33-755 Homework 5

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1. Bell States and Teleportation

Suppose Alice and Bob share the fully-entangled state $\left|B^1\right\rangle$, and Alice is to teleport an unknown state $|\psi\rangle$ to Bob by measuring her half (b) of the entangled state along with $|\psi\rangle$.

(a) Assume that $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathcal{H}_a$. Express the state $|\Psi\rangle \equiv |\psi\rangle \otimes \left|B^1\right\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$, using the tensor product basis.

$$\begin{aligned} |\Psi\rangle &= |\psi\rangle \otimes \left|B^1\right\rangle & (1)\\ &= \alpha|0\rangle + \beta|1\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) & (2)\\ &= \frac{1}{\sqrt{2}}[\alpha|001\rangle + \alpha|010\rangle + \beta|101\rangle + \beta|110\rangle] & (3)\end{aligned}$$

(b) The same state can be expressed in the basis of Bell states $\{B^k\}$ on $\mathcal{H}_a \otimes \mathcal{H}_b$ as

$$|\Psi\rangle = \frac{1}{2}\sum_{k=0}^{3} B^k \otimes V_k|\psi\rangle \tag{4}$$

where $\{V_k\}$ is a set of unitary maps from $\mathcal{H}_a$ to $\mathcal{H}_c$ (i.e. $V_k|\psi\rangle \in \mathcal{H}_c$). Express $V_k|\psi\rangle$ in the basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_c$, for $k = 0, 1, 2, 3$.

Note that there are two ways to express $|\Psi\rangle$ on a basis of the Bell states:

$$|\Psi\rangle = (\left|B^0\right\rangle \otimes \beta|0\rangle + \alpha|1\rangle) + (\left|B^1\right\rangle \otimes \alpha|0\rangle + \beta|1\rangle) \tag{5}$$

and a second one using the antisymmetric states:

$$|\Psi\rangle = (\left|B^2\right\rangle \otimes -\beta|0\rangle + \alpha|1\rangle) + (\left|B^3\right\rangle \otimes \alpha|0\rangle - \beta|1\rangle) \tag{6}$$

Adding these together will give us twice the total state $|\Psi\rangle$, which is where the $\frac{1}{2}$ in the formula comes from:

$$\begin{aligned} |\Psi\rangle =& \frac{1}{2}\left[\left(\left|B^0\right\rangle \otimes \beta\,|0\rangle + \alpha\,|1\rangle \right) + \left(\left|B^1\right\rangle \otimes \alpha\,|0\rangle + \beta\,|1\rangle \right) \right. \\ & \left. + \left(\left|B^2\right\rangle \otimes -\beta\,|0\rangle + \alpha\,|1\rangle \right) + \left(\left|B^3\right\rangle \otimes \alpha\,|0\rangle - \beta\,|1\rangle \right) \right] \end{aligned} \tag{7}$$

By equating the right halves of the product spaces to $V_k\,|\psi\rangle$, we find:

$$V_0\,|\psi\rangle = V_0(\alpha\,|0\rangle + \beta\,|1\rangle) = \beta\,|0\rangle + \alpha\,|1\rangle \tag{8}$$

$$\Rightarrow V_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{9}$$

$$V_1\,|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{10}$$

$$\Rightarrow V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$

$$V_2\,|\psi\rangle = -\beta\,|0\rangle + \alpha\,|1\rangle \tag{12}$$

$$\Rightarrow V_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

$$V_3\,|\psi\rangle = \alpha\,|0\rangle - \beta\,|1\rangle \tag{14}$$

$$\Rightarrow V_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{15}$$

$$\tag{16}$$

(c) Alice measures the combination of $|\psi\rangle$ and b, in the basis of Bell states on $\mathcal{H}_a \otimes \mathcal{H}_b$ mentioned above, yielding a specific outcome $k(0 \leq k \leq 3)$. She then e-mails the result k to Bob, who must apply the unitary operator U_k to his half (c) of the original entangled state $\left|B^1\right\rangle$ in order to complete the teleportation process. What are these operators U_k?

When Alice measures her state and sends the result, she, in effect, puts the system into the state described above. Bob's particle is now in the state $V_k\,|\psi\rangle$, so he needs to perform the inverse of V_k to retrieve Alice's message state $|\psi\rangle$. Because the matrices are unitary, the inverse is just the transpose:

$$U_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{17}$$

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{18}$$

$$U_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{19}$$

$$U_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

(d) Check your result by transmitting the basis state $|\psi\rangle = |0\rangle$. Show that whatever bell state k Alice measures on her pair of bits in $\mathcal{H}_a \otimes \mathcal{H}_b$, Bob will obtain $|0\rangle \in \mathcal{H}_c$ after applying U_k to $|c\rangle$.

Alice will be sent a particle in a superposition of $|0\rangle$ and $|1\rangle$. The total state will therefore be

$$|\Psi\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|001\rangle + |010\rangle) \tag{21}$$

When she measures the first two particles (the ones she actually has) she will get, with equal likelihood, $|00\rangle$ or $|01\rangle$. We can rewrite this state in terms of the Bell state expansion above:

$$|\Psi\rangle = \frac{1}{2}[\left|B^0\right\rangle \otimes |1\rangle + \left|B^1\right\rangle \otimes |0\rangle + \left|B^2\right\rangle \otimes |1\rangle + \left|B^3\right\rangle \otimes |0\rangle] \tag{22}$$

Therefore, if she measures $|00\rangle$, she can tell Bob either $k = 0$ or $k = 2$, since she could be in either of those Bell states. Regardless of her choice, Bob will be able to perform the operator U_0 or U_2 on his particle, which must now be in the $|1\rangle$ state, and this operation will give him the original $|0\rangle$ state. Similarly, if she measures $|01\rangle$, she can tell him $k = 1$ or $k = 3$ and he will be able to use those operators on his particle, which must be in the $|0\rangle$ state, to get $|0\rangle$.

2. Infinite Dimensional Commutation Relation

Let two operators A and B satisfy the commutation relation $[B, A] = \imath I$. Explain why the Hilbert space must be infinite dimensional.

Suppose these operators acted on a finite Hilbert space of dimension n. We would then know that

$$\text{Tr}(AB) = \text{Tr}(BA) \tag{23}$$

so

$$\text{Tr}([B, A]) = \text{Tr}(BA - AB) = \text{Tr}(BA) - \text{Tr}(AB) = 0 \tag{24}$$

However, $\text{Tr}(\imath I) = n\imath$ for an n-dimensional space, so the space cannot have a finite dimension. In an infinite dimensional space, the operators need not have a finite trace, and therefore the trace of the commutator is meaningless ($\text{Tr}(AB) \to \infty \Rightarrow \text{Tr}([B, A]) = \infty - \infty$ is undefined), so it is perfectly reasonable to have $[B, A] = \imath I$.