PROGRESS NOTES

Autologistic model: condition on sum

Given the conditionals: for $j = 1, 2, \ldots, J$

$$\operatorname{logit} \Pr(L_j = 1 | L_{\{-j\}}) = \beta_j + \theta_j \sum_{k \neq j} l_k \tag{0.1}$$

we have

$$\begin{aligned}
\Pr(L_j = l | L_1, \ldots, L_{j-1}, L_{j+1} = 0, \ldots, L_J = 0) &= \frac{\exp[l(\beta_j + \theta_j \sum_{k<j} l_k)]}{1 + \exp[\beta_j + \theta_j \sum_{k<j} l_k]} \\
\Pr(L_j = 0 | L_1, \ldots, L_{j-1}, L_{j+1} = 0, \ldots, L_J = 0) &= \frac{1}{1 + \exp[\beta_j + \theta_j \sum_{k<j} l_k]}
\end{aligned}$$

Then

$$\frac{\Pr(L_j = l | L_1, \ldots, L_{j-1}, L_{j+1} = 0, \ldots, L_J = 0)}{\Pr(L_j = 0 | L_1, \ldots, L_{j-1}, L_{j+1} = 0, \ldots, L_J = 0)} = \exp[l(\beta_j + \theta_j \sum_{k<j} l_k)]$$

By Brook's lemma, the joint distribution is determined up to a proportionality constant by the conditionals.

$$\begin{aligned}
\Pr(L_1 = l_1, \ldots, L_J = l_J) &= \prod_{j=1}^{J} \frac{\Pr(L_j | L_1, \ldots, L_{j-1}, L_{j+1} = 0, \ldots, L_J = 0)}{\Pr(L_j = 0 | L_1, \ldots, L_{j-1}, L_{j+1} = 0, \ldots, L_J = 0)} \Pr(L_1 = 0, \ldots, L_J = 0) \\
&\propto \prod_{j=1}^{J} \exp[l_j(\beta_j + \theta_j \sum_{k<j} l_k)] \\
&= \exp\{\sum_{j=1}^{J} \beta_j l_j + \sum_{j=2}^{J} \sum_{k=1}^{j-1} \theta_j l_j l_k\}
\end{aligned}$$

Let the proportionality(normalizing) constant be

$$A(\boldsymbol{\beta}, \boldsymbol{\theta}) = \Big\{ \sum_{l^* \in \{0,1\}^J} \exp[\sum_{j=1}^{J} \beta_j l_j^* + \sum_{j=2}^{J} \sum_{k=1}^{j-1} \theta_j l_j^* l_k^*] \Big\}^{-1}$$

Then the joint distribution defined by the autologistic model is

$$\Pr(L_1 = l_1, \ldots, L_J = l_J) \quad = \quad A(\boldsymbol{\beta}, \boldsymbol{\theta}) \exp\{\sum_{j=1}^{J} \beta_j l_j + \sum_{j=2}^{J} \sum_{k=1}^{j-1} \theta_j l_j l_k\} \tag{0.2}$$

Apparently, this is a special form of the quadratic exponential model in Zhao (1990).

For example, when $J = 3$, the joint density is

$$\Pr(L) = A(\boldsymbol{\beta}, \boldsymbol{\theta}) \exp\{\beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \theta_2 l_1 l_2 + \theta_3 l_1 l_3 + \theta_3 l_2 l_3\}$$

Note that θ_1 is not involved in the joint density (0.2). This suggests that in the pair-wise autologistic model (0.1), $\boldsymbol{\theta} \in \mathbb{R}^J$ leads to over-specification. For a well defined joint density function, the $\operatorname{logit} \Pr(L_j = 1 | L_{\{-j\}})$ derived from (0.2) must be equal to the model in (0.1).

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$$\begin{aligned}
\text{(0.3)} \quad \operatorname{logit}\Pr(L_j=1|L_{\{-j\}}) &= \log \frac{\exp\{\beta_j + \sum_{k\neq j}\beta_k l_k + \sum_{r\neq j}\sum_{k\neq j, k<r}\theta_r l_r l_k + \theta_j \sum_{k=1}^{j-1} l_k + \sum_{k=j+1}^{J}\theta_k l_k\}}{\exp\{\sum_{k\neq j}\beta_k l_k + \sum_{r\neq j}\sum_{k\neq j,k<r}\theta_r l_r l_k\}} \\
&= \beta_j + \theta_j \sum_{k=1}^{j-1} l_k + \sum_{k=j+1}^{J}\theta_k l_k \\
&= \beta_j + \theta_j \sum_{k=1}^{j-1} l_k + \theta_j \sum_{k=j+1}^{J} l_k
\end{aligned}$$

Therefore, for $j = 1, 2, \ldots, J-1$, the following constrant must be satisfied for any value of $l_j \in \{0, 1\}$

$$\theta_j \sum_{k=j+1}^{J} l_k = \sum_{k=j+1}^{J} \theta_k l_k$$

By induction, we can show that all $\{\theta_j\}_{j=1}^{J}$ must have the save value.

Autologistic model: pair-wise saturated

Given the conditionals: for $j = 1, 2, \ldots, J$

$$\operatorname{logit}\Pr(L_j = 1|L_{\{-j\}}) = \beta_j + \sum_{k\neq j}\theta_{jk} l_k \tag{0.4}$$

we have

$$\begin{aligned}
\Pr(L_j = l|L_1,\ldots,L_{j-1},L_{j+1}=0,\ldots,L_J=0) &= \frac{\exp[l(\beta_j + \sum_{k<j}\theta_{jk}l_k)]}{1+\exp[\beta_j + \sum_{k<j}\theta_{jk}l_k]} \\
\Pr(L_j = 0|L_1,\ldots,L_{j-1},L_{j+1}=0,\ldots,L_J=0) &= \frac{1}{1+\exp[\beta_j + \sum_{k<j}\theta_{jk}l_k]}
\end{aligned}$$

Then

$$\frac{\Pr(L_j = l|L_1,\ldots,L_{j-1},L_{j+1}=0,\ldots,L_J=0)}{\Pr(L_j = 0|L_1,\ldots,L_{j-1},L_{j+1}=0,\ldots,L_J=0)} = \exp[l(\beta_j + \sum_{k<j}\theta_{jk}l_k)]$$

By Brook's lemma, the joint distribution is determined up to a proportionality constant by the conditionals.

$$\begin{aligned}
\Pr(L_1 = l_1,\ldots,L_J=l_J) &= \prod_{j=1}^{J}\frac{\Pr(L_j|L_1,\ldots,L_{j-1},L_{j+1}=0,\ldots,L_J=0)}{\Pr(L_j = 0|L_1,\ldots,L_{j-1},L_{j+1}=0,\ldots,L_J=0)}\Pr(L_1=0,\ldots,L_J=0) \\
&\propto \prod_{j=1}^{J}\exp[l_j(\beta_j + \sum_{k<j}\theta_{jk}l_k)] \\
&= \exp\{\sum_{j=1}^{J}\beta_j l_j + \sum_{j=2}^{J}\sum_{k=1}^{j-1}\theta_{jk} l_j l_k\}
\end{aligned}$$

This model leads to the same conditional logit as in (0.4), thus it is a valid model.
Note that this model is equivalent to the quadratic exponential model in Zhao (1990).