Partitions and Permutations

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Partitions

The word *partition* is shared by (at least) two different concepts, although both refer to the process of dividing an object into smaller sub-objects.

▶ Integer Partitions
A partition of an integer n is a way to write it as a sum of smaller integers, such as

$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$

▶ Set Partitions
A partition of a *set* is a way to divide it into a number of subsets, such as

$\{1,2,3\}, \quad \{1,2\}\{3\}, \quad \{1,3\}\{2\}, \quad \{1\}\{2,3\}, \quad \{1\}\{2\}\{3\}$

Lexicographic order

We will represent a partition of an integer n by a sequence $a_1 a_2 \ldots$, where $a_1 \geq a_2 \geq \cdots > 0$. Thus the partitions of 7 would be represented as follows.

1111111	211111	2211
2221	3111	3211
322	331	4111
421	43	511
52	61	7

In this table the partitions are given in *lexicographic order*.

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Reverse Lexicographic Order

The simplest algorithm to generate partitions actually generates
them in *reverse lexicographic order*, starting with the partition n
and ending with

► Find the largest index i such that $a_i \neq 1$; suppose that
$a_i = a + 1$ so that

$$\alpha = \beta(a+1)11\ldots1.$$

► Replace the suffix $(a + 1)11\ldots1$ by $aa\ldots ar$ where $r < a$,
thus obtaining

$$\text{succ}(\alpha) = \beta aa\ldots ar.$$

Example 1

What is the successor of $\alpha = 22211$?

▶ The largest i such that $a_i \neq 1$ is $i = 3$

a_1	a_2	a_3	a_4	a_5
2	2	2	1	1

▶ Replace the suffix 211 with 1111

a_1	a_2	a_3	a_4	a_5	a_6
2	2	2	1	1	
2	2	1	1	1	1

Therefore we get

$$\operatorname{succ}(22211) = 221111.$$

Example 2

What is the successor of $\alpha = 3311$?

- ▶ The largest i such that $a_i \neq 1$ is $i = 2$

$$
\begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
3 & 3 & 1 & 1
\end{array}
$$

- ▶ Replace the suffix 311 with as many 2s as possible, and a remainder if necessary.

$$
\begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
3 & 3 & 1 & 1 \\
3 & 2 & 2 & 1
\end{array}
$$

Therefore we get

$$\operatorname{succ}(3311) = 3221.$$

Fixed number of parts

Suppose we are interested in generating the number of partitions
of n into exactly k parts, for example if $n = 11$ and $k = 4$ we get

$$\begin{array}{cccc}
3332 & 4322 & 4331 & 4421 \\
5222 & 5321 & 5411 & 6221 \\
6311 & 7211 & 8111 &
\end{array}$$

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Successors

The easiest algorithm for generating partitions of fixed size k starts with the partition

$$(n - k + 1)1111$$

and then computes the successor of $\alpha = a_1 a_2 \dots a_k$ as follows:

- Let i be the smallest index such that $a_i < a_1 - 1$.
- Assign $a_i + 1$ to each of a_2, a_3, ..., a_i and then set a_1 as needed to maintain a partition of n.

The algorithm terminates if there are no values i such that $a_i < a_1 - 1$; in this case each part of the partition is $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$.

Example

Suppose that $n = 15$ and $k = 5$ and we want the successor of
$\alpha = 66111$.

▶ The smallest index i such that $a_i < 5$ is $i = 3$

$$
\begin{array}{ccccc}
a_1 & a_2 & a_3 & a_4 & a_5 \\
6 & 6 & 1 & 1 & 1
\end{array}
$$

▶ Set a_2 and a_3 to the value $a_3 + 1$ which is 2

$$
\begin{array}{ccccc}
a_1 & a_2 & a_3 & a_4 & a_5 \\
* & 2 & 2 & 1 & 1
\end{array}
$$

▶ Set a_1 to the required value to maintain a partition of $n = 15$

$$
\begin{array}{ccccc}
a_1 & a_2 & a_3 & a_4 & a_5 \\
9 & 2 & 2 & 1 & 1
\end{array}
$$

Colex ordering

This ordering of partitions with a fixed number of parts is not
reverse lexicographic, but rather *colexicographic*.

Lexicographic	Reverse lex	Colexicographic
3322	7111	7111
3331	6211	6211
4222	5311	5311
4321	5221	4411
4411	4411	5221
5221	4321	4321
5311	4222	3331
6211	3331	4222
7111	3322	3322

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Conjugate permutations

Suppose that f and g are two permutations in the symmetric group $\mathrm{Sym}(n)$. Then they are said to be *conjugate* if

$$f = h^{-1}gh$$

for some $h \in \mathrm{Sym}(n)$.

Example

The permutations $f = (1,2,3)(4,5)$ and $g = (1,3,4)(2,5)$ are conjugate because

$$(1,2,3)(4,5) = (2,3,4)\ (1,3,4)(2,5)\ (2,4,3)$$

and therefore we may take $h = (2,4,3)$ in the above definition.

Conjugacy is an equivalence relation

▶ Every permutation is conjugate to itself

$$f = e^{-1}fe$$

▶ If f is conjugate to g, then g is conjugate to h

$$f = h^{-1}gh \Longrightarrow g = hfh^{-1}$$

▶ If a is conjugate to b and b conjugate to c, then a is conjugate to c

$$a = h^{-1}bh \text{ and } b = g^{-1}cg \Longrightarrow a = h^{-1}g^{-1}cgh = (gh)^{-1}c(gh).$$

Conjugacy Classes

This means that the permutations fall into *disjoint* classes called
conjugacy classes such that two permutations are conjugate if and
only if they lie in the same class.

e	$(3,4)$	$(1,2)(3,4)$	$(2,3,4)$	$(1,2,3,4)$
	$(2,3)$	$(1,3)(2,4)$	$(2,4,3)$	$(1,2,4,3)$
	$(2,4)$	$(1,4)(2,3)$	$(1,2,3)$	$(1,3,4,2)$
	$(1,2)$		$(1,2,4)$	$(1,3,2,4)$
	$(1,3)$		$(1,3,2)$	$(1,4,3,2)$
	$(1,4)$		$(1,3,4)$	$(1,4,2,3)$
			$(1,4,2)$	
			$(1,4,3)$	

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Cycle Structure

The *cycle structure* of a permutation is the number of cycles of each length in its cycle decomposition.

Theorem
Two permutations are conjugate in $\mathrm{Sym}(n)$ if and only if they have the same cycle structure.

We need to prove two things

- ▶ If two permutations are conjugate, then they have the same cycle structure, and

- ▶ If two permutations have the same cycle structure, then they are conjugate.

Conjugate permutations have the same cycle structure

If the permutation g maps

$$i \rightarrow ig$$

then $h^{-1}gh$ maps

$$ih \rightarrow igh.$$

Therefore for every cycle

$$(a, b, \ldots, c)$$

in the cycle decomposition of g, there is a corresponding cycle

$$(ah, bh, \ldots, ch)$$

in the cycle decomposition of $f = h^{-1}gh$.

Permutations with the same cycle structure are conjugate

If two permutations f and g have the same cycle structure, then we can find a conjugating permutation h such that $f = h^{-1}gh$.

For each cycle

$$(f_1, f_2, \ldots f_k)$$

of f there is a corresponding cycle

$$(g_1, g_2, \ldots, g_k)$$

of g.

Then define h by the rule

$$h : g_i \to f_i.$$

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Example

Let
$$f = (1, 4, 5)(2, 7)(3, 6) \qquad g = (2, 4, 6)(1, 5)(3, 7)$$

Then define h (in image notation) as follows:

$$h = \left(\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 4 & 7 & 5 & 6 \end{array} \right)$$

Then $h = (1, 2)(5, 7, 6)$ and it is easy to check that

$$f = h^{-1}gh$$

and so h is a conjugating permutation.

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In GAP

```
gap> f := (1,4,5)(2,7)(3,6);
(1,4,5)(2,7)(3,6)
gap> g := (2,4,6)(1,5)(3,7);
(1,5)(2,4,6)(3,7)
gap> h := PermList([2,1,3,4,7,5,6]);
(1,2)(5,7,6)
gap> h^-1*g*h;
(1,4,5)(2,7)(3,6)
gap> f = last;
true
gap>
```

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Cycle structure

The cycle structure of a permutation can be represented in a number of different ways. One way would be to simply list the lengths of the cycles. For example, if $f \in \mathrm{Sym}(10)$ has cycle decomposition

$$f = (1, 2, 4)(5, 6)(8, 9, 10)$$

then its cycle structure could be written as

$$33211$$

In this representation, the cycle structure of a permutation is simply a partition of n, the degree of the permutation!

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Partitions and Conjugacy Classes

We have therefore essentially proved the following:

Theorem
The number of conjugacy classes of $\mathrm{Sym}(n)$ is equal to the number of partitions of n.

For $\mathrm{Sym}(4)$ the correspondence is

Partition	Representative
1111	e
211	$(1,2)$
22	$(1,2)(3,4)$
31	$(1,2,3)$
4	$(1,2,3,4)$

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Counting partitions

Let $p(n)$ be the number of partitions of n, and let $p_k(n)$ be the number of partitions of n that have k parts.
Then

$$p(n) = \sum_{k=1}^{n} p_k(n).$$

We can write these numbers out in a triangle reminiscent of Pascal's triangle:

$$p_1(1)$$
$$p_1(2) \quad p_2(2)$$
$$p_1(3) \quad p_2(3) \quad p_3(3)$$
$$p_1(4) \quad p_2(4) \quad p_3(4) \quad p_4(4)$$
$$\cdots \quad \cdots \quad \cdots$$

Top of the triangle

By direct calculation we see that the triangle starts

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 1 & & 1 & \\
1 & & 3 & & 3 & & 2 & & 1 & & 1 \\
 & \cdots & & & \cdots & & & \cdots & & &
\end{array}
$$

The total number of partitions $p(n)$ of each number n is obtained by summing the entries in each row of this triangle.

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A recurrence

We can express the values of $p_k(n)$ in terms of smaller such numbers by splitting up the partitions of n according to the number of parts that are equal to 1. So if P is the set of all partitions of n with k parts, then define P_0, P_1, ..., P_k by

$$P_i = \{\pi \in P \mid \pi \text{ has exactly } i \text{ 1s}\}$$

Then clearly

$$|P| = \sum_{i=0}^{k} |P_i|.$$

Therefore we need to count the number of partitions in each set P_i.

Partitions with i 1s

Given a partition $\pi \in P_i$, consider the partition obtained by subtracting one from each part of π. This is a partition of $n - k$ with $k - i$ parts, and conversely any partition of $n - k$ with $k - i$ parts yields a partition in P_i if we add 1 to each part, and adjoin i parts equal to 1.

Therefore

$$|P_i| = p_{k-i}(n - k).$$

Example

The partitions of 10 into 4 parts with two 1s are

$$4411 \quad 5311 \quad 6211$$

and the partitions of 6 into 2 parts are

$$33 \quad 42 \quad 51.$$

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Putting it all together
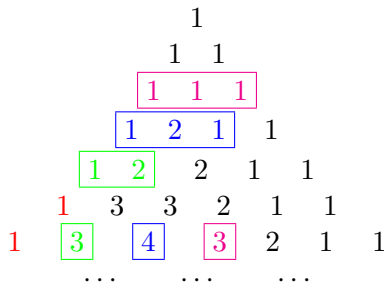
Therefore

$$p_k(n) = \sum_{i=0}^{k} p_{k-i}(n-k)$$

and replacing i by $k - i$ we get

$$p_k(n) = \sum_{i=0}^{k} p_i(n-k).$$

This sum involves terms of the form $p_0(x)$; this value must be taken to be 0 unless $x = 0$ in which case $p_0(0)$ is defined to be 1.

Extending the triangle

We will use this to extend the triangle to $n = 7$

$$
\begin{array}{ccccccccccccc}
& & & & & & 1 & & & & & & \\
& & & & & 1 & & 1 & & & & & \\
& & & & \boxed{1} & \boxed{1} & \boxed{1} & & & & & & \\
& & & \boxed{1} & \boxed{2} & \boxed{1} & & 1 & & & & & \\
& & \boxed{1} & \boxed{2} & & 2 & & 1 & & 1 & & & \\
& 1 & & 3 & & 3 & & 2 & & 1 & & 1 & \\
1 & & \boxed{3} & & \boxed{4} & & \boxed{3} & & 2 & & 1 & & 1 \\
& & \cdots & & \cdots & & \cdots & & & & & &
\end{array}
$$

The blue value $p_3(7)$ is obtained by adding the first 3 values on the line for $n = 7 - 3 = 4$, thus

$$4 = 1 + 2 + 1.$$

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Size of conjugacy classes

Given a partition of n, how many permutations have that cycle structure?

For example, how big is the conjugacy class of $\mathrm{Sym}(7)$ with cycle structure 322? A permutation with this cycle structure has the form

$$(a, b, c)(d, e)(f, g).$$

There are 7! ways of assigning the numbers 1, 2, ..., 7 to the 7 positions, but many of them yield the same permutation.

For example

$$(1, 2, 3)(4, 5)(6, 7) \text{ and } (2, 3, 1)(6, 7)(4, 5)$$

are the same.

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Overcounting

For the conjugacy class of $\mathrm{Sym}(7)$ with cycle structure 322, there are

- ▶ 3 ways to write the cycle (a, b, c)
- ▶ 2 ways to write the cycle (d, e)
- ▶ 2 ways to write the cycle (f, g)
- ▶ 2 ways to order the two 2-cycles

Therefore every permutation arises in

$$3 \times 2 \times 2 \times 2 = 24$$

ways, and so the total number of distinct permutations in this conjugacy class is

$$7!/24 = 210.$$

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In general

In general, suppose that a partition has c_i parts equal to i, so that

$$c_1 + 2c_2 + 3c_3 + \cdots + nc_n = n.$$

Then the number of permutations in the conjugacy class with this cycle structure is

$$\frac{n!}{(\prod_i c_i! i^{c_i})}.$$

Proof.

Each cycle of length i can be written in i different ways, and the c_i cycles of length i can be written in $c_i!$ different orders. Hence the denominator of this expression counts the number of different ways of writing the permutation.

Example

Check that this all works for $n = 5$ where there are 7 partitions:

Partition	c_1	c_2	c_3	c_4	c_5	Overcount	Number
11111	5	0	0	0	0	120	1
2111	3	1	0	0	0	12	10
221	1	2	0	0	0	8	15
311	2	0	1	0	0	6	20
32	0	1	1	0	0	6	20
41	1	0	0	1	0	4	30
5	0	0	0	0	1	5	24
							120

Stirling Numbers

The *Stirling numbers* are the two series of numbers $s(n, k)$ and $S(n, k)$ defined as follows:

- Stirling numbers of the 1st kind:
  $(-1)^{n-k} s(n, k)$ is the number of permutations of degree n with k cycles.

- Stirling numbers of the 2nd kind:
  $S(n, k)$ is the number of partitions of an n-set into k non-empty parts.

First, the second kind

It is easy to list all the set partitions for small n and k:

- $n = 1$
  - $k = 1$: 1
- $n = 2$
  - $k = 1$: 12
  - $k = 2$: 1|2
- $n = 3$
  - $k = 1$: 123
  - $k = 2$: 12|3, 13|2, 1|23
  - $k = 3$: 1|2|3
- $n = 4$
  - $k = 1$: 1234
  - $k = 2$: 1|234, 134|2, 124|3, 123|4, 12|34, 13|24, 14|23
  - $k = 3$: 12|3|4, 13|2|4, 14|2|3, 1|23|4, 1|24|3, 1|2|34
  - $k = 4$: 1|2|3|4

The triangle

We can write the numbers out in a triangle (similar to Pascal's triangle) as follows:

$$S(1,1)$$
$$S(2,1) \quad S(2,2)$$
$$S(3,1) \quad S(3,2) \quad S(3,3)$$
$$S(4,1) \quad S(4,2) \quad S(4,3) \quad S(4,4)$$
$$\cdots \quad \cdots \quad \cdots$$

From the previous slide we get

$$1$$
$$1 \quad 1$$
$$1 \quad 3 \quad 1$$
$$1 \quad 7 \quad 6 \quad 1$$
$$\cdots \quad \cdots \quad \cdots$$

A recurrence

We can express the Stirling number $S(n, k)$ in terms of smaller Stirling numbers by noting that a set partition of $\{1, \ldots, n\}$ with k parts is obtained either by

- Adding n as a singleton to a partition of $\{1, \ldots, n-1\}$ with $k-1$ parts, or
- Putting n into one of the cells of a partition of $\{1, \ldots, n-1\}$ with k parts

Counting the number of set partitions of each type yields

$$S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

Illustration

For $(n, k) = (4, 3)$, this formula is

$$S(4, 3) = S(3, 2) + 3 \times S(3, 3)$$

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
 & 1 & 3 & 3 \times 1 & \\
1 & & 7 & 6 & 1 \\
\cdots & & \cdots & & \cdots
\end{array}
$$

and the corresponding set partitions in the two groups are

- 12|3|4, 13|2|4, 1|23|4
- 14|2|3, 1|24|3, 1|2|34

An explicit formula

We can find an explicit formula for $S(n, k)$ by using the principle of inclusion and exclusion:

Let A denote the set of all functions

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$$

Now, let A_i be the set

$$A_i = \{f \in A \mid f^{-1}(i) = \emptyset\}.$$

In other words, A_i is the set of functions whose *range* does not include i.

Principle of Inclusion/Exclusion

If A_1, A_2, ..., A_k are all subsets of a set A then for any index set $\mathcal{I} \subseteq \{1, 2, \ldots, k\}$, define

$$A_{\mathcal{I}} = \bigcap_{i \in \mathcal{I}} A_i$$

(where we take $A_{\emptyset} = X$).

Principle of Inclusion/Exclusion

The number of elements of X that do not belong to any of the sets A_1, A_2, ..., A_k is given by

$$\sum_{\mathcal{I} \subseteq \{1,2,\ldots,k\}} (-1)^{|\mathcal{I}|} |A_{\mathcal{I}}|.$$

Set partitions with k parts

By PIE, the number of functions that map $\{1, 2, \ldots, n\}$ *onto* $\{1, 2, \ldots, k\}$ is given by:

$$\sum_{\mathcal{I} \subseteq \{1,2,\ldots,k\}} (-1)^{|\mathcal{I}|} |A_{\mathcal{I}}|.$$

For each possible size $0 \leq j \leq k$ there are $\binom{k}{j}$ possible subsets $\mathcal{I}$ of size j, and each of them makes the same contribution to this sum. More precisely, if $|\mathcal{I}| = j$ then

$$|A_{\mathcal{I}}| = (k - j)^n$$

because this just counts the number of functions that avoid a particular set of j elements.

The formula

Putting this together, we see that the number of functions from $\{1, \ldots, n\}$ *onto* $\{1, \ldots, k\}$ is given by

$$\sum_{j=0}^{j=k} (-1)^j \binom{k}{j} (k-j)^n$$

which (on replacing j by $k-j$) is equal to

$$\sum_{j=1}^{j=k} (-1)^{k-j} \binom{k}{j} j^n.$$

Now each such function determines a partition of $\{1, 2, \ldots, n\}$ into k parts, but we have counted each partition $k!$ times and so

$$S(n, k) = \frac{1}{k!} \sum_{j=1}^{j=k} (-1)^{k-j} \binom{k}{j} j^n.$$

Bell numbers

The *total number* of partitions of a set of size n into any number
of non-empty parts is called the Bell number $B(n)$.
Thus

$$B(n) = \sum_{k=1}^{k=n} S(n, k).$$

The first few Bell numbers are

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975$$

GAP has built-in functions `Stirling1(n,k)`, `Stirling2(n,k)`
and `Bell(n)` giving these numbers.

Now, the first kind

The triangle of Stirling numbers of the first kind

$$
\begin{array}{ccccccc}
& & & s(1,1) & & & \\
& & s(2,1) & & s(2,2) & & \\
& s(3,1) & & s(3,2) & & s(3,3) & \\
s(4,1) & & s(4,2) & & s(4,3) & & s(4,4) \\
& \cdots & & \cdots & & \cdots &
\end{array}
$$

starts as follows:

$$
\begin{array}{cccc}
& & 1 & \\
& -1 & & 1 \\
& 2 & -3 & 1 \\
-6 & 11 & -6 & 1 \\
\cdots & \cdots & & \cdots
\end{array}
$$

A recurrence

We can find a recurrence expressing $s(n, k)$ in terms of smaller
Stirling numbers, by noting that a permutation of degree n with k
cycles can be obtained either by

- Adjoining element n as a fixed point to a permutation of
  degree $n - 1$ with $k - 1$ cycles, or
- Inserting n into one of the cycles of a permutation of degree
  $n - 1$ with k cycles.

and then counting the number of permutations in each category.

Finding $s(4, 2)$

The permutations of degree 4 with 2 cycles consist of

▶ The permutations of degree 3 with 1 cycle, with 4 adjoined as a fixed point

$$(1, 2, 3)(4) \qquad (1, 3, 2)(4)$$

▶ The permutations of degree 3 with 2 cycles, with 4 inserted into one of the cycles

$(1, 2)(3)$	$(1, 3)(2)$	$(1)(2, 3)$
$(1, 4, 2)(3)$	$(1, 4, 3)(2)$	$(1, 4)(2, 3)$
$(1, 2, 4)(3)$	$(1, 3, 4)(2)$	$(1)(2, 4, 3)$
$(1, 2)(3, 4)$	$(1, 3)(2, 4)$	$(1)(2, 3, 4)$

There are 2 in the first group and $9 = 3 \times 3$ in the second group, thus giving us 11 altogether.

The formula

This arguments shows us that

$$|s(n,k)| = |s(n-1,k-1)| + (n-1)|s(n-1,k)|$$

but it does not take into account the sign of $s(n,k)$ (that is, whether it is positive or negative).

Now $(-1)^{n-k}$ is equal to $(-1)^{(n-1)-(k-1)}$ and so taking the signs into consideration we get

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k).$$

An astonishing connection

The reason for associating signs with the Stirling numbers is related to the following astonishing connection between these two sets of numbers, illustrated here for $n = 4$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} = I_4$$