Discretization

> *Since there are some problems about mathjax, I also upload the pdf version /Large Sample / 5 Discretization.*

Weyl Inequality

The non-increasing ordered singular value of $A, B \in \mathbb{R}^{m\times n}$,
$\max_{k=1,\dots,\min(m,n)} |\sigma_k(A) - \sigma_k(B)| \le \|A - B\|_{\text{op}} = \sigma_{\max}(A - B)$

Hint: $\sigma_k(A) = \min_{\dim(V)=n-k+1} \max_{v\in V; \|v\|=1} v^T Av$, in addition,
$\sigma_k(A) = \max_{\dim(V)=k} \min_{x\in V; \|x\|_2=1} \|Ax\|_2$

Covariance Estimation

If $X_1, \dots, X_n \in \text{SG}_d(\sigma^2)$, then $\forall\delta \in (0,1), \exists C > 0$

$$\mathbb{P}\left(\|\widehat{\Sigma} - \Sigma\|_{\text{op}} \le \sigma^2 C \min\left\{\sqrt{\frac{d + \log(2/\delta)}{n}}, \frac{d + \log(2/\delta)}{n}\right\}\right) \ge 1 - \delta$$

where $\widehat{\Sigma} = \frac{1}{n}\sum_{i=1}^n X_i X_i^{\text{T}}$.

Proof Sketch:

- Discretization (Use y close enough to x^* s.t. $\|A\|_{\text{op}} = |x^{*\text{T}} Ax^*|$):
  $\|A\|_{\text{op}} = \max_{x\in\mathbb{S}^{n-1}} |x^{\text{T}} Ax| \le (1 - 2\epsilon)^{-1} \max_{y\in\mathcal{N}_\epsilon}(y^{\text{T}} Ay)$
- Use summation bound maximum:
  $\max_{y\in\mathcal{N}_\epsilon}(y^{\text{T}} Ay) \le \sum_{y\in\mathcal{N}_\epsilon}(y^{\text{T}} Ay)$
- Bound the cardinality of $\mathcal{N}_\epsilon$ by volume comparison theorem;
  $\sum_{\mathcal{N}_\epsilon}(Y^{\text{T}} AY) \to |\mathcal{N}_\epsilon| Y^{\text{T}} AY$
- Use the concentration inequality to bound $Y^{\text{T}} AY$.

Reference

- Discretization
- Singular Value