High-Dimensional Bayesian

Bayesian Lasso

Lasso minimizes the penalized least squares to achieve sparsity

$$\min_{oldsymbol{eta}} \sum_{i=1}^n rac{1}{2}ig(y_i - \mathbf{x}_i^Toldsymbol{eta}ig)^2 + \lambda \sum_{j=1}^p |eta_j|$$

The objective function is equivalent to maximizing

$$\prod_{i=1}^{n} \underbrace{\exp\Bigl(-ig(y_{i} - \mathbf{x}_{i}^{T}oldsymbol{eta}ig)^{2}/2\Bigr)}_{likelihood} imes \underbrace{\exp\Biggl(-\lambda \sum_{j=1}^{p} |eta_{j}|\Bigr)}_{prior}$$

It's not easy to directly sample from the posterior. But it can be written as a mixture of normals, mixed with exponential distribution,

$$rac{a}{2}e^{-a|z|} = \int_0^\infty rac{1}{\sqrt{2\pi s}} e^{-z^2/(2s)} imes rac{a^2}{2} e^{-a^2s/2} ds$$

Hence, the full model is

$$egin{aligned} \mathbf{y} \mid \mathbf{\mu}, \mathbf{X}, oldsymbol{eta}, \sigma^2 &\sim N\left(\mathbf{\mu}\mathbf{1} + \mathbf{X}oldsymbol{eta}, \sigma^2\mathbf{I}
ight) \ eta_j \mid s_j^2, \sigma^2 &\sim N\left(\mathbf{0}, \sigma^2 s_j^2
ight) \ s_j^2 \mid au \sim rac{ au^2}{2}e^{- au^2 s_j^2/2} \ \sigma^2 &\sim \pi\left(\sigma^2
ight) \end{aligned}$$

The Horseshoe Prior

To be completed.

Bayesian Robust Regression

We want to simulate the uncertainty with t distribution(heavy tail) in this case.

$$f(t) = rac{\Gamma((v+1)/2)}{\sqrt{\pi v}\Gamma(v/2)} ig(1+x^2/vig)^{-rac{v+1}{2}}$$

Analogous to the Bayesian Lasso, it's not a easy task to sample from the posterior directly.

Here, we introduce an integral (mixture of $x\sim\mathcal{N}(0,\sigma^2)$ and $1/\sigma^2\sim\Gamma(v/2,v/2)$):

$$\int_0^\infty rac{\sqrt{ au}}{\sqrt{2\pi}} \mathrm{exp}ig(- au x^2/2ig) imes rac{(v/2)^{v/2}}{\Gamma(v/2)} au^{v/2-1} e^{-v au/2} d au$$

Thus, for robust regression

$$y_i = \mathbf{x}_i^T oldsymbol{eta} + \epsilon_i, v \sim t_v(0)$$

The full model is:

$$egin{aligned} y_i &\sim N\left(\mathbf{x}_i^Toldsymbol{eta}, \sigma_i^2
ight) \ 1/\sigma_i^2 &\sim \operatorname{Gamma}(v/2, v/2) \ oldsymbol{eta} &\sim N\left(\mathbf{0}, \delta^{-2}\mathbf{I}
ight) \end{aligned}$$

Then, the full conditional density for each variables is:

$$egin{aligned} au_i \mid oldsymbol{eta}, \mathbf{y}, \mathbf{X} \sim \operatorname{Gamma}\!\left(rac{v+1}{2}, rac{v+ig(y_i - \mathbf{x}_i^Toldsymbol{eta}ig)^2}{2}
ight) \ oldsymbol{eta} \mid au_1, \cdots, au_n, \mathbf{y}, \mathbf{X} \sim N\left(oldsymbol{\mu}_n, \Sigma_n^2
ight) \end{aligned}$$

where

$$oldsymbol{\mu}_n = \mathrm{A}^{-1}\mathbf{b}, \; \Sigma_n = \mathrm{A}^{-1}, \; \mathrm{A} = \sum_{i=1}^n au_i \mathbf{x}_i \mathbf{x}_i^T + \delta^2 \mathbf{I}, \; \mathbf{b} = \sum_{i=1}^n au_i y_i \mathbf{x}_i$$

Note: MH algorithm is also realizable in this case.