DP Formulation

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1 Model Formulation

There are n products offered, and each product corresponds to one resource. Let $N = \{1, 2, ..., n\}$ be the set of products. At time t, the price for product j is $r_{t,j}$. The consumption matrix is an n×n diagonal matrix. The jth column A^j is the incidence vector for product j. There are T discrete time periods that are counted forward, so period T is the last period.

In each period, there is one customer arrival with probability λ, and no customer arrival with probability $1 - \lambda$. When a customer arrives, the firm must decide how to price all products (if they are available). Let $r_t \subseteq R(x)$ be the pricing policy, and $R(x)$ is the set of all possible pricing policies. Given the pricing policy r_t, the customer chooses the product j with probability $P_j(r_t)$, and makes no purchase with probability $1 - \sum_j P_j(r_t)$.

The state x at the beginning of any period t is an n-vector of unsold products. So the state space is $X = \{0, 1, ..., c_1\} * \{0, 1, ..., c_2\} * ... * \{0, 1, ..., c_n\}$, c_n is the capacity of product n. Let $v_t(x)$ be the maximum total expected revenue over periods, starting at state x at the beginning of period t. The optimality equation is

$$V_t(x) = \max_{r_t \subseteq R(x)} \left\{ \sum_{j=1}^{n} \lambda P_j(r_t) \left(r_{t,j} + V_{t+1} \left(x - A^j \right) \right) \right. + \left(\lambda P_0(r_t) + 1 - \lambda \right) V_{t+1}(x) \} \tag{1}$$

Affine approximation:

$$v_t(x) \approx \theta_t + \sum_{j=1}^{n} V_{t,j} x_j \tag{2}$$

Plugging into the linear programming:

$$\min_{\theta, V} \theta_1 + \sum_{j=1}^{n} V_{1,j} c_j \tag{3}$$

$$\theta_t - \theta_{t+1} + \sum_{j=1}^{n} \left(V_{t,j} x_j - V_{t+1,j} \left(x_j - \lambda P_j(r_t) \right) \right) \geq \sum_{j=1}^{n} \lambda P_j(r_t) r_{t,j} \quad \forall t, x, r_t \subseteq R(x) \tag{4}$$

The boundary conditions are

$$v_{T+1}(x) = 0, \forall x \tag{5}$$

$$v_t(0) = 0, \forall t \tag{6}$$

Number of variables: $T + T * n$ (They are continuous variables)

Number of constraints: $T * 3^n * c_1^n$ (For each product, we have 3 optional prices and $c_1 = c_2 = ... = c_n$)

2 Model Setting

2.1 MNL Choice model

Input: pricing policy r_t

$$P_j(r_t) = \frac{exp(a_j - \beta r_{t,j})}{1 + \sum_{i=1}^{n} exp(a_i - \beta r_{t,i})} \tag{7}$$

Output: choice probability for each product $P_1(r_t), P_2(r_t), ..., P_n(r_t)$