Chris Piech
CS 109

Problem Set #5
Nov 9, 2018

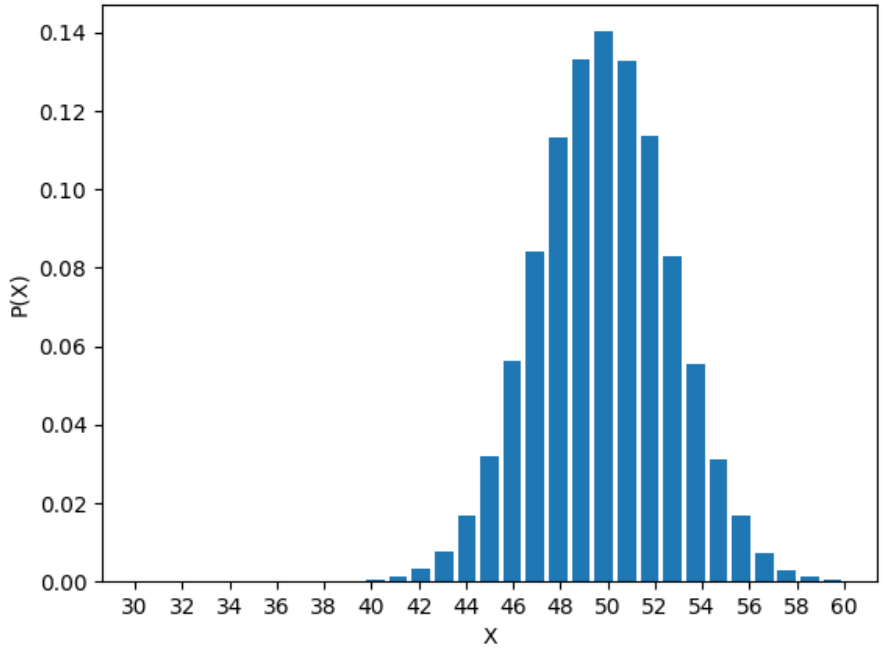
Problem Set #5 Solutions

With problems by Mehran Sahami and Chris Piech

1. a. Given a prior $Beta(a, b)$, after n successes and m failures your posterior distribution is $Beta(a + n, b + n)$. Thus we have a posterior belief of $Beta(2 + 7, 2 + 2)$ = **Beta(9,4)**.

   b. $P(X > 0.5) = 1 - F_x(0.5) = 0.927$

2. a. Bar graph of simulated probabilities:



   b. By the Central Limit Theorem, the sum of 100 IID variables can be approximated as a normal. Let $X_i \sim Uniform(0, 1)$ be a single random variable and $X = \sum_{i=1}^{100} X_i$. Then

$$X \sim N(n \cdot E[X_i], n \cdot Var(X_i)) = N(100 \cdot 0.5, 100 \cdot \frac{1}{12}) = \boldsymbol{N(50, 8.33)}$$

   You could have also gotten to this answer by calculating the $E[X]$ and $Var(X)$. $E[X]$ is the sum of the expectations of all the X_i values = 100 * 0.5 = 50. Since each of the X_i are independent $Var(X)$ is the the sum of the variances of all the X_i values = $100 * Var(X_i) = 100 * 1/12 = 8.33$. We want students to justify using a normal by citing the central limit theorem.

c.

$$\begin{aligned} P(47.5 < X < 48.5) &= P\left(\frac{47.5-50}{\sqrt{100/12}} < Z < \frac{48.5-50}{\sqrt{100/12}}\right) \\ &= \phi(-0.520) - \phi(-0.867) \\ &= (1-\phi(0.520)) - (1-\phi(0.867)) = \phi(0.867) - \phi(0.520) \\ &\approx 0.807 - 0.6985 \approx 0.11 \end{aligned}$$

Which is very similar to the results I got from sampling. Of note, 100,000 calculations means that these numbers should only agree (consistently) to two decimal places.

3. a. Let X_i be the amount of money that person i gives.
   $E[X_i] = 0.2 * 1 + 0.35 * 5 + 0.3 * 10 + 0.05 * 20 =$ **5.95**
   b. $E[X_i^2] = 0.2 * 1^2 + 0.35 * 5^2 + 0.3 * 10^2 + 0.05 * 20^2 = 58.95$
   $Var(X_i) = E[X_i^2] - E[X_i]^2 = 58.95 - 5.95 * 5.95 =$ **23.5475**
   c. Let Y be the total amount of money that the band earns. Y = sum of all the X_is. By the central limit theorem, since the X_is are IID and the number of X_is is more than 20, we expect Y to be normally distributed.
   $Y \sim N(n * E[X_i], n * Var(X_i))$
   $Y \sim$ ***N*(297.5, 1177.35)**
   d.

$$\begin{aligned} P(Y > 350) &= 1 - P(Y < 350) \\ &= 1 - P\left(Z < \frac{350-297.5}{\sqrt{1177.35}}\right) \\ &= 1 - \phi(1.53) \approx \mathbf{0.06} \end{aligned}$$

(better with a continuity correction... but you get the same answer)

4. $Cov(X,Y) = E[XY] - E[Y]E[X]$.
   As we have seen from class, $E[X] = 3.5$
   $E[Y] = E[X^2] = \sum_x \frac{1}{6}x^2 = \frac{1}{6}(1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = 15.1667$
   $E[XY] = E[X^3] = \sum_x \frac{1}{6}x^3 = \frac{1}{6}(1 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3) = 73.5$
   Finally, $Cov(X,Y) = 73.5 - 3.5 \cdot 15.1667 =$ **20.4**

5. $2X \sim N(2 * 1, 2^2 * 2) = N(2, 8)$ by linear transformation of a normal RV. Using convolution, you get that $2X + Y \sim N(2 + 1, 8 + 2) =$ ***N*(3, 10)**.

6. Think of the dice roll as a multinomial with $n = 6$ and $p_1 = ... = p_6 = \frac{1}{6}$:

$$\frac{P(X_1 = 1, ..., X_6 = 1)}{P(X_6 = 6)} = \frac{\binom{n}{1,...,1} p_1 * ... * p_6}{\binom{n}{6} p_6^6} = \frac{\binom{6}{1,...,1}\left(\frac{1}{6}\right)^1 ... \left(\frac{1}{6}\right)^1}{\binom{6}{6}\left(\frac{1}{6}\right)^6} = \frac{6!\left(\frac{1}{6}\right)^6}{1\left(\frac{1}{6}\right)^6} = 6!$$

7. a. E[X] = $E[X] = \sum_{i=1}^{6} P(X = i)E[X]$ = 1/6(6 + 5 + 4 + 3)+1/6($\sum_{i=1}^{2} i + E[X]$)
   Solving for E[X] yields: E[X] = 1/6(6 + 5 + 4 + 3 + 2 + 1) + 2/6E[X]
   2/3E[X] = 1/6(6 + 5 + 4 + 3 + 2 + 1)
   E[X] = 1/4(6 + 5 + 4 + 3 + 2 + 1)=21/4 = **5.25**
   b. Note that $Y \sim Geo(p = 4/6 = 2/3)$, as we will repeatedly roll the die until we get a 3, 4, 5 or 6 (out of 6 possible values that could be rolled). Since the expectation of a Geometric random variable is 1/p, we get E[Y] = 1/(2/3) = **3/2**.

8. Using the definition of variance:

$$Var(Y) = E[Y^2] - E[Y]^2$$

We already know from class $E[Y] = 15$. To calculate $E[Y^2]$, we can use the law of total expectation:

$$E[Y^2] = \sum_{i \in \{1,2,3\}} E[Y^2 \mid X = i]p(X = i)$$

$$= \frac{1}{3}\Big(3^2 + E[(5 + Y)^2] + E[(7 + Y)^2]\Big)$$

A few algebra steps gives the following equation:

$$E[Y^2] = \frac{1}{3}(443 + 2E[Y^2]) \implies E[Y^2] = 443$$

Now we can plug this back into the original equation to get the variance:

$$Var(Y) = 443 - 15^2 = 218$$

9. (a) What is the approximate probability that Program A completes in less than 950 seconds? Define a new random variable $X_A = \sum_{i=1}^{20} x_i$ where x_i is a random variable representing the time for the ith algorithm to finish running. Since they're IID, we can apply the Central Limit Theorem to approximate the X_A distribution:

$$X_A \sim \mathcal{N}(20 * 50, 20 * 100) \implies X_A \sim \mathcal{N}(1000, 2000)$$

Now we can calculate $P(X_A < 950)$:

$$P(X_A < 950) = \Phi\Big(\frac{950 - 1000}{\sqrt{2000}}\Big) = \Phi(-1.12) = 1 - \Phi(1.12) \approx 0.1314$$

(b) What is the approximate probability that Program B completes in less than 950 seconds? Following the same steps outlined in part (a) above give our approximate X_B distribution:

$$X_B \sim \mathcal{N}(1040, 4000)$$

Now we can calculate $P(X_B < 950)$:

$$P(X_B < 950) = \Phi\Big(\frac{950 - 1040}{\sqrt{4000}}\Big) \approx 0.0778$$

(c) What is the approximate probability that Program A completes in less time than B? Define a new random variable $X_{A-B} = X_A - X_B$. Since the sum/difference of random variables is also normal,

$$X_{A-B} \sim \mathcal{N}(1000 - 1040, 2000 + 4000)$$

$$X_{A-B} \sim \mathcal{N}(-40, 6000)$$

If program A finishes in less time than program B, then $X_{A-B} < 0$. We can calculate the probability of this:

$$P(X_{A-B} < 0) = \Phi\Big(\frac{0 - (-40)}{\sqrt{6000}}\Big) = \Phi(0.516) \approx 0.697$$

10. a. A few reasons: Some students might do activity1 and activity2 and their knowledge of the first activity might affect their performance in the second activity, the populations in activity1 and activity 2 could be entirely different, there could be a confounding variable of time (e.g. if activity2 occurs during Christmas time and activity1 doesn't), etc. Since the sample of students who get the two treatments are different, you will not get a legitimate comparison.

b. A few reasons: education systems might be different, access to resources might be different, time zones are different, etc. Since the sample of students who get the two treatments are different, you will not get a legitimate comparison.

11. A/B testing. Over a two-week period, Coursera randomly assigns each student to either be given activity1 (group A), or activity2 (group B). The activity that is shown to each student and the student's measured learning outcomes can be found in the file: learningOutcomes.csv

a. What is the difference in sample means of learning outcomes between students who were given activity1 and students who were given activity2?

$$\bar{X}_1 - \bar{X}_2 = -8.2$$

b. Calculate a p-value for the observed difference in means reported in part (a). In other words: assuming the learning outcomes for students who had been given activity1 and activity2 were identically distributed, what is the probability that you could have sampled two groups of students such that you could have observed a difference of means as extreme, or more extreme, than the one calculated from your data? Describe any code you used to calculate your answer.

The p-value can be calculated using bootstrapping. Here is our code:

```
def nullHypothesis(setA, setB, observedDiff):
    universalSet = copy.deepcopy(setA)
```

```
    universalSet.extend(setB)
    count = 0
    for i in range(10000):
        sampleA = sample(universalSet, len(setA))
        sampleB = sample(universalSet, len(setB))
        diff = abs(np.mean(sampleB) - np.mean(sampleA))
        if diff >= observedDiff: count += 1
    return count / 10000.0
```

The resulting p-value is 0.04

c. File background.csv stores the background of each user. Student backgrounds fall under three categories: more experience, average experience, less experience. For each of the three backgrounds calculate a difference in means in learning outcome between activity1 and activity2, and the p-value of that difference. Describe your methodology.

For students with **less** experience
Activity1 is better than Activity2
Difference in means = 26.0
P-Value < 0.001
Based on the 311 students with less experience

For students with **average** experience
Activity2 is better than Activity1
Difference in means = 25.0
P-Value < 0.01
Based on the 229 students with average experience

For students with **more** experience
Activity2 is better than Activity1
Difference in means = 28.4
P-Value < 0.0001
Based on the 511 students with average experience

All of these values were calculated using bootstrapping, but only for the population of students with the given background.

Our actual Python implementation of this question is included below for reference:

```python
'''
Implementation of coding portion of Question 11 of
HW #5 for CS 109: Probability for Computer Scientists

Author: Sam Schwager
Last edited: 12/5/2018
'''

import numpy as np
import pandas as pd


def calculate_p_value(sample_1, sample_2, num_iters, verbose=True):
    # State which activity is preferred based on our observations
    if verbose: print("Activity 1 preferred") if np.mean(sample_1) > np.mean(sample_2) else print("Activity 2 preferred")

    # We'll compare the absolute means from our bootstrapped
    # samples to this observed difference in means
    observed_mean_diff = abs(np.mean(sample_1) - np.mean(sample_2))

    if verbose: print ("Absolute observed mean difference is: " + str(observed_mean_diff))

    # We'll draw bootstrap samples from universal_set
    universal_set = np.concatenate((sample_1, sample_2))
    count = 0

    for i in range(num_iters):
        resample_1 = np.random.choice(universal_set, sample_1.shape[0])
        resample_2 = np.random.choice(universal_set, sample_2.shape[0])

        if abs(np.mean(resample_1) - np.mean(resample_2)) > observed_mean_diff: count += 1

    return float(count) / num_iters


def main():
    # Read in learning outcomes and replace the strings corresponding
    # to each activity with an appropriate integer value
    learning_outcomes = pd.read_csv("learningOutcomes.csv", header=None)\
                          .replace(["activity1", "activity2"], [1, 2])

    # Get numpy array with first column corresponding to activity
    # and second column corresponding to learning outcome
    learning_outcomes_arr = learning_outcomes.values[:, 1:3]

    # Select rows where first column corresponds to desired activity
    activity_1_outcomes = learning_outcomes_arr[learning_outcomes_arr[:,0] == 1]
    activity_2_outcomes = learning_outcomes_arr[learning_outcomes_arr[:,0] == 2]

    # Use numpy array slicing to calculate absolute mean difference
    mean_diff = abs(np.mean(activity_1_outcomes[:, 1]) - np.mean(activity_2_outcomes[:, 1]))
    print ("Part a: The absolute difference in means is " + str(mean_diff))

    mean_diff_p_val = calculate_p_value(activity_1_outcomes[:, 1], activity_2_outcomes[:, 1], 10000, verbose=False)
    print ("Part b: The p-value for the observed difference in means from part a is " + str(mean_diff_p_val))

    # Read in background data for part c
    backgrounds = pd.read_csv("background.csv", header=None)

    # Create sets of less, average, and more experienced IDs
    less_experience_IDs = set(backgrounds.loc[backgrounds[1] == "less"].loc[:, 0])
    avg_experience_IDs = set(backgrounds.loc[backgrounds[1] == "average"].loc[:, 0])
    more_experience_IDs = set(backgrounds.loc[backgrounds[1] == "more"].loc[:, 0])

    # Select rows of learning outcomes meeting given experience conditions
    less_experience_outcomes = learning_outcomes[learning_outcomes[0].isin(less_experience_IDs)]
    avg_experience_outcomes = learning_outcomes[learning_outcomes[0].isin(avg_experience_IDs)]
    more_experience_outcomes = learning_outcomes[learning_outcomes[0].isin(more_experience_IDs)]

    # Get numpy arrays with first column corresponding to activity
    # and second column corresponding to learning outcome
    less_arr = less_experience_outcomes.values[:, 1:3]
    avg_arr = avg_experience_outcomes.values[:, 1:3]
    more_arr = more_experience_outcomes.values[:, 1:3]

    print ("There are " + str(less_arr.shape[0]) + " students with less experience")
    print ("There are " + str(avg_arr.shape[0]) + " students with average experience")
    print ("There are " + str(more_arr.shape[0]) + " students with more experience")

    less_pval = calculate_p_value(less_arr[less_arr[:,0] == 1][:, 1], less_arr[less_arr[:,0] == 2][:, 1], 10000)
    avg_pval = calculate_p_value(avg_arr[avg_arr[:,0] == 1][:, 1], avg_arr[avg_arr[:,0] == 2][:, 1], 10000)
    more_pval = calculate_p_value(more_arr[more_arr[:,0] == 1][:, 1], more_arr[more_arr[:,0] == 2][:, 1], 10000)

    print ("Part c (i): The p-value for the observed difference in means for less-experienced users is " + str(less_pval))
    print ("Part c (ii): The p-value for the observed difference in means for average-experienced users is " + str(avg_pval))
    print ("Part c (iii): The p-value for the observed difference in means for more-experienced users is " + str(more_pval))


if __name__ == "__main__":
    main()
```

12. a.
```
grades = []
f = open(file, 'r')
for line in f:
    grades.append(int(line))
```

The sample mean is: mean = np.mean(grades)
Answer: 69.2296

b. Using bootstrapping:
```
means = []
for i in range(10000):
    sample = np.random.choice(grades, 5, replace=True)
    mean = np.mean(sample)
    means.append(mean)
```

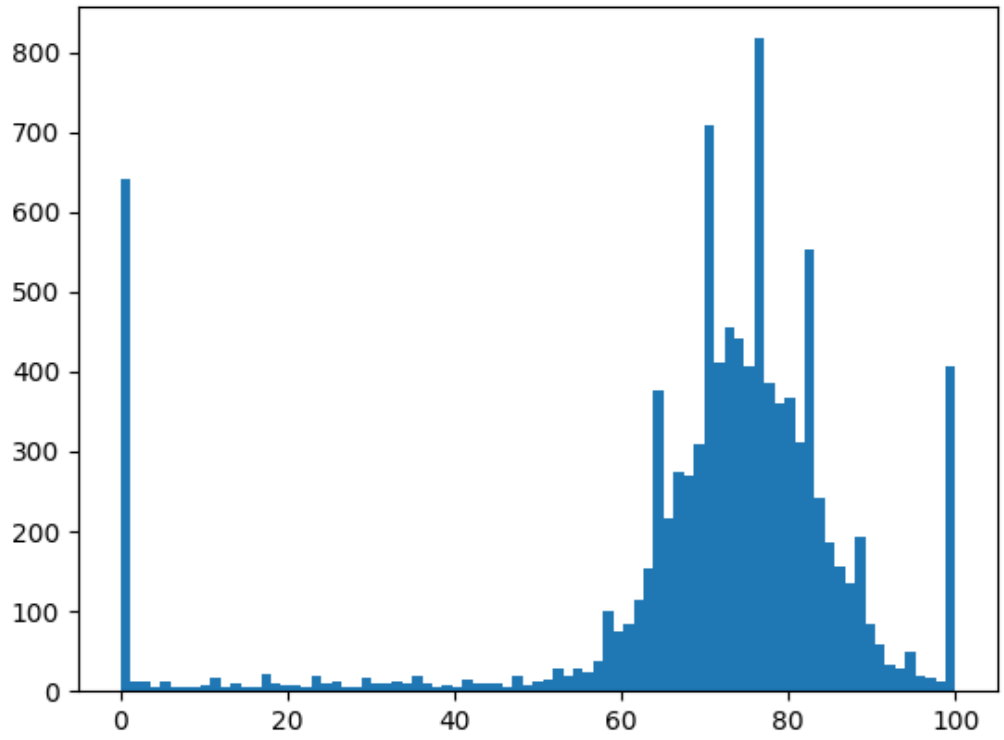
The variance of the mean is: np.var(means)
Answer: 101.26

c. Using bootstrapping:
```
medians = []
for i in range(100000):
    sample = np.random.choice(grades, 5, replace=True)
    median = np.median(sample)
    medians.append(median)
```

The variance of the median is: np.var(medians)
Answer: 56.021

d. Yes. Expected mean of 5 grades is: np.mean(means) = 69.17 and expected median of 5 grades is np.mean(medians) = 73.61.

e. You could conclude that the median is better simply because it has a lower variance. A better argument can be made by looking at the histogram of grades given to the control experiment:
```
import matplotlib.pyplot as plt
plt.hist(grades, bins='auto')
plt.show()
```



By looking at the histogram you get to observe an important phenomena in peer grades. There is a notable number of 0s and a notable number of 100s! You cannot just get rid of the 100s, because it is possible that that was a legitimate grade. The true grade of this assignment is 75, which you can deduce from the peak of the Gaussian looking distribution in the middle. The median of 5 grades is much closer, in expectation, to the true grade.

13. a. $\theta_1 \sim \text{Beta}(a = 6, b = 3)$
    b. here is our pseudocode:

```
a1, b1 = 1, 1
a2, b2 = 1, 1
for i in range(100):
  v1 = sampleBeta(a1, b1)
  v2 = sampleBeta(a2, b2)
  chosen = armax([v1, v2])
  worked = giveDrug(chosen)
  if chosen == 1 and worked: a1 += 1
  if chosen == 1 and not worked: b1 += 1
  if chosen == 2 and worked: a2 += 1
  if chosen == 2 and not worked b2 += 1
```

    c. If X is a beta, $E[X] = \frac{a}{a+b}$:

$E[\theta_1] = \frac{11}{22} = 1/2$
$E[\theta_2] = \frac{76}{82} \approx 0.93$

14. (a) Here is our pseudocode to calculate `probFlu()` using **Joint Sampling**:

```
def bernoulli(p):
    if random() < p return 1 else return 0

def generateOneSample():
    sample = {}
    sample["stress"] = bernoulli(0.5)
    sample["exposure"] = bernoulli(0.1)
    sample["cold"] = bernoulli(probCold(sample["stress"],
                                        sample["exposure"])
    sample["flu"] = bernoulli(probFlu(sample["stress"],
                                      sample["exposure"])
    for i in range(1, 11):
        symptom = "X" + str(i)
        probSymptom = probSymptom(i, sample["cold"],
                                     sample["flu"])
        sample[symptom] = bernoulli(probSymptom)
    return sample

def probFlu():
    numExposureAndX2 = 0
    numFluExposureAndX2 = 0
    for i in range(1000000):
        sample = generateOneSample()
        if sample["exposure"] == 1 and sample["X2"] == 1:
            numExposureAndX2 += 1
            if sample["flu"] == 1:
                numFluExposureAndX2 += 1
    return numFluExposureAndX2 / numExposureandX2
```

(b) We can use the three sets of insights given to us in the problem statement to realize that the joint distribution over all variables can be expressed as such: $P(S = s, E = e, C = c, F = f, \text{Symptoms}) = P(S = s)P(E = e)P(C = c|S = s, E = e)P(F = f|S = s, E = e) \prod_j P(X_j = x_j|F = f, C = c)$
We also use the definition of conditional probability to rewrite the quantity we want to calculate as $P(F = 1|E = 1, X_2 = 1) = \frac{P(F=1,E=1,X_2=1)}{P(E=1,X_2=1)}$
Notice that given the definition of our joint distribution, we can calculate these quantites by marginalizing out all unseen (hidden) variables. That is, $P(F = 1, E = 1, X_2 = 1) = \sum_s \sum_c \sum_{X_1} \sum_{X_3} \sum_{X_4}, ..., \sum_{X_{10}} P(S = s, E = 1, C = c, F = 1, \text{Symptoms})$ and
$P(E = 1, X_2 = 1) = \sum_f \sum_s \sum_c \sum_{X_1} \sum_{X_3} \sum_{X_4}, ..., \sum_{X_{10}} P(S = s, E = 1, C = c, F = f, \text{Symptoms})$. Putting this all together, here is our pseudocode to calculate this quan-

tity without using sampling (also known as exact inference):

```
def bernoulli(p):
    if random() < p return 1 else return 0
def jointProb(s,e,c,f,symptoms):
    prob = 1
    prob *= 0.5 if s == 1 else 0.5
    prob *= 0.1 if e == 1 else 0.9
    prob *= probCold(s,e) if c == 1 else 1 - probCold(s,e)
    prob *= probFlu(s,e) if f == 1 else 1 - probFlu(s,e)
    for i in range(10):
        p = probSymptom(i+1,f,c)
        prob *= p if symptoms[i] == 1 else 1-p
    return prob
def calculateNumerator():
    totalSum = 0.0
    possibleVals = [0,1]
    for s in possibleVals:
        for c in possibleVals:
            for x1 in possibleVals:
                for x3 in possibleVals:
                    ...
                        for x10 in possibleVals:
                            x = [x1,1,x3,...,x10]
                            totalSum += jointProb(s,1,c,1,x)
def calculateDenominator():
    totalSum = 0.0
    possibleVals = [0,1]
    for f in possibleVals:
        for s in possibleVals:
            for c in possibleVals:
                for x1 in possibleVals:
                    for x3 in possibleVals:
                        ...
                            for x10 in possibleVals:
                                x = [x1,1,x3,...,x10]
                                totalSum += jointProb(s,1,c,f,x)
def probFlu():
    return calculateNumerator() / calculateDenominator()
```