Summary of Short-term Research Objectives

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1 Model Specification

Let L be a K-dimensional Bernoulli random variable denoting the true state. Consider the general log linear model:

$$f(l;\Theta) = \exp\{\Theta_1^T l + \Theta_2^T u_2 + \ldots + \Theta_K^T u_K - A^*(\Theta)\}$$

where U_k is a $\binom{K}{k} \times 1$ vector of k-way cross-products, $k = 1, \ldots, K$, and $\Theta = (\Theta_1, \ldots, \Theta_K)$ contains the the natural parameters, which is a $(2^K - 1) \times 1$ vector.

Model restrictions, let $\tilde{l} = (l, u_2, \ldots, u_K)^T$, and $S = \sum_{j=1}^K L_j = s$ has some fixed pmf

$$\begin{aligned} \pi(s) :=& P(S = s) \\ =& \frac{1}{A(\Theta)} \sum_{\tilde{l}:S=s} \exp\{\Theta^T \tilde{l}\}\ , s = 0, 1, \ldots, K \qquad (1) \\ A(\Theta) =& \sum_{\tilde{l}:l \in \{0,1\}^K} \exp\{\Theta^T \tilde{l}\} \qquad (2) \end{aligned}$$

2 Posterior Distribution

2.1 General Model

For General model, $\tilde{L}$ is a square matrix with dimension $J_1 = 2^K - 1$. Recall that

$$
\begin{aligned}
A(\Theta) =& \frac{1}{\pi(0)} \\
\pi(s) =& \frac{1}{A(\Theta)} \sum_{\tilde{l}:S=s} \exp\{\Theta^T \tilde{l}\}\ , s = 1, \ldots, K \qquad (3) \\
\mu_k =& \frac{1}{A(\Theta)} \sum_{\tilde{l}:l_k=1} \exp\{\Theta^T \tilde{l}\}\ , k = 1, \ldots, K \qquad (4)
\end{aligned}
$$

Define intermediate parameter $\phi_j = \exp(\theta^T \tilde{l}_j) > 0$, $\Theta = (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(K)}), j = 1, \ldots, J_1$ and two $K \times J_1$ sub-design matrices B, C, where $B[k,j] = 1(\sum_{s=1}^{K} \tilde{L}[j,s] = k)$, $C[k,j] = \tilde{L}[j,k]$. Thus (3) and (4) become

$$
\begin{aligned}
\vec{\phi} >& 0 \qquad (5) \\
B\vec{\phi} =& \vec{\pi}/\pi(0) \qquad (6) \\
C\vec{\phi} =& \vec{\mu}/\pi(0) \qquad (7)
\end{aligned}
$$

Note that B and C are not independent constraints and should be compatible so that $\binom{B}{C}$ has rank $2K - 1$. Explicitly, μ and π must satisfy

$$
\sum_{k=1}^{K} \mu_k = \sum_{k=1}^{K} k\pi_k
$$

Based on [1][2], we can sample $\vec{\phi}$ from Uniform distribution subject to the above linear constraints efficiently and robustly. Then Θ are the solutions to the linear system (J_1 equations with J_1 unknowns):

$$
\tilde{L}\Theta = \log \vec{\phi}
$$

Then the posterior distribution of (μ, π) given data L becomes:

$$
\begin{aligned}
P(\mu, \pi | L) \propto& P(L, \mu, \pi) \\
=& P(L|\mu, \pi) P(\mu, \pi) \\
=& \int P(L, \phi | \mu, \pi) d\phi P(\mu, \pi) \\
\propto& \int P(L|\phi) P(\phi|\mu, \pi) d\phi P(\mu, \pi) \\
\approx& \sum P(L|\phi_h) P(\pi|\mu) P(\mu)
\end{aligned}
$$

where $P(L|\phi_h)$ is the log linear model density evaluated at ϕ_h, which is sampled uniformly from the feasible region subject to constrains (5-7).
$P(\mu)$ is a multivariate logit Normal density with mean 0 and standard deviation Σ.
$P(\pi|\mu) = P(\pi_0|\mu) P(\pi_1, .., \pi_K | \pi_0, \alpha)$, where $P(\pi_0|\mu)$ is a uniform density on $(0, 1 - \max(\mu))$ and $P(\pi_1, ..\pi_K|\pi_0, \alpha)$ is a Stick Breaking prior with shape parameter α on $(0, 1 - \pi_0)$.

2.1.1 Sampling Algorithm

Algorithm 1 Usample(n, μ, π_0, π_{0-})

if neither μ, π_{0-} is NULL **then**
    sample ϕ uniformly from feasible region defined by

$$\begin{aligned} \phi >& 0 \\ B\phi =& \pi_{0-}/\pi_0 \\ C\phi =& \mu/\pi_0 \end{aligned}$$

else if only μ is NULL **then**
    sample ϕ uniformly from feasible region defined by

$$\begin{aligned} \phi >& 0 \\ B\phi =& \pi_{0-}/\pi_0 \end{aligned}$$

else
    sample ϕ uniformly from feasible region defined by

$$\begin{aligned} \phi >& 0 \\ C\phi =& \mu/\pi_0 \end{aligned}$$

return n samples of ϕ

Algorithm 2 SampleByBlock($n_{Iter}, n_{Burn}, L, \mu_{init}, \pi_{init}, \sigma$)

Initialize $\mu^{(0)} = \mu_{init}$, $\pi^{(0)} = \pi_{init}$
for $t \in 1$ to $(n_{Iter} + n_{Burn})$ **do**
    $\phi =$ Usample$(1, \mu =$ NULL, $(\pi_0, \pi_{0-}) = \pi^{(t-1)})$
    $\mu^* = \pi_0^{(t-1)} C\phi$
    $\alpha = \frac{P(L,\mu^*,\pi^{(t-1)})}{P(L,\mu^{(t-1)},\pi^{(t-1)})}$
    $\mu^{(t)} = \mu^*$ with probability $\max(1, \alpha)$; other wise, $\mu^{(t)} = \mu^{(t-1)}$
    $\pi_0^* =$ logit$^{-1}[logit(pi_0^{(t-1)}) + \epsilon]$, where $\epsilon \sim$ N$(0, \sigma)$.
    $\phi =$ Usample$(1, \mu = \mu^{(t)}, \pi_0 = \pi_0^*, \pi_{0-} =$ NULL$)$
    $\pi_{0-}^* = \pi_0^* B\phi$
    $\pi^* = (\pi_0^*, \pi_{0-}^*)$
    $\alpha = \frac{P(L,\mu^{(t)},\pi^*)}{P(L,\mu^{(t)},\pi^{(t-1)})}$
    $\pi^{(t)} = \pi^*$ with probability $\max(1, \alpha)$; other wise, $\pi^{(t)} = \pi^{(t-1)}$
return $\mu^{(n_{Burn}+1)}$ to $\mu^{(n_{Burn}+n_{Iter})}$ and $\pi^{(n_{Burn}+1)}$ to $\pi^{(n_{Burn}+n_{Iter})}$

2.2 Measurement Error Model

Let M be the observed measurement, and L be the latent true state, with γ and δ representing the TPR and FPR.

$$\begin{aligned}
P(\mu,\pi,\gamma,\delta|M) &\propto P(M,\mu,\pi,\gamma,\delta) \\
&= P(M|\mu,\pi,\gamma,\delta)P(\mu,\pi,\gamma,\delta) \\
&= \sum_{L\in\mathbb{L}} P(M,L|\mu,\pi,\gamma,\delta)P(\mu,\pi,\gamma,\delta) \\
&= \sum_{L\in\mathbb{L}} \big[P(M|L,\gamma,\delta)P(L|\mu,\pi)\big]P(\mu,\pi)P(\gamma)P(\delta)
\end{aligned}$$

where

$$\begin{aligned}
P(M|L,\gamma,\delta) &= \prod_{k=1}^{K} P(M_k|L_k,\gamma,\delta) \ \ \text{(conditional independence assumption)} \\
&= \prod_{k=1}^{K} (\gamma^{L_k}\delta^{1-L_k})^{M_k}[(1-\gamma)^{L_k}(1-\delta)^{1-L_k}]^{1-M_k}
\end{aligned}$$

$P(\gamma)$ and $P(\delta)$ are beta priors, and $P(L|\mu,\pi)P(\mu,\pi)$ is defined in the same way as in previous section.

2.2.1 Sampling Algorithm

Reference

1. Smith RL . Efficient Monte-Carlo Procedures for Generating Points Uniformly Dis- tributed over Bounded Regions. Operations Research, 32(6), 12961308 (1984).
2. Van den Meersche, Karel, K. E. R. Soetaert, and D. J. Van Oevelen. ”xsample (): an R function for sampling linear inverse problems.” Journal of Statistical Software 30 (2009).