Lecture 7: Representations of SU(2)

Wednesday, January 29, 2020

From last lecture, we were examining the (irreducible) representations of SU(2). We found that $-(2j+1) \leq m \leq (2j+1)$ and that the dimensionality of any representation of this form is

$$\dim(R) = 2j+1 \quad j \in \frac{\mathbb{Z}}{2}$$

$$J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle$$

$$J_z |jm\rangle = \hbar m |jm\rangle j$$

and

$$J_\pm |m\rangle = c_\pm |m \pm 1\rangle$$

where

$$c_\pm = \hbar\sqrt{(j \pm m)(j \pm m + 1)}$$

Consider the 3-dimensional representation ($j = 1$). We can write down the matrix elements of any given group element:

$$\langle m'| J_x |m\rangle = \langle m'| \frac{1}{2}(J_+ + J_-) |m\rangle = c\delta_{m',m+1} + c'\delta_{m',m-1}$$

We also discussed the unitary operator which comes from exponentiating the group elements and defined these as the Wigner matrices:

$$U(\hat{\mathbf{n}}, \theta) = e^{-\imath \frac{\hat{\mathbf{n}} \cdot \vec{\mathbf{J}}}{\hbar}\theta}$$

$$\langle jm| U(\hat{\mathbf{n}}, \theta) |jm'\rangle = D^{(j)}_{mm'}(\hat{\mathbf{n}}, \theta)$$

We also showed that

$$D^{(j)}_{m'm} |jm\rangle = |jm'\rangle$$

so

$$J^2 |jm'\rangle = J^2 D_{m'm} |jm\rangle = D^{(j)}_{m'm}\hbar^2 j(j+1) |jm\rangle = \hbar^2 j(j+1) D^{(j)}_{m'm} |jm\rangle = \hbar^2 j(j+1) |jm'\rangle$$

The Wigner matrices form an irreducible representation of SU(2):

$$D^{(j)}_{mm'}(R_1) D^{(j)}_{m'm''}(R_2) = D^{(j)}_{mm''}(R_1 R_2)$$

0.0.1 Euler Angles

Any rotation can be written as a sum of rotations about three axes. By convention, we call the magnitudes of the rotations (α, β, γ), where the rotations are over the axes $\hat{\mathbf{z}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ again in that order. We can write the Wigner matrices in terms of Euler angles:

$$D^{(j)}_{m'm}(\alpha, \beta, \gamma) = \langle m'| e^{-\imath\alpha J_z/\hbar} e^{-\imath\beta J_y/\hbar} e^{-\imath\gamma J_z/\hbar} |m\rangle = e^{-\imath(\alpha m' + \gamma m)} \underbrace{\langle m'| e^{-\imath\beta J_y/\hbar} |m\rangle}_{d^j_{m'm}(\beta)}$$

0.1 Orbital Angular Momentum

Let's now look at the observable $\vec{\mathbf{L}}$. We can carry some similar terms over from the discussion of $\vec{\mathbf{J}}$. Eigenstates will be written as

$$|lm\rangle \quad -(2l+1) \leq m \leq 2l+1$$

$$\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}(\theta, \varphi)$$

We want to write our eigenvectors in terms of the axis of rotation $\hat{\mathbf{n}}$:

$$\langle\hat{\mathbf{n}}| L_z |lm\rangle = \hbar m \langle\hat{\mathbf{n}}|lm\rangle$$

Define

$$F_{l,m}(\theta,\varphi) = \langle\hat{\mathbf{n}}|lm\rangle$$

Consider

$$\begin{aligned}
\langle\hat{\mathbf{n}}| R_z(\delta\varphi) |lm\rangle \overbrace{\to}^{\varphi\to 0} &\langle\hat{\mathbf{n}}| \left(I - \imath\frac{L_z}{\hbar}\delta\varphi\right) |lm\rangle \\
&= \langle\hat{\mathbf{n}}|lm\rangle - \imath\frac{\delta\varphi L}{\hbar}\langle\hat{\mathbf{n}}| L_z |lm\rangle \\
\langle\theta,\varphi| R_z(\delta\varphi) |lm\rangle = & \\
\langle\theta,\varphi+\delta\varphi| \approx \langle\theta,\varphi| - \frac{\partial}{\partial\varphi}\langle\theta,\varphi|\,\delta\varphi = &
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle\hat{\mathbf{n}}|lm\rangle - \imath\frac{\delta\varphi}{\hbar}\langle\hat{\mathbf{n}}| L_z |lm\rangle &= \langle\hat{\mathbf{n}}|lm\rangle - \delta\varphi\frac{\partial}{\partial\varphi}\langle\hat{\mathbf{n}}|lm\rangle \\
\langle\hat{\mathbf{n}}| L_z |lm\rangle = \hbar m\langle\hat{\mathbf{n}}|lm\rangle &= -\imath\hbar\frac{\partial}{\partial\varphi}\langle\hat{\mathbf{n}}|lm\rangle
\end{aligned}$$

The solutions to this differential equation are the spherical harmonics:

$$F_{lm} \to Y_{lm}(\theta,\varphi) \implies -\imath\hbar\frac{\partial}{\partial\varphi}Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

However, this only clears up the φ dependence. Now we need to figure out how θ works:

$$L^2 Y_{lm} = \hbar^2 l(l+!)Y_{lm}$$

We can write

$$L_x = -\imath\hbar\left[-\sin(\varphi)\frac{\partial}{\partial\theta} - \cot(\theta)\cos(\varphi)\frac{\partial}{\partial\varphi}\right]$$

and

$$L_y = -\imath\hbar\left[\cos(\varphi)\frac{\partial}{\partial\theta} - \cot(\theta)\sin(\varphi)\frac{\partial}{\partial\varphi}\right]$$

so

$$L^2 = \left[-\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\varphi^2}\right]$$

In certain cases, the Wigner matrices are actually equivalent to the spherical harmonics. Consider $|\hat{\mathbf{n}}\rangle = D(R)|\hat{\mathbf{z}}\rangle$. If we use our Euler rotation convention, the γ rotation is about $\hat{\mathbf{z}}$, but we are acting on $|\hat{\mathbf{z}}\rangle$, so this rotation does nothing:

$$D(R)|\hat{\mathbf{z}}\rangle = D(\alpha=\varphi,\beta=\theta,0)|\hat{\mathbf{z}}\rangle$$

Let's insert the identity:

$$|\hat{\mathbf{n}}\rangle = \sum_{lm} D(R)|lm\rangle\langle lm|\hat{\mathbf{z}}\rangle$$

Next, project onto $|l'm'\rangle$:

$$\langle l'm'|\hat{\mathbf{n}}\rangle = \sum_{lm}\langle l'm'| D(R)|lm\rangle\langle lm|\hat{\mathbf{z}}\rangle$$

Rotation matrices don't change the length of the vector, so

$$\begin{aligned}
\langle l'm'|\hat{\mathbf{n}}\rangle &= \sum_{m}\langle lm'| D(R)|lm\rangle\langle lm|\hat{\mathbf{z}}\rangle \\
&= \sum_{m} D^{(l)}_{m'm}(R)\underbrace{\langle lm|\hat{\mathbf{z}}\rangle}_{Y^*_{lm}(\theta=0,\varphi)}
\end{aligned}$$

Note

$$e^{\imath L_z \varphi} |\hat{\mathbf{z}}\rangle = |\hat{\mathbf{z}}\rangle \implies L_z |\hat{\mathbf{z}}\rangle = 0 \quad \text{and} \quad L_Z |m = 0\rangle = 0$$

Therefore

$$\langle lm' | \hat{\mathbf{n}} \rangle = D^{(l)}_{m'0}(R) Y^*_{l0}(\theta = 0, \varphi) = Y^*_{lm'}(\theta, \varphi)$$

We already know the φ-dependence:

Aside

The professor is not implying anything by raising the l-index (no Condon-Shortley phase)

$$L_z Y^l_m = \hbar m Y^l_m = -\imath\hbar \frac{\partial}{\partial\varphi} Y^l_m \implies Y^l_m \sim e^{\imath m \varphi} F(\theta)$$

Therefore

$$Y^*_{l0}(\theta = 0, \varphi) = Y^*_{l0}(\theta = 0, \varphi = 0) = \text{const.}$$

since all the φ-dependence only happens when $m \neq 0$.

For homework, we will show that

$$Y^l_0(0, 0) = \sqrt{\frac{2l+1}{4\pi}}$$

Finally, this means that

$$Y^*_{lm'}(\theta, \varphi) = D_{m'0}(\alpha = \varphi, \beta = \theta, \gamma = 0) \sqrt{\frac{2l+1}{4\pi}}$$

We have found that

$$D^{(l)}_{m'0}(\varphi, \theta, 0) = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm'}(\theta, \varphi)$$