Slice Sampler

Simple Slice Sampler

The generation from a distribution with density $f(x)$ is equivalent to uniform generation on the subgraph of f,

$$\mathscr{S}(f) = \{(x, u); 0 \le u \le f(x)\}$$

and f need only be known up to a normalizing constant.

We consider using a *random walk* on $\mathscr{S}(f)$ and this is slice sampling.

Algorithm

1. Move from (x, u) to (x, u') by

$$u' \mid x \sim \text{ Uniform } \{(\{u : u \le f_1(x)\}\}$$

2. Move from (x, u') to (x', u') by

$$x \mid u' \sim \text{ Uniform } (\{x : u' \le f_1(x)\})$$

<u>Note</u>: The uniform distribution on $\mathscr{S}(f)$ is indeed stationary for both steps. This algorithm will work well only if the exploration of the subgraph of f_1 by the corresponding random walk is fast enough.

Example

Simple slice sampler

Generate from the density $f(x) = (1/2)e^{-\sqrt{x}}\mathbf{1}(x > 0)$

- $U|x \sim \mathcal{U}\left(0, \frac{1}{2}e^{-\sqrt{x}}\right)$;
- $X|u \sim \mathcal{U}\left(0, [\log(2u)]^2\right)$

Truncated normal distribution

Generate from the density $f(x) \propto f_1(x) = \exp\left(-(x+3)^2/2\right)\mathbb{I}_{[0,1]}(x)$

- $U|x \sim \mathcal{U}\left(0, \exp\{-(x+3)^2/2\}\right)$
- $X|u \sim \mathcal{U}\{y; \exp\{-(y+3)^2/2\} \ge u\} \cap [0, 1]$

The General Slice Sampler

Expressing the distribution of u is always difficult, it inspires us to find a way to simplify this.

Suppose there is a decomposition of f:

$$f(x) \propto \prod_{i=1}^{k} f_i(x)$$

Algorithm

At iteration $t+1$, simulate

1. $\omega_1^{(t+1)} \sim U_{[0, f_1(x^{(t)})]}$

   $\ldots$

k. $\omega_k^{(t+1)} \sim U_{[0, f_k(x^{(t)})]}$

k+1. $x^{(t+1)} \sim U_{A^{(t+1)}}$, with

$$A^{(t+1)} = \left\{ y; f_i(y) \geq \omega_i^{(t+1)}, i = 1, \ldots, k \right\}$$

Note: When f_k's are simple, the expressions is tractable, but it happens that the intersection set is small.

Example

A 3D slice sampler

Generate from the density:

$$f(x) = \exp\left(-x^2/2\right) \times (1 + \cos(\pi x)) \times \mathrm{I}(x \in [-0.5, 0.5])$$

- Step 1: generate $\omega | x$
  - $\omega_1 \sim U[0, \exp(-x^2/2)]$
  - $\omega_2 \sim U[0, 1 + \cos(\pi x)]$
- Step 2: generate $x' | \omega$
  - $I_1 = [-\sqrt{-2\log(\omega_1)}, \sqrt{-2\log(\omega_1)}]$
  - $I_2 = [\frac{\arccos(\omega_2' - 1)}{\pi}, \infty)$
  - $I_3 = [-0.5, 0.5]$
  - $x' \sim \mathbf{1}_{I_1 \cap I_2 \cap I_3}(x)$