ECE410

Linearization of Nonlinear Systems

Objective

This handout explains the procedure to linearize a nonlinear system around an equilibrium point. An example illustrates the technique.

1 State-Variable Form and Equilibrium Points

A system is said to be in *state-variable form* if its mathematical model is described by a system of n first-order differential equations and an algebraic output equation:

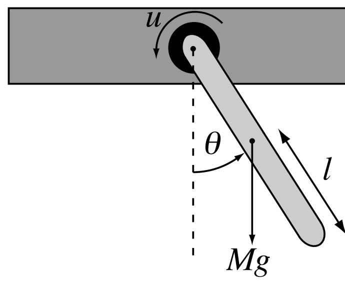
$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u), \end{aligned} \tag{1}$$

where $x = [x_1 \ \cdots x_n]^\top$, $u = [u_1 \ \cdots u_m]^\top$, and $y = [y_1 \ \cdots y_p]^\top$. When f and h are nonlinear functions of x and u, then we say that the system is nonlinear. In this course we will work exclusively with linear systems, i.e., systems for which (1) becomes

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{2}$$

where A is an $n \times n$ real matrix, B is $n \times m$, C is $p \times n$, and D is $p \times m$. Sometimes, physical systems are described by nonlinear models such as (1), and the tools we will learn in this course can not be employed to design controllers. However, if a nonlinear system operates around an *equilibrium point*, i.e., around a configuration where the system is at rest, then it is possible to study the behavior of the system in a neighborhood of such point.

Example 1 (A simple pendulum). Consider the dynamics of the pendulum depicted below, where u denotes an input torque provided by a DC motor.



The equation of motion for this system is

$$\begin{aligned} I\frac{d^2\theta}{dt^2} + Mgl\sin\theta &= u \\ y &= \theta, \end{aligned} \tag{3}$$

where I is the moment of inertia of the pendulum around the pivot point, and y is the output of the system, i.e., the variable one wants to control. Consider now the equivalent state-variable representation of (3), obtained by choosing $x_1 = \theta$ and $x_2 = \dot{\theta}$,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{Mgl}{I}\sin x_1 + \frac{u}{I} \\ y &= x_1 \end{aligned} \tag{4}$$

The model (4) has precisely the form (1), where in this case $x = [x_1, x_2]^\top$ and

$$f(x,u) = \begin{bmatrix} x_2 \\ -\frac{Mgl}{I}\sin x_1 + \frac{u}{I} \end{bmatrix}, \qquad h(x,u) = x_1.$$

Since f contains the term $\sin x_1$, the system (4) is nonlinear. Observe that when $(x,u) = (0,0)$, $\dot{x} = f(x,u) = 0$ which implies that $x(t) = [x_1(t)\ x_2(t)]^\top$ is constant for all t. In other words, if the pendulum is in the vertical downward position with no angular velocity (i.e., $x_1 = x_2 = 0$), and with no input torque (i.e., $u = 0$), then the pendulum stays in the vertical downward position for all time (i.e., $x(t)$ is constant for all t). For this reason, the configuration $x = [0\ 0]^\top$ is referred to as an *equilibrium point.*

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We now generalize the intuition developed in the previous example by defining the notion of an equilibrium point.

Definition 1 *(Equilibrium Point)* Consider a system in state-variable form (1). Suppose that u is set to be a constant value $\bar{u}$. Then, $\bar{x}$ is said to be an **equilibrium point of (1) with** $u = \bar{u}$ if $f(\bar{x}, \bar{u}) = [0\ 0\ \ldots 0]^\top$.

Example 2 (Pendulum - continued) Back to the pendulum example, suppose we turn off the DC motor, that is, we set $u = \bar{u} = 0$. Let's use the definition above to find all corresponding equilibria. We set $f(x,u)\Big|_{u=\bar{u}=0} = [0\ 0]^\top$, in other words,

$$\begin{aligned} x_2 &= 0 \\ -\frac{Mgl}{I}\sin x_1 &= 0 \end{aligned}$$

The solutions are

$$\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \quad k \text{ integer.}$$

Physically, this means that the pendulum is at equilibrium whenever the angle θ is either 0 (pendulum pointing downward) or π (pendulum pointing upward), and the angular velocity $\dot{\theta}$ is zero. Qualitatively, the equilibrium $\bar{x} = [0\ 0]^\top$ is stable, while the equilibrium $\bar{x} = [\pi\ 0]^\top$ is unstable.

Now suppose we turn on the DC motor in such a way that it produces a desired constant torque $u = \bar{u} \neq 0$. The corresponding equilibria must satisfy the equation $f(x,\bar{u}) = [0\ 0]^\top$, i.e.,

$$\begin{aligned} x_2 &= 0 \\ -\frac{Mgl}{I}\sin x_1 + \frac{\bar{u}}{I} &= 0. \end{aligned}$$

Note that, setting $u = \bar{u} = Mgl\sin\bar{x}_1$, the state

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ 0 \end{bmatrix}$$

is an equilibrium point of the pendulum. Physically, that means that by imparting a suitable constant torque to the pendulum one can make the pendulum be at rest at any desired angle $\bar{x}_1$. For instance, by imparting a torque $u = \bar{u} = Mgl$, the configuration $x_1 = \pi/2$, $x_2 = 0$ is an equilibrium of the pendulum. Is such configuration stable or unstable? We will see the answer to this question later on in the course.

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2 Linearization

Although almost every physical system contains nonlinearities, oftentimes its behavior *within a certain operating range of an equilibrium point* can be reasonably approximated by that of a linear model. One reason for approximating the nonlinear system (1) by a linear model of the form (2) is that, by so doing, one can apply the systematic linear control design techniques introduced in this course. Keep in mind, however, that a linearized model is valid only when the system operates in a sufficiently small range around an equilibrium point.

Given the nonlinear system (1) and an equilibrium point $\bar{x} = [\bar{x}_1 \ \cdots \ \bar{x}_n]^\top$ obtained when $u = \bar{u}$, we define a coordinate transformation as follows. Denote

$$\begin{aligned}\tilde{x} &:= x - \bar{x},\\ \tilde{u} &:= u - \bar{u},\\ \tilde{y} &:= y - h(\bar{x}, \bar{u}).\end{aligned}$$

The new coordinates $\tilde{x}$, $\tilde{u}$, and $\tilde{y}$ represent the variations of x, u, and y from their equilibrium values. You have to think of these as a new state, new control input, and new output respectively.

The linearization of (1) at $\bar{x}$ is given by

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u}\\ \tilde{y} &= C\tilde{x} + D\tilde{u},\end{aligned} \tag{5}$$

where

$$A = \left[\frac{\partial f}{\partial x}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{x},\bar{u}) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial f_n}{\partial x_n}(\bar{x},\bar{u}) \end{bmatrix},$$

$$B = \left[\frac{\partial f}{\partial u}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial f_1}{\partial u_m}(\bar{x},\bar{u}) \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial f_n}{\partial u_m}(\bar{x},\bar{u}) \end{bmatrix},$$

$$C = \left[\frac{\partial h}{\partial x}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial h_1}{\partial x_n}(\bar{x},\bar{u}) \\ & \vdots & \vdots \\ \frac{\partial h_p}{\partial x_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial h_p}{\partial x_n}(\bar{x},\bar{u}) \end{bmatrix},$$

$$D = \left[\frac{\partial h}{\partial u}\right]_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial h_1}{\partial u_m}(\bar{x},\bar{u}) \\ & \vdots & \vdots \\ \frac{\partial h_p}{\partial u_1}(\bar{x},\bar{u}) & \cdots & \frac{\partial h_p}{\partial u_m}(\bar{x},\bar{u}) \end{bmatrix}.$$

Remark 1: The linearization (5), also referred to as a *small-signal model*, is valid only in a sufficiently small neighborhood of the equilibrium point $\bar{x}$. Notice that, as expected, (5) has the linear structure (2).

Remark 2: Note that the matrices A, B, C, D have *constant* coefficients in that all partial derivatives are evaluated at the numerical values $(\bar{x}, \bar{u})$.

Example 3 (Linearization of the pendulum system). We return to the pendulum example. Recall that the state-variable model is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{Mgl}{I}\sin x_1 + \frac{u}{I} \\ y &= x_1 \end{aligned}$$

Consider the equilibrium point, obtained by setting $u = \bar{u} = 0$, corresponding to the vertical upward position and no control input, i.e., $\bar{x} = [\pi \ 0]^\top$. Following the procedure outlined above, we define

$$\tilde{x} = x - \bar{x} = \begin{bmatrix} x_1 - \pi \\ x_2 \end{bmatrix}, \qquad \tilde{u} = u - 0 = u, \qquad \tilde{y} = y - \pi,$$

and we form the matrices containing partial derivatives

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{Mgl}{I}\cos x_1 & 0 \end{bmatrix}, \ \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix}, \ \frac{\partial h}{\partial x} = [1, 0], \ \frac{\partial h}{\partial u} = 0.$$

Next, we evaluate the matrices above at $(\bar{x}_1, \bar{x}_2, \bar{u}) = (\pi, 0, 0)$ and we write the linearized model

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{I} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \tilde{u} \\ \tilde{y} &= [1 \ 0] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}. \end{aligned}$$

Notice how the linearized model is expressed in terms of a new state $\tilde{x}$, new control input $\tilde{u}$, and new output $\tilde{y}$. These represent variations of x, u, and y from their equilibrium values. The linearized model above is only valid in a small neighborhood of the equilibrium $\bar{x} = [\pi \ 0]^\top$, that is, it is only valid when the components $\tilde{x}_1$ and $\tilde{x}_2$ of the vector $\tilde{x}$, are small. Physically this can be rephrased as follows: the linearized model of the pendulum at the vertical upward position is only valid when the angle θ is in a small neighborhood of π and the angular speed $\dot{\theta}$ is small.

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