turns out to be real anyway).

Therefore,

$$\frac{\mathrm{d}P_\omega}{\mathrm{d}\Omega} = \frac{Z_0}{k^2}\frac{1}{2}\hat{n}\cdot \sum_{l,m,l',m'} \left[a_E(l,m)\vec{\mathbb{X}}_{lm} + a_M(l,m)\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{lm}\right] \\ \times\left(\left[a_E^*(l',m')\vec{\mathbb{X}}^*_{l'm'} + a_M^*(l',m')\hat{\mathbf{n}}\times\vec{\mathbb{X}}^*_{l'm'}\right]\times\hat{\mathbf{n}}\right)$$

This is not exactly the most appealing form for this equation. We can rewrite

$$\hat{\mathbf{n}}\cdot\left[\left(\vec{\mathbf{H}}_\omega\times\hat{\mathbf{n}}\right)\times\vec{\mathbf{H}}^*_\omega\right] = \vec{\mathbf{H}}_\omega\cdot\vec{\mathbf{H}}^*_\omega$$

Doing this will still give us double summations, but we can integrate this expression over the sphere. To do this, the following identity is useful:

Lemma 0.35.1.

$$\vec{\mathbb{X}}^*_{l'm'}\cdot\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{lm}\right)\mathrm{d}\Omega = 0$$

Therefore,

$$P_\omega = \frac{1}{2}\frac{Z_0}{k^2}\sum_{l,m,l',m'}\int \mathrm{d}\Omega[a_E^* a_E \vec{\mathbb{X}}^*_{lm}\vec{\mathbb{X}}_{l'm'} \\ + \left(a_E^* a_M + a_M a_E^*\right)\vec{\mathbb{X}}^*_{lm}\cdot\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{l'm'}\right) \\ + a_M^* a_M^*\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}_{lm}\right)\cdot\left(\hat{\mathbf{n}}\times\vec{\mathbb{X}}^*_{l'm'}\right)]$$

The integral over the first term reduces to δ-functions, the middle term vanishes, and the final term also reduces to δ-functions, so

$$P_\omega = \frac{1}{2}\frac{Z_0}{k^2}\sum_{l,m}\left[|a_E(l,m)|^2 + |a_M(l,m)|^2\right]$$

Recall Maxwell's equations in this region:

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}}_\omega = \frac{\rho_\omega}{\epsilon_0}$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{H}}_\omega = \vec{\mathbf{J}}_\omega - \epsilon_0 \imath\omega\vec{\mathbf{E}}_\omega$$
$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}}_\omega - \imath k Z_0\vec{\mathbf{H}}_\omega = \vec{\mathbf{0}}$$
$$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{H}}_\omega + \frac{\imath k}{Z_0}\vec{\mathbf{E}}_\omega = \vec{\mathbf{J}}_\omega$$

since

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{J}}_\omega = \imath\omega\rho_\omega$$

Therefore we can write

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}}_\omega = -\frac{1}{\imath\omega\epsilon_0}\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{J}}_\omega \implies \vec{\boldsymbol{\nabla}}\cdot\underbrace{\vec{\mathbf{E}}_\omega + \frac{1}{\imath\omega\epsilon_0}\vec{\mathbf{J}}_\omega}_{\vec{\mathbf{E}}'_\omega} = \vec{\mathbf{0}}$$

where

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}}'_\omega = \vec{\mathbf{0}} = \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{H}}_\omega$$

Therefore, we find that

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}}_\omega = \frac{\imath k}{Z_0}\left[\vec{\mathbf{E}}'_\omega - \frac{\imath Z_0}{k}\vec{\mathbf{J}}_\omega\right] = \vec{\mathbf{J}}_\omega$$

so

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}}_\omega + \frac{\imath k}{Z_0}\vec{\mathbf{E}}'_\omega = \vec{\mathbf{0}}$$

and

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}}'_\omega - \imath k Z_0 \vec{\mathbf{H}}_\omega = \frac{\imath Z_0}{k}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega$$

Why are we doing this? We want to be able to determine a_M and a_E from the source components. If we take the curl of the previous equations, we find

$$-\nabla^2\vec{\mathbf{H}}_\omega + \frac{\imath k}{Z_0}\left[\imath k Z_0 \vec{\mathbf{H}}_\omega + \frac{\imath Z_0}{k}\vec{\boldsymbol{\nabla}} \times JVec_\omega\right] = \vec{\mathbf{0}}$$

or

$$\left(\nabla^2 + k^2\right)\vec{\mathbf{H}}_\omega = -\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega$$

From the other equation, we find

$$\left(\nabla^2 + k^2\right)\vec{\mathbf{E}}'_\omega = -\frac{\imath Z_0}{k}\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega)$$

Observe that

Lemma 0.35.2.

$$\nabla^2(\vec{\mathbf{x}} \cdot \vec{\mathbf{F}}) = 2\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{F}} + (\nabla^2\vec{\mathbf{F}}) \cdot \vec{\mathbf{x}}$$

If we apply this to our fields, which are divergence free, we find that

$$(\nabla^2 + k^2)(\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega) = -\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega$$

and

$$(\nabla^2 + k^2)(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}'_\omega) = -\frac{\imath Z_0}{k}\vec{\mathbf{x}} \cdot \left(\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega)\right)$$

We can rewrite the second equation as

$$(\nabla^2 + k^2)(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}'_\omega) = \frac{Z_0}{k}\mathbb{L} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega)$$

We can solve these equations:

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega = \frac{1}{4\pi}\int \frac{e^{\imath k|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|}$$

which we can expand as

$$\sum_{l,m} (\imath k) j_l(kr') h_l^{(1)}(kr) Y_{lm}(\Omega) Y_{l'm'}^*(\Omega')$$

so

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega = \sum_{l,m} (\imath k) \int j_l(kr') Y_{lm}^*(\Omega') [-\imath \vec{\mathbb{L}} \cdot \vec{\mathbf{J}}_\omega](\vec{\mathbf{x}}') \, \mathrm{d}^3 x' \, h_l^{(1)}(kr) Y_{lm}(\Omega)$$

and

$$Z_0 a_E(l,m) h_l^{(1)}(kr) = -\frac{k}{\sqrt{l(l+1)}} \int Y_{lm}^*(\Omega')(\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega) \, \mathrm{d}\Omega$$

Lecture 49: Radiation, Continued

Wednesday, December 04, 2019

Recall the source-free solutions (away from the source):

$$\vec{\mathbf{E}}_\omega = Z_0 \left(\vec{\mathbb{L}} \psi + \frac{\imath}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}} \chi \right)$$

$$\vec{\mathbf{H}}_\omega = \vec{\mathbb{L}} \chi - \frac{\imath}{k} \vec{\boldsymbol{\nabla}} \times \vec{\mathbb{L}} \psi$$

We want to connect these to the region where the source is present. Recall that last time, we expanded

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega = \sum_{l,m} \frac{1}{k} \sqrt{l(l+1)} Y_{lm} h_l^{(1)}(kr) a_M(l,m)$$

and

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{E}}_\omega = \sum_{l,m} -\frac{Z_0}{k} \sqrt{l(l+1)} Y_{lm} h_l^{(1)}(kr) a_E(l,m)$$

When we looked at the solutions for the source term last time, we found that

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{H}}_\omega = \left[\imath (\imath k) \int j_l(kr') Y_{lm}(\Omega') (\vec{\mathbb{L}} \cdot \vec{\mathbf{J}}_\omega)(\vec{\mathbf{x}}') \, \mathrm{d}^3 x' \right] h_l^{(1)}(kr) Y_{lm}(\Omega)$$

and a similar expression for $\vec{\mathbf{E}}_\omega$. We then find that

$$a_E(l,m) = \frac{\imath k}{\sqrt{l(l+1)}} \int j_l(kr) Y_{lm}^*(\Omega) \left(\vec{\mathbb{L}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega \right) \mathrm{d}^3 x$$

$$a_M(l,m) = -\frac{k^2}{\sqrt{l(l+1)}} \int j_l(kr) Y_{lm}^*(\Omega) \left(\vec{\mathbb{L}} \cdot \vec{\mathbf{J}}_\omega \right) \mathrm{d}^3 x$$

Recall that

$$P_\omega = \frac{Z_0}{k^2} \sum_{l,m} \left(|a_m|^2 + |a_E|^2 \right)$$

so we can now find the power radiated by any source. We have not made any approximations yet, and these equations work as long as we are not inside the source region. The next stage is to try to make connections with other things we already know. We can take limits of the Hankel functions and expand them as exponentials. We shouldn't expand the spherical Bessel functions inside the integral unless we assume $kr << 1$. Recall the expansions:

$$j_l(kr) \mapsto \frac{(kr)^l}{(2l+1)!!}\left[1 - \cdots\right]$$

when $kr << 1$. If the wavelength is much greater than the length scale of the source (low frequencies), we can use this approximation.

Examine the term in a_M:

$$\vec{\mathbb{L}} \cdot \vec{\mathbf{J}}_\omega = \imath \vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{x}} \times \vec{\mathbf{J}}_\omega)$$

Recall

$$\vec{\mathbf{m}} = \frac{1}{2} \int (\vec{\mathbf{x}} \times \vec{\mathbf{J}})\, \mathrm{d}^3 x$$

Additionally,

$$\vec{\mathbb{L}} \cdot \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}_\omega\right) = \imath \nabla^2 (\vec{\mathbf{x}} \cdot \vec{\mathbf{J}}_\omega) - \frac{\imath}{r} \partial_r \left(r^2 \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}}_\omega\right)$$

Now we can write

$$a_M(l,m) = \frac{k^2}{\imath \sqrt{l(l+1)}} \int Y^*_{lm}(\Omega) j_l(kr) \vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{x}} \times \vec{\mathbf{J}}_\omega)\, \mathrm{d}^3 x$$

The other coefficient is not as simple, as there will be a derivative with respect to r, so we need to do some integration by parts to make it in a nice form.

$$a_E(l,m) = \frac{\imath k}{\sqrt{l(l+1)}} \int j_l(kr) Y^*_{lm}(\Omega) \left\{\imath \nabla^2 (\vec{\mathbf{x}} \cdot \vec{\mathbf{J}}_\omega) - \frac{\imath}{r} \partial_r (r^2 \imath \omega \rho_\omega)\right\} \mathrm{d}^3 x$$

We then integrate by parts, using the fact that $\omega = ck$ and $\mathrm{d}^3 x = r^2\, \mathrm{d}r\, \mathrm{d}\Omega$:

$$a_E(l,m) = \frac{ck^2}{\sqrt{l(l+1)}} \int Y^*_{lm} \partial_r (r j_l(kr)) \rho_\omega\, \mathrm{d}^3 x + \frac{k^3}{\sqrt{l(l+1)}} \int Y^*_{lm}(\Omega) j_l(kr) (\vec{\mathbf{r}} \cdot \vec{\mathbf{J}}_\omega)\, \mathrm{d}^3 x$$

Again, these expressions are still exact. We have yet to make any approximations. Now we will expand the spherical Bessel functions and make some approximations to see what we get. Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x j_l(x)\right] = x j_{l-1}(x) - l j_l(x)$$

so

$$\frac{\mathrm{d}}{\mathrm{d}(kr)}\left[kr j_l(kr)\right] = \frac{(kr)(kr)^{l-1}}{[2(l-1)+1]!!} - \frac{l(kr)^l}{(2l+1)!!} = \frac{(l+1)}{(2l+1)!!}(kr)^l$$

Quote

"They turned out to be more real than the Real numbers" - Turgut, on the imaginary numbers

When $\frac{d}{\lambda} << 1$, we find

$$a_E(l,m) \mapsto \frac{ck^{l+2}}{\imath(2l+1)!!}\sqrt{\frac{l+1}{l}}\int \underbrace{r^l Y^*_{lm}\rho_\omega}_{Q_{lm}} \,\mathrm{d}^3x$$

where we are ignoring the term that goes like k^3. We can also expand the magnetic term:

$$a_M(l,m) \mapsto \frac{k^{3+l}}{\sqrt{l(l+1)}(2l+1)!!}\int r^l Y^*_{lm}(\vec{\mathbf{r}}\cdot\vec{\mathbf{J}}_\omega)\,\mathrm{d}^3x$$

This assumes that the source is completely specified; $\vec{\mathbf{J}}_\omega$ is known. Of course, in a real scenario, the radiation field will effect the conductor itself, and in practice there are almost no self-contained solutions. In almost all problems, you must assume a current distribution.

LECTURE 50: RELATIVITY

Friday, December 06, 2019

0.36 Relativity

The theory of relativity is formulated on the fact/observation that light moves at a finite speed an that speed is the same for all observers. We define a four-position as the regular position with an additional component $x^0 = ct$. By this definition,

$$(x^0)sr - \vec{\mathbf{x}}\cdot\vec{\mathbf{x}} = 0 = \mathrm{d}s^2$$

for light.

In general, $\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x'^\nu}\,\mathrm{d}x'^\nu$. If we transform $\mathrm{d}s^2 \mapsto a(\vec{\mathbf{v}})\,\mathrm{d}s^2$ under some shift to another inertial frame, we find that $\mathrm{d}s^2 \mapsto a(|\vec{\mathbf{v}}|)\,\mathrm{d}s^2$ and $a(|\vec{\mathbf{v}}_1|)a(|\vec{\mathbf{v}}_2|)\,\mathrm{d}s^2 = a(|\vec{\mathbf{v}}_1+\vec{\mathbf{v}}_2|)\,\mathrm{d}s^2$ so $a(|\vec{\mathbf{v}}|\to 0) = 1$, which implies $\mathrm{d}s^2$ is an invariant under Lorentz transformations.

We can write

$$\mathrm{d}s^2 = \mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\eta_{\mu\nu}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

is the Minkowski metric. This defines the Lorentz group $SO(3)$ since $\Lambda^\mu_\nu \Lambda^\sigma_\lambda \eta_{\mu\sigma} = \eta_{\nu\lambda}$.

We can also show that the most general linear transformation which preserves $\mathrm{d}s^2$ is

$$\begin{bmatrix} x'^0 \\ x'^1 \end{bmatrix} = \begin{bmatrix} \cosh(x) & -\sinh(x) \\ -\sinh(x) & \cosh(x) \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}$$

so that

$$-\frac{v}{c} = \frac{\mathrm{d}x'^1}{\mathrm{d}x'^0} = -\tanh(x)$$

which gives us the transformations

$$x'^0 = \gamma(x^0 - vx^1)$$

$$x'^1 = \gamma(x^1 - vx^0)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

How do we connect this to electrodynamics? Let's introduce a 4-vector source

$$J^\mu = (c\rho, \vec{\mathbf{J}})$$

4-vectors are geometric objects which transform like $\mathrm{d}x^\mu$ under Lorentz transforms:

$$a'^\mu = \Lambda^\mu_\lambda a^\lambda$$

We can write the 4-velocity as $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ where $c\,\mathrm{d}\tau = \mathrm{d}s$ is the proper time. In any other frame, $\mathrm{d}\tau = \sqrt{1 - \frac{v^2}{c^2}}\,\mathrm{d}t$. We can also define the 4-momentum $p^\mu = mu^\mu$. As it turns out, we can write moving charges as

$$\rho = \sum q_i \delta(\vec{\mathbf{x}} - \vec{\mathbf{v}}_i(t))$$

and

$$\vec{\mathbf{J}} = \sum q\vec{\mathbf{v}}_i \delta(\vec{\mathbf{x}} - \vec{\mathbf{v}}_i(t))$$

Recall the charge conservation law

$$\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0$$

or

$$\partial_\mu J^\mu = 0$$

We can also write a 4-potential

$$A^\mu = \left(\frac{\Phi}{c}, \vec{\mathbf{A}}\right)$$

which implies that the Lorentz gauge which we used is actually just

$$\partial_\mu A^\mu = 0$$

Recall that using this, we found the wave equations

$$\nabla^2\vec{\mathbf{A}} - \frac{1}{c^2}\partial_t\vec{\mathbf{A}} = -\mu_0\vec{\mathbf{J}}$$

and

$$\nabla^2\Phi - \frac{1}{c^2}\partial_t\Phi = -\frac{\rho}{\epsilon_0}$$

This wave operator is really

$$\nabla^2 - \frac{1}{c^2}\partial_t^2 = \partial_\mu\partial^\mu = \Box$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

so

$$\Box A^\nu = -\mu_0 J^\nu$$

If we define $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and recall that to raise and lower indices, we use

$$A_\mu \equiv \eta_{\mu\nu}A^\nu$$

$F_{\mu\nu}$ is a 2-tensor ($F'^{\mu\nu} = \Lambda^\mu_\lambda\Lambda^\nu_\sigma F^{\lambda\sigma}$). We can show that

$$F^{0i} = E^i$$

and

$$\epsilon_{kij}F^{ij} = B_k$$

Using this 2-tensor, we can show that Maxwell's equations are simply

$$\partial_\mu F^{\mu\lambda} = \mu_0 J^\lambda$$

We can define the dual of this tensor as

$$*F^{\mu\lambda} = \frac{1}{2}\epsilon^{\mu\lambda\sigma\alpha}F_{\sigma\alpha}$$

then

$$\partial_\mu * F^{\mu\lambda} = 0$$

which describes the fact that the magnetic field has no sources.

If we write

$$A^\nu = -\mu_0(\Box^{-1})J^\nu$$

we can show that the inverse of the d'Alambertian is

$$\Box^{-1} = \frac{\delta\left(t - t' - \frac{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}{c}\right)}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|} = \Theta(x^0 - x'^0)\delta((x-x')^2)$$

Finally, the Lorentz force is defined as

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}u_\nu$$

Conclusion

This concludes the lecture series taught by Dr. Osman Teoman Turgut at Carnegie Mellon University during the Fall of 2019. I hope these notes will prove useful to current and future students of the course. Please note that I am missing lectures 20 and 45, and I don't quite remember what we discussed those days, but on the bright side, it's all in Jackson. I cannot guarantee the accuracy of these notes, since, asside for the discussion of Green's Theorem, they were all typed (quickly) during class with minimal proofreading afterwards (thanks Michael Saavedra).
- Dene Hoffman