Ergodic Theory

Ergodicity describes the phenomenon that the time average of one relization (sample path) of stationary process converges to the expectation of it at $t = 0$. It transforms a probabilistic problem to a deterministic problem.

The general ergodicity properties of Gaussian processes can be inferred already from its covariance function.

Ergodicity is the equivalence of

$$\mathbb{E}(x) = \int_\Omega x(\omega) dP(\omega) \qquad \frac{1}{n}\sum_1^n x(T^{k-1}\omega)$$

Ergodic Theorem in $\mathcal{L}^2$

Ergodic Example in $\mathcal{L}^2$

$\{x(t)\}$ is a stationary stochastic process with covariance function $r(t)$. Then if $\mathbb{E}(x(t)) = 0$,

$$\frac{1}{T}\int_0^T r(t)\mathrm{d}t \to 0 \quad \text{implies} \quad \frac{1}{T}\int_0^T x(t)\mathrm{d}t \overset{q.m}{\to} 0$$

as $T \to \infty$.

Continuity at $t = 0$

If the spectral distribution F is a step function and the spectral representation $x(t) = \int e^{i\omega t} dZ(\omega)$ is available:

$$\begin{aligned}
&\lim_{T\to\infty} \tfrac{1}{T}\textstyle\int_0^T r(t)e^{-i\omega_k t}\mathrm{d}t = \Delta F_k \\
&\lim_{T\to\infty} \tfrac{1}{T}\textstyle\int_0^T |r(t)|^2\mathrm{d}t = \sum_k (\Delta F_k)^2 \\
&\lim_{T\to\infty} \tfrac{1}{T}\textstyle\int_0^T x(t)e^{-i\omega_k t}\mathrm{d}t = \Delta Z_k
\end{aligned}$$

- If the spectral distribution is continuous at $t = 0$, then $\frac{1}{T}\int_0^T r(t)\mathrm{d}t \to 0$ and hence implies ergodic;
- If the spectral distribution is discontinuous at $t = 0$, then $\frac{1}{T}\int_0^T r(t)\mathrm{d}t \to \Delta F_k$ and hence implies non-ergodic.

Non-Ergodic Example in $\mathcal{L}^2$

Suppose $\omega_k > 0$, $x(t) = \sum_k A_k \cos(\omega_k t + \phi_k)$. Then, for $x(t)^2$

- $\mathbb{E}(x(0)^2) = \sum \mathbb{E}(A_k^2/2)$
- $\frac{1}{T}\int_0^T x(t)^2 dt \to \sum A_k^2/2 \neq \sum \mathbb{E}(A_k^2/2)$

Stationarity and Transformation

In this part, stationarity means strictly stationary.

Measure Preserving Transformation: $P\left(T^{-1}A\right) = P(A)$ for all $A \in \mathscr{F}$, then T is *measure preserving transformation* and P is *invariant*.

From measure presering transformation to a stationary sequance

Take a random variable $x(\omega)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and T is measure presering. Let $x_k(\omega) = x(T^{k-1}\omega)$ and then the sequence is strictly stationary.

From stationary sequance to a measure presering transformation

The *shift transformation* $T : T\omega = (x_2, x_3, \cdots)$ is a measure preserving transformation, if $\{x_k\}$ is a strictly stationary sequence.

The Ergodic Theorem(Transformation View)

In this part, we want to explore the ergodic theorem in the transformation view. For a fixed initial outcome ω, the time average

$$\frac{1}{n}\sum_{1}^{n} x\left(T^{k-1}\omega\right)$$

With the relationship of measure preserving transformation and strictly stationary sequence, what we care about is

$$S_n/n = \frac{1}{n}\sum_{1}^{n} x_k$$

As the form suggested, this is the law of large numbers in stochastics process.

Invariant Set/r.v & Ergodic Transformation

Consider $(\Omega, \mathcal{F}, \mathbb{P})$ and measure preserving transformation $T : \Omega \to \Omega$

Invariant set $A \in \mathscr{F}$: $T^{-1}A = A$

Family of invariant ses $\mathscr{S}$: $\mathscr{J} = \{\text{invariant sets } A \in \mathscr{F}\}$

Invariant random variable $x \in (\Omega, \mathscr{F}, \mathbb{P})$: $x(\omega) = x(T\omega)\ a.s. \Leftrightarrow x \in \mathscr{S}$

<u>Note</u>: The equivalent definition of invariant random variable links it to invariant sets.

Ergodic: A measure preserving transformation T on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *ergodic* $\Leftrightarrow \forall A \in \mathscr{S} = \mathbb{P}(A) = 0, 1 \Leftrightarrow$every (bounded) invariant random variable x is $a.s.$ a constant.

Note: For measure preserving transformation, such limit S_n/n always exist. The next question is what the limiting distribution or limiting value it is.

Brikhoof Ergodic Theorem

Let T be a measure preserving transformation on $(\Omega, \mathscr{F}, P)$. Then, for any random variable x with $E(|x|) < \infty$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{0}^{n-1} x\left(T^k \omega\right) = E(x \mid \mathscr{J}), a.s$$

Note: This implies the limiting behavior depends on initial state.

Measure Preserving Ergodic Transformation

If T is additionally ergodic, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{0}^{n-1} x\left(T^k \omega\right) = E(x), a.s$$

Infinite Expectation

If x is non-negative with $\mathbb{E}(x) = \infty$ then $S_n/n \to \infty$ if T is ergodic.

Ergodic Non-Random Walks

Take a set $A \in F$ and consider its indicator function $\chi_A(\omega)$. If T is ergodic,

$$\frac{1}{n} \sum_{0}^{n-1} \chi_A\left(T^k \omega\right) \stackrel{a.s.}{\rightarrow} P(A)$$

The Ergodic Theorem(Process View)

Stationary Sequence

Let $\{x_n\}$ be a strictly stationary *sequence*

Invariant set $A \in \mathscr{F}$: $A = \{(x_n, x_{n+1}, \ldots) \in B\}, n \geq 1, b \in \mathscr{B}_\infty$

Family of invariant ses $\mathscr{S}$: $\mathscr{J} = \{\text{invariant sets } A \in \mathscr{F}\}$

Invariant random variable $\phi \in (\mathbb{R}^\infty, \mathscr{B}_\infty)$: $z = \phi\left(x_n, x_{n+1}, \ldots\right), n \geq 1$

Ergodic: The sequence is called *ergodic* $\Leftrightarrow \forall A \in \mathscr{S} = \mathbb{P}(A) = 0, 1$.

Ergodic Theorem

1. If $\{x_n, n \in \mathbb{N}\}$ is a stationary sequence with $E(|x_1|) < \infty$, and $\mathscr{J}$ denotes the σ -field of invariant sets, then

$$\frac{1}{n} \sum_1^n x_k \stackrel{a.s.}{\rightarrow} E\left(x_1 \mid \mathscr{J}\right), a.s$$

2. If $\{x_n, n \in \mathbb{N}\}$ is stationary and ergodic, then

$$\frac{1}{n} \sum_1^n x_k \stackrel{a.s.}{\rightarrow} E\left(x_1\right), a.s$$

Transitivity

If $\{x_n, n \in \mathbb{N}\}$ is stationary and ergodic, and $\phi\left(x_1, x_2, \ldots\right)$ is measurable on $\left(\mathbb{R}^{\infty}, \mathscr{B}_{\infty}\right)$, then the process $y_n=\phi\left(x_n, x_{n+1}, \ldots\right)$ is also stationary and ergodic.

Stationary Process

For continuous time processes $\{x(t), t \in \mathbb{R}\}$, the *shift transformation* U_τ is

$$\left(U_\tau x\right)(t)=x(t+\tau)$$

Measure preserving transformation: If $x(t)$ is strictly stationary.

Invariant set B: If $\mathbb{P}(B \neq U_\tau B)=0$

Family of invariant ses $\mathscr{S}$: $\mathscr{J}=\{\text{invariant sets } A \in \mathscr{F}\}$

Ergodic: The process is called *ergodic* $\Leftrightarrow \forall A \in \mathscr{S}=\mathbb{P}(A)=0,1$.

Ergodic Theorem

1. For any stationary process $\{x(t), t \in \mathbb{R}\}$ with $E(|x(t)|)<\infty$ and integrable sample paths, as $T \rightarrow \infty$

$$\frac{1}{T} \int_0^T x(t) \mathrm{d} t \stackrel{a.s.}{\rightarrow} E(x(0) \mid \mathscr{J})$$

2. If further $\{x(t), t \in \mathbb{R}\}$ is ergodic, then

$$\frac{1}{T} \int_0^T x(t) \mathrm{d} t \stackrel{a.s.}{\rightarrow} E(x(0))$$

Ergodic Gaussian Sequences and Processes

Ergodicity for Gaussian stationary processes is characterized by their covariance function r(t) in continuous or discrete time.

Ergodicity for Gaussian stationary processes

Let $\{x(t), t \in \mathbb{R}\}$ be stationary and Gaussian with $E(x(t)) = 0$, $V(x(t)) = 1$ and covariance function $r(t)$.

1. $x(t)$ is ergodic $\Leftrightarrow F(\omega)$ is continuous everywhere.
2. If $r(t) \to 0$ as $t \to \infty$, then $x(t)$ is ergodic.

Note: A sufficient condition is $f(\omega)$ exists. That means $F(\omega) = \int_{-\infty}^{\omega} f(x)\mathrm{d}x$ is continuous.

Reference

- Stationary Stochastic Processes: Theory and Applications