Functions with Bounded Differences

> *Since there are some problems about mathjax, I also upload the pdf version /Large Sample / 3 Functions with Bounded Differences.*

Basic Definition

- $X_1, \cdots, X_n$ are independent;
- $\mathbb{E}_i = \mathbb{E}(|X_1, \cdots . X_i)$;
- $Z = f(X_1, \ldots, X_n)$;
- $\Delta_i = \mathbb{E}_i Z - \mathbb{E}_{i-1} Z$;
- $Z - \mathbb{E}Z = \sum_{i=1}^{n} \Delta_i$;
- $\mathbb{E}^{(i)} = \mathbb{E}(|X_1, \cdots . X_{i-1}, X_{i+1}, \cdots, X_n)$
- Bounded difference with parameter $\{c_i\}$'s

$$\sup_{x_1, \ldots, x_n, x_i' \in \mathcal{X}} \left| f(x_1, \ldots, x_n) - f\left(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\right) \right| \leq c_i, 1 \leq i \leq n$$

- Martingale difference(MGD) sequence $\{(X_i, \mathcal{F}_i)\}$
    - $\{\mathcal{F}_i\}$ is a filtration.

    b. X_i is integrable.

    c. $\mathbb{E}(X_i \mid \mathcal{F}_{i-1}) = 0 \forall i \in \mathbb{N}$
- MGD σ_i^2-sub-Gaussian

$$\mathbb{E}\{\exp(\lambda X_i) \mid \mathcal{F}_{i-1}\} \leq \exp\left(\lambda^2 \sigma_i^2 / 2\right) \forall i \in \mathbb{N}$$

Efron-Stein Inequality

Lemma

$$\operatorname{Var}(Z) = \mathbb{E}\left\{\left(\sum_{i=1}^{n} \Delta_i\right)^2\right\} = \sum_{i=1}^{n} \mathbb{E}\left(\Delta_i^2\right)$$

Note: Independence is not necessary here.

Efron-Stein Inequality

Use the notation above

$$\mathrm{Var}(Z) \leq \sum_{i=1}^{n} \mathbb{E}\left\{\left(Z-\mathbb{E}^{(i)} Z\right)^{2}\right\} \stackrel{\text{def}}{=} v$$

If $X_1', \cdots, X_n'$ are independent copies and denote
$Z_i' = f\left(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n\right)$, then

$$\begin{aligned} v &= \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}\left\{\left(Z-Z_i'\right)^{2}\right\} = \sum_{i=1}^{n} \mathbb{E}\left\{\left(Z-Z_i'\right)_{+}^{2}\right\} \\ &= \sum_{i=1}^{n} \mathbb{E}\left\{\left(Z-Z_i'\right)_{-}^{2}\right\} = \inf_{Z_i} \sum_{i=1}^{n} \mathbb{E}\left\{\left(Z-Z_i\right)^{2}\right\} \end{aligned}$$

Martingale Difference Sequence

If $\{X_i\}$ is σ_i^2 -sub-Gaussian MGD, then for any $t \geq 0$,

$$\mathbb{P}\left\{\sum_{i=1}^{n}\left(X_i-\mathbb{E} X_i\right) \geq t\right\} \vee \mathbb{P}\left\{\sum_{i=1}^{n}\left(X_i-\mathbb{E} X_i\right) \leq -t\right\} \leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^{n} \sigma_i^2}\right)$$

If $f: \mathcal{X}^n \rightarrow \mathbb{R}$ satisfies c_i -bounded difference, then for any $t \geq 0$

$$\mathbb{P}\left\{f\left(X_{1:n}\right)-\mathbb{E} f\left(X_{1:n}\right) \geq t\right\} \leq \exp\left(-\frac{2 t^2}{\sum_{i=1}^{n} c_i^2}\right)$$

Reference

- Functions with Bounded Differences