Summary of Short-term Research Objectives

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1 Model Specification

Let L be a K-dimensional Bernoulli random variable denoting the true state. Consider the general log linear model:

$$f(l;\Theta) = \exp\{\Theta_1^T l + \Theta_2^T u_2 + \ldots + \Theta_K^T u_K - A^*(\Theta)\}$$

where U_k is a $\binom{K}{k} \times 1$ vector of k-way cross-products, $k = 1, \ldots, K$, and $\Theta = (\Theta_1, \ldots, \Theta_K)$ contains the the natural parameters, which is a $(2^K - 1) \times 1$ vector.

Model restrictions, let $\tilde{l} = (l, u_2, \ldots, u_K)^T$, and $S = \sum_{j=1}^K L_j = s$ has some fixed pmf

$$\begin{aligned} \pi(s) :=& P(S = s) \\ =& \frac{1}{A(\Theta)} \sum_{\tilde{l}:S=s} \exp\{\Theta^T \tilde{l}\}\ ,\ s = 0, 1, \ldots, K \qquad (1) \\ A(\Theta) =& \sum_{\tilde{l}:l\in\{0,1\}^K} \exp\{\Theta^T \tilde{l}\} \qquad (2) \end{aligned}$$

2 Research Objectives

Suppose $\{L_i\}$ are i.i.d random variables with density $P(L_i;\Theta)$ defined above, our goal is to find a re-parameterization from Θ to $(\boldsymbol{\pi},\boldsymbol{\gamma})$ such that $P(L_i;\Theta)=P(L_i;\boldsymbol{\pi},\boldsymbol{\gamma})$, where $\boldsymbol{\pi}=(\pi_0,\pi_1,\ldots,\pi_s,\ldots,\pi_K)^T$, with $\sum_{s=0}^{K}\pi_s=1$.

3 Results

3.1 Definition of New Parameters

Using previous notations, with some properly defined $\boldsymbol{\gamma}$, we have:

$$\begin{aligned}P(L_i;\boldsymbol{\pi},\boldsymbol{\gamma}) =&P(L_i,S_i;\boldsymbol{\pi},\boldsymbol{\gamma})\\ =&P(L_i|S_i;\boldsymbol{\pi},\boldsymbol{\gamma})P(S_i;\boldsymbol{\pi},\boldsymbol{\gamma})\end{aligned}$$

where $P(L_i|S_i;\boldsymbol{\pi},\boldsymbol{\gamma})=P(L_i|S_i;\boldsymbol{\gamma})$ because $1(S_i=s)$ is the sufficient statistic for π_s, furthermore S_i is the sufficient statistic for $\boldsymbol{\pi}$. Also $P(S_i;\boldsymbol{\pi},\boldsymbol{\gamma})=\pi_{S_i}$ by definition.

Therefore, we have

$$P(L_i;\boldsymbol{\pi},\boldsymbol{\gamma}) =P(L_i|S_i;\boldsymbol{\gamma})\pi_{S_i}$$

Then we define the following parameters:

$$\begin{aligned}\gamma_{j_1,\ldots,j_s} =&P(L_{ij_1}=\ldots=L_{ij_s}=1|S_i=s) \qquad (3)\\ \boldsymbol{\gamma} =&(\gamma_1,\gamma_2,\ldots,\gamma_{12},\ldots,\gamma_{1\ldots K})^T,\text{ where }\sum_j\gamma_j=\sum_{j\neq j'}\gamma_{jj'}=\sum_{j\neq j'\neq j''}\gamma_{jj'j''}=\ldots=\gamma_{1\ldots K}=1\end{aligned}$$

Therefore $(\boldsymbol{\pi},\boldsymbol{\gamma})^T$ is a vector of length 2^K+K with degrees of freedom 2^K-1.

Let $J_i=\{j:L_{ij}=1\}$. We have,

$$P(L_i;\boldsymbol{\pi},\boldsymbol{\gamma})=\gamma_{J_i}\pi_{S_i} \qquad (4)$$

The relation between $(\boldsymbol{\pi},\boldsymbol{\gamma})$ and Θ is defined by equation (2) together with:

$$\gamma_{J_i}=\frac{\exp(\Theta^T\tilde{l}_i)}{\sum_{l:l^T1=S_i}\exp(\Theta^T\tilde{l})} \qquad (5)$$

with 2^K-1-K degrees of freedom.

Therefore (1), (3) and (5) together define 2^K-1 non-linear equations for 2^K-1 unknowns. If there exists a unique root for the above non-linear system, then there is a one-to-one mapping between $(\boldsymbol{\pi},\boldsymbol{\gamma})$ and Θ, which provides the re-parameterization.

3.2 Find the Re-parameterization