Martingale

> *Since there are some problems about mathjax, I also upload the pdf version /Large Sample / 2 Martingale.*

Basic Definition

- Under measurable space $(\Omega, \mathcal{F})$:
- Filtration $\{\mathcal{F}_n\}$: $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_\infty$.
- $\{\mathcal{F}_n\}$-adapted r.s. M_n: $M_n \in \mathcal{F}_n$
- Martingale $\{M_n\}$:
  - $\{\mathcal{F}_n\}$-adapted
  - $\mathbb{E}|M_n| < \infty$
  - $\mathbb{E}\left(M_{n+1} \mid \mathcal{F}_n\right) = M_n \quad \forall n$
- Sub-Martingale $\{M_n\}$:
  - $M_n \leq \mathbb{E}\left(M_{n+1} \mid \mathcal{F}_n\right) \quad \forall n$
- Super-Martingale $\{M_n\}$:
  - $M_n \geq \mathbb{E}\left(M_{n+1} \mid \mathcal{F}_n\right) \quad \forall n$
- $\{\mathcal{F}_n\}$-predictable r.s.H_n: $H_n \in \mathcal{F}_{n-1}$.
- Martingale X transformation(through predictable H) $(H \cdot X)$:
  - $(H \cdot X)_n = \sum_{m=1}^n H_m\left(X_m - X_{m-1}\right)$
  - $(H \cdot X)_0 = 0$
- Stopping Time T: $[T = n] \in \mathcal{F}_n$.

Proporties

Transformation of martingale

- Martingle M_n + ϕ convex + $\mathbb{E}\left(\left|\phi\left(M_n\right)\right|\right) < \infty \rightarrow \phi(M_n)$ ia a sub-martingale;
- Sub-martingle M_n + ϕ non-decreasing&convex + $\mathbb{E}\left(\left|\phi\left(M_n\right)\right|\right) < \infty \rightarrow \phi(M_n)$ ia a sub-martingale;
- Example:
  - Martingle M_n +$\mathbb{E}|M_n|^p < \infty \rightarrow |M_n|^p$ is a sub-martingale;
  - Sub-martingale $M_n \rightarrow (M_n - a)_+$ is a sub-martingale;
  - Super-martingale $M_n \rightarrow M_{n \wedge a}$ is a super-martingale.
- Martingale X_n+predictable&bounded $H_n \rightarrow (H \cdot X)_n$ is a martingale;
- Sub/Super-martingle X_n+predictable&bounded $H_n \geq 0 \rightarrow (H \cdot X)_n$ is a sub/super-martingale;
- Sub/Super-martingale X_n+T is a stopping time $\rightarrow M_{n \wedge T}$ is a sub/super-martingle.
  - Denote $H_n = \mathbf{1}_{T \geq n}$ predictable;
  - Consider $(H \cdot M) = M_{n \wedge T} - M_0$.

Doob Sub-martingale Decomposition

For any sub-martingale $\{(X_n, \mathcal{F}_n, \mathbb{P}), n \geq 0\}$, there is a unique decomposition

$$X_n = M_n + A_n$$

where

- $\{(M_n, \mathcal{F}_n, \mathbb{P}), n \geq 0\}$ is a martingale;
- $\{A_n, n \geq 0\}$ is predictable and non-decreasing.

Hint: Denote $d_j := X_j - \mathbb{E}[X_j \mid \mathcal{F}_{j-1}], d_0 := X_0$, then

- $M_n := \sum_{j=0}^n d_j$
- $A_n := X_n - M_n$

Stopping Times

Wald's Equation

- $X_1, X_2, X_3, \cdots$ are $i.i.d.$ +$\mathbb{E}|X_i| < \infty$ + T is a stopping time + $\mathbb{E}(T) < \infty$. Let then$S_n = \sum_{i=1}^n X_i$,
$$\mathbb{E}S_T = \mathbb{E}X\mathbb{E}T$$
- $X_1, X_2, X_3, \cdots$ are $i.i.d.$ +$\mathcal{E}X_n = 0$+$\mathbb{E}|X_i|^2 = \sigma^2 < \infty$ + T is a stopping time + $\mathbb{E}(T) < \infty$. Let $S_n = \sum_{i=1}^n X_i$, then
$$\mathbb{E}S_T^2 = \sigma^2\mathbb{E}T$$

Martingale Convergence Theorem

If X_n is a non-negative super-martingale, the it converges to a a.s. finite X_∞ and

$$\mathbb{E}(X_\infty) \leq \mathbb{E}(X_0)$$

Levy's 0-1 Law

Suppose X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_n$ is a filtration, then

$$\mathbb{E}(X \mid \mathcal{F}_k) \to \mathbb{E}(X \mid \mathcal{F}_\infty), \quad k \to \infty$$

a.s, and L_1.

Doob's Maximal Inequality

Suppose Y_n is a sub-martingale and $b \in \mathbb{R}$, denote $M_N = \max_{0 \leq n \leq N} Y_n$, then

$$b\mathbb{P}(M_N \geq b) \leq \mathbb{E}\left(Y_N \mathbf{1}_{\{M_N \geq b\}}\right) \leq \mathbb{E}(Y_N)$$

Hint: Denote $T(\omega) = \inf\{n : Y_n(\omega) \geq b\}$ or N if it's empty and decompose $\mathbb{E}Y_T \leq \mathbb{E}Y_N$ w.r.t to the set $[M_n \leq b] \cup [M_n \geq b]$.

Remark: This inequality controls the maximum probability w.r.t the summation.

Reference

- Martingale: Lecture Notes