Particle Physics

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Contents

Lecture 1: Particle Physics
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0.1 Some Basics

The fine-structure constant—the fundamental coupling of the EM field to the electron:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$$

The length scale we will often use is the Fermi: $10^{-15}\text{m} = 1\text{fm}$. We also use barns for cross sections: $1\text{b} = 100\text{fm}^2$, but typically we use millibarns and smaller.

In natural units, $\hbar = c = 1$, so $197.3\text{MeV} \cdot \text{fm} = 1$ and $\alpha = \frac{e^2}{4\pi}$. Also $1\text{Å} = 10^5\text{fm} = 5 \times 10^5\text{GeV}^{-1}$ and $1\text{b} = 100\text{fm}^2 =$ <++>

0.1.1 Fermions and Bosons

Fermions have half-integer spin and anti-symmetric wave functions under interchange of particles. Because of this, two of them can't occupy the same state (hence the Pauli exclusion principle). Bosons have integer spin and symmetric wave functions, so no exclusion principle.

The universe is made of fermions interacting by exchanging bosons (and occasional boson field excitations). The elementary fermions are the (anti)leptons (electrons, neutrinos, etc.) and the quarks. $q\bar{q}$ pairs are mesons, qqq groups are baryons, and all of these are called hadrons. The elementary bosons are (potentially) the graviton, γ, g, Z^0, $W^{\pm}$, and the Higgs.

0.1.2 Spectroscopic Notation

For some reason we still label orbital angular momentum with letters: s-wave, p-wave, d, f, g, h, ... for $L = 0$, 1, 2, ...

We also refer to $L = 0, 1, 2$ as scalar, vector, and tensor states in that order. Labeling by parity, we can also have pseudo-scalar/vector/tensor particles: 0^+, 1^-, and 2^+ are the regular versions, the switched signs are the pseudo versions.

0.1.3 Decay Width

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \int B(\omega) e^{-\imath\omega t}\, \mathrm{d}\omega$$

Physically, this implies $\Delta\omega\Delta t \geq 1/2$ or $\Delta E \delta t \geq \hbar/2$. If we happen to have a wave function $|\psi(t)|^2 \propto e^{-t/\tau}$, we say that the decay width (full width at half maximum) is $\Gamma = \frac{1}{\tau} = \frac{\hbar c}{e\tau}$ because the Fourier transform gives

$$|B(\omega)|^2 \propto \frac{(\Gamma/2)^2}{(\omega - \omega_0)^2 + (\Gamma/2)^2}$$

a Lorentzian/Breit-Wigner distribution.

0.2 Overview of the Standard Model

QED or the EM interactions involve γ coupling to charges:

[horizontal=a to b] i1 – [fermion] a – [fermion] i2, a – [photon] b, ;

The graviton theoretically couples to mass:

[horizontal=a to b] i1 – [fermion] a – [fermion] i2, a – [photon] b, ;

We don't yet have a working quantum theory of gravity.

The weak force bosons (Z^0 and $W^{\pm}$) couple to the weak charge of every known Fermion. These vertices are the same as EM, but don't necessarily conserve charge or flavor. In general for all diagrams, no vertex has more than three lines coming from it. In comparison, the weak force coupling constant is actually much larger than α, but the mass of the bosons are huge in comparison (nucleon size). The only way neutrinos can interact is by weak interactions, and that involves producing a high mass particle out of nothing but energy, so they don't interact very often. Initially, there was a B-boson coupled to hypercharge and three W-bosons coupled to weak isospin. At some time in the early universe, the electroweak symmetry was spontaneously broken, so the first two W-bosons formed the $W^{\pm}$-bosons in a linear combination, and the γ and Z^0 bosons were formed from the final two weak force bosons in a transformation involving the Weinberg angle/weak mixing angle. Experimentally, $\sin^2(\theta_W) = 0.22290 \pm 0.00030$.

LECTURE 2:
Wednesday, September 02, 2020

The photon couples only to charge, which is a linear combination of $T_3 = \frac{1}{2}Y_W = q$. The photon will not couple to the Higgs boson.

Spontaneous symmetry breaking results in massless Goldstone bosons, but $m_W > 80$GeV and $m_{Z^0} > 91$GeV. The Higgs field was zero at the beginning of the universe, but the field has a sombrero potential, so when it left the top of the potential, it entered a state where there were a finite number of Higgs bosons, giving the $W^\pm$ and Z^0 their mass while m_γ remains zero.

W bosons can change quark flavors.

0.3 The Strong Force

The gluon (g) couples to the "color" charge, the QCD charge carried by quarks and gluons. The nuclear force is a residual interaction between colorless hadrons because they are made of colored objects. The Van der Waals force is to the electromagnetic force as the nuclear force is to the strong force. The nuclear force is responsible for the energy of fusion, fission, and radioactive decay.

0.4 Dark Matter and Dark Energy

We don't know.

0.5 Quarks

Quarks are never seen alone. The strong charge seems to be confined (although we can't prove it), so only colorless objects can exist. Colorless means rgb combinations or $r\bar{r}$, $g\bar{g}$, or $b\bar{b}$ (the anti-colors are often called cyan ($\bar{r}$), magenta ($\bar{g}$), and yellow ($\bar{b}$)).

The first "generation" of these particles are the up and down quarks, the electron, and electron neutrino.

> "Who ordered that?"
>
> —Rabi, on the muon

The second generation of particles includes the charm and strange quarks and the muon and its corresponding neutrino.

The third generation includes the top and bottom quarks, the tau, and the tau neutrino.

The Z^0 decay width has been precisely measured by LEP and can be explained by calculable partial widths of all known quark-antiquark, lepton-antilepton, gamma-gamma, and neutrino-antineutrino decays. Of course, the neutrinos were originally thought to be massless, but just adding the other contributions leaves some extra stuff which must be accounted for by the neutrinos.

There's no pattern in the distribution of the quark masses (probably).

Pro Tip

Find a pattern here for a free Nobel Prize$^{\text{TM}}$.

Besides the usual baryons and mesons, there are other possible hadrons with more quarks (pentaquarks and so on), as well as hybrid states (states with "active glue") and glueballs (states made of only gluons)—plus a sea of virtual gluons and virtual quark pairs (these states just describe the valence quarks).

There are then a ton of different particles with different quark contents (and even some with the same quark contents but different internal structure i.e. wave functions).

The Standard Model includes QCD, which works well in perturbative expansions at high energy (the tree-level diagrams have the highest contribution). Unfortunately, on the other end, predictions of hadron properties require calculations of quark interactions at low energy, where the coupling constant ~ 1. This means perturbative expansions don't work well because they don't necessarily (and usually don't at all) converge.

LECTURE 3:
Friday, September 04, 2020

Simulations support early qualitative arguments that widely-separated quark pairs are connected by a gluonic "flux tube". The force between the quarks is constant and the energy stored increases apparently without end as you stretch the two quarks apart until there's enough energy to make another quark-antiquark pair out of the vacuum. If a quark is hit hard, a jet of hadrons is observed.

0.6 Weak Interaction of Constituent Fermions

β-decay: $n \to p + e^- + \bar{\nu}_e$ or (in nucleus) $p \to n + e^+ + \nu_e$.

In 1956, Lee and Yang realized that parity conservation hadn't been tested for the Weak force but had been demonstrated for the Strong and EM forces. Chien-Shiung and Wu discovered that there was an observable asymmetry consistent with ν_e being all left-handed, meaning e^+ was mostly right-handed. The same correlation was found for the antiparticles, so this first fact violated parity conservation while the second violated charge conjugation. Furthermore, it was thought that CP was conserved until Conin and Fitch showed in 1964 that K^0 decays can violate CP. Later there was CP violation in B decays and in neutrinos.

W-exchange flips particles within weak-isospin doublets (leptons and their neutrinos, quarks within generations).

For leptons and quarks, Z^0 exchange gives the same particle. Most importantly, there are no flavor-changing neutral currents. For leptons, the W bosons still cant change flavor, only charge.

We can construct a unitary matrix called the CKM matrix which predicts the probabilities of flavor change, but the values are empirical. The matrix really converts u, d, s states to u', d', s' states, which are linear combinations of the former. Notably $s \to u$ is suprressed by a factor of $|V_{us}|^2 \sim 0.05$, which explains why s quarks decay so slowly.

0.7 Neutrinos

> "Neutrinos have no mass so we're done" —Quinn

Ray Davis, in the 60–80's, did some experiments to measure chlorine decays (to argon, by getting hit by neutrinos). He kept seeing less argon than was predicted. Additional experiments showed that non-electron-neutrinos coming were coming from the sun, and other flavored neutrinos had conflicting fluxes on either side of the earth. Eventually, the idea that neutrinos can oscillate was born. Based on these results, a unitary matrix can be written to describe a change of base to mass eigenstates rather than the weak flavor eigenstates of neutrinos. It can be shown that the neutrino mass-squared differences can be calculated, but measurements are not precise enough to measure the mass of an individual state.

Lecture 4: Interaction of Radiation with Matter

Wednesday, September 09, 2020

0.7.1 Ionization

Also known as "why radiation is dangerous". If a charged particle passes through matter, it will ionize nearby atoms as they are passed. The ionization will take energy away from the radiation. Some will go to the atom and some to the electron being attracted or repelled. The highest energy transfer in a single interaction is

$$T_{\max} = \frac{2m_e c^2 \beta^2 \gamma^2}{1 + 2\gamma m_e/m + (m_e/m)^2}$$

For heavy particles ($m >> m_e$), the rate of energy loss per unit thickness is approximated by the Bethe-Bloch formula:

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle = -kZ_R^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \frac{\ln(2) m_e c^2 \beta^2 \gamma^2 T_{\max}}{I^2} - \beta^2 - \frac{\delta(\gamma\beta)}{2} \right]$$

which is equal to

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle = -kZ_R^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \frac{\ln(2) m_e c^2 \beta^2 \gamma^2}{I^2} - \beta^2 \right]$$

if $\gamma m_e << m$. In both equations, $k = 4\pi N_A r_e^2 m_e c^2 = 0.307 \mathrm{MeV cm^2/mol}$, I is the mean ionization potential, A is the relative atomic mass, and Z_R is the Z of the radioactive particle. Z/A is usually about 1/2 for light elements, smaller for lead and nearly 1 for hydrogen.

At very high energy, this formula no longer works and radiative effects become important.

The spectrum of ionized electrons is

$$\frac{\partial^2 N}{\partial T \partial x} \sim \frac{1}{2} k Z_R^2 \frac{Z}{A} \frac{1}{\beta^2} \frac{1}{T^2}$$

for $I << T < T_{\max}$. Note the $\frac{1}{T^2}$ dependence.

0.7.2 Multiple Scattering

Particle tracks are deflected due to many small-angle Coulomb interactions off of nuclei. This gives an approximate Gaussian distribution about a straight path with a width

$$\omega_0 = \frac{13.6\mathrm{MeV}}{\beta c p} Z_R \sqrt{\frac{x}{x_0}} (1 + 0.038 \ln(x/x_0))$$

where x_0 is the radiation length of the material.

0.7.3 Bremsstrahulng

An electron (or another high energy lepton or pion) can emit radiation as the particle is accelerated in the near-field of another particle.

0.7.4 Photon Radiation

At low energy (less than 1MeV), this is dominated by ionization, Raleigh scattering, and Compton scattering. At medium energy, Compton dominates, and at high energy (> 100MeV), this is dominated by pair-production in the field of the nucleus (also in the field of atomic electrons). The mean free path for pair production is given by $\sim \frac{9}{7}x_0$. For cosmically high energies ($> 10^{11}$GeV), hadronic processes (photo-nuclear effects) dominate.

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Friday, September 11, 2020

0.7.5 Electromagnetic Cascades

If a high energy photon, electron, positron, (or really high energy muon) enters a thick material, it may initiate an cascade of radiation (or shower).

0.7.6 Cherenkov Radiation

This is emitted if a particle travels faster than the speed of light in a material (this is perfectly allowed and predicted by even Classical EM). A handy way to remember the angle of the radiation is:

$$\cos(\theta_c) = \frac{c/n}{\beta c} = \frac{1}{n\beta}$$

Additionally, the number of photons per unit distance is given by

$$\frac{\partial^2 N}{\partial E \partial x} = \frac{\alpha Z_R^2}{\hbar c} \sin^2(\theta_c) \propto 1 - \frac{1}{\beta^2 n^2(E)}$$

or

$$\frac{\partial^2 N}{\partial \lambda \partial x} = \frac{2\pi\alpha Z_R^2}{\lambda^2}\left(1 - \frac{1}{\beta^2 n^2(\lambda)}\right)$$

Because of this dependence, Cherenkov radiation tends to be UV or blue.

0.7.7 Transition Radiation

Charged particles crossing a transition between materials with different indices of refraction will radiate. For very high γ and many layers, the emissions can be x-rays.

0.7.8 Hadronic Showers

If a hadron enters a thick material, it may initiate a hadronic shower. The scale for these interactions is set by the nuclear interaction length. The probability for a hadron to travel a distance x without a nuclear interaction is

$$\Pr(x) = e^{-(x/\lambda_{\text{nuc}})}$$

where

$$\lambda_{\text{nuc}} = \frac{1}{\rho_A \sigma_{\text{nuc}}} = \frac{A}{N_A \rho \sigma_{\text{nuc}}}$$

where ρ_A is the atomic density.

0.8 Experiment Triggers

Experiments require triggers to select events of interest from the huge amount of interactions.

- First-level triggers use fast detectors, hardware discriminators, and fast hardware logic
- Second-level triggers may add information from slower devices and use digitized results from fast-readout electronics to make more sophisticated decisions
- Higher-level triggers that decide what data gets written to the disk

0.9 Detectors

(A) Scintillators collect ionization energy and convert it into light through some intrinsic property of a material or a material mixed with fluorescent particles.

Light collection usually uses total internal reflection to lead light into a photo-detector.

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Monday, September 14, 2020

(B) Geiger Counters take advantage of the ionized electron field creating a "gas avalanche" of electrons by accelerating those electrons with a powered field.

(C) Tracking Detectors come in various forms. Photographic emulsion (film) was the first form of this detector. Henri Becquerel discovered that pitch blend sitting on photographic plates exposed the plate, despite the black paper protecting it. The next step in these detectors was to make stacks of these films. There's no triggering, so multiple tracks are superimposed, but the tracking itself is very good, high resolution, and good $\mathrm{d}E\,/\,\mathrm{d}x$ measurement. High-altitude balloon flights measured cosmic rays and discovered the meson (mountaintop experiments).

Bubble/Cloud/Spark Chambers identify ionized trails by the instabilities they caused. Cloud chambers use super-saturated vapor, bubble chambers have a super-heated liquid, and spark chambers use gas in an electric field near break-down. External stereo-cameras could be triggered to photograph interesting events.

Wire Chambers use thin (gold-plated tungsten) wires. The early version, a multiwire proportional chamber, could tell which wire was near the track and the energy deposited.

Drift Chambers use drift time of electrons relative to a start time from an external scintillator to determine how far the track was from the sense wire.

Micro-Pattern Detectors are conductors coated in insulators dotted with small holes. The detector works by sensing the large electric field generated when a particle goes through a hole.

Time Projection Chambers involve a 2D active surface (like a Gas Electron Multiplier) and a volume wrapped in a field cage (low electric field). In low-rate environments, this is useful because the 2D surface can detect ionized particles which fall down and reconstruct the 3D path based on how long it took for them to fall.

(Micro)Strip detectors are usually made of silicon or germanium. These are just reverse biased diodes which don't conduct unless something knocks charge into the conduction band. Beyond this, you can also make pixel detectors rather than strips. More electronics means more cost, and there's no gain, so they are very sensitive to noise.

(D) Cherenkov Detectors are often used in particle identification, In Threshold Cherenkov Detectors, lighter particles emit radiation while heavier ones don't. Another form of detector works by having lighter particles emitting a narrower cone of light which doesn't match the correct angle for total internal refraction in the detector. Finally, there are Ring Imaging Cherenkov detectors, where one or two radiators and perhaps some optics and position sensing light detectors measure speeds event by even from the ring diameter of the radiation.

(E) Electro-Magnetic Calorimeters measure the amount of energy deposited by hadrons from electron/photon showers.

(F) Hadron Calorimeters try to take advantage of hadron showers. They typically can't actually stop the particles, so you alternate scintillators with "absorbers" which initiate EM showers.

LECTURE 7: THE EFFECT OF A MAGNETIC FIELD
Wednesday, September 16, 2020

A particle of charge q and momentum $\vec{\mathbf{p}}$ in a uniform magnetic field $\vec{\mathbf{B}}$ moves in an arc of radius $R = \left|\frac{p_\perp}{Bq}\right|$.

Magnetic "optics" are possible because of magnetic quadrupoles. A doublet of these quadrupoles can be designed to focus a beam in both directions.

LECTURE 9:
Monday, September 21, 2020

In the last lecture, we mentioned that if a system of N particles was (or will be, or could be) a single particle, the invariant mass, M_I is the rest mass of that particle. For example, the Higgs is seen as an enhancement at $M_I 125\text{GeV}$ in decays of $Z^0 Z^0$, $W^+ W^-$, $\bar{q}q$, $\tau^+\tau^-$, $\mu^+\mu^-$ and $\gamma\gamma$.

Consider the reaction $B + T \to C + D + E + \cdots + Z$ where B is the beam and T is the target.

A common exercise is to find the lab-frame threshold energy for a particular reaction. The way to do this is to find the minimum invariant mass of the final state in the center of mass frame:

$$M_{I,\text{min}} = M_C + M_D + M_E + \cdots + M_Z$$

if they are all at rest. This is okay in the COM frame, but not in the lab frame, since the beam brought in momentum so the momentum of the final state can't be zero.

$$\begin{aligned} M_I^2 &= E^2_{\text{tot, lab}} - p^2_{\text{tot, lab}} \\ &= (E_B + E_t)^2 - (\vec{\mathbf{p}}_B + \cancelto{0}{\vec{\mathbf{p}}_T}) \\ &= E_B^2 + 2M_T E_B + M_T^2 - p_B^2 \\ &= M_B^2 + 2M_T E_B + M_T^2 \end{aligned}$$

0.10 Four-Derivatives

For a boost along z,

$$z = \gamma z' + \gamma\beta ct'$$

$$ct = \gamma ct' + \gamma\beta z'$$

$$\frac{\partial}{\partial z'} = \left(\frac{\partial z}{\partial z'}\right)\frac{\partial}{\partial z}\bigg|_{t,x,y} + \left(\frac{\partial ct}{\partial z'}\right)\frac{\partial}{\partial ct} = \gamma\frac{\partial}{\partial z} + \gamma\beta\frac{\partial}{\partial ct}$$

and

$$\frac{\partial}{\partial ct'} = \frac{\partial z}{\partial ct'}\frac{\partial}{\partial z} + \frac{\partial ct}{\partial ct'}\frac{\partial}{\partial ct} = \gamma\beta\frac{\partial}{\partial z} + \gamma\frac{\partial}{\partial ct}$$

$$\begin{pmatrix} \frac{\partial}{\partial ct'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial ct} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

This transforms as a covariant 4-vector, so we write it as $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial ct}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$. We can also define the contravariant derivative $\partial^\mu = g^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right)$.

0.10.1 Generalization of Laplacian

$$\Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We want to generalize this to the d'Alembertion:

$$\Box = \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \partial^\mu \partial_\mu$$

0.11 Mandelstam Variables

For an s-channel interaction, $s \equiv q^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2$.

For a t-channel interaction, $t \equiv q^2 = (p_1 - p_3)^2 = (p_2 - p_4)^2$.

For a u-channel interaction, $u \equiv q^2 = (p_1 - p_4)^2 = (p_2 - p_3)^2$. The only difference between this and the t-channel is that we make a choice of which particle is p_3 and p_4. Usually this definition is based on which particles are forward/backward-peaked in the center of mass frame.

The textbook says that u-channels only apply when there are identical particles in the final state. This doesn't seem to be correct. For example, in high-energy neutrino-electron scattering, if the exchange particle is a W boson, this is a t-channel reaction which is distinct from the exchange of a Z boson in a u-channel.

Lecture 10: Non-Relativistic Quantum Mechanics
Wednesday, September 23, 2020

0.12 Non-Relativistic Quantum Mechanics

Take a plane wave $\psi(x,t) = Ae^{\imath c \vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t}$. Any wave function can be written as a linear combination of plane waves. DeBroglie postulated that $\vec{\mathbf{p}} = \hbar \vec{\mathbf{k}}$ and $E = \hbar\omega$ for plane waves.

$$P_\gamma = \frac{E}{c} = \frac{h\frac{c}{\lambda}}{c} = \frac{2\pi\hbar}{\lambda} = \hbar k$$

$$E_\gamma = \hbar\nu = 2\pi\hbar\nu = \hbar\omega$$

$$v_G = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}(E/\hbar)}{\mathrm{d}p/\hbar} = \frac{\mathrm{d}}{\mathrm{d}p}\frac{p^2}{2m} = \frac{p}{m}$$

Each observable A is associated with an operator $\hat{A}$, meaning the possible values of that observable are the eigenvalues of this operator:

$$\hat{A}\psi_a = a\psi_a$$

For real observables, $\hat{A}$ is Hermitian: $\hat{A}^\dagger = \hat{A}$. The plane wave suggests that $\hat{\vec{\mathbf{p}}} = -\imath\vec{\boldsymbol{\nabla}}$ and $\hat{H} = \imath\partial_t$. We assume the Hamiltonian is the energy operator since that's how it works classically. The time evolution of a wave function is given by the Schrödinger equation:

$$\imath\hbar\partial_t\psi(\vec{\mathbf{r}}, t) = \hat{H}\psi(\vec{\mathbf{r}}, t)$$

For a single particle,

$$\hat{H} = \frac{\hat{\vec{\mathbf{p}}}^2}{2m} + \hat{V}(\vec{\mathbf{r}}) = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

Attempting to find separable solutions of the form $\psi(\vec{\mathbf{r}}, t) = \psi(\vec{\mathbf{r}})\varphi(t)$ gives $\imath\hbar\frac{1}{\varphi(t)}\frac{\partial}{\partial t}\varphi(t) = \frac{1}{\psi(\vec{\mathbf{r}})}\hat{H}\psi(\vec{\mathbf{r}})$. Solving this, we find $\varphi = Ae^{-\imath Et/\hbar}$. This give us the time-independent Schrödinger equation:

$$\hat{H}\psi(\vec{\mathbf{r}}) = E\psi(r)$$

Solving this gives the allowed wave functions and energies.

Any wave function at $t = 0$ can be written as a linear combination of energy eigenstates. The time-evolution is know to be

$$a(\vec{\mathbf{r}}, t) = \sum_i w_i\psi_i(\vec{\mathbf{r}})e^{-\imath E_i t/\hbar}$$

Including electromagnetism transforms $\vec{\mathbf{p}} \to -im\hbar\vec{\boldsymbol{\nabla}} - e\vec{\mathbf{A}}$.

Lecture 11: Probability Density and Current

Friday, September 25, 2020

The probability density of finding a particle at a specific place and time is $|\psi|^2 = \psi^*\psi \equiv \rho(\vec{\mathbf{r}}, t)$. If ψ is normalized, $\int \rho\,\mathrm{d}^3r = 1$, or for multidimensional wave functions, we use the more general Dirac notation:

$$|\psi\rangle^\dagger |\psi\rangle \equiv \langle\psi|\psi\rangle = \int \rho\,\mathrm{d}^3r$$

Any conserved quantity must obey the continuity equation:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} + \frac{\partial\rho}{\partial t} = 0$$

where $\vec{\mathbf{j}}$ is current density (charge flow per unit area per unit time)

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = -\int_S \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}}$$

where S is a 2D surface which encloses the volume V and $\mathrm{d}\vec{\mathbf{s}}$ is chosen to point outward from the surface.

$$\int_V \frac{\mathrm{d}}{\mathrm{d}t}\rho\,\mathrm{d}V = -\int_V \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}\,\mathrm{d}V$$

by the divergence theorem. Of course, this is true for any V, so just remove those integrals! □

To identify $\vec{\mathbf{j}}$, take the free-space Schrödinger equation and multiply by ψ^*:

$$\psi^* \times \left(-\frac{1}{2m}\nabla^2\psi = \imath\frac{\partial\psi}{\partial t} \right)$$

$$\psi \times \left(-\frac{1}{2m}\nabla^2\psi^* = -\imath\frac{\partial\psi^*}{\partial t} \right)$$

Subtracting these equations, we get

$$\begin{aligned} -\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) &= \imath\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) \\ -\frac{1}{2m}\left(\vec{\boldsymbol{\nabla}}\cdot\left(\psi^*\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\psi^*\right)\right) &= \imath\frac{\partial\rho}{\partial t} \\ \frac{\partial\rho}{\partial t} = -\vec{\boldsymbol{\nabla}}\cdot\left(\frac{\imath\hbar}{2m}(\psi^*\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\psi^*)\right) &= -\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{j}} \end{aligned}$$

so

$$\vec{\mathbf{j}} = \frac{\imath\hbar}{2m}\left(\psi^*\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\psi^*\right)$$

For a plane wave in volume V,

$$\psi(\vec{\mathbf{r}}, t) = \frac{1}{\sqrt{V}}e^{\imath(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}$$

where $\hbar\vec{\mathbf{k}} = \vec{\mathbf{p}}$ and $\hbar\omega = E$.

Inside the volume, $\rho = \frac{1}{V}$ throughout the volume.

The current density will be

$$\begin{aligned} \vec{\mathbf{j}} &= -\frac{\imath\hbar}{2mV}\left(e^{-(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}(\imath\vec{\mathbf{k}})e^{\imath(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)} - e^{\imath(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)(-\imath\vec{\mathbf{k}})}e^{-\imath(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}\right) \\ &= -\frac{\imath\hbar}{2mV}\left(2\imath\vec{\mathbf{k}}\right) \\ &= \frac{\hbar\vec{\mathbf{k}}}{mV} = \frac{\vec{\mathbf{p}}}{m}\frac{1}{V} \\ &= \rho\vec{\mathbf{v}} \end{aligned}$$

0.12.1 Time-Dependence

If $\hat{A}\left|a_i\right\rangle = a_i\left|a_i\right\rangle$, choose an orthonormal set of $\left|a_i\right\rangle$. Then

$$\left|\psi\right\rangle = \sum_i \omega_i\left|a_i\right\rangle$$

and if $\left|\psi\right\rangle$ is normalized, $\left|\omega_i\right|^2$ is the probability of finding the system in the state $\left|a_i\right\rangle$ with $a = a_i$ if you measure $\hat{A}$, so

$$\left\langle\psi\middle|\psi\right\rangle = \sum_i\left|\omega_i\right|^2$$

The average value measured for a is $\bar{a} = \sum_i\left|\omega_i\right|^2 a_i \equiv \left\langle\psi\middle|\hat{A}\middle|\psi\right\rangle \equiv \left\langle\hat{A}\right\rangle$.

$$\hat{H}\left|\psi\right\rangle = \hbar\imath\partial_t\left|\psi\right\rangle \implies \left\langle\psi\right|\hat{H} = -\hbar\imath\partial_t\left\langle\psi\right|$$

and

LECTURE 12:
Monday, September 28, 2020

$$\partial_t \left\langle \hat{A} \right\rangle = \langle\psi|\frac{\hat{H}}{-\hbar\imath}\hat{A}|\psi\rangle + \langle\psi|\hat{A}\frac{\hat{H}}{\hbar\imath}|\psi\rangle = \frac{\imath}{\hbar}\left\langle \left[\hat{H},\hat{A}\right]\right\rangle$$

so if $\hat{A}$ commutes with $\hat{H}$, then $\left\langle \hat{A} \right\rangle$ is a constant, even if $|\psi\rangle$ isn't an eigenstate.

If $|\psi\rangle$ is an eigenstate of $\hat{H}$ and $\hat{B}$ does not commute with $\hat{H}$, then

$$\partial_t \left\langle \hat{B} \right\rangle = \frac{\imath}{\hbar}\left\langle \left[\hat{H},\hat{B}\right]\right\rangle = \imath\left|\left\langle E\hat{B} - \hat{B}E\right\rangle\right| = E\left\langle \hat{B} - \hat{B}\right\rangle = 0$$

If $|\varphi_0\rangle$ is not an eigenfunction of $\hat{H}$ or $\hat{B}$, then

$$|\varphi_0\rangle = \sum_i c_i\,|\psi_i\rangle$$

so

$$|\varphi(t)\rangle = \sum_i c_i\,|\psi_i\rangle\, e^{-\imath E_i t}$$

The probability of being in the state b_k is

$$\begin{aligned}
\Pr(b_k) &= |\langle b_k|\varphi(t)\rangle|^2 \\
&= \left|\sum_i c_i\,\langle b_k|\psi_i\rangle\, e^{-\imath E_i t}\right|^2 \\
&= \left|\sum_i c_i b_{ik} e^{-\imath E_i t}\right|^2
\end{aligned}$$

The probability of measuring b_k undergoes a complicated evolution as the relative phases of the energy eigenstates causes them to "beat" against each other.

Example. In neutrino oscillation, there are (as far as we know) three energy eigenstates and three flavor eigenstates. The flavor states beating against each other are the cause for oscillations. ◇

0.12.2 Compatible Observables

If $\left[\hat{A},\hat{B}\right] = 0$, for a non-degenerate eigenstate $|\varphi\rangle$ of $\hat{A}$,

$$\hat{A}(\hat{B}\,|\varphi\rangle) = \hat{B}\hat{A}\,|\varphi\rangle\,\hat{B}a\,|\varphi\rangle = a(\hat{B}\,|\varphi\rangle)$$

so $\hat{B}\,|\varphi\rangle$ is an eigenstate of $\hat{A}$, so $\hat{B}\,|\varphi\rangle = b\,|\varphi\rangle$ so $|\varphi\rangle$ is a simultaneous eigenstate of $\hat{A}$ and $\hat{B}$.

If $|\varphi_i\rangle$ are degenerate eigenstates,

$$\hat{A}\,|\varphi_i\rangle = a\,|\varphi_i\rangle$$

then

$$\hat{A}(\hat{B}\,|\varphi_i\rangle) = \hat{B}\hat{A}\,|\varphi_i\rangle = a(\hat{B}\,|\varphi_i\rangle)$$

so $\hat{B}\,|\varphi_i\rangle$ is an eigenstate of $\hat{A}$. Therefore, it must be a linear combination of $|\varphi_i\rangle$'s:

$$\hat{B}\,|\varphi_i\rangle = \sum_{j=1}^{k} c_{ij}\,|\varphi_i\rangle$$

We can diagonalize c_{ij}, so $\hat{B}\,|\varphi_l'\rangle = b_l\,|\varphi_l'\rangle$, where $|\varphi_l'\rangle$ are linear combinations of $|\varphi_j\rangle$ and are eigenstates of both $\hat{A}$ and $\hat{B}$.

If $\hat{C}$ commutes with $\hat{A}$ and $\hat{B}$, there exists a linear combination of $\hat{A}$ eigenstates which are also $\hat{B}$ and $\hat{C}$ eigenstates. The states can be labeled by their simultaneous eigenvalues: $|a_i, b_k, c_j\rangle$ and so on, for other commuting operators. If $\hat{A}$ and $\hat{B}$ don't commute, there is no such linear combination, and so they do not share simultaneous eigenstates.

Definition 0.12.1.

$$\Delta a \equiv \sqrt{\left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2}$$

It can be shown that $\Delta a \Delta b \geq \frac{1}{2}\left|\left[\hat{A}, \hat{B}\right]\right|^2$.

Example.

$$[\hat{x}, \hat{p}] = \imath\hbar \implies \Delta x \Delta p \geq \frac{\hbar}{2}$$

◇

Lecture 13:
Wednesday, September 30, 2020

0.13 Orbital Angular Momentum

Classically, $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$, so quantum mechanically, one would assume $\hat{L} = \hat{r} \times \hat{p}$. Therefore, $\vec{\mathbf{L}}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$ and so on.

We already saw that $[\hat{x}, \hat{p}_x] = \imath\hbar$ and $[\hat{x}, \hat{p}_y] = \cdots = 0$ (the off-diagonals commute). It's trivial to derive the commutation relations for angular momentum:

$$[L_i, L_j] = \imath\hbar\epsilon_{ijk}L_k$$

This same relation applies to other angular-momentum-like operators (operators with Lie algebras) like $\hat{\vec{\mathbf{S}}}$, $\hat{\vec{\mathbf{J}}}$ and $\hat{\vec{\mathbf{I}}}$.

If we define the Casimir operator $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, $\left[\hat{L}^2, \hat{L}_i\right] = 0\ \forall i$. We can choose to work in a basis of simultaneous eigenstates of $\hat{L}^2$ and any (single) projection. The typical choice of projection is $\hat{L}_z$. We can then choose eigenstates labeled $|\lambda, m\rangle$ where

$$\hat{L}^2 |\lambda, m\rangle = (\hbar)\lambda |\lambda, m\rangle$$
$$\hat{L}_z |\lambda, m\rangle = (\hbar)m |\lambda, m\rangle$$

We can define raising and lowering operators $\hat{L}_\pm = \hat{L}_x \pm \imath\hat{L}_y$ which commute with $\hat{L}^2$.

$$\left[\hat{L}_z, \hat{L}_\pm\right] = \pm(\hat{L}_x \pm \imath\hat{L}_y) = \pm\hat{L}_\pm$$

Additionally (homework), $\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z + \hat{L}_z^2$. Therefore

$$\hat{L}^2(\hat{L}_\pm |\lambda, m\rangle) = \hat{L}_\pm(\hat{L}^2 |\lambda, m\rangle) = \lambda\hat{L}_\pm |\lambda, m\rangle$$

so $\hat{L}_\pm |\lambda, m\rangle$ is still an eigenstate of $\hat{L}^2$ with eigenvalue λ. We can then use the commutation relation to show that

$$\hat{L}_z\hat{L}_+ = \hat{L}_+\hat{L}_z \pm \hat{L}_\pm$$

so

$$\hat{L}_z(\hat{L}_+ |\lambda, m\rangle) = (\hat{L}_+\hat{L}_z + \hat{L}_+) |\lambda, m\rangle = (\hat{L}_+ m + \hat{L}_+) |\lambda, m\rangle = (m+1)(\hat{L}_+ |\lambda, m\rangle)$$

so we can conclude that $\hat{L}_+ |\lambda, m\rangle$ is still an eigenstate of $\hat{L}_z$, but with eigenvalue $m+1$ unless $\hat{L}_+ |\lambda, m\rangle = 0$. The same property can be shown with $\hat{L}_-$.

Let l be the max value of m which cannot be raised. Then

$$\hat{L}^2 |\lambda, l\rangle = (\hbar^2) l(l+1) |\lambda, l\rangle$$

so $\lambda = l(l+1)$. Therefore, we will change our notation to $|l, m\rangle$ where $l(l+1)$ is the $\hat{L}^2$ eigenvalue of $|l, m\rangle$. By the lowering operator relation, there exists a minimum $m_{\text{min}} = -l$, so $2l$ must be an integer so that m can step between $+l$ and $-l$.

We can now ask what the raising and lowering operators actually do to the states. We know they increase or decrease m, but in general

$$\hat{L}_{\pm} |l, m\rangle = a^{\pm}_{l,m} |l, m \pm 1\rangle$$

To find $a_{l,m}$, take the Hermitian conjugate and find the inner product:

$$a^{\pm}_{l,m} = (\hbar)\sqrt{l(l+1) - m(m \pm 1)}$$

Lecture 14: Fermi's Golden Rule

Friday, October 02, 2020

0.14 Fermi's Golden Rule

Once you know the matrix elements for a transition, how can you determine the rate of the transition? Let $|k\rangle$ be an orthonormal eigenbasis of an unperturbed Hamiltonian (no transitions should happen):

$$\hat{H}_0 |k\rangle = E_k |k\rangle$$

If we add a perturbation $\hat{H} = \hat{H}_0 + \hat{H}'$, the Schrödinger equation tells us

$$\imath\hbar\partial_t |\psi\rangle = (\hat{H}_0 + \hat{H}') |\psi\rangle$$

where $|\psi\rangle = \sum_k c_k(t) |k\rangle e^{-\imath E_k t}$. The coefficients are time-dependent because the states are not necessarily eigenstates of the perturbed Hamiltonian. Therefore,

$$\imath\hbar \sum_k \frac{\mathrm{d}c_k}{\mathrm{d}t} |k\rangle e^{-\imath E_k t} + \sum_k E_k c_k |k\rangle e^{-\imath E_k t} = \sum_k c_k E_k |k\rangle e^{-\imath E_k t} + \sum_k H' c_k |k\rangle e^{-\imath E_k t}$$
$$\imath\hbar \sum_k \frac{\mathrm{d}c_k}{\mathrm{d}t} |k\rangle e^{-\imath E_k t} = \sum_k H' c_k |k\rangle e^{-\imath E_k t}$$

Suppose the initial state is $|\psi\rangle = |k\rangle$ with $k = i$. Then $c_k(0) = \delta_{ik}$. As a first approximation, assume that $c_i(t) \approx 1$ and $c_{k\neq i}(t)$ is negligible:

$$\hbar\imath \sum_i \frac{\mathrm{d}c_k}{\mathrm{d}t} |k\rangle e^{-\imath E_k t} = H' |i\rangle e^{-\imath E_k t}$$

Multiply both sides by some other energy eigenstate state $\langle f|$:

$$\hbar\imath \sum_i \frac{\mathrm{d}c_f}{\mathrm{d}t} e^{-\imath E_f t} = \langle f|H'|i\rangle e^{-\imath E_i t}$$

so

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} = \frac{-\imath}{\hbar} \langle f|H'|i\rangle e^{\imath(E_f - E_i)t}$$

To first order, the transition matrix element T_{fi} is $T_{fi} = \langle f|H'|i\rangle$.

If H' turns on at $t = 0$, then at $t = T$,

$$c_f(T) = -\frac{\imath}{\hbar} \int_0^T T_{fi} e^{\imath(E_f - E_i)t} \, \mathrm{d}t$$

If H' has no explicit time dependence,

$$c_f(T) = -\frac{\imath}{\hbar} \langle f|H'|i\rangle \int_0^T e^{\imath(E_f - E_i)t}$$

Therefore

$$\Pr(i \to f) = |c_f(t)|^2 = \frac{|T_{fi}|^2}{\hbar^2} \left(\int_0^T e^{\imath(E_f-E_i)t}\,\mathrm{d}t \right) \left(\int_0^T e^{-\imath(E_f-E_i)t'}\,\mathrm{d}t' \right)$$

This makes the transition rate

$$\Gamma_{fi} = \frac{\Pr(f \to i)}{T} = \frac{4\sin^2(E_f - E_i)t/2}{T(E_f - E_i)^2} |T_{fi}|^2$$

We can extend $T \to \infty$ to show that

$$\mathrm{d}\Gamma_{fi} = 2\pi |T_{fi}|^2 \delta(E_f - E_i)$$

In a continuum of states, we need to integrate over the density of states:

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i)$$

To next order,

$$\hbar\imath \frac{\mathrm{d}c_f}{\mathrm{d}t} e^{-\imath E_f t} = -\frac{\imath}{\hbar} \sum_{k\neq i} \langle f|H'|k\rangle\, e^{-\imath E_k t} \langle k|H'|i\rangle \frac{e^{\imath(E_k - E_i)t} - 1}{\imath(E_k - E_i)}$$

Lecture 15:

Monday, October 05, 2020

$$\frac{\mathrm{d}c_f}{\mathrm{d}t} = -\frac{\imath}{\hbar} \left(\langle f|\hat{H}'|i\rangle + \sum_{k\neq i} \frac{\langle f|\hat{H}'|k\rangle \langle k|\hat{H}'|i\rangle}{E_i - E_k} \right) e^{\imath(E_f - E_i)t}$$

From here,

$$T_{fi} = \langle f|\hat{H}'|i\rangle + \sum_{k\neq i} \frac{\langle f|\hat{H}'|k\rangle \langle k|\hat{H}'|i\rangle}{E_i - E_k}$$

where

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i) \qquad \text{(Fermi's Golden Rule)}$$

0.15 Decay Rates and Cross Sections

$$\rho(E_i) = \left.\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_i} = \int \frac{\mathrm{d}n}{\mathrm{d}E} \delta(E_i - E)\,\mathrm{d}E$$

so

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\,\mathrm{d}n$$

0.15.1 Phase Space and Normalization

In the Born approximation, we treat initial and final states as momentum eigenstates

$$\psi(x,t) = Ae^{\imath(\vec{\mathbf{p}}\cdot\vec{\mathbf{r}}-Et)}$$

To normalize within a cube of side a, $A = \frac{1}{a^{3/2}}$ with periodic boundary conditions: $\psi(x+a,y,z) = \psi(x,y,z)$ or rigid/open boundaries $\psi(a,y,z) = 0$. The allowed states that match the boundary conditions will be an array of $m_i \in \mathbb{Z}$: $\vec{\mathbf{p}} = (m_x, m_y, m_z)\frac{2\pi}{a}$.

The volume of phase space for $(2N_{\max})^3$ states is $(2N_{\max})^3\left(\frac{2\pi}{a}\right)^3$, so the volume of $\vec{\mathbf{p}}$-space per state is $\frac{(2\pi)^3}{V}$. Since the components can be positive or negative, to find all states with momentum between p and $p+\mathrm{d}p$, we can count states within a spherical shell of radius p and thickness $\mathrm{d}p$:

$$\mathrm{d}n = 4\pi p^2\frac{\mathrm{d}p}{\frac{(2\pi)^3}{V}}$$

so

$$\frac{\mathrm{d}n}{\mathrm{d}p} = \frac{4\pi p^2}{(2\pi)^3}V$$

Let's make the normalization volume $V = 1$ so $a = 1$. For each independent momentum,

$$\rho(E) = \mathrm{d}n\,E = \frac{4\pi p^2}{(2\pi)^3}\left|\frac{\mathrm{d}p}{\mathrm{d}E}\right|$$

For a decay to N particles, $\vec{\mathbf{p}} = \sum_{i=1}^{N}\vec{\mathbf{p}}_i$, so one $\vec{\mathbf{p}}_i$ is not independent:

$$\vec{\mathbf{p}}_N = \vec{\mathbf{p}}_a - \sum_{i=1}^{N-1}\vec{\mathbf{p}}_i$$

The total number of states for the N particles is

$$\mathrm{d}n = \prod_{i=1}^{N-1}\mathrm{d}n_i = \prod_{i=1}^{N-1}\frac{\mathrm{d}^3p_i}{(2\pi)^3}$$

or

$$\begin{aligned}
\mathrm{d}n &= \prod_{i=1}^{N-1}\frac{\mathrm{d}^3p_i}{(2\pi)^3}\delta^3(\vec{\mathbf{p}}_a - \sum_{i=1}^{N}\vec{\mathbf{p}}_i)\,\mathrm{d}^3\vec{\mathbf{p}}_N \\
&= (2\pi)^3\prod_{i=1}^{N}\frac{\mathrm{d}^3p_i}{(2\pi)^3}\delta^3(\vec{\mathbf{p}} - \sum_{i=1}^{N}\vec{\mathbf{p}}_i)
\end{aligned}$$

0.15.2 Lorentz-Invariant Phase Space

Volume is not Lorentz invariant. The direction of motion is length contracted. To correct for this, we introduce ψ' with a different normalization. While $\langle\psi|\psi\rangle = 1$, we will have $\langle\psi'|\psi'\rangle = 2E$.

LECTURE 16:
Wednesday, October 07, 2020

We can define the Lorentz-invariant matrix elements of $\hat{H}'$ as

$$\begin{aligned}
M_{fi}^{(1)} &= \langle\psi_a',\psi_b',\ldots,\psi_z'|\hat{H}'|\psi_1',\psi_2',\ldots,\psi_n'\rangle \\
&= (2E_a2E_b\ldots2E_z\times2E_12E_2\ldots2E_n)^{1/2}\times\langle\psi_a,\ldots,\psi_z|\hat{H}|\psi_1,\ldots,\psi_n\rangle
\end{aligned}$$

For a decay $a \to 1 + 2$

$$
\begin{aligned}
\Gamma_{fi} &= 2\pi \int |T_{fi}|^2 \delta(E_a - E_1 - E_2)\, \mathrm{d}n \\
&= (2\pi)^4 \int |T_{fi}|^2 \delta(E_a - E_1 - E_2)\delta^3(\vec{\mathbf{p}}_a - \vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2) \times \frac{\mathrm{d}^3 p_1}{(2\pi)^3}\frac{\mathrm{d}^3 p_2}{(2\pi)^3} \\
&= \frac{(2\pi)^4}{2E_a} \int |M_{fi}|^2 \delta(E_a - E_1 - E_2)\delta^3(\vec{\mathbf{p}}_a - \vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2) \times \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E1}\frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2}
\end{aligned}
$$

These final two terms are Lorentz-invariant phase space for each of the final particles. To see this, consider a boost in the z-direction:

$$
\mathrm{d}^3 p' = \mathrm{d}p'_x\,\mathrm{d}p'_y\,\mathrm{d}p'_z = \mathrm{d}p_x\,\mathrm{d}p_y\,\frac{\mathrm{d}p'_z}{\mathrm{d}p_z}\,\mathrm{d}p_z = \frac{\mathrm{d}p'_z}{\mathrm{d}p_z}\,\mathrm{d}^3 p
$$

where $\frac{\mathrm{d}p'_z}{\mathrm{d}p_z} = \gamma\left(1 - \beta\frac{\mathrm{d}E}{\mathrm{d}p_z}\right)$. Since $E = \sqrt{\vec{\mathbf{p}}^2 + m^2}$, $\frac{\mathrm{d}E}{\mathrm{d}p_z} = \frac{p_z}{\sqrt{p^2+m^2}} = \frac{p_z}{E}$. Therefore, $\mathrm{d}^3 p' = \frac{E'}{E}\,\mathrm{d}^3 p$ is not Lorentz-invariant, but $\frac{\mathrm{d}^3 p'}{E'} = \frac{\mathrm{d}^3 p}{E}$ is.

$\frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}$ can be written more "naturally" as

$$
\begin{aligned}
\int \frac{\delta(E_i^2 - p_i^2 - m_i^2)}{\delta(f(x))}\,\mathrm{d}E_i &= \int \frac{\delta(E_i^2 - p_i^2 - m_i^2)}{\delta(x - x_0)/\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}}\,\mathrm{d}E_i \\
&= \int \frac{\delta(E_i - \sqrt{p_i^2 - m_i^2})}{2E_i}\,\mathrm{d}E_i
\end{aligned}
$$

so

$$
\frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i} = \int \mathrm{d}E_i\,\frac{\mathrm{d}^3 p_i}{(2\pi)^3}\delta(E_i^2 - p_i^2 - m_i^2)
$$

so

$$
\iiint \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i} f = \int \mathrm{d}^4 p_i\,\frac{\delta((p_i^\mu)^2 - m_i^2)}{(2\pi)^3} f
$$

Using this,

$$
\Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\int |M_{fi}|^2 \delta^4(p_a^\mu - p_1^\mu - p_2^\mu)\prod_{i=1}^{2}\frac{\delta(p_i^2 - m_i^2)}{(2\pi)^3}\,\mathrm{d}^4 p_i
$$

0.15.3 Decay to Multiple Final Channels

Take, for example, Λ-decay:

$$
\begin{aligned}
\Lambda &\to p\pi^- \quad 0.64 \\
&\to n\pi^0 \quad 0.36 \\
&\to n\gamma \quad 0.0017 \\
&\to p\pi^-\gamma \quad 0.0003
\end{aligned}
$$

If Γ_i is the rate of decay to channel i, then the total rate of decay is $\Gamma = \sum_i \Gamma_i$, $\tau = 1/\Gamma$, and the branching ratio of channel i, $\mathrm{BR}_i = \Gamma_i/\Gamma$.

0.15.4 Two-Body Decay

In the center of momentum,

$$
\Gamma_{fi} = \frac{1}{8\pi^2 m_a}\int |M_{fi}|^2 \delta(m_a - E_1 - E_2)\delta^3(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2)\frac{\mathrm{d}^3 p1}{2E_1}\frac{\mathrm{d}^3 p_2}{2E_2}
$$

Now we can integrate over $\vec{\mathbf{p}}_2$:

$$\Gamma_{fi} = \frac{1}{8\pi^4 m_a} \int |M_{fi}|^2 \frac{\delta(m_a - E_1 - E_2)}{4E_1 E_2} \, \mathrm{d}^3 p_1$$

Using $\mathrm{d}^3 p_1 = p_1^2 \, \mathrm{d}r_1 \, \mathrm{d}\Omega_1$,

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |M_{fi}|^2 \underbrace{\delta(m_a - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2})}_{f(p_1)} \frac{p_1^2 \, \mathrm{d}r_1 \, \mathrm{d}\Omega_1}{4E_1 E_2}$$

We know that $f(p_1) = 0$ for $p_1 = p_1^* = \frac{1}{2m_a}\sqrt{(m_a^2 - (m_1+m_2)^2)(m_a^2 - (m_1 - m_2))^2}$.

LECTURE 17: INTERACTION CROSS SECTIONS
Friday, October 09, 2020

$$f(p_1) = m_a - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}$$
$$\left.\frac{\mathrm{d}f}{\mathrm{d}p_1}\right|_{p_1^*} = p_1^* \frac{E_2 + E_1}{E_1 E_2}$$

so

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |M_{fi}|^2 \frac{\delta(p_1 - p_1^*)}{p_1^*(E_1 + E_2)} E_1 E_2 \frac{{p_1^*}^2}{4E_1 E_2} \, \mathrm{d}p_1 \, \mathrm{d}\Omega^*$$

So for any 2-body decay,

$$\Gamma_{fi} = \frac{p_1^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 \, \mathrm{d}\Omega^*$$

0.16 Interaction Cross Sections

0.16.1 Beam Hitting a Target

If we look at a beam hitting a stationary target, we say that the interaction probability is

$$f = \frac{N_{\text{atom}}}{A} \sigma_{\text{int}} = \frac{N_{\text{atom}} T}{V} \sigma_{\text{int}} = \rho_{\text{atom}} T \sigma_{\text{int}}$$

where T is the target thickness. The interaction rate is therefore

$$R = \frac{\mathrm{d}N_{\text{beam}}}{\mathrm{d}t} \rho_{\text{atom}} T \sigma_{\text{int}}$$

0.16.2 Colliding Beams

For one beam particle with speed v_A colliding with $\rho_{b,\text{atom}}$ particles per unit volume moving at speed v_b in the opposite direction, the effective target thickness "seen" by beam a in time $\mathrm{d}t$ is $\mathrm{d}T = (v_a + v_b)T$. Therefore the interaction probability is

$$f = \rho_{b,\text{atom}} (v_a + v_b) \, \mathrm{d}t \, \sigma_{\text{int}}$$

The interaction rate per beam particle is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \rho_{b,\text{atom}} (v_a + v_b) \sigma$$

For aligned beams of uniform density, if particles of both beams are confined within a length L and area A, then the interaction rate is

$$\Gamma_{fi} = (LA\rho_{a,\text{atom}}) \rho_{b,\text{atom}} (v_a + v_b) \sigma = \mathcal{L} \times \sigma$$

where $\mathcal{L}$ is the instantaneous luminosity (events/femtobarns/second).

0.16.3 Lorentz Invariant Flux

The rate in volume V is $\Gamma_{fi} = (v_a + v_b)\rho_a\rho_b\sigma V$. If $\rho_a = \frac{1}{V} = \rho_b$ (normalized to one particle per unit volume), then

$$\Gamma_{fi} = (v_a + v_b)\sigma$$

so

$$\sigma = \frac{\Gamma_{fi}}{v_a + v_b} = (2\pi)^4 \frac{1}{2E_a}\frac{1}{2E_b}\int |M_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\delta^3(\vec{\mathbf{p}}_a + \vec{\mathbf{p}}_b - \vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2)\frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2}$$

We claim that the factor $\frac{1}{(v_a+v_b)\times 2E_a \times 2E_b} \equiv \frac{1}{F}$ is Lorentz invariant:

$$F = 4E_aE_b(v_a + v_b) = 4E_aE_b\left(\frac{p_a}{E_a} + \frac{p_b}{E_b}\right) = 4(E_bP_a + E_aP_b)$$

Therefore,

$$F^2 = 16(E_b^2p_a^2 + E_b^2p_b^2 + 2E_aE_bp_ap_b)$$

Meanwhile,

$$(p_a^\mu p_b^\mu)^2 = (E_aE_b + \underbrace{p_ap_b}_{-\vec{\mathbf{p}}_a\cdot\vec{\mathbf{p}}_b})^2 = E_a^2E_b^2 + p_a^2p_b^2 + 2E_aE_bp_ap_b$$

So we can rewrite the final term:

$$\begin{aligned} F^2 &= 16(E_b^2p_a^2 + E_a^2p_b^2 + (p_a^\mu \cdot p_b^\mu)^2 - E_a^2E_b^2 - p_a^2p_b^2) \\ &= 16((p_a^\mu \cdot p_b^\mu)^2 - (E_a^2 - p_a)^2(E_b^2 - p_b^2)) \\ &= 4((p_a^\mu \cdot p_b^\mu) - m_a^2m_b^2) \end{aligned}$$

which is Lorentz invariant.

If F and Γ_{fi} are Lorentz invariant, so is σ:

LECTURE 18: CROSS SECTIONS, CONT.
Monday, October 12, 2020

Since the cross section is Lorentz invariant, we can do calculations in any frame. Typically, the center of mass frame is the most convenient. In this frame, two incoming beam particles have momentum p_i^* and they are scattered at some angle with momentum p_f^*. We can write the flux as

$$F = 4E_a^*E_b^*\left(\frac{p_i^*}{E_a^*} + \frac{p_i^*}{E_b^*}\right) = 4p_i^*(E_a^* + E_b^*) = p_i^*\sqrt{s}$$

so

$$\sigma = \frac{1}{(2\pi)^2}\frac{1}{4p_i^*\sqrt{s}}\int |M_{fi}|^2\delta(\sqrt{s} - E_1 - E_2)\delta^3(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2)\frac{\mathrm{d}^3p_1}{2E_1}\frac{\mathrm{d}^3p_2}{2E_2}$$

This is the same integral we did for 2-body decay with $m_a = \sqrt{s}$:

$$\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\int |m_{fi}|^2\,\mathrm{d}\Omega^*$$

0.16.4 Differential Cross-Sections

Often, it is more interesting to not complete the integration over the final state kinematics and leave the cross-section in differential form. For example, if we didn't integrate over d^3p_1 we would be left with $\mathrm{d}^3p_1 = p_1^2\,\mathrm{d}p_1\,\mathrm{d}\Omega_1 = p_1^2\frac{\mathrm{d}p_1}{\mathrm{d}E_1}\,\mathrm{d}E_1\,\mathrm{d}\Omega_1$ such that

$$\mathrm{d}\sigma = \int \mathrm{d}^3p_2\,(\cdots)\left(p_1^2\frac{\mathrm{d}p_1}{\mathrm{d}E_1}\right)\mathrm{d}E_1\,\mathrm{d}\Omega_1$$

so
$$\frac{\partial^2\sigma}{\partial E_1 \partial\Omega_1} = \int \mathrm{d}^3 p_2 \, (\cdots) \left(p_1^2 \frac{\mathrm{d}p_1}{\mathrm{d}E_1} \right)$$
This is called the double-differential cross-section.

Then, if we know the luminosity of the experiment, $\mathcal{L}\frac{\partial^2\sigma}{\partial E_1 \partial\Omega_1}$ gives us the scattering rate into $\mathrm{d}\Omega_1$ with energy E_1 to $E_1 + \mathrm{d}E_1$. Fore a two-body final state, E_1 is a function of θ_1, so we measure $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

For a two-body final state in the center of mass,
$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |M_{fi}|^2 \, \mathrm{d}\Omega^*$$
so
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |M_{fi}|^2$$
We want to then find the cross-section in the lab frame (for a fixed target). We can get the Lorentz-invariant cross-section in terms of $t = (p_1 - p_3)^2 = (p_1^* - p_3^*)^2$:
$$t = m_1^2 + m_3^2 - 2(E_1^* E_3^* - p_1^* p_3^* \cos(\theta^*))$$
so
$$\mathrm{d}t = 2p_1^* p_3^* \, \mathrm{d}\cos(\theta^*) = 2p_i^* p_f^* \, \mathrm{d}\cos(\theta^*)$$
(note t is not time).

We are trying to find
$$\mathrm{d}\Omega^* = \mathrm{d}\cos(\theta^*) \, \mathrm{d}\varphi^* = \frac{\mathrm{d}t \, \mathrm{d}\varphi}{2p_i^* p_f^*}$$
so
$$2p_i^* p_f^* \frac{\partial^2\sigma}{\partial t \partial\varphi} = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |M_{fi}|^2$$
and
$$\frac{\partial^2\sigma}{\partial t \partial\varphi} = \frac{1}{128\pi^2 s {p_i^*}^2} |M_{fi}|^2$$
This is now Lorentz-invariant.

0.16.5 3-Body Final States and Dalitz Plots

Consider two particles colliding to create three final-state particles:
$$(p_a + p_b) = p_1 + p_2 + p_3 = \sqrt{s}$$
so
$$s = p_1^2 + p_2^2 + p_3^2 + 2p_1 p_2 + 2p_1 p_3 + 2p_2 p_3$$
Define $m_{12}^2 = (p_1 + p_2)^2$ and so on:
$$m_{12}^2 + m_{23}^2 + m_{13}^2 = s + m_1^2 + m_2^2 + m_3^2$$
which is always a constant.

$$\mathrm{d}\sigma = \frac{(2\pi)^4}{N_a + N_b} \frac{1}{2E_a} \frac{1}{2E_b} |M_{fi}|^2 \prod_{i=1}^{3} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i} \delta^3(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_3) \delta(\sqrt{s - E_1 - E_2 - E_3})$$
If $|m_{fi}|^2$ is isotropic in the center of mass (no angular dependence, e.g. A spin-zero intermediate state) or if we average over spins, it can be shown that integration over seven of the final state coordinates gives
$$\mathrm{d}\sigma = \frac{(2\pi)^4}{N_a + N_b} \frac{1}{2E_a} \frac{1}{2E_b} \frac{1}{(2\pi)^3 32 s} \overline{|m_{fi}|^2} \, \mathrm{d}m_{12}^2 \, \mathrm{d}m_{23}^2$$

If we plot our observed events on axes of m_{23}^2 vs. m_{12}^2, this will give a uniform distribution if $\overline{|M_{fi}|^2}$ is constant. Otherwise, there will be structure which tells you about the behavior of the transition matrix element.

Lecture 19: Lab-Frame Differential Cross-Sections
Wednesday, October 14, 2020

Recall that in any frame,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s p_i^{*2}}|M_{fi}|^2$$

For electron scattering, neglect m_e. p_i^* follows from $\sqrt{s} = E_{\text{tot}} = p_i^* + \sqrt{m_T^2 + p_i^{*2}}$. Moving stuff around,

$$p_i^* = \frac{s - m_T^2}{2\sqrt{s}} \tag{$*$}$$

Also see equation 3.38 in the book for the general form.

In the lab frame, $p_1 = (E_1, 0, 0, E_1)$, $p_2 = (m_T, 0, 0, 0)$, and $p_3 = (E_3, 0, E_3\sin(\theta), E_3\cos(\theta))$ so $s = (p_1 + p_2)^2 = 0 + m_T^2 + 2E_1 m_T$. Plugging this into $*$,

$$p_i^* = \frac{E_1 m_t}{\sqrt{s}} \tag{$**$}$$

Recall that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{\mathrm{d}t}{\mathrm{d}\Omega}\right|\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos(\theta))}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

where $t = (p_1 - p_3)^2 = -2E_1E_3(1 - \cos(\theta))$. We often define $Q^2 \equiv -t = 4EE'\sin^2(\theta/2) \equiv -q^{\mu 2}$.

Let $\omega \equiv E_4 - E_2 = \sqrt{m_T^2 + \vec{\mathbf{q}}^2}$ so

$$(m_T + \omega)^2 = m_T^2 + \vec{\mathbf{q}}^2 = m_T^2 + \omega^2 + 2m_T\omega$$

where $\vec{\mathbf{q}}^2 - \omega^2 = 2m_T\omega = -t = Q^2$, so

$$\omega = \frac{Q^2}{2m_T} = \frac{2E_1E_3\sin^2(\theta/2)}{m_T} = E_1 - E_3$$

so

$$E_1 = E_3\left(1 + \frac{2E_1}{m_t}\sin^2(\theta/2)\right)$$

or

$$E_3 = \frac{E_1}{1 + \frac{E_1}{m_T}(1 - \cos(\theta))}$$

ze needed that so we can find $\frac{\mathrm{d}t}{\mathrm{d}(\cos(\theta))}$ where $t = (p_2 - p_4)^2 = 2m_T^2 - 2m_TE_4$ assuming that $p_4^2 = m_T$, which is only true for elastic collisions. Using this, we can write $t = 2m_T(E_3 - E_1)$, so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos(\theta))} = 2m_T\frac{\mathrm{d}E_3}{\mathrm{d}(\cos(\theta))} = 2m_T\frac{-E_1}{\left(1 + \frac{E_1}{m_T}(1 - \cos(\theta))\right)^2}\left(-\frac{E_1}{m_T}\right) = 2E_3^2$$

Therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}(2E_3^2)\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{2}\left(\frac{1}{64\pi s p_i^{*2}}|M_{fi}|^2\right)$$

From $**$, we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{1}{\left(m_T + 2E_1\sin^2(\theta/2)\right)^2}|M_{fi}|^2$$

0.17 Nuclear Physics

For nuclei with N neutrons and Z protons, we define the atomic number as $A \equiv N + Z$. Usually, we label using A and Z: ^{7}Be has $A = 7$ and $Z = 4$ (from the element name) such that $N = 3$.

Isotopes have the same Z but different A. Isobars have the same A but different Z and N. The mass of any nucleon is $m = Nm_n + Zm_p - B$, where B is the binding energy.

LECTURE 20: NUCLEAR PHYSICS, CONT.
Wednesday, October 21, 2020

From our previous lecture, we said that the mass of the nucleus could be written

$$M_n = Nm_n + Zm_p - B$$

This is based on the "liquid drop model", where we treat nuclear matter as an incompressible liquid ($R \propto A^{1/3}$).

Strong (nuclear) interactions are short-ranged, so they saturate: $B_1 = a_v A$. However, nucleons on the surface are not as tightly bound (since they are not surrounded by nucleons): $B_2 = a_v A - a_s A^{2/3}$.

0.17.1 Coulomb Repulsion

B is reduced by repulsion of two protons from $Z - 1$ other protons with distance $R \propto A^{1/3}$:

$$\Delta B \propto \sum_{i=1}^{Z} \frac{e(Z-1)e}{R} \propto \frac{Z(Z-1)}{A^{1/3}}$$

so

$$B_3 = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}}$$

0.17.2 Asymmetry Term

We expect $N > Z$ for heavy nuclei. The advantage is reduced as N grows by some asymmetry term (otherwise ^{238}H would be common). Treating the nucleus as two degenerate Fermi gasses (one of neutrons and another of protons), the Pauli exclusion principle says that if $N >> Z$, extra neutrons added are at a higher energy than extra protons (the Fermi energy for the neutrons will be higher than that of the protons if there are more neutrons than protons).

The Fermi pressure for particles on a lattice is $p = \frac{h}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$. $Z = \frac{4}{3}\pi n_{\mathcal{F}_p}^3$ so $n_{\mathcal{F}_p} \propto Z^{1/3}$ and $n_{\mathcal{F}_n} \propto N^{1/3}$.

Therefore, the Fermi energy will be

$$E_{\mathcal{F}_p} = \frac{p_{\mathcal{F}_p}^2}{2m} = \frac{1}{2m}\left(\frac{h}{L}\right)^2 \left(\frac{3}{4\pi} Z\right)^{2/3}$$

so

$$E_{\text{tot}_p} = 4\pi \int_0^{n_{\mathcal{F}_p}} E(n) n^2 \, \mathrm{d}n = 4\pi \int \frac{1}{2m}\left(\frac{h}{L} n\right) n^2 \, \mathrm{d}n \propto \frac{1}{L^2} n_{\mathcal{F}_p}^5 \propto \frac{Z^{5/3}}{A^{2/3}}$$

For N neutrons and Z protons vs. $\frac{A}{2}$ neutrons and $\frac{A}{2}$ protons,

$$\Delta E_{\text{tot}} \propto \frac{1}{A^{2/3}} N^{5/3} + \frac{1}{A^{2/3}} Z^{5/3} - 2\frac{1}{A^{2/3}}\left(\frac{A}{2}\right)^{5/3}$$

Let $N = \frac{A}{2}(1+\lambda)$ and $Z = \frac{A}{2}(1-\lambda)$ ($\lambda \equiv \frac{N-Z}{A}$). Then

$$\Delta E_{\text{tot}} \propto \frac{1}{A^{2/3}}\left(\frac{A}{2}\right)^{5/3}\left((1+\lambda)^{5/3}+(1-\lambda)^{5/3}-2\right)$$

If $\lambda \ll 1$,

$$\begin{aligned}\Delta E &\propto A\left[\left(1+\frac{5}{3}\lambda+\frac{5}{3}\left(\frac{5}{3}-1\right)\lambda^2\right)+\left(1-\frac{5}{3}\lambda+\frac{5}{3}\left(\frac{5}{3}-1\right)\lambda^2\right)-2\right]\\ &\propto A\lambda^2 = A\left(\frac{N-Z}{A}\right)^2 = \frac{(N-Z)^2}{A}\end{aligned}$$

so

$$B_4 = B_3 - a_{\text{sym}}\frac{(N-Z)^2}{A}$$

Finally, beyond the liquid-drop model plus the Fermi gas model, there are pairing correlations. Nuclei are more tightly bound if N is even and Z is even. They are less bound if one of them is odd, and even less if both are:

$$B_5 = a_v A - a_s A^{2/3} - a_c\frac{Z(Z-1)}{A^{1/3}} - a_{\text{sym}}\frac{(N-Z)^2}{A} - a_{\text{pairity}}\delta(N,Z)$$

where

$$\delta = \begin{cases} 1 & \text{if} \quad N \quad \text{odd} \quad Z \quad \text{odd} \\ 0 & \text{if} \quad A \\ -1 & \text{if} \quad N \quad \text{even} \quad Z \quad \text{even}\end{cases}$$

An empirical fit (for $A > 20$) yields (in MeV):

a_v	a_s	a_c	a_{sym}	a_{pairity}
15.76	17.8	0.711	23.7	$11.18/\sqrt{A}$

This predicts the curve of binding energy roughly and also predicts the β-stability curve and proton/neutron drip lines. The liquid drop model also qualitatively explains fusion.

LECTURE 21: NUCLEAR SHAPES

Monday, October 26, 2020

0.18 Nuclear Shapes

Nuclei are spherical for low-J ground states. Nuclei in high-J ground states are prolate (stretched-sphere) or oblate (flattened-sphere), which modifies the Coulomb energy. The liquid drop model is also applied to high-excitation, short-lived collective excitations:

$$r = \sum_{l,m} a_{l,m}Y_{lm}(\Omega)$$

We call the first two excitations giant monopole ("Breathing"): $J^\pi = 0^+$, $E \sim 80\text{MeV}/A^{1/3}$ and giant dipole: $J^\pi = 1^-$ with $E \sim 77\text{MeV}/A^{1/3}$. Beyond this, there are quadrupole moments with $J^\pi = 2^+$ discrete excitations around $1-2$MeV.

0.19 The Shell Model

For hydrogenic atoms, one unit of angular orbital momentum l has the same energy as one unit of radial excitation, which leads to the approximate degeneracy of $2s$ and $1p$ states in a non-perturbative model. Compare this to a 3D harmonic oscillator, where two units of l requires the same energy as one unit of radial excitation. The actual potential due to the distribution of nucleons is more like a "rounded" square well. Nucleons move like "free" particles in this potential because they can't scatter. The Pauli principle blocks changes in wave function. However, nucleon scattering experiments reveal strong $\vec{\mathbf{L}} \cdot \vec{\mathbf{S}}$ coupling. This is not due to magnetic effects, but rather due to coupling to exchanged mesons. This splits each l-level into two j-levels with $j = l \pm \frac{1}{2}$ where the larger j is pushed to lower energy. This splitting is the reason for some of the "magic numbers" we've been seeing with certain values of A and Z, such as 2 and 82. This complicated mixing leads to what are called bands or shells of levels with gaps in between. Shells are filled for Z (or N) equal to $2, 8, 20, 28, 50, \sim 82, \sim 126$. These numbers correspond to the elements (in order) helium, oxygen, calcium (two isotopes), nickel (two isotopes), and lead (82 protons and 126 neutrons). The further you move away from these magic numbers, the more difficult it is to predict properties about energy levels.

Single-particle excitations of highest shell-model nucleons explain many observed states, such as excitations of one particle beyond a magic number or two-body couplings of excited nucleon and unexcited nucleon states. These single particle excitations de-excite by γ-emission when possible or by electron conversion (ϵ-rays) if no radiative transition is possible.

Lecture 22: Radiative Transitions
Wednesday, October 28, 2020

0.20 Radiative Transitions

If we have an electric dipole (called an E1 transition), all you need to know is the spin and pairity. $M_{fi} = \langle\psi_f|\, H' \,|\psi_i\rangle$, where H' could be written in terms of spherical harmonics, where it could be said to have a "spin" and "pairity".

For electric dipoles, $J^\pi = 1^-$, so the selection rules require $J_f = J_i, J_i \pm 1$, but you cannot transition from $J_i = 0$ to $J_f = 0$. $\pi_f = -\pi_i$.

For magnetic dipoles (M1), $J^\pi = 1^+$, so the J-rules are the same as electric dipoles, but $\pi_f = +\pi_i$.

For electric quadrupoles (E2), $J^\pi = 2^+$, so $J_f = J_i, J_i \pm 1, J_i \pm 2$, but you can not transition from $J_i = 0$ to $J_f = 0, 1$ or from $J_i = 1$ to $J_f = 0$. $\pi_f = +\pi_i$.

Magnetic quadrupoles (M2) are 2^-, electric octopoles (E3) are 3^-, and magnetic octopoles (M3) are 3^+, and the transition rules are fairly straightforward.

0.21 Isobaric Analog States

The strong force approximately treats neutrons and protons as identical. Small differences in the mass (energy) for same $A = N + Z$ are because $m_n > m_p$ (by about 0.0013 parts per 1), the electrostatic force treats the proton differently (this can be big for large Z, but doesn't effect the relative positions of the levels), and the Pauli principle.

We can examine combinations of two nucleons. With opposite spin states, the three possible combinations are called isobaric states, and are unbound. If we look at parallel spin states, we find one bound state (the deuteron). The singlet-triplet structure is analagous to coupling two spin-1/2 particles. Imagine a 3-space (not coordinate space) in which the 3-component of 'isobaric spin' of a nucleon decides whether it was a proton ($|I = 1/2, I_3 = +1/2\rangle$) or a neutron ($|1/2, -1/2\rangle$). Coupling two of them would give a singlet ($|0,0\rangle$) state and triplet ($|1,\{-1,0,1\}\rangle$) states (which are expected to have approximately the same mass because the strong force treats them approximately the same if they are all a spin singlet).

We can do a similar construction for three nucleons. Note that if we have three of the same type of nucleon, one of them must exist in a higher energy state due to the Pauli principle. Otherwise, tritium and helium-3 will exist in a ground and excited state. The four excited states here form an isobaric analogue quadruplet, and the two ground states form a doublet:

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$$

In more recent years, this isobaric spin has been renamed to isotopic spin. In isobar notation, the neutron typically has $I_3 = -1/2$, but in isotopic, this is labeled $T_3 = +1/2$, so the convention is flipped.

Lecture 23: Isospin

Friday, October 30, 2020

We can define two projections I_1 and I_2 along axes perpendicular to the I_3 axis with $[I_i, I_j] = \imath\epsilon_{ijk}I_k$. Then $I^\pm = I_1 \pm \imath I_2$ raise and lower I_3.

Starting from the "stretch" configuration of A nucleons (e.g. $Z = A$ protons with $I_3 = A/2$), we can write the isospin state as

$$\left|\frac{A}{2},\frac{A}{2}\right\rangle = \bigotimes_A \left|\frac{1}{2},\frac{1}{2}\right\rangle$$

I^- allows us to write down all of the $\left|\frac{A}{2}, I_3\right\rangle$ states. We can then write down the $\left|\frac{A}{2}-1, \frac{A}{2}-1\right\rangle$ state ($A-1$ protons and one neutron) by orthogonality with $\left|\frac{A}{2}, \frac{A}{2}-1\right\rangle$. Then I^- allows us to generate all $I = \frac{A}{2} - 1$ states, and so on.

This is just like writing states of coupled angular momentum, and we can use the same Clebsh-Gordon coefficients.

Once the quark model was advanced, I^- was generalized from $I^- |p\rangle = |n\rangle$ to $I^- |u\rangle = |d\rangle$ (and $I^- \left|\bar{d}\right\rangle = -|\bar{u}\rangle$). This means that we think of $I^-|p\rangle = I^-|duu\rangle = |ddu\rangle = |n\rangle$. Acting this on a neutron gives $I^-|n\rangle = |ddd\rangle = 0$ by the Pauli principle.

This extends to all hadrons containing light quarks:

$$\left|\pi^+\right\rangle = \left|u\bar{d}\right\rangle \implies I^-\left|\pi^+\right\rangle = \frac{1}{\sqrt{2}}(\left|d\bar{d}\right\rangle - |u\bar{u}\rangle) = \left|\pi^0\right\rangle$$

and $I^-\left|\pi^0\right\rangle = |d\bar{u}\rangle = |\pi^-\rangle$.

These three pions are therefore predicted to have identical strong interactions, aside from slight differences in the down and up quark masses and electromagnetic interactions. Isospin not only explains the relations of multiple hadrons which differ only by up and down quarks, but it also predicts the relative strength of the strong interaction of states which differ by isospin. For example, the spin-3/2 Δ isobar has 4 charge states:

Name	Quark content	I_3
Δ^{++}	uuu	$+3/2$
Δ^{+}	uud	$+1/2$
Δ^{0}	udd	$-1/2$
Δ^{-}	ddd	$-3/2$

Take scattering π^- on a p (at Δ resonance):

$$\pi^- + p \to \Delta^0$$

either elastically (resulting in the same particles) or with charge exchange (decaying to $\pi^0 n$). From CG coefficients, we can write

$$|J, M\rangle = \sum_{M=m_1+m_2} \langle J, M | j_1, m_1, j_2 m_2\rangle |j_1, m_1\rangle |j_2, m_2\rangle$$

[TODO I missed stuff here, need to go back and rewatch lecture]

0.22 Relativistic Quantum Mechanics

In SR, $E^2 = p^2 + m^2$ and we know that $E = \frac{p^2}{2m} + V$ gave us the Schrödinger equation. We can try the same trick by replacing these kinematic variables with operators:

$$\hat{E}^2 \psi = \hat{p}^2 \psi + m^2 \psi$$

with $\hat{p} = -\imath \vec{\boldsymbol{\nabla}}$ and $\hat{E} = \imath \partial_t$.

We can rewrite this as

$$(\partial^\mu \partial_\mu + m^2)\psi = 0 \tag{Klein-Gordon Equation}$$

Solutions to this are

$$\psi(\vec{\mathbf{r}}, t) = A e^{\imath(\vec{\mathbf{p}} \cdot \vec{\mathbf{r}} - Et)}$$

Unfortunately, $E = \pm\sqrt{p^2 + m^2}$, but we can't have negative-energy states or the resulting wave functions won't be complete.

$$\psi^*(\text{K.G.}) - \psi(\text{K.G.})^* = \frac{\partial}{\partial t} \underbrace{\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)}_{-\imath\rho} = \vec{\boldsymbol{\nabla}} \cdot \underbrace{(\psi^* \vec{\boldsymbol{\nabla}} \psi - \psi \vec{\boldsymbol{\nabla}} \psi^*)}_{\imath \vec{\mathbf{j}}}$$

$\rho = 2|A|^2 E$, so how can the probability density be negative? We can only use the K.G. equation in quantum field theories of spin-0 fields.

Dirac wanted a first-order equation in derivatives which would be the square root of the K.G. equation. Assume that

$$\hat{E}\psi = (\vec{\alpha} \cdot \vec{\mathbf{p}} + \beta m)\psi$$

where

$$(\vec{\alpha} \cdot \vec{\mathbf{p}} + \beta m) = p^2 + m^2 \tag{$*$}$$

Then

$$\imath \partial_t \psi = (-\imath \alpha_x \partial_x - \imath \alpha_y \partial_y - \imath \alpha_z \partial_z + \beta m)\,\psi$$

Operating a second time should give the K.G. equation. We require $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I$ and $\alpha_j \beta + \beta \alpha_j = 0$ and $\alpha_i \alpha_j + \alpha_j \alpha_i = 0$, so these α and β terms cannot just be scalars.

Lecture 24: The Dirac Equation

Monday, November 02, 2020

Las time we said that the equivalence of the square of the Dirac equation and the Klein-Gordon equation requires $\alpha_{x,y,z}^2 = \beta^2 = I$ and $\{\alpha_j, \beta\} = 0$ and $\{\alpha_i, \alpha_j\} = 0$ (anti-commute). These must be matrices or operators. For matrices, $\text{Tr}(A) \equiv \sum_i A_{ii}$, and $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$, so

$$\text{Tr}(\alpha_i) = \text{Tr}(\alpha_i \beta\beta) = \text{Tr}(\beta\alpha\beta) = -\text{Tr}(\alpha_i\beta\beta) = -\text{Tr}(\alpha_i)$$

so $\text{Tr}(\alpha_i) = 0$. Similarly, $\text{Tr}(\beta) = 0$. Additionally, we can show that $\alpha_i^2, \beta^2 = I$ implies that the eigenvalues of α_i and β are $\lambda = \pm 1$.

Finally, the trace of a matrix is equal to the sum of the eigenvalues, so the dimension of these matrices must be even (2x2, 4x4, etc.), and $H = \vec{\alpha} \cdot \hat{\vec{\mathbf{p}}} + \beta m$ must be Hermitian so α_i and β must be anti-commuting, Hermitian, traceless matrices. These are the same properties as the Pauli matrices. Only three exist in 2×2 space, so these must be at least 4×4 matrices. Let's try the simplest case. We define the wave function in this four-dimensional space as the Dirac Spinor,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

One example of matrices which satisfy the properties we want are the "Dirac-Pauli" representation:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

where $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Any unitary transformation of these matrices is also a good representation. The results of the Dirac equation do not depend on representation.

0.23 Angular Momentum with the Dirac Equation

$$\begin{aligned} \left[\hat{H}_D, \hat{\vec{\mathbf{L}}}\right] &= \left[\vec{\alpha} \cdot \hat{\vec{\mathbf{p}}} - \beta m, \hat{\vec{\mathbf{r}}} \times \hat{\vec{\mathbf{p}}}\right] \\ &= \left[\vec{\alpha} \cdot \hat{\vec{\mathbf{p}}}, \hat{\vec{\mathbf{r}}} \times \hat{\vec{\mathbf{p}}}\right] \end{aligned}$$

For example,

$$\begin{aligned} \left[\hat{H}_D, \hat{L}_x\right] &= \left[\vec{\alpha} \cdot \hat{\vec{\mathbf{p}}}, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right] \\ &= 0 + \alpha_y[\hat{p}_y, \hat{y}]\hat{p}_z + \alpha_z[\hat{p}_z, -\hat{z}]\hat{p}_y \\ &= -\imath(\alpha_y \hat{p}_z - \alpha_z \hat{p}_y) = -\imath([\vec{\alpha} \times \hat{\vec{\mathbf{p}}}]_x) \end{aligned}$$

and the same for y and z, so in general

$$\left[\hat{H}_D, \vec{\mathbf{L}}\right] = -\imath \vec{\alpha} \times \hat{\vec{\mathbf{p}}} \neq 0!$$

This seems like a big problem.

Consider an operator $\hat{\vec{\mathbf{S}}} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$.

Then

$$\left[\alpha_i, \hat{S}_j\right] = 0$$

so

$$\left[\hat{H}_D, \hat{S}_x\right] = \imath \vec{\alpha} \times \hat{\vec{\mathbf{p}}}$$

So whatever $\vec{\mathbf{S}}$ is still isn't conserved. However, let $\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{S}}$.

$$\left[\vec{\mathbf{H}}_D, \vec{\mathbf{J}}\right] = 0$$

so $\vec{\mathbf{J}}$ is conserved. We can identify $\vec{\mathbf{S}}$ as the intrinsic angular momentum (spin) carried by whatever particles the Dirac equation describes. We can calculate $\vec{\mathbf{S}}^2 = \frac{3}{4} I_4$, and the $\hat{S}_i$ commutation rules follow from the σ_i commutation rules: $\left[\hat{S}_i, \hat{S}_j\right] = \epsilon_{ijk}\hat{S}_k$, so all the typical angular momentum relations apply to spin. Then we get that $s(s+1) = \frac{3}{4}$, or $s = 1/2$ so the Dirac equation describes spin-1/2 particles.

Lecture 25: Covariant Form of the Dirac Equation
Wednesday, November 04, 2020

0.24 Magnetic Moment

Consider an interaction with a $\vec{\mathbf{B}}$-field. In this situation, $\vec{\mathbf{p}} \to \vec{\mathbf{p}} - q\vec{\mathbf{A}}$ and $E \to q\Phi$, so the Dirac equation becomes

$$(\vec{\alpha} \cdot (\vec{\mathbf{p}} - q\vec{\mathbf{A}}) + \beta m)\psi = (E - q\Phi)\psi$$

$\vec{\mathbf{p}}_i = \partial_{\dot{q}_i}\mathcal{L}$ shows, in the non-relativistic limit, that $u = -\frac{q}{2m}(\vec{\sigma} \cdot \vec{\mathbf{B}}) = -\mu \cdot \vec{\mathbf{B}}$ where $\vec{\mu} = \frac{q}{2m}\vec{\sigma} = \frac{q}{m}\vec{\mathbf{S}} = g\frac{q}{2m}\vec{\mathbf{S}}$ where g is the gyromagnetic ratio. In Dirac theory, this is exactly 2 for *point* spin-1/2 particles (neglecting higher-order effects).

0.25 Covariant (Standard) Form of the Dirac Equation

$$\begin{aligned}
(\vec{\alpha} \cdot \hat{\vec{\mathbf{p}}} + \beta m)\psi = \hat{E}\psi = \imath\partial_t\psi \\
0 &= (-\vec{\alpha} \cdot \hat{\vec{\mathbf{p}}} - \beta m + \imath\partial_t)\psi \\
&= (\imath\alpha_x\partial_x + \imath\alpha_y\partial_y + \imath\alpha_z\partial_z - \beta m + \imath\partial_t)\,\psi \\
&= \left(\imath\underbrace{\beta\alpha x}_{\gamma^1}\partial_x + \imath\underbrace{\beta\alpha_y}_{\gamma^2}\partial_y + \imath\underbrace{\beta\alpha_z}_{\gamma^3}\partial_z - mI + \imath\underbrace{\beta}_{\gamma^0}\partial_t\right)\psi \\
&= (\imath\gamma^\mu\partial_\mu - m)\psi \\
&\equiv (\imath\not{\partial} - m)\psi = 0
\end{aligned}$$

This looks invariant, but γ^μ is *not* a 4-vector, but rather a set of constants.

From properties of $\vec{\alpha}$ and β, $(\gamma^0)^2 = \beta^2 = I$ and for $k = 1,2,3$, $(\gamma^k)^2 = \beta\alpha_k\beta\alpha_k = -\alpha_k\beta\beta\alpha_k = -I$.

On this week's homework, we show that $[\gamma^\mu, \gamma^\nu] = \delta_{\mu\nu}$.

$\gamma^0 = \beta$ is Hermitian, so $(\gamma^k)^\dagger = (\beta\alpha_k)^\dagger = \alpha_k\beta = -\beta\alpha_k = -\gamma^k$, so γ^k are anti-Hermitian.

In the Dirac-Pauli representation,

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

0.26 Covariant Current and Adjoint Spinor

We saw that $\rho = \psi^\dagger\psi = \psi^\dagger\gamma^0\gamma^0\psi$ and $\vec{\mathbf{j}} = \psi^\dagger\vec{\alpha}\psi = \psi^\dagger\gamma^0\vec{\gamma}\psi$, so the 4-current can be written

$$j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$$

This makes the continuity equation $\partial_\mu j^\mu = 0$.

We can also define the adjoint spinor $\bar{\psi} = \psi^\dagger\gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$. Then $j^\mu = \bar{\psi}\gamma^{\mu\psi}$.

0.27 Free Particle Solutions of the Dirac Equation

Let's look for plane-wave solutions with well-defined energy and momentum of the form

$$\psi(\vec{\mathbf{r}}, t) = u(E, P)e^{-\imath(\vec{\mathbf{r}}\cdot\vec{\mathbf{p}} - Et)}$$

where $u(E, \vec{\mathbf{p}})$ is the 4-component Dirac spinor and not a function of $\vec{\mathbf{r}}$ and t. Putting this through the Dirac equation, we find

$$\begin{aligned} 0 &= (\imath\gamma^\mu\partial_\mu - m)\psi \\ &= (\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)\psi \end{aligned}$$

Divide out the exponential on both sides (all the space and time dependence), and we're left with

$$(\gamma^\mu p_\mu - m)u(E, \vec{\mathbf{p}}) = 0$$

This is the equation for a spinor for a particle with E and $\vec{\mathbf{p}}$, or

$$(\not{p} - m)u = 0$$

At rest, $\psi = ue^{-\imath Et}$, so the spinor form of the Dirac equation reduces to

$$E\gamma^0 u = mu$$

$$E\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = m\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

Solutions of this form are $A_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ or $A_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ with $E = m$ *or* $A_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ or $A_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ with $E = -m$.

These are eigenstates of $S_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$, so the first two solutions are spin-up and spin-down $E > 0$ solutions. The second two are also spin-up and spin-down, but $E < 0$.

We can also write down the time behavior.

$$\psi(t) = \begin{pmatrix} A_1 \\ A_2 \\ 0 \\ 0 \end{pmatrix}e^{-\imath mt} + \begin{pmatrix} 0 \\ 0 \\ A_3 \\ A_4 \end{pmatrix}e^{+\imath mt}$$

0.27.1 General Plane-Wave Free-Particle Solutions

[TODO] I missed stuff here.

LECTURE 26: SPINOR SOLUTIONS

Friday, November 06, 2020

In our last class, we solved the spatial part of the wave functions, but we still need to find solutions to the spinor. We will write this as $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$ where u_A and u_B are both two-component spinors.

$$\begin{pmatrix} (E-M)I_2 & -\vec{\sigma}\cdot\vec{\mathbf{p}} \\ \vec{\sigma}\cdot\vec{\mathbf{p}} & -(E+M)I_2 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

so

$$u_A = \frac{\vec{\sigma}\cdot\vec{\mathbf{p}}}{E-M}u_B \qquad u_B = \frac{\vec{\sigma}\cdot\vec{\mathbf{p}}}{E+M}u_A$$

If we chose a basis with $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$u_B = \frac{1}{E+M}\begin{pmatrix} p_z & p_x - \imath p_y \\ p_x + \imath p_y & -p_z \end{pmatrix} u_A$$

From this, we get two (of four independent) solutions, depending on which u_A we put in:

$$u_1(E,\vec{\mathbf{p}}) = A_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+M} \\ \frac{p_x+\imath p_y}{E+M} \end{pmatrix}$$

$$u_2(E,\vec{\mathbf{p}}) = A_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-\imath p_y}{E+M} \\ \frac{-p_z}{E+M} \end{pmatrix}$$

Similarly, we could start with $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and get solutions where

$$u_A = \frac{1}{E-M}\begin{pmatrix} p_z & p_x - \imath p_y \\ p_x + \imath p_y & -p_z \end{pmatrix} u_B$$

This gives us the other two solutions:

$$u_3(E,\vec{\mathbf{p}}) = A_3 \begin{pmatrix} \frac{p_z}{E-M} \\ \frac{p_x+\imath p_y}{E-M} \\ 1 \\ 0 \end{pmatrix}$$

$$u_4(E,\vec{\mathbf{p}}) = A_4 \begin{pmatrix} \frac{p_x-\imath p_y}{E-M} \\ \frac{-p_z}{E-M} \\ 0 \\ 1 \end{pmatrix}$$

For $\vec{\mathbf{p}} = 0$, u_1 and u_2 are reduced to the rest solutions with $E = M$, and u_3 and u_4 are reduced to the rest solutions with $E = -M$. Since, in general, all spinors (in free space) satisfy $E^2 = p^2 + m^2 \implies E = \pm\sqrt{p^2+m^2}$, we identify u_1 and u_2 as the $E = \sqrt{p^2+m^2} > 0$ solutions and u_3 and u_4 as the $E = -\sqrt{p^2+m^2} < 0$.

0.28 Antiparticles and the Dirac Sea

Negative-energy solutions present a problem as all electrons would be expected to drop in energy to these negative solutions, and since we can go arbitrarily high in excited states, we should be able to also go arbitrarily low with these negative states, so they would drop without limit, releasing infinite energy from each electron.

Dirac postulated that the negative energy states are all occupied (by an infinite number of electrons, which released an infinite amount of energy when they filled those states). If we assume this is okay, then we have an occupied infinite sea of negative energy states and the Pauli principle prevents excess electrons from descending into the sea.

This then creates a further prediction that a high-energy γ should be able to excite one of these "sea" electrons out of one of the highest-energy states ($\sim -m_e$) to one of the lowest positive-energy levels ($\sim +m_e$), leaving a hole in the sea. Nothing like this had ever been observed, so Dirac tried to come up with reasons for why this is. First, he believed these negative energy electrons were protons. As soon as pair production was discovered, it turned out that this actually could be done. The "hole" in the sea can be interpreted as an e^+, and the sea accounts for e^+e^- pair creation and annihilation.

0.29 The Feynman-Stückelberg Interpretation

Negative energy states can be interpreted as particles moving backwards in time:

$$e^{-\imath Et} = e^{+\imath |E| t} = e^{-\imath |E|(-t)}$$

or as a positive-energy antiparticle moving forward in time:

$$e^{-\imath(-E)t}$$

Figure 0.29.1: Electron Annihilation

0.29.1 Anti-Particle Spinors

u_3 and u_4 are spinors for negative-energy particles propagating backwards in time, so $\vec{\mathbf{p}}$ in the spinor is negative of physical momentum of the antiparticle (and $E_{e^-} = -E_{e^+}$).

Define anti-particle spinors v with $v_1(E, \vec{\mathbf{p}}) = u_4(-E, -\vec{\mathbf{p}})$, where the first E and $\vec{\mathbf{p}}$ correspond to the physical energy and momentum of the antiparticle, while the second correspond to that of a regular particle with negative energy.

$$v_1(E, \vec{\mathbf{p}}) = u_4(-E, -\vec{\mathbf{p}}) = A_1' \begin{pmatrix} \frac{p_x - \imath p_y}{E+M} & \frac{-p_z}{E+M} \\ 0 \\ 1 \end{pmatrix}$$

and

$$v_2(E, \vec{\mathbf{p}}) = u_3(-E, -\vec{\mathbf{p}}) = A_2' \begin{pmatrix} \frac{p_z}{E+M} & \frac{p_x + \imath p_y}{E+M} \\ 1 \\ 0 \end{pmatrix}$$

0.29.2 Normalization

For Lorentz-invariance, we need to normalize $\psi \propto E$ (we ended up normalizing it to $2E$ to match the $\delta^4(p^2 - m^2)$ phase-space convention).

Let $2E = u_1^\dagger u_1 = |A_1|^2 \left(1 + \frac{p_z^2}{(E+M)^2} + \frac{p_x^2 + p_y^2}{(E+M)^2}\right) = \frac{E^2 + M^2 + 2EM + p^2}{(E+M)^2} |A_1|^2$.

Lecture 27: Normalization of Dirac Spinors

Monday, November 09, 2020

In the last lecture, we ended with

$$2E = u_1^\dagger u_1 = \frac{E^2 + M^2 + 2EM + p^2}{(E+M)^2} |A_1|^2 = \frac{2E(E+M)}{(E+M)^2} |A_1|^2$$

so

$$A_1 = \sqrt{E+M}$$

and similarly, $A_2 = A_1' = A_2' = \sqrt{E+M}$.

0.30 $\vec{\mathbf{S}}$ for an Anti-Proton Spinor

For an anti-proton state written in terms of physical E and $\vec{\mathbf{p}}$ of anti-particles $\psi = N(E, \vec{\mathbf{p}}) e^{-\imath(\vec{\mathbf{p}} \cdot \vec{\mathbf{r}} - Et)}$. Acting the Hamiltonian on this, we get

$$\hat{H}\psi = -E\psi$$

and

$$\hat{\vec{\mathbf{p}}}\psi = -\imath \vec{\boldsymbol{\nabla}} \psi = -\vec{\mathbf{p}}\psi$$

so we need to create some modified operators which give the proper E and $\vec{\mathbf{p}}$ (not the negative of these):

$$\hat{H}^{(v)} = -\imath \partial_t \qquad \hat{\vec{\mathbf{p}}}^{(v)} = +\imath \vec{\boldsymbol{\nabla}}$$

so

$$\hat{\vec{\mathbf{L}}}^{(v)} = \vec{\mathbf{r}} \times \hat{\vec{\mathbf{p}}}^{(v)} = -\hat{\vec{\mathbf{L}}}$$

We want a spin operator $\hat{\vec{\mathbf{S}}}^{(v)}$ such that $\left[\hat{H}_D, \vec{\mathbf{L}}^{(v)} + \vec{\mathbf{S}}^{(v)}\right]$, so $\hat{\vec{\mathbf{S}}}^{(v)} = -\hat{\vec{\mathbf{S}}}$, so $v_1 \propto u_4$ is a spin-up antiparticle, whereas u_4 was a spin-down, negative-energy particle.

0.31 Charge Conjugation

The C-operator interchanges matter and antimatter particles. Using minimal substitution, $\hat{E} \to \hat{E} - q\Phi$ and $\vec{\mathbf{p}} \to \vec{\mathbf{p}} - q\vec{\mathbf{A}}$ causes $p_\mu \to p_\mu - qA_\mu$ and $\imath\partial_\mu \to \imath\partial_\mu - qA_\mu$.

The charge conjugation operator gives $Cu_1 \to v_1$ and $Cu_2 \to v_2$.

0.32 Spin and Helicity

For particles at rest, $u_1(E,0) = A_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $u_2(E,0) = A_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are eigenstates of

$$\hat{S}_z = \frac{1}{L} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

In general ($\vec{\mathbf{p}} \neq 0$), u_1 and u_2 are not eigenvectors of S_z. For particles moving in the $\pm\hat{\mathbf{z}}$ direction,

$$u_1 = A_1 \begin{pmatrix} 1 \\ 0 \\ \pm\frac{p}{E+m} \\ 0 \end{pmatrix}, \qquad u_2 = A_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \mp\frac{p}{E+m} \end{pmatrix}$$

and

$$v_1 = A_1' \begin{pmatrix} 0 \\ \mp\frac{p}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v_2 = A_2' \begin{pmatrix} \pm\frac{p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are eigenstates of S_z.

$$\hat{S}_z u_{1,2}(E,0,0,\pm p) = \pm_{(1,2)} \frac{1}{2} u_{1,2}(E,0,0,\pm p)$$

and

$$\hat{S}_z^v v_{1,2}(E,0,0,\pm p) = \pm_{(1,2)} \frac{1}{2} v_{1,2}(E,0,0,\pm p)$$

0.32.1 Helicity

Since $\left[\hat{H}_D, \hat{S}_z\right] \neq 0$, it is not generally possible to define a basis of simultaneous eigenstates of $\hat{H}_D$ and $\hat{S}_z$. We define helicity, h, as

$$\hat{h} = \frac{\hat{S} \cdot \hat{\vec{\mathbf{p}}}}{p} = \frac{1}{2p} \begin{pmatrix} \vec{\sigma} \cdot \vec{\mathbf{p}} & 0 \\ 0 & \vec{\sigma} \cdot \vec{\mathbf{p}} \end{pmatrix}$$

Then, $\left[\hat{H}_D, \hat{h}\right] = 0$ since $\hat{H}_D = \vec{\alpha} \cdot \vec{\mathbf{p}} + \beta m$. Let $\hat{h}u = \lambda u$ where $u = \left(u_A // u_D\right)$ ($u_{A,D}$ are 2-spinors). Therefore we define

$$\vec{\sigma} \cdot \vec{\mathbf{p}} u_{A,B} = 2p\lambda u_{A,B}$$

so

$$\underbrace{(\vec{\sigma} \cdot \vec{\mathbf{p}})(\vec{\sigma} \cdot \vec{\mathbf{p}})}_{p^2} u_A = 4p^2\lambda^2 u_A$$

so $\lambda = \pm\frac{1}{2}$ are the eigenvalues of helicity. We define $\lambda = \frac{1}{2}$ to be "right-handed" helicity.

$$(\vec{\sigma}\cdot\vec{\mathbf{p}})u_A = (E+m)u_B$$

so $u_B = 2\lambda\frac{p}{E+m}u_A$.

Let $\vec{\mathbf{p}} = (p\sin(\theta)\cos(\varphi), p\sin(\theta)\sin(\varphi), p\cos(\theta))$ so

$$\vec{\sigma}\cdot\vec{\mathbf{p}} = \begin{pmatrix} p\cos(\theta) & p\sin(\theta)e^{-\imath\varphi} \\ p\sin(\theta)e^{\imath\varphi} & -p\sin(\theta) \end{pmatrix}$$

With $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$, this becomes

$$p\begin{pmatrix} & \sin(\theta)e^{-\imath\varphi} \\ \sin(\theta)e^{\imath\varphi} & -\cos(\theta)) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2p\lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

so

$$a(\cos(\theta) - 2\lambda) + b\sin(\theta)e^{-\imath\varphi} = 0$$

or

$$\frac{b}{a} = \frac{2\lambda - \cos(\theta)}{\sin(\theta)}e^{\imath\varphi}$$

For right-handed helicity,

$$\frac{b}{a} = \frac{\sin(\theta/2)}{\cos(\theta/2)}e^{\imath\varphi}$$

From $u_D = +\frac{p}{E+m}u_A$, we have

$$u_\uparrow = A_\uparrow\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{\imath\varphi} \\ \frac{p}{E+m}\cos(\theta/2) \\ \frac{p}{E+m}\sin(\theta/2)e^{\imath\varphi} \end{pmatrix}$$

with

$$2E = u_\uparrow^\dagger u_\uparrow = |A_\uparrow|^2\left(1 + \frac{p^2}{E^2+m^2+2Em}\right) = |A_\uparrow|^2\frac{2E}{E+m}$$

so $A_\uparrow = \sqrt{E+m}$.

Lecture 28: Parity in Dirac Fermions

Wednesday, November 11, 2020

With $c \equiv \cos(\theta/2)$ and $s \equiv \sin(\theta/2)$, right-handed spinors can be written

$$u_\uparrow\sqrt{E+M}\begin{pmatrix} c \\ se^{\imath\varphi} \\ \frac{p}{E+M}c \\ \frac{p}{E+M}se^{\imath\varphi} \end{pmatrix}$$

and left-handed as

$$u_\downarrow\sqrt{E+M}\begin{pmatrix} -s \\ ce^{\imath\varphi} \\ \frac{p}{E+M}s \\ \frac{-p}{E+M}ce^{\imath\varphi} \end{pmatrix}$$

For the antiparticle (v) states, the right-handed helicity has

$$\frac{\vec{\mathbf{S}}^{(v)}\cdot\vec{\mathbf{p}}}{p}v_\uparrow = \frac{1}{2}v_\uparrow \implies \frac{\vec{\mathbf{S}}\cdot\vec{\mathbf{p}}}{p}v_\uparrow = -\frac{1}{2}v_\uparrow$$

so

$$v_{\uparrow} = \sqrt{E+M}\begin{pmatrix} \frac{p}{E+M}s \\ -\frac{p}{E+M}ce^{\imath\varphi} \\ -s \\ ce^{\imath\varphi} \end{pmatrix} \qquad v_{\downarrow} = \sqrt{E+M}\begin{pmatrix} \frac{p}{E+M}c \\ \frac{p}{E+M}se^{\imath\varphi} \\ c \\ se^{\imath\varphi} \end{pmatrix}$$

0.33 Intrinsic Parity of (Dirac) Fermions

The parity operator, $\hat{\pi}$, reverses x, y, and z axes. Suppose $\hat{\pi}\psi = \psi'$. Then $\hat{\pi}\psi' = \hat{\pi}^2\psi = I\psi = \psi$.

If ψ satisfies the Dirac equation, $\imath\gamma^\mu\partial_\mu\psi - m\psi = 0$ so $\psi' = \hat{\pi}\psi$ should satisfy

$$\imath\left(\gamma^1\partial_{x'} + \gamma^2\partial_{y'} + \gamma^3\partial_{z'}\right)\psi' - m\psi = -\imath\gamma^0\partial_t\psi'$$

where $x' = -x$ and so on. We can write ψ as $\hat{\pi}\psi'$, so

$$\imath\left(\gamma^1\hat{\pi}\partial_x\psi' + \gamma^2\hat{\pi}\partial_y\psi' + \gamma^3\hat{\pi}\partial_z\psi'\right) - m\hat{\pi}\psi' = -\imath\gamma^0\hat{\pi}\partial_t\psi'$$

Multiply by γ^0 and write x as $-x'$ and so on:

$$\imath\left(-\gamma^0\gamma^1\hat{\pi}\partial_{x'}\psi' - \gamma^0\gamma^2\hat{\pi}\partial_{y'}\psi' - \gamma^0\gamma^3\hat{\pi}\partial_{z'}\psi'\right) - m\gamma^0\hat{\pi}\psi' = -\imath\gamma^0\gamma^0\hat{\pi}\partial_t\psi'$$

The commutator gives us $-\gamma^0\gamma^i = +\gamma^i\gamma^0$, so we can make that substitution across the board. This gives us the same equation as before, but with $\gamma^0\hat{\pi} = kI$ where $k \in \mathbb{Z}$. This means that $\hat{\pi} = {\gamma^0}^2\hat{\pi} = \gamma^0(kI) = k\gamma^0$. Finally, $I = \hat{\pi}^2 = k^2{\gamma^0}^2 = k^2 I$ so $k = \pm 1$. By convention, we choose $k = +1$ so $\hat{\pi} = \gamma^0$.

Acting this operator on u_1 and u_2 (at rest), we find that $\hat{\pi}u_{1,2} = \gamma^0 u_{1,2} = u_{1,2}$, so fermions have positive parity. For anti-particles at rest, $\hat{\pi}v_{1,2} = -v_{1,2}$, so antifermions have negative parity.

0.34 Pseudovectors and Pseudoscalars

Notice that under parity transformations, vectors formed from cross-products remain unchanged: $\mathbf{A} = \vec{\mathbf{v}}\times\vec{\mathbf{v}} \to -\vec{\mathbf{v}}\times-\vec{\mathbf{v}} = \mathbf{A}$. Similarly, dotting a vector and pseudovector generates a pseudoscalar, which *does* switch sign under a parity transformation. Matrix elements could have pseudovectors and pseudoscalar components, so it's possible to create matrix elements which act differently under parity. As it turns out, the electromagnetic and strong forces don't contain such components, but the weak force can be experimentally shown to violate parity because it contains such matrix elements.

0.35 Interaction by Particle Exchange

In non-relativistic quantum mechanics,

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i}\frac{\langle f|V|i\rangle\ \langle j|V|i\rangle}{E_i - E_j}$$

In relativistic quantum mechanics, we replace the static potential, V, by interaction via exchanged particles. The first term has no meaning, while higher-order terms can be interpreted as two interactions of exchanged particles.

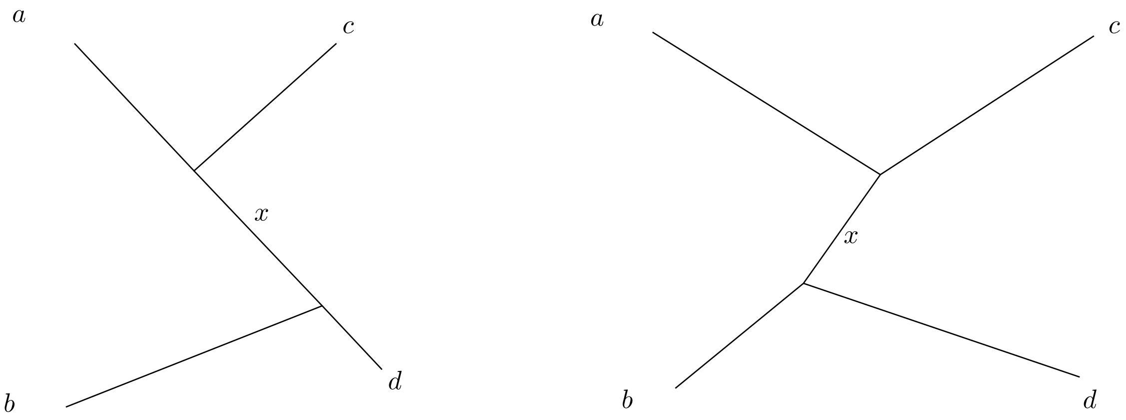


Figure 0.36.1: Time Ordering

0.36 Time-Ordered Perturbation Theory

We start by picturing two orderings of $a + b \to c + d$ by exchange of particle x:

LECTURE 29: PROPAGATORS AND VIRTUAL PARTICLES
Friday, November 13, 2020

The second-order contribution to the transition matrix for $a + b \to c + d$ through the particle x is

$$T_{fi}^{ab} = \frac{\langle f|V|j\rangle \; \langle j|V|i\rangle}{E_i - E_j} = \frac{\langle cd|V|cbx\rangle \; \langle bcx|V|ab\rangle}{(E_a - E_b) - (E_b + E_c + E_x)} = \frac{\langle d|V|x + b\rangle \; \langle c + x|V|a\rangle}{E_a - (E_c + E_x)}$$

Note that $E_j \neq E_i$, and this is allowed for a short period of time by the energy-time uncertainty relation. Another way of thinking of this is that E_x is not well-defined if it is short-lived, so $E_j = E_c + E_b + E_x$ is not well-determined and *could* equal E_i.

The invariant matrix element M_{ji} is found by normalizing all particles (in i or in j) to $\sqrt{2E}$.

$$V_{ji} = \langle c + x|V|a\rangle = \frac{M_{a\to a+x}}{(2E_a 2E_c 2E_x)^{1/2}}$$

or really,

$$\langle bcx|V|ab\rangle = \frac{M_{ab\to bcx}\cancel{\langle b|b\rangle}}{(2E_a \cancel{(2E_b)^2} 2E_c 2E_x)^{1/2}}$$

For the simplest (constant scalar) interaction for $M_{a\to c+x}$,

$$V_{ji} = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$

$$V_{fj} = \frac{g_b}{(2E_b 2E_d 2E_x)^{1/2}}$$

so

$$\begin{aligned} T_{fi}^{ab} &= \frac{\langle dc|V|cbx\rangle \; \langle xcb|V|ab\rangle}{E_a - E_c - E_x} \\ &= \frac{1}{E_a - E_c - E_x} \frac{g_a g_b}{2E_x (2E_a 2E_b 2E_c 2E_d)^{1/2}} \end{aligned}$$

So the Lorentz-invariant matrix element is

$$M_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab} = \frac{g_a g_b}{2E_x(E_a - E_c - E_x)}$$

Similarly,

$$M_{fi}^{ba} = \frac{g_a g_b}{2E_x(E_b - E_d - E_x)}$$

so

$$M_{fi} = M_{fi}^{ab} + m_{fi}^{ba} = \frac{g_a g_b}{2E_x}\left[\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x}\right]$$

We know that $E_a + E_b = E_c + E_d$ so $E_b - E_d = E_c - E_a = -(E_a - E_c)$, so

$$\begin{aligned} M_{fi} &= \frac{g_a g_b}{2E_x}\left[\frac{1}{(E_a - E_c) - E_x} - \frac{1}{(E_a - E_c) + E_x}\right] \\ &= \frac{g_a g_b}{\cancel{2E_x}}\left[\frac{\cancel{2E_x}}{(E_a - E_c)^2 - E_x^2}\right] \end{aligned}$$

Finally $E_x^2 = \vec{\mathbf{p}}_x^2 + m_x^2 = (\vec{\mathbf{p}}_a - \vec{\mathbf{p}}_c)^2 + m_x^2$, so

$$\begin{aligned} M_{fi} &= \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{\mathbf{p}}_a - \vec{\mathbf{p}}_c)^2 - m_x^2} \\ &= \frac{g_a g_b}{q^2 - m_x^2} \end{aligned}$$

and so a propagator is born:

$$\frac{1}{q^2 - m_x^2}$$

always appears when we exchange a particle with rest mass m_x. In QED, we will be only exchanging photons with $m_\gamma = 0$, but this isn't true for weak interactions. However, the weak interaction masses are large, which makes this propagator small, which is why it is "weak".

0.37 Virtual Particles

Feynman diagrams represent the sum over all time-orderings, which may require $\bar{x}$ to be exchanged rather than x (not true for photons since they are their own antiparticle). 4-momentum must be conserved at every vertex, but E is not conserved at vertices of time-ordered diagrams. $q = p_a - p_c = p_d - p_b$, so $q^2 \neq m_x^2$ for Feynman diagrams. Then x is off the "mass-shell". We call this a virtual particle, which may be a mathematical construct resulting from representing interacting particles in a basis of eigenstates of the unperturbed Hamiltonian.

For s-channel annihilation, $q = p_1 + p_2 = p_3 + p_4$ so $q^2 = s > 0$. Therefore, x is time-like.

For t-channel and u-channel scattering, $q = p_1 - p_3 = p_4 - p_2$, so it can be shown that $q^2 = t < 0$.

0.38 Quantum Electrodynamics

From the above discussion, the Lorentz-invariant matrix element for $a + b \to c + d$ by exchange of x can be written as

$$M = \frac{\langle\psi_c|V|\psi_a\rangle\ \langle\psi_d|V|\psi_b\rangle}{q^2 - m_x^2}$$

For QED (photon-exchange), $m_x = 0$.

Lecture 30: Feynman Rules for QED

Monday, November 16, 2020

For QED, we can write the free-photon field A_μ in terms of the four-vector polarization of state λ, $\varepsilon_\mu^{(\lambda)}$,

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{\imath(\vec{\mathbf{p}}\cdot\vec{\mathbf{r}} - Et)}$$

For a real photon (not virtual), $\vec{\varepsilon} \perp \vec{\mathbf{p}}$ (transverse, as opposed to virtual protons, which can have a longitudinal component). For a real photon propagating in the z-direction, we have two possible polarization vectors,

$$\varepsilon^{(1)} = (0,1,0,0) \qquad \varepsilon^{(2)} = (0,0,1,0)$$

Any other polarizations are linear combinations of these. For example, circular polarization is $\frac{1}{\sqrt{2}}(\varepsilon^{(1)} \pm \imath\varepsilon^{(2)})$.

The interaction of the charge q and the EM field by a four-vector potential $A^\mu = (\Phi, \vec{\mathbf{A}})$ is found by minimal substitution: $\partial_\mu = \partial_\mu + \imath q A_\mu$ where $A_\mu = (\Phi, -\vec{\mathbf{A}})$ while $\partial_\mu = \left(\frac{\partial}{\partial t}, +\vec{\boldsymbol{\nabla}}\right)$. The free-particle Dirac equation tells us

$$(\imath\gamma^\mu\partial_\mu - m)\psi = 0$$

so this becomes

$$(\gamma^\mu\partial_\mu\psi + \imath q\gamma^\mu A_\mu\psi + \imath m\psi) = 0$$

(having multiplied through by $-\imath$). If we then multiply by $\imath\gamma^0$, we get

$$\imath\partial_t\psi + \imath\gamma^0\vec{\gamma}\cdot\vec{\boldsymbol{\nabla}}\psi - q\gamma^0\gamma^\mu A_\mu\psi - m\gamma^0\psi = 0$$

or

$$\imath\partial_t\psi = \hat{H}\psi$$

where

$$\hat{H} = \underbrace{(m\gamma^0 - \imath\gamma^0\vec{\gamma}\cdot\vec{\boldsymbol{\nabla}})}_{H^0} + \underbrace{q\gamma^0\gamma^\mu A_\mu}_{H^1}$$

$$\langle\psi(p_3)|H^1|\psi(p_1)\rangle = u_e^\dagger(p_3)\underbrace{Q_e}_{-1} e\gamma^0\gamma^\mu\varepsilon_\mu^{(\lambda)*}u_e(p_1)$$

is the interaction matrix element for an electron, p_1, scattering by interaction with the EM field to electron p_3. If we write these as plane waves, $e^{\imath(\vec{\mathbf{p}}_1\cdot\vec{\mathbf{r}} - E_1 t)}$, $e^{-\imath(\vec{\mathbf{p}}_3\cdot\vec{\mathbf{r}} - E_3 t)}$, and $e^{-\imath(\vec{\mathbf{q}}\cdot\vec{\mathbf{r}} - \omega t)}$, these cancel iff $\vec{\mathbf{q}} = \vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_3$ and $\omega = E_1 - E_3$ (four-momentum conservation).

For $e^-\tau^- \to e^1\tau^1$, the τ^--γ vertex interaction is $u_\tau^\dagger(p_4)Q_\tau e\gamma^0\gamma^\nu\varepsilon_\nu^{(\lambda)}u_\tau(p_2)$. M is found by summing over the time-orderings (the propagator!) and all internal polarizations (in this case, the virtual photon polarizations):

$$M = \sum_\lambda u_e^\dagger(p_3)Q_e e\gamma^0\gamma^\mu u_e(p_1)\epsilon_\mu^{(\lambda)*}\frac{1}{q^2}\varepsilon_\nu^{(\lambda)}u_\tau^\dagger(p_4)Q_\tau e\gamma^0\gamma^\nu u_\tau(p_2)$$

According to the textbook, this reduces to $\sum_\lambda \epsilon_\mu^{(\lambda)*}\epsilon_\nu^\lambda = -g_{\mu\nu}$, so

$$M = Q_e Q_\tau e^2 \underbrace{\left[\underbrace{\bar{u}_e}_{u^\dagger\gamma^0}(p_2)\gamma^\mu u_e(p_1)\right]}_{j_e^\mu} - \frac{g_{\mu\nu}}{q^2}\left[\bar{u}_\tau(p_4)\gamma^\nu u_\tau(p_2)\right]$$

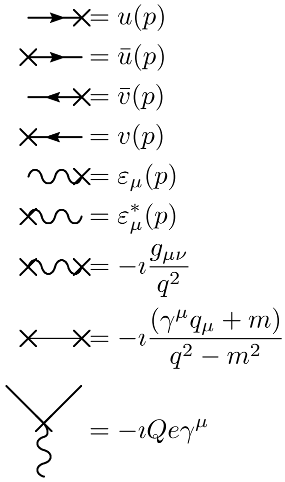
The four-current j^μ is a covariant four-vector, so

$$M = Q_e Q_\tau e^2 j_e^\mu \frac{g_{\mu\nu}}{q^2} j_\tau^\nu = -Q_e Q_\tau e^2 j_e^\mu j_{\tau\mu}\frac{1}{q^2} = -Q_e Q_\tau e^2 \frac{j_e\cdot j_\tau}{q^2}$$

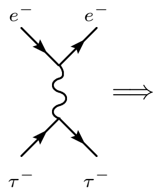
Then, if spins are not determined, take an average over initial spins and sum over final spins.

0.39 Feynman Rules for QED

The product of these factors will be $-\imath m$. We say that initial-state particles are $u(p)$ (arrows entering a vertex) and $\bar{u}(p)$ (arrows leaving a vertex). For an initial-state antiparticle, we have $\bar{v}(p)$ (backwards arrows entering a vertex) and for final-state antiparticles we have $v(p)$ (backward arrows leaving a vertex). Finally, for initial-state photons, we have $\varepsilon_\mu(p)$ (squiggle entering a vertex) and $\varepsilon^*_\mu(p)$ (squiggle leaving a vertex). The photon propagator is $-\imath\frac{g_{\mu\nu}}{q^2}$ (a squiggle between two vertices), and the fermion propagator is $-\imath\frac{(\gamma^\mu q_\mu + m)}{q^2 - m^2}$. Finally, we write the QED vertex as $-\imath Q e\gamma^\mu$:

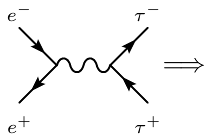


When presented with a Feynman diagram, we can use these Feynman rules to write out all the parts of the matrix element. For the electron-tauon scattering described above, we have



$$
\begin{aligned}
-\imath M &= [\bar{u}_e(p_3)(\underbrace{\imath e\gamma^\mu}_{-\imath Q_e} e\gamma^\mu)u_e(p_1)] - \imath\frac{g_{\mu\nu}}{q^2}[\bar{u}_\tau(p_4)(\underbrace{\imath e\gamma^\nu}_{-\imath Q_\tau e\gamma^\nu})u_\tau(p_2)] \\
m &= -e^2[\bar{u}_e(p_3)\gamma^\mu u_e(p_1)]\frac{g_{\mu\nu}}{q^2}[\bar{u}_\tau(p_4)\gamma^\nu u_\tau(p_2)]
\end{aligned}
$$

For another example, consider electron-positron annihilation into tauon-antitauon pair production. Now



$$
-\imath M = [\bar{v}_e(p_2)(\imath e\gamma^\mu)u_e(p_1)]\left(\frac{-\imath g_{\mu\nu}}{q^2}\right)[\bar{u}_\tau(p_3)(\imath e\gamma^\nu)v_\tau(p_4)]
$$

In general, you if you ever draw two arrows facing each other, you've messed up. Start and the end of a fermion line and move backwards along the arrows. The first arrow you move backwards along is an adjoint (you then have to decide if it's a particle or antiparticle).

Lecture 31: Missing Lecture
Friday, November 20, 2020

[Missed Lecture]

LECTURE 32: MUON-ELECTRON INTERACTIONS
Friday, November 20, 2020

For the anti-muon, the right-handed helicity is

$$v_{\uparrow}(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}$$

and the left-handed spinor is

$$v_{\downarrow} = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

where $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

This allows us to calculate j_e for each of the four initial-state helicity combinations (which we average) and j_μ for each of the four final-state helicity combinations (which we sum).

0.40 Muon and Electron Currents

The matrix element for a particular helicity combination of $e^-e^+ \to \mu^-\mu^+$ can be written $M_{fi} = -\frac{e^2}{s}(j_e \cdot j_\mu)$ where $j_e^\mu = \bar{v}(p_2)\gamma^\mu u(p_1)$ and $j_\mu^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$ are calculated with the combinations of right and left-handed helicity spinors found above.

In general, combinations of $\bar{\psi}\gamma^\mu\varphi$ can be evaluated explicitly using the Dirac-Pauli representation as

$$j^0 = \bar{\psi}\gamma^0\varphi = \psi^\dagger\gamma^0\gamma^0\varphi = \psi^\dagger\varphi$$
$$j^1 = \bar{\psi}\gamma^1 = \psi^\dagger\gamma^0\gamma^1\varphi = \psi_0^*\varphi_3 + \psi_1^*\varphi_2 + \psi_2^*\varphi_1 + \psi_3^*\varphi_0$$

and so on.

To calculate the matrix elements, we need to take into account all 16 helicity combinations between pairs of right or left-handed particles going to pairs of right/left-handed particles. We must find j_e and j_μ for each of the four possible combinations. For example, if the final state has $\mu_\uparrow^-\mu_\downarrow^+$,

$$\begin{aligned} j_\mu^0 &= \bar{u}_\uparrow(p_2)\gamma^0 v_\downarrow(p_4) \\ &= E\begin{pmatrix} c & s & c & s \end{pmatrix}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix} \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} j_\mu^1 &= \bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) \\ &= E(-c^2 + s^2 - c^2 + s^2) \\ &= 2E(s^2 - c^2) \\ &= -2E\cos(\theta) \end{aligned}$$

Doing all of these, we find that

$$j_{\mu,RL} = 2E\begin{pmatrix}0 & -\cos(\theta) & \imath & \sin(\theta)\end{pmatrix}$$

You can also show that

$$j_{\mu,RR} = \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix} = j_{\mu,LL}$$

These matrix elements only vanish in the relativistic limit, so they aren't necessarily zero at lower energies.

$$j_{\mu,LR} = 2E\begin{pmatrix}0 & -\cos(\theta) & -\imath & \sin(\theta)\end{pmatrix}$$

We could similarly calculate j_e by brute force:

$$[\bar{u}\gamma^\mu v]^\dagger = v^\dagger(\gamma^\mu)^\dagger\gamma^0 u$$

If $\mu = 0$, $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$. If $\mu \neq 0$, we still get $\gamma^0\gamma^\mu$, so

$$[\bar{u}\gamma^\mu v]^\dagger = v^\dagger\gamma^0\gamma^\mu u = \bar{v}\gamma^\mu u$$

We can then find the electron current, $j_e^\nu = \bar{v}_e\gamma^\nu u_e$ by taking the Hermitian conjugate of j_μ^ν and setting $\theta = 0$:

$$\begin{aligned}
j_{e,RL} &= 2E\begin{pmatrix}0 & -1 & -\imath & 0\end{pmatrix}\\
j_{e,LR} &= 2E\begin{pmatrix}0 & -1 & \imath & 0\end{pmatrix}
\end{aligned}$$

Now we have all the currents, so we can calculate the matrix element!

0.41 Electron-Muon Production Cross-Section

Of the 16 possible helicity combinations, only 4 have non-zero currents: $RL \to RL$, $RL \to LR$, $LR \to RL$, and $LR \to LR$. For the first,

$$M_{RL\to RL} = -\frac{e^2}{s(=(2E)^2)} j_e^\mu g_{\mu\nu} j_\mu^\nu = -e^2(-\cos(\theta) - 1) = e^2(1 + \cos(\theta))$$

$s = (2E)^2$ in the center of mass frame or for symmetric colliders. The same can be said about $M_{LR\to LR}$ by parity.

$$M_{RL\to LR} = M_{LR\to RL} = e^2(1 - \cos(\theta))$$

Since helicity states are orthogonal,

$$\begin{aligned}
|M_{RL\to RL} + M_{LR\to LR} + M_{RL\to LR} + M_{LR\to RL}|^2 &= |M_{RL\to RL}|^2 + |M_{LR\to LR}|^2 + |M_{RL\to LR}|^2 + |M_{LR\to RL}|^2\\
&= e^4\left[2(1+\cos(\theta))^2 + 2(1-\cos(\theta))^2\right]\\
&= 2e^4\left[2 + 2\cos^2(\theta)\right]
\end{aligned}$$

We have to average over the four initial combinations, even though two of them are zero:

$$\left\langle |M_{fi}|^2 \right\rangle = e^4(1 + \cos^2(\theta))$$

where $e^2 = 4\pi\alpha$, so

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\cancelto{1}{\frac{p_f^*}{p_i^*}}\left\langle |M_{fi}|^2 \right\rangle\\
&= \frac{\alpha^2}{4s}\left(1 + \cos^2(thetai)\right)
\end{aligned}$$

since $\frac{p_f^*}{p_i^*} \to 1$ in the relativistic limit.

LECTURE 33:
Monday, November 23, 2020

When we left off, we had found the average matrix element for $e^+e^- \to \mu^+\mu^-$ and the differential cross-section (and they work pretty okay, especially if we then add in electroweak corrections. We can now find the total cross section:

$$\begin{aligned}\sigma_{\text{tot}} &= \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega \\ &= 2\pi\int_{-1}^{1}\frac{alpah^2}{4s}(1+\cos^2(\theta))\,\mathrm{d}(\cos(\theta)) \\ &= \frac{\pi\alpha^2}{2s}\left(2+\frac{2}{3}\right) \\ &= \frac{4}{3}\frac{\pi\alpha^2}{s}\end{aligned}$$

This turns out to work pretty well.

0.42 Lorentz-Invariant Average Matrix Element

We have $\left\langle |M_{fi}|^2 \right\rangle = e^4(1+\cos^2(\theta))$ with

$$\begin{aligned}p_1 &= (E,0,0,E) \\ p_2 &= (E,0,0,-E) \\ p_3 &= (E,E\sin(\theta),0,E\cos(\theta)) \\ p_4 &= (E,-E\sin(\theta),0,-E\cos(\theta))\end{aligned}$$

so

$$\frac{(p_1\cdot p_3)^2+(p_1\cdot p_4)^2}{(p_1\cdot p_2)^2} = \frac{1+\cos^2(\theta)}{2}$$

Therefore,

$$\left\langle |M_{fi}|^2 \right\rangle = 2e^4\frac{(p_1\cdot p_3)^2+(p_1\cdot p_4)^2}{(p_1\cdot p_2)^2}$$

with $s=(p_1+p_2)^2 = m_1^2+m_2^2+2p_1\cdot p_2 \approx 2p_1\cdot p_2$ and $t=(p_1-p_3)^2\approx -2p_1\cdot p_3$ and $u=(p_1-p_4)^2\approx -2p_1\cdot p_4$, so

$$\left\langle |M_{fi}|^2 \right\rangle = 2e^4\frac{t^2+u^2}{s^2}$$

0.43 Spin in electron-positron Annihilation

The non-vanishing j_e contributions are from a right-handed electron and left-handed positron $|S,S_z\rangle = |1,+1\rangle$ and a left-handed electron and right-handed positron with $|S,S_z\rangle = |1,-1\rangle$. The non-vanishing j_μ contributions have the muon going off at an angle which is not necessarily the z-axis, but we can look at the components along their axis: $|S,S_z\rangle_\theta = |1,\pm1\rangle$. It is tempting to think that $|1,1\rangle_{90^\circ}$ is orthogonal to $|1,1\rangle_z$, but when you perform the math on this, you'll find that the dot product of these vectors is 1/2. In general,

$$_\theta\left\langle 1,1|1,1\right\rangle_z = \frac{1}{2}(1+\cos(\theta)) = M_{LR\to LR}$$

[Notes Incomplete, need to finish later]

Lecture 34: Helicity and Chirality

Monday, November 30, 2020

In the relativistic approximation, helicity and chirality are the same. Without it,

$$u_\uparrow = N \begin{pmatrix} c \\ se^{\imath phi} \\ kc \\ kse^{\imath\varphi} \end{pmatrix}$$

and

$$\gamma^5 u_\uparrow = N \begin{pmatrix} kc \\ kse^{\imath\varphi} \\ c \\ se^{\imath\varphi} \end{pmatrix}$$

so

$$(1 \pm \gamma^5) u_\uparrow = N \left(\begin{pmatrix} c \\ se^{\imath phi} \\ kc \\ kse^{\imath\varphi} \end{pmatrix} \pm \begin{pmatrix} kc \\ kse^{\imath\varphi} \\ c \\ se^{\imath\varphi} \end{pmatrix} \right)$$

Then

$$P_R u_\uparrow = \frac{1}{2}(1+\gamma^5)u_\uparrow = \frac{1}{2}(1+k)N \begin{pmatrix} c \\ se^{\imath\varphi} \\ c \\ se^{\imath\varphi} \end{pmatrix} = \frac{1}{2}(1+k)u_R \frac{N}{\sqrt{E}}$$

and

$$P_L u_\uparrow = \frac{1}{2}(1-\gamma^5)u_\uparrow = \frac{1}{2}(1-k)u_L \frac{N}{\sqrt{E}}$$

We know that the sum of the right-handed component and left-handed component of a spinor must equal the spinor, so

$$u_\uparrow = \frac{1}{2}\left[(1+k)u_R + (1-k)u_L\right] \frac{N}{\sqrt{E}}$$

In there relativistic limit, $k \to 1$ and $N \to \sqrt{E}$, so $u_\uparrow \to u_R$.

We are skipping the section (6.5) on *"Trace Techniques"*. In this section, the textbook shows that the sums over all the initial and final state spins and averages can be written in terms of a trace of a four-by-four matrix:

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m$$

where $\not{p} \equiv \gamma^\mu p_\mu$, and similarly,

$$\sum_{r=1}^{2} v_r(p)\bar{v}_r(p) = \not{p} - m$$

Then, for example,

$$j_e^\mu \cdot j_f^\nu = \mathrm{Tr}((p_2 - m_2)\gamma^\mu(p_1 + m_1)\gamma^\nu)$$

which can be simplified using handy trace identities:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}\big(\gamma^5\big) = \mathrm{Tr}\big(\gamma^5\gamma^\mu\gamma^\nu\big) = 0$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

Also, the trace of σ^5 times any odd number of γ matrices is 0 and

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\imath\varepsilon^{\mu\nu\rho\sigma}$$

We don't need to know any of these trace identities for this class.

0.44 Crossing Symmetry

In section 6.5.6, we revisit crossing symmetry. Recall the calculation we did for electron-positron annihilation into fermion-antifermion. We could also examine the "unrelated" diagram of electron-fermion scattering (the same diagram rotated).

For the original annihilation diagram, it is clear that

$$j_e \cdot j_f \propto \bar{v}_e(p_2)\gamma^\mu u_e(p_1)\bar{u}_f(p_3)\gamma^\nu v(p_4)$$

For the scattering diagram, we have

$$j_e \cdot j_f \propto \bar{u}_e(p_3)\gamma^\mu u_e(p_1)\bar{u}_f(p_4)\gamma^\nu u_f(p_2)$$

We can try to match up the terms in each of these equations. Above, we showed that we can insert complete sets of particle or antiparticle states as the identity. Clearly $p_1 \to p_1$ and $p_3 \to p_4$, but $\not{p} - m \to -(\not{p} + m)$ gives us $p_2 \to -p_3$ and $p_4 \to -p_2$.

Under these transformations,

$$s = (p_1 + p_2)^2 \to (p_1 - p_3)^2 = t'$$

(where t' is t for the scattering diagram),

$$t = (p_1 - p_3)^2 \to (p_1 - p_4)^2 = u'$$

and

$$u = (p_1 - p_4)^2 \to (p_1 + p_2)^2 = s'$$

For $e^-e^+ \to f\bar{f}$, we found that

$$\left\langle |m_{fi}|^2 \right\rangle = 2e^4 \frac{t^2 + u^2}{s^2}$$

Using this symmetry argument, we can predict that the matrix element for $e^- f \to e^- f$ is

$$\left\langle |m_{fi}|^2 \right\rangle = 2e^4 \frac{u'^2 + s'^2}{t'^2}$$

0.45 Elastic Electron Scattering

0.45.1 Rutherford & Mott Scattering

These forms of scattering are electron-proton scattering where the electron does not have enough energy to really cause any effect on the proton, so the recoil is negligible ($Q^2 = -q^2 \ll m_p^2$).

Again, let

$$\kappa \equiv \frac{p_e}{E_e + m_e}$$

and

$$N = \sqrt{E + m_e}$$

so that

$$u_\uparrow(p_1) = N\begin{pmatrix}1\\0\\\kappa\\0\end{pmatrix} \qquad u_\downarrow(p_1) = N\begin{pmatrix}0\\1\\0\\-\kappa\end{pmatrix}$$

Define the scattering angle as $(\theta, \varphi) = (\theta, 0)$ and $c \equiv \cos(\theta/2)$ and $s \equiv \sin(\theta/2)$, such that

$$u_\uparrow(p_3) = N\begin{pmatrix}c\\s\\\kappa c\\\kappa s\end{pmatrix} \qquad u_\downarrow(p_3) = N\begin{pmatrix}s\\c\\\kappa s\\-\kappa c\end{pmatrix}$$

Calculating $j_{e,\alpha\beta} = \bar{u}_\alpha(p_3)\gamma^\mu u_\beta(p_1)$ explicitly, we find

$$\begin{aligned} j_{e,\uparrow\uparrow} &= N^2\Big((\kappa^2+1)c \quad 2\kappa s \quad 2\imath\kappa s \quad 2\kappa c\Big) \\ j_{e,\downarrow\downarrow} &= N^2\Big((\kappa^2+1)c \quad 2\kappa s \quad -2\imath\kappa s \quad 2\kappa c\Big) \\ j_{e,\downarrow\uparrow} &= N^2\Big((1-\kappa^2)s \quad 0 \quad 0 \quad 0\Big) \\ j_{e,\uparrow\downarrow} &= N^2\Big((\kappa^2-1)s \quad 0 \quad 0 \quad 0\Big) \end{aligned}$$

The non-zero helicity-flip for small κ disappears as $\kappa \to 1$.

If we neglect recoil, we have $\kappa_p = \kappa_{p'} = 0$, so in the lab frame

$$u_\uparrow(p_2) = \sqrt{2m_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_\downarrow(p_2) = \sqrt{2m_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

We also take the recoil direction into account, although the momentum is negligible, as $(\theta, \varphi) = (\eta, \pi)$.

Lecture 35:

Wednesday, December 02, 2020

Continuing from last lecture,

$$u_\uparrow(p_4) = \sqrt{2m_p}\begin{pmatrix}c_\eta\\-s_\eta\\0\\0\end{pmatrix} \qquad u_\downarrow(p_4) = \sqrt{2m_p}\begin{pmatrix}-s_\eta\\-c_\eta\\0\\0\end{pmatrix}$$

where $c_\eta = \cos(\eta/2)$ and $s_\eta = \sin(\eta/2)$. Then

$$\begin{aligned} j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} &= 2m_p\Big(c_\eta \quad 0 \quad 0 \quad 0\Big) \\ j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} &= -2m_p\Big(s_\eta \quad 0 \quad 0 \quad 0\Big) \end{aligned}$$

Then

$$\begin{aligned} \Big\langle |m_{fi}|^2 \Big\rangle &= \frac{1}{4}\frac{e^2}{q^2}\sum_{\alpha,\beta,\gamma,\delta} j_{e_{\alpha\beta}} \cdot j_{p_{\gamma\delta}} \\ &= \frac{4m_p^2 m_e^2 e^4(\gamma_e+1)}{q^2}\left((1-\kappa^2)^2 + 4\kappa^2 c^2\right) \end{aligned}$$

Using $\kappa = \frac{p_e}{E_e+m_e} = \frac{\gamma\beta_e}{\gamma+1}$ and $Q^2 = -q^2 = 4p^2\sin^2(\theta/2)$, we have

$$\left\langle |m_{fi}|^2 \right\rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4(\theta/2)} \left[\beta^2\gamma^2\cos^2(\theta/2) + 1\right]$$

In Rutherford scattering, $\beta \ll 1$ ($E_e < 100$keV), so we can neglect the first term in square brackets:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{16E_k^2\sin^4(\theta/2)}$$

where $E_k = \frac{p^2}{2m_e}$, where p is the momentum of the electron.

In Mott scattering, we consider relativistic electrons (still with a recoil-less target), so $\gamma \gg 1$. This puts E_k somewhere between 5MeV and $\sim$ 50MeV. In this case,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_k^2\sin^4(\theta/2)}\cos^2(\theta/2)$$

This result is often written σ_{Mott}, which is bad notation because it's technically a differential cross-section.

0.46 Relativistic Elastic Scattering

We will now consider relativistic scattering from a target particle (can be a proton, neutron, nucleus, etc.) without neglecting target recoil.

$$\begin{aligned}
p_1 &= \begin{pmatrix} E_1 & 0 & 0 & E_1 \end{pmatrix} \\
p_2 &= \begin{pmatrix} m_t & 0 & 0 & 0 \end{pmatrix} \\
p_3 &= \begin{pmatrix} E_3 & 0 & E_3\sin(\theta) & E_3\cos(\theta) \end{pmatrix} \\
p_4 &= \begin{pmatrix} E_4 & \vec{\mathbf{p}}_4 \end{pmatrix} = \begin{pmatrix} m_t + w & \vec{\mathbf{q}} \end{pmatrix}
\end{aligned}$$

Start by treating the target as a point Dirac particle. Assuming single-arm, unpolarized scattering. We've already done the matrix element calculation:

$$\left\langle |m_{fi}|^2 \right\rangle = \frac{8e^4}{q^4}\left((p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m_t^2(p_1\cdot p_3)\right)$$

where $p_1 \cdot p_2 = m_t E_1$ and $p_2 \cdot p_3 = m_t E_3$.

Since $p_4 = p_2 + (p_1 - p_3)$, we can write

$$p_3 \cdot p_4 = p_3\cdot p_2 + p_3\cdot p_1 - \cancelto{0}{p_3^2} = m_t E_3 + E_1E_3(1-\cos(\theta))$$

and similarly

$$p_1 \cdot p_4 = m_t E_1 + E_1E_3(1-\cos(\theta))$$

For elastic scattering, $w \equiv E_1 - E_3 = \frac{Q^2}{2m_t}$.

$$\left\langle |m_{fi}|^2 \right\rangle = \frac{m_t^2 e^4}{E_1E_3\sin^4(\theta/2)}\left[\cos^2(\theta/2) + \frac{Q^2}{2m_t^2}\sin^2(\theta/2)\right]$$

Then we can determine the lab-frame differential cross-section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{m_t E_1}\right)^2 \left\langle |m_{fi}|^2 \right\rangle$$

so

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)}\frac{E_3}{E_1}\left(\cos^2(\theta/2) + \frac{Q^2}{2m_t^2}\sin^2(\theta/2)\right) \\ &= \sigma_{\text{Mott}}\eta_{\text{recoil}}\left(1 + 2\tau + \tan^2(\theta/2)\right)\end{aligned}$$

where $\eta_{\text{recoil}} = E_3/E_1 = \frac{1}{1+\frac{2E_1}{m_t}\sin^2(\theta/2)}$ and $\tau = \frac{Q^2}{4m_t}$.

0.47 Separation of Form Factors

The most general 4-current at the target which is consistent with Lorentz-invariance, parity conservation, etc. is:

$$j_t^\mu = \bar{u}_t(p_4)\left[F_1(Q^2)\gamma^\mu + F_2(Q^2)\frac{\imath\sigma^{\mu\nu}q^\nu}{2m_t}\right]u_t(p_2)$$

where $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$. These F_1 and F_2 are called Dirac form-factors. If the target were a point particle, they would be 1. Calculating matrix elements with this general current (assuming single-photon exchange) and the differential cross section as above, we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sigma_{\text{Mott}}\eta_{\text{recoil}} \times \left[F_1^2(Q^2) + \tau F_2^2(Q^2) + 2\tau(F_1(Q^2) + F_2(Q^2))^2\tan^2(\theta/2)\right]$$

We can define the Sachs (electric and magnetic) form-factors as

$$G_M(Q^2) \equiv F_1(Q^2) + F_2(Q^2)$$

and

$$G_E(Q^2) \equiv F_1(Q^2) - \tau F_2(Q^2)$$

so

$$\begin{aligned}\frac{G_E^2 + \tau G_M^2}{1+\tau} &= \frac{(F_1^2 + \tau^2 F_2^2 - 2\tau F_1 F_2) + \tau(F_1^2 + F_2^2 + 2F_1F_2)}{1+\tau} \\ &= \frac{F_1^2(1+\tau) + \tau F_2^2(1+\tau)}{1+\tau} \\ &= F_1^2 + \tau F_2^2\end{aligned}$$

so

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \sigma_{\text{Mott}}\eta_{\text{recoil}} \times \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2\tan^2(\theta/2)\right] \\ &= \sigma_{\text{Mott}}\eta_{\text{recoil}}\frac{1}{1+\tau}\left[G_E^2 + \tau(1 + 2(1+\tau))\tan^2(\theta/2)\right]\end{aligned}$$

We can then define $\epsilon = \left[1 + 2(1+\tau)\tan^2(\theta/2)\right]^{-1}$, which is the virtual photon's longitudinal polarization, to get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sigma_{\text{Mott}}\frac{\eta_{\text{recoil}}}{\epsilon(1+\tau)}\left[\epsilon G_E^2 + \tau G_M^2\right]$$

Lecture 36:

Friday, December 04, 2020

If you were to plot $\epsilon(1+\tau)\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}/(\eta\sigma) = [\epsilon G_E^2 + \tau G_m^2]$ vs. ϵ, you would find that the slope is G_E^2 and the intercept is τG_M^2. This Rosenbluth separation allows separation of G_E and G_M (and hence F_1 and F_2) by making multiple measurements of the cross section at the same Q^2 but at different E_1 and θ, which means different ϵ.

0.48 Non-Relativistic Interpretation of the Form Factor

If $\vec{\mathbf{q}}^2 = w^2 - q^2 = w^2 + Q^2 = \left(\frac{Q^2}{2m_t}\right)^2 + Q^2 = Q^2 + \left(1 + \frac{Q^2}{(2m_t)^2}\right)$, then if $Q \ll 2m_t$, $\vec{\mathbf{q}}^2 \approx Q^2$.

Non-relativistically, for charge $Q_t\rho(\vec{\mathbf{r}})$,

$$V(\vec{\mathbf{r}}) = \int \frac{Q_t\rho(\vec{\mathbf{r}}')}{4\pi|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \, \mathrm{d}^3 r'$$

In the Born Approximation (plane wave initial and final states),

$$\begin{aligned}
\mathcal{M}_{fi} &= \langle \psi_f | V(r) | \psi_i \rangle \\
&= \int e^{-\imath \vec{\mathbf{p}}_3 \cdot \vec{\mathbf{r}}} V(r) e^{\imath \vec{\mathbf{p}}_1 \cdot \vec{\mathbf{r}}} \, \mathrm{d}^3 r \\
&= \int \mathrm{d}^3 r \, e^{\imath \vec{\mathbf{q}} \cdot \vec{\mathbf{r}}} \int \mathrm{d}^3 r' \, \frac{Q_t \rho(r')}{4\pi|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \\
&= \int \mathrm{d}^3 r \int \mathrm{d}^3 r' \, e^{\imath \vec{\mathbf{q}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}')} e^{\imath \vec{\mathbf{q}} \cdot \vec{\mathbf{r}}'} \frac{Q_t \rho(r')}{4\pi|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}
\end{aligned}$$

Now let $\vec{\mathbf{R}} \equiv \vec{\mathbf{r}} - \vec{\mathbf{r}}'$.

$$\mathcal{M}_{fi} = \underbrace{\int \mathrm{d}^3 R \, e^{\imath \vec{\mathbf{q}} \cdot \vec{\mathbf{R}}} \frac{Q_t}{4\pi R}}_{\text{point-charge } \mathcal{M}_{fi}} \times \underbrace{\int \mathrm{d}^3 r' \, \rho(r') e^{\imath \vec{\mathbf{q}} \cdot \vec{\mathbf{r}}'}}_{G_E(\vec{\mathbf{q}}^2)}$$

Therefore,

$$\begin{aligned}
G_E(\vec{\mathbf{q}}^2) &= \int \rho(r) e^{\imath \vec{\mathbf{q}} \cdot \vec{\mathbf{r}}} \, \mathrm{d}^3 r \\
&= 2\pi \int_0^\infty \mathrm{d}r \int_{-1}^1 \mathrm{d} \underbrace{\eta}_{\cos(\theta)} \, e^{\imath q r \eta} \rho(r)
\end{aligned}$$

Really,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_0 \frac{\epsilon G_E^2 + \tau G_M^2}{\epsilon(1+\tau)}$$

where the non-relativistic G_M is similar to a Fourier transform of magnetization density (due to spins and currents). Still, G_E and G_M can be separated by Rosenbluth (or other ways).

In the limit as $|\vec{\mathbf{q}}| \to 0$, $G_E \to \frac{Q_N}{e}$, which is 1 for the proton and 0 for the neutron. Likewise, $G_M \to \frac{\mu_{n/p}}{\mu_N}$ which is 2.793 for the proton and -1.913 for the neutron, where $\mu_N = \frac{e\hbar}{2m_N} \ll \mu_B \equiv \frac{e\hbar}{2m_e}$, which is not what Dirac would predict (he would say this is equal to G_E in this limit).

Lecture 37:

Monday, December 07, 2020

The proton radius as measured by hydrogen and muon spectroscopy may or may not agree, depending on recent (2019) findings. For nuclei, where G_E may dominate, $\rho(r)$ can be easily reconstructed with an inverse Fourier transform constrained to fit the charge distribution. For nucleons, the large magnetic form factor is consistent with the Fourier transform of an exponential distribution:

$$G_M^{n/p} \propto G_{\text{dipole}} \equiv \frac{1}{(1 - Q^2/Q_0^2)^2}$$

with $Q_0 = 0.84\text{GeV}$, so $G_M^p = \frac{\mu_p}{\mu_N} G subb\text{dipole} \sim 2.793 G_{\text{dipole}} = (1+\kappa) G_{\text{dipole}}$ where $\kappa = 1.793$ is called the anomalous magnetic moment. Similarly, $G_M^n = \frac{\mu_n}{\mu_N} G_{\text{dipole}} \sim -1.913 G_{\text{dipole}}$.

There is a small contribution from G_E^p (especially small when $\tau G_M^2 \gg G_E^2$) which *seemed* to be well-described by $G_E^p = G_{\text{dipole}}$. For the neutron, G_E^n must be zero for $\vec{\mathbf{q}}^2 - 0$. This suggests a simple structure of $\rho(r) = Ae^{-r/a}$ with a cusp at the origin. However, this formulation assumed single-photon exchange.

The, an alternate technique to determine these form factors was developed measuring transfer of polarization from a polarized electron beam. This derivation automatically gives the ratio between the form factors with no Rosenbluth separation needed. This experiment (JLab Hall A) gives seems to show that G_E is not a dipole. Two-photon exchange makes a several percent change in the cross-section. Also, these effects happen to have the same ϵ-dependence as expected due to G_M. The overall cross-section is about right, and so is G_M, but extracting G_E from Rosenbluth separation doesn't quite give the right answer.

Relativistically, the form factors are not Fourier transforms of ρ and m, except it can be shown that in the "Breit frame" (a.k.a. the brick-wall frame), in which $E_3 = E_1$, G_E is exactly a Fourier transform of ρ. However, the Breit frame is different for each Q^2, so the Fourier transform can't be inverted. There is no simple interpretation of the form factors in the relativistic (target recoil) regime. For high Q^2,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \sigma_{\text{Mott}}\eta_{\text{recoil}}\frac{Q^2}{2m_t}G_M^2\tan^2(\theta/2)$$

If $G_M^p \propto G_{\text{dipole}} \propto \frac{1}{Q^4}$ for large Q^2, then the cross section is $\propto \frac{1}{Q^6}$ so it falls off rapidly with Q^2 because of finite nucleon size. If deep inelastic doesn't follow this dipole approximation, then it would suggest that the electron is scattering off something smaller (or a point particle).

0.49 Deep Inelastic Scattering

For $Q^2 = -q^2 = 4E_1E_3\sin^2(\theta/2)$, we define the Bjorkan x as

$$x \equiv \frac{Q^2}{2p_2 \cdot q}$$

In the lab frame, this is

$$x = \frac{Q^2}{2m_t\omega}$$

For elastic scattering, $\omega = \frac{Q^2}{2m_t}$, so plugging that in we get $x = 1$. For inelastic scattering, $\omega > \frac{Q^2}{2m_t} \implies x < 1$. Specifically, $W^2 = p_4^2 = (q + p_2)^2 = q^2 + 2p_2 \cdot q + p_2^2$ so $W^2 + Q^2 - m_p^2 = 2p_2 \cdot q$:

$$x = \frac{Q^2}{Q^2 + (W^2 - m_p^2)}$$

where $W^2 - m_p^2 = 0$ for elastic scattering, otherwise it is > 0, so $0 < x < 1$, and $x = 0$ only if $Q^2 = 0$.

Define inelasticity as

$$y = \frac{p_2 \cdot q}{p_2 \cdot p_1}$$

In the lab frame, $p_2 = (m_t, 0, 0, 0)$ so

$$y = \frac{\omega}{E_1}$$

Let's also define

$$\nu = \frac{p_2 \cdot q}{m_t}$$

In the lab frame, $p_2 \cdot q = m_t\omega$ so

$$\nu = \omega$$

is the energy-transfer in the lab frame. x, y, ν are all Lorentz-invariant.

LECTURE 38: DEEP INELASTIC SCATTERING
Wednesday, December 09, 2020

In the last class, we defined the Lorentz-invariant variables (Bjorkan) x, (inelasticity) y, and ν. Inelastic kinematics can be described (Lorentz-invariantly) in terms of any two independent choices of

$$Q^2 = -q^2 \quad x = \frac{Q^2}{2p_2 \cdot q} \quad y = \frac{p_2 \cdot q}{p_2 \cdot p_1} \quad \nu = \frac{p_2 \cdot q}{m_t}$$

For fixed s (single beam energy), all events at any θ and E_3 have the same $s = (p_1 + p_2)^2 = m_e^2 + m_t^2 + 2p_1 \cdot p_2 \approx m_t + 2p_1 \cdot p_2$, so $p_1 \cdot p_2 = \frac{1}{2}(s - m_p^2)$ (here we are considering electron-proton scattering).

Therefore, $y = \frac{p_2 \cdot q}{p_2 \cdot 1} = \frac{2p_2 \cdot q}{(s-m_p^2)} = \frac{2m_p}{s-m_p^2}\nu$, so y and ν are not independent for fixed s. Therefore, we can define kinematics by any two of Q^2, x, y, and ν except for the pair y and ν.

At low Q^2 (really, at large x), in the lab frame,

$$\begin{aligned} w^2 &= (p_2 + q)^2 + p_2^2 + 2p_2 \cdot q + q^2 \\ &= m_t^2 + 2p_2 \cdot (p_1 - p_3) - Q^2 \\ &= m_t^2 + 2m_t(E_1 - E_3) - 4E_1 E_3 \sin^2(\theta/2) \\ &= (m_t^2 + 2E_1 m_t) - (2m_t + 4E_1 \sin^2(\theta/2))E_3 \end{aligned}$$

which is linear in E_3. Different scattered-electron energies correspond to different final-state excitations. The continuum of states at high W is deep-inelastic scattering. $\frac{\partial^2 \sigma}{\partial \Omega \partial E_3}$ vs. Q^2 for higher W^2 has a small dependence on Q^2. Constant form factor indicates scattering from something small or point-like within the proton.

0.50 Structure Functions

Again, starting from a target current operator of mixture γ^μ and $\frac{\imath \sigma^{\mu\nu} q_\nu}{m_p}$ with coefficients that depend on Q^2 and now x, it can be shown that

$$\frac{\partial^2 \sigma}{\partial x \partial Q} = \frac{4\pi\sigma^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right)\frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2)\right]$$

These are not to be confused with $F_1(Q^2)$ and $F_2(Q^2)$, the Dirac and Pauli elastic form factors. By measuring at different beam energies but the same x and Q^2, then the form factors here will not change. Two different, known, linear combinations of these form factors can be measured because y changes, so F_1 and F_2 can be extracted separately.

For larger-Q^2 deep inelastic scattering ($Q^2 \gg m_p y^2$), this simplifies with just $1 - y$ multiplying F_2/x.

0.50.1 Bjorken Scaling

At large Q^2 (larger than a few GeV-squared), F_1 and F_2 become almost independent of Q^2. This suggests scattering from point-like constituents. We typically write these as $F_{\{1,2\}}(x, Q^2) \to F_{\{1,2\}}(x)$

0.50.2 Callen-Gross Relation

At large Q^2, $F_1(x)$ and $F_2(x)$ are not independent:

$$F_2(x) = 2xF_1(x)$$

which again, is consistent with electron scattering from a point particle (of spin 1/2), a Dirac particle with fixed magnetic moment. All of this suggests a smaller constituent particle of the proton, the quark.

0.51 Electron-Quark Scattering

We will be talking about scattering from a "free" quark. While we don't really see free quarks ever, QCD has asymptotic freedom, which basically means if you hit a quark hard enough, it doesn't interact with the other quarks very much. This differs from electron-muon scattering only in the charge:

$$\left\langle |\mathcal{M}_{fi}|^2 \right\rangle = 2Q_q^2 e^4 \left(\frac{s^2+u^2}{t^2} \right) \approx 2Q_q^2 e^4 \frac{(p_1 \cdot p_2)^2 + (p_1 \cdot p_2)^2}{(p_1 \cdot p_3)^2}$$

In the center-of-mass frame,

$$\begin{aligned} p_1 &= (E, 0, 0, E) \\ p_2 &= (E, 0, 0, -E) \\ p_3 &= (E, E\sin(\theta^*), 0, E\cos(\theta^*)) \\ p_4 &= (E, -E\sin(\theta^*), 0, -E\cos(\theta^*)) \end{aligned}$$

where $E \equiv \frac{\sqrt{s}}{2}$ (really E^*). Then,

$$\left\langle |\mathcal{M}_{fi}|^2 \right\rangle = 2Q_q^2 e^4 \frac{4E^4 + E^4(1+\cos(\theta^*))^2}{E^4(1-\cos(\theta^*))^2}$$

so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{Q_q^2 e^4}{8\pi^2 s} \frac{1 + \frac{1}{4}(1+\cos(\theta^*))^2}{(1-\cos(\theta^*))^2}$$

QED conserves chirality, which means that it conserves helicity in the relativistic limit. The 1 in the numerator comes from the contribution from $RR \to RR$ and $LL \to LL$ which is zero because $S_z = 0$. The $\frac{1}{4}(1+\cos(\theta^*))^2 = \frac{1}{2}(\cos^2(\theta^*/2))$ comes from the $RL \to RL$ and $LR \to LR$ contributions with $|S_z| = 1$.

LECTURE 39:
Friday, December 11, 2020

From the final result of the previous lecture, we can write down the Lorentz-invariant cross-section:

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}q^2} &= \frac{1}{64\pi s p_i^{*2}} \left\langle |\mathcal{M}_{fi}|^2 \right\rangle \\ &= \frac{Q_q^2 e^4}{32\pi s p_i^*} \frac{s^2+u^2}{t^2} \\ &= \frac{Q_q^2 e^4}{8\pi q^4} \frac{s^2+u^2}{s^2} \end{aligned}$$

Recall that $u + s + t = \sum_i m_i^2 = 0$ in this case, so $u = -s - t = -s - q^2$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2 Q_q^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s} \right)^2 \right] \tag{1}$$

0.52 Quark-Parton Model

If, in the "infinite momentum frame" a struck quark carries a fraction ξ of the 4-momentum, $p_q = (\xi E_2, 0, 0, \xi E_2)$, after collision, $p_q^2 = (\xi p_2 + q)^2 = (\xi p_2)^2 + 2\xi p_2 \cdot q + q^2 = m_q^2$. If we ignore the mass term and the ξ^2 term (suppose they are very small), then

$$\xi = \frac{-q^2}{2\pi_2 \cdot q} = \frac{Q^2}{2p_2 \cdot q} = x$$

x, for any electron-proton scattering, is a fraction of p_2 carried by the struck quark (in the infinite momentum frame), while s for the electron-proton system is $s = (p_1 + p_2)^2 \sim 2p_1 \cdot p_2$. s_q for the system is $s_q = (p_1 + xp_2)^2 \sim x2p_1 \cdot p_2 = xs$.

Similarly, $y_q = \frac{p_q \cdot q}{p_q \cdot p_1} = \frac{xp_2 \cdot q}{xp_2 \cdot p_1} = y$, and $x_q = 1$ because scattering from quarks is always elastic. Then, (1) becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} = \frac{2\pi\alpha^2 Q_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{xs}\right)\right]$$

with $q^2 = -Q^2 = -(s_q - m_q^2)x_q y_q$, since $y = \frac{2m_p}{s - m_p^2}\nu$ and $x = \frac{Q^2}{2m_p\nu}$. Therefore, $\frac{q^2}{s_q} = \frac{q^2}{xs} = -x_q y_q = -y$.

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}q^2} &= \frac{2\pi\alpha^2 Q_q^2}{q^4}\left[1 + (1-y)^2\right] \\ &= \frac{4\pi\alpha^2 Q_q^2}{Q^4}\left[1 - y + \frac{y^2}{2}\right] \end{aligned} \tag{2}$$

0.52.1 Parton Distribution Functions

Let the probability of up or down quarks within the proton having momentum-fraction x (in the ∞-momentum frame) be $u^p(x)\,\mathrm{d}x \equiv u(x)\,\mathrm{d}x$ or $d^p(x)\,\mathrm{d}x \equiv d(x)\,\mathrm{d}x$ for the up and down quarks and barred versions for the antiquarks.

$$\frac{\partial^2\sigma^p}{\partial x \partial Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_i Q_i^2 q_i(x)$$

Compare this with the high Q^2 limit in terms of $F_{1,2}(x, Q^2)$:

$$\frac{\partial^2\sigma}{\partial x \partial Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2}{x} + y^2 F_1\right]$$

so

$$F_2(x, Q^2) = 2xF_1 = x\sum_i Q_i^2 q_i(x)$$

This predicts Bjorken scaling, since this is not dependent on Q^2, and it also predicts the Callen-Gross relation, $F_2 = 2xF_1$.

0.52.2 Determination of PDFs

Neglecting the s and $\bar{s}$ component,

$$\begin{aligned} F_2^{ep}(x) &= x\sum_i Q_i^2 q_i(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\bar{u}(x) + \frac{1}{9}\bar{d}(x)\right) \\ F_2^{en}(x) &= x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\bar{d}(x) + \frac{1}{9}\bar{u}(x)\right) \end{aligned}$$

Then

$$\int_0^1 F_2^{ep}(x)\,\mathrm{d}x = \frac{4}{9}f_u + \frac{1}{9}f_d$$

where $f_u \equiv \int_0^1 [xu(x) + x\bar{u}(x)]$ and $f_d \equiv \int_0^1 [xd(x) + x\bar{d}(x)]$, the fraction of the proton momentum carried by up/down quarks.

$$\int_0^1 F_2^{en}(x)\,\mathrm{d}x = \frac{4}{9}f_d + \frac{1}{9}f_u$$

From SLAC measurements, $\int F_2^{ep}(x)\,\mathrm{d}x = 0.18$ and $\int F_2^{en}(x)\,\mathrm{d}x = 0.12$ so $f_u = 0.36$ and $f_d = 0.18$. Note that this doesn't add up to 1, so there is a large portion of the momentum which is presumably carried by gluons (g) or s and $\bar{s}$ (strange quarks).