Mixing and Asymptotic Independence

In this part, we care about:

- Predictability
- Asymptotic Independence and hence generalized central limit theorem.

Singularity and regularity

To start with, singularity measures the new information added with increasing observation interval; while regularity measures the infomation $x(t)$ originate.

Note that $\mathscr{H}(x,t) = \mathscr{S}(x(s); s \leq t)$ and define $\mathscr{H}(x,-\infty) = \cap_{t \leq t_0} \mathscr{H}(x,t), t \to -\infty$. $\mathscr{H}(x,t)$ can be intepreted as the information of $x(s), s \leq t$ and $\mathscr{H}(x,-\infty)$ is the ancient information(far from now).

Definition

For process $\{x(t), t \in \mathbb{R}\}$ is **singular** if $\mathscr{H}(x,-\infty) = \mathscr{H}(x)$; is **regular** if $\mathscr{H}(x,-\infty) = \mathbf{0}$.

Note: Singularity is also called purely determinism, that is, all future can be obtained from arbitary far back past.

The Cramér-Wold decomposition

Every stochastic process $\{x(t), t \in \mathbb{R}\}$, with $E\left(|x(t)|^2\right) < \infty$, exists a decomposition of two *uncorrelated* processes

$$x(t) = y(t) + z(t)$$

where $\{y(t), t \in \mathbb{R}\}$ is regular and $\{z(t), t \in \mathbb{R}\}$ is singular.

$AR(1)$ **Prcess with added term**

For $x(t) = ax(t-1) + e + e(t)$, it can be decomposed as $y(t) + z(t)$

$$y(t) = \sum_{k=0}^{\infty} a^k e(t-k) \qquad z(t) = \frac{1}{1-a} e$$

where $y(t)$ is regular and $z(t)$ is singular.

Spectrum Decomposition

The spectrum distribution function can be decomposed as

$$F(\omega) = F^{(s)}(\omega) + F^{(ac)}(\omega) + F^{(d)}(\omega)$$

where $F^{(ac)}(\omega) = \int_{\infty}^{\omega} f(x)dx$, $f(\omega) = F'(\omega)$, $F^{(d)}(\omega) = \sum_{\omega_k \le \omega} \Delta F_k$ and $F^{(s)}(\omega)$ is the singular part.

Conditions for Stationary Sequences

Fristly, note the integral $P = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\omega)\mathrm{d}\omega$ is either finite or $-\infty$.

Then, for a stationary sequence $\{x_n, n \in \mathbb{N}\}$

- If $P = -\infty$, then x_n is singular.
- If $P > -\infty$, and the spectrum is *absolutely continuous* with $f(\omega) > 0$ for almost all ω, i.e. $F(\omega) = F^{(ac)}(\omega)$, then x_n is regular.
- If $P > -\infty$, but $F(\omega)$ is either discontinuous, or is continuous with non-vanishing singular part, $F^{(s)}(\omega) \neq 0$, then X_n is neither singular nor regular.

Note: A stationary sequence $x(t)$ is regular $\Leftrightarrow x_t = \sum_{k=-\infty}^{t} h_{t-k}y_k$ where y_k is uncorrelated.

Some Special Case

- If the spectrum is *discrete* with a finite number of jumps, then $f(\omega) = 0 \ a.s.$ and $P = -\infty$, so the process is singular;
- If the spectrum is *absolutely continuous* with density $f(\omega)$, singularity
  and regularity depends on whether $f(\omega)$ comes close to 0 or not.
- If $f(\omega) \ge c > 0$ for $-\pi < \omega \le \pi$, then the integral is finite and the process is regular.

Conditions for Stationary Processes

Define the integral $Q = \int_{-\infty}^{\infty} \frac{\log f(\omega)}{1+\omega^2}\mathrm{d}\omega$.

Then, for a stationary process $\{x(t), t \in \mathbb{R}\}$

- If $Q = -\infty$, then $x(t)$ is singular.
- If $Q > -\infty$, and the spectrum is absolutely continuous, then $x(t)$ is regular.

Note: For a statioanry process $\{x(t)\}$, there is a singular-regular decomposition $x(t) = x^{(s)}(t) + \int_{u=-\infty}^{t} h(t-u)\mathrm{d}\xi(u)$, where $\{\zeta(t), t \in \mathbb{R}\}$ is a process with uncorrelated increments.

Discussion

Consider three process with covariance function and spectral density

- $r_1(t) = e^{-t^2/2}$, $f_1(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2}$, $Q = -\infty$
- $r_2(t) = \frac{sin(t)}{t}$, $f_2(\omega) = \frac{1}{2}\mathbf{1}_{\{\omega:|\omega|\le 1\}}$, $Q = -\infty$
- $r_3(t) = \exp(-\alpha|t|)$, $f_3(\omega) = \frac{\alpha}{\pi(\alpha^2+\omega^2)}$, $Q > -\infty$

For the first two cases, $Q = -\infty$, which means $x(t)$ is singular and hence it is *predictable*. However, from the perspective of covariance function $r(t) \to 0, t \to \infty$, when the time varies with relatively long distance, the r.v. $x(s+t)$ and $x(s)$ are nearly irrelative (If $x(t)$ is in additional Gaussian, they are nearly independent.)

Therefore, independent doesn't implies unpredictability. It also depends on the convergence rate of covariance function: for the case one $e^{-t^2/2}$ is too fast; for the case two $\frac{sin(t)}{t}$ is too slow.

Mixing

Global Mixing for General Process

$\{x(t)\}$ is stochastic process defined on $\mathscr{M}_a^b$, t is arbitary and $A \in \mathscr{M}_{-\infty}^t$. $U^{-k}B = \{x(\cdot); x(k+\cdot) \in B\} \in \mathscr{M}_k^\infty$.

- **uniform mixing**: $\exists\ \phi(n)$ s.t. $\phi(n) \to 0$ as $n \to \infty$, $B \in \mathscr{M}_{t+n}^\infty$

$$|P(A \cap B) - P(A)P(B)| \le \phi(n)P(A)$$

- **strong mixing**: $\exists\ \alpha(n)$ s.t. $\alpha(n) \to 0$ as $n \to \infty$, $B \in \mathscr{M}_{t+n}^\infty$

$$|P(A \cap B) - P(A)P(B)| \le \alpha(n)$$

- **ergodic mixing**: $B \in \mathscr{M}_t^\infty$,

$$\lim_{k\to\infty} P\left(A \cap U^{-k}B\right) = P(A)P(B)$$

- **weak mixing**: $B \in \mathscr{M}_t^\infty$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \left|P\left(A \cap U^{-k}B\right) - P(A)P(B)\right| = 0$$

<u>Note</u>: From the expression, we know that mixing evaluates the level of independence w.r.t the distance n.

<u>Note</u>: Uniform mixing $\Rightarrow$ Strong mixing $\Rightarrow$ Ergodic mixing

Global Mixing for Gaussian Process

Let $\{x(t), t \in \mathbb{Z}\}$ be a stationary Gaussian process.

- **uniformly mixing**: $r(t) = 0$ for $|t| > m$, m-dependent.
- **strongly mixing**: $f(\omega) \ge c > 0$ on $-\pi < \omega \le \pi$
- **weakly mixing**: $\Leftrightarrow$ ergodic $\Leftrightarrow$ the spectral distribution function is continuous.

Reference

- Stationary Stochastic Processes