Metric Entropy

> *Since there are some problems about mathjax, I also upload the pdf version /Large Sample / 4 Metric Entropy.*

Some Definition

Suppose (Θ, ρ) is a metric space.

- For $\epsilon > 0$, $\{\theta^i\}_{i=1}^N$ is a ϵ-**covering** of Θ if $\forall \theta \in \Theta, \rho(\theta, \theta^i) \le \epsilon$.
- **covering number** $N(\epsilon, \Theta, \rho)$: The cardinality of the minimum ϵ-covering of Θ.
- **metric entropy** $H(\epsilon, \Theta, \rho)$: $H(\epsilon, \Theta, \rho) = \log N(\epsilon, \Theta, \rho)$
- For $\delta > 0$, $\{\theta^i\}_{i=1}^N$ is a δ-**covering** of Θ if $\forall i \neq j, \rho(\theta^i, \theta^j) > \delta$.
- **packing number** $M(\delta, \Theta, \rho)$: The cardinality of the maximum δ-packing of Θ.

Suppose $\mathcal{F}$ is a functional family with measure μ on $\mathcal{X}$

- For $\epsilon > 0$, $\{[l_i, u_i]\}_{i=1}^M$ is an ϵ-**bracketing** if $\forall f \in \mathcal{F}$, $l_i(x) \le f(x) \le u_i(x), \forall x \in \mathcal{X}$ and $\|l_i - u_i\|_{L_p(\mu)} \le \epsilon$.
- **bracketing number** $N_{[]}(\epsilon, \mathcal{F}, L_p(\mu))$: The cardinality of the minimum ϵ-bracketing of $\mathcal{F}$.

Proporties

- $M(2\epsilon, \Theta, \rho) \le N(\epsilon, \Theta, \rho) \le M(\epsilon, \Theta, \rho)$
- $N(\epsilon, \mathcal{F}, L_p(\mu)) \le N_{[]}(2\epsilon, \mathcal{F}, L_p(\mu))$
- When μ is a probability measure: $N_{[]}(\epsilon, \mathcal{F}, L_p(\mu)) \le N(\epsilon/2, \mathcal{F}, L_\infty) = N_{[]}(\epsilon, \mathcal{F}, L_\infty)$

Volume comparison lemma

Consider $\Theta = \{\theta \in \mathbb{R}^d : \|\theta\| \le r\}$ with $\rho(x, y)=|x-y| , then

$$\left(\frac{r}{\epsilon}\right)^d \le N(\epsilon, \Theta, \rho) \le \left(1 + \frac{2r}{\epsilon}\right)^d$$

also

$$d \log \frac{r}{\epsilon} \leq \log N(\epsilon, \Theta, \rho) = H(\epsilon, \Theta, \rho) \leq d \log\left(1 + \frac{2r}{\epsilon}\right)$$

Sketch proof:

- Lower bound: Compare the volume of Θ and ϵ-ball;
- Upper bound: Compare the volume of $\Theta + \epsilon B$ with ϵ-ball (via packing);

Main Theorem(ULLN)

Bracketing Number

Consider $X_1, \cdots, X_n \stackrel{i.i.d}{\sim} \mathbb{P}$ with $N_{[]}\left(\epsilon, \mathcal{F}, L_1(\mathbb{P})\right) < \infty, \forall \epsilon > 0$. Then,

$$\sup_{f \in \mathcal{F}} |P_n f - \mathbb{P} f| \xrightarrow{\mathbb{P}} 0. \quad \text{ULLN}$$

Covering number

If $\mathcal{F}$ is a functional class with envelop function $F \in L_1(\mathbb{P})$. $\forall M > 0$ and $\epsilon > 0$, we have $\log N\left(\epsilon, \mathcal{F}_M, L_1\left(P_n\right)\right) = o_p(n)$. Then,

$$\|P_n - \mathbb{P}\|_{\mathcal{F}} \xrightarrow{\mathbb{P}} 0$$

Discussion: The condition imposed on bracketing number is stricter than that on covering number, but bracketing number is harder to calculate than covering number.

Reference

- Metric Entropy