Convergence

> *Since there are some problems about mathjax, I also upload the pdf version /Large Sample/ 8 U Statistics.*

Definition

Assume a known function h in r arguments is

- permutation symmetric;
- A unbiased estimator of parameter θ: $\theta = \mathbb{E}h(X_1, X_2, \cdots, X_r)$.

A **U Statistics** with **kernel** h is defined as:

$$U = \frac{1}{\binom{n}{r}} \sum_{\beta} h\left(X_{\beta_1}, \ldots, X_{\beta_r}\right)$$

Property

U statistics

- is permutation symmetric in $X_1, X_2, \cdots, X_n$;
- has smaller variance than $h(X_1, X_2, \cdots, X_r)$;
- $U = \mathrm{E}\left(h\left(X_1, \ldots, X_r\right) \mid X_{(1)}, \ldots, X_{(n)}\right)$
- **Asymptotic Normality**: If $Eh^2\left(X_1, \ldots, X_r\right) < \infty$, then $\sqrt{n}(U - \theta - \hat{U}) \xrightarrow{\mathbf{P}} 0$. Consequently,
  the sequence $\sqrt{n}(U - \theta)$ is asymptotically normal with mean 0 and variance $r^2\zeta_1$, where, with $X_1 \ldots\ldots X_r, X_1' \ldots . X_r'$ denoting i.i.d. variables,

$$\zeta_1 = \operatorname{cov}(h\left(X_1, X_2 \ldots\ldots X_r\right), h\left(X_1, X_2', \ldots, X_r'\right))$$

Reference

- Asymptotic Statistics