Question 1

Part A

RHS: There are 5^n ways to divide n balls to 5 buckets

LHS: For every i, choose i out of n balls. Then, divide i balls into 4 buckets , divide $n-1$ balls into 1 bucket. Multiply them, and you get the formula for dividing n balls into 5 buckets.

$$\sum_{i=0}^{n} \binom{n}{i} = 5^n$$

$$\sum_{i=k}^{n} 1^{n-i} 4^i = 5^n$$

Q. E. D

Part B

We choose $k+1$ numbers from $1, 2, ..., n+1$ numbers, the number of ways we can do this is:

$$\binom{n+1}{k+1}$$

We will assume the largest possible number chosen is $i+1$, where $i = n$.

Now, the number of ways to choose the other k numbers is:

$$\binom{n}{k}$$

Instead, assume $i = n-1$. This would mean the ways to choose the remaining k is:

$$\binom{n-1}{k}$$

Using this logic, we can now decrease i until $i = k$. The largest number we can have is $k+1$. Now, the ways to select the rest is

$$\binom{n}{k}$$

Finally, we sum these probabilities to retrieve:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + ... + \binom{k}{k} = \sum_{i=k}^{n} \binom{j}{k}$$

Q. E. D

Question 2

Part A

We want to solve the following recurrence:

$$T(n) = 3T(\frac{n}{4}) + n!$$

We know that $n! = O(n^n)$ and case 3 of the master theorem applies, therefore:

$$f(n) = \Omega(n^{\log_4 3 + c})\ \forall\ c$$

We want to show that:

$$3f(\frac{n}{4}) \leq c * f(\frac{n}{4}),\ 0 < c < 1$$

Let: $n = 4k$, then:

$$3(\frac{4k}{4})! \leq c * (4k)!$$

$$3k! \leq \frac{3}{4}(4k)!$$

Since the condition holds:

$$\boxed{T(n) = \Theta(n!)}$$

Part B

We want to solve the following recurrence:

$$T(n) = 6T(\frac{n}{3}) + n^2 \log n$$

We know that case 3 of the master theorem applies, therefore: $f(n) = \Omega(n^{log_6 3+\epsilon}), \epsilon > 0$
Choosing $\epsilon = 0.387$, $f(n) = \Omega(n^{log 0.613+0.387})$

We can show that:

$$6(\frac{n}{3})^2 log(\frac{n}{3}) \leq c * n^2 \log n$$
$$\frac{2n^2}{3} log(\frac{n}{3}) \leq \frac{2}{3} n^2 logn$$

Since the condition holds:

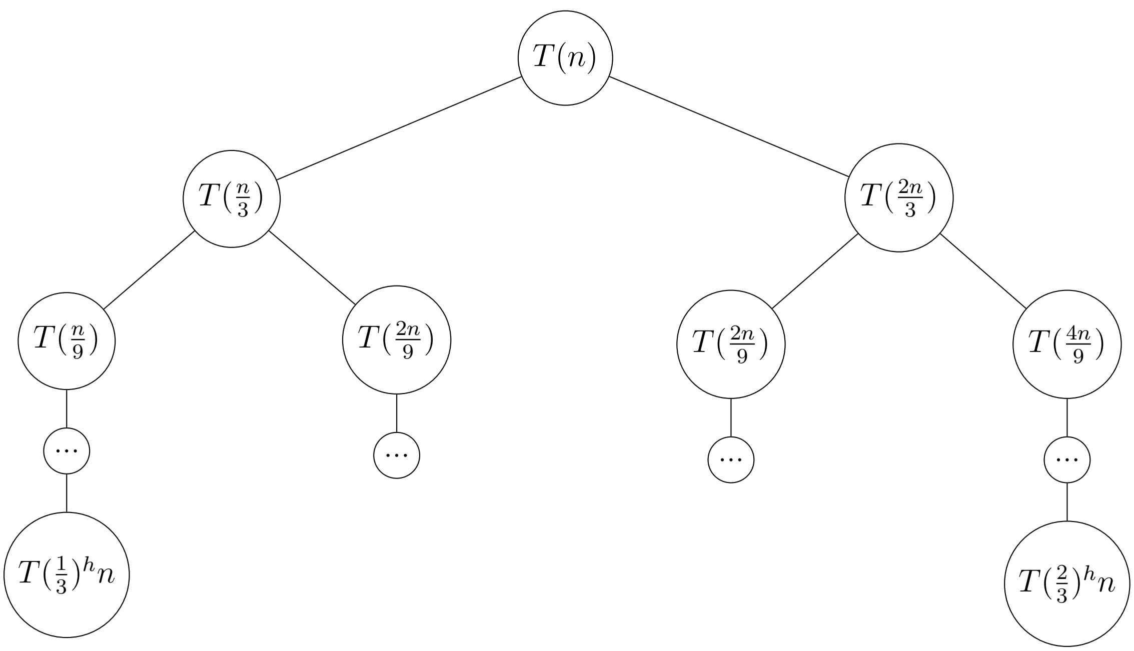
$$\boxed{T(n) = \Theta(n^2 logn)}$$

Part C

We want to solve the following recurrence:

$$T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + \Theta(n)$$

We will draw the following tree:



Let h be the height of the tree.

$$(\frac{2}{3})^h * n = 1$$

$$n = (\frac{3}{2})^h$$

$$h = log_{\frac{3}{2}} n = O(\log n)$$

Each level of the tree has complexity $\Theta(n)$.

$$T(n) = h * \Theta(n)$$

$$T(n) = O(\log n) * \Theta(n)$$

$$\boxed{T(n) = O(n \log n)}$$

Question 3

Our ordering is shown below:

$$\sum_{i=2}^{n}(\frac{1}{i-1}-\frac{1}{i+1})+2=(1+\frac{1}{200n})^{200n}=(1-\log\frac{1}{1-\frac{1}{n}})^{n}$$

$$<<\log^{*}(\log^{*}n)<<\log^{*}\frac{n}{2}<<\log^{(200)}n<<\log^{*}2^{n}<<\lfloor\log\log n!\rfloor<<\log^{2}n$$

$$<<n^{\frac{1}{\log\log n}}<<n^{\frac{\log\log n}{\log n}}<<n=n^{1+\frac{1}{\log n}}<<\sum_{i=1}^{n}\log i<<n(\log n)^{2}<<n^{4.5}-(n-2)^{4.5}$$

$$<<n^{5}<<2^{\log^{*}n}<<2^{n}<<e^{n}<<n^{\log\log n}\equiv(\log n)^{\log n}<<(\log n)^{\frac{n}{2}}$$

Question 4

Part A

We will prove the following with induction:

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

where F_n is the n^{th} Fibonacci number, which are defined as:

$$F_n = \begin{cases} 1 & n = 1 \\ 1 & n = 2 \\ F_{n-2} + F_{n-1} & \text{otherwise} \end{cases}$$

and

$$\phi = \frac{1+\sqrt{5}}{2},\ \psi = \frac{1-\sqrt{5}}{2}$$

Proof

Basis. For $n = 1$ and $n = 2$:

$$F_1 = \frac{\phi - \psi}{\phi - \psi} = 1,\ F_2 = \frac{\phi^2 - \psi^2}{\phi - \psi} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

Hypothesis. Assume

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi},\ F_{n-1} = \frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi}$$

Inductive step. Prove $F_{n+1} = \frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi}$

$$\begin{aligned} F_{n+1} &= F_{n-1} + F_n \\ &= \frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi} + \frac{\phi^n - \psi^n}{\phi - \psi} \\ &= \frac{\phi^{n-1} - \psi^{n-1} + \phi^n - \psi^n}{\phi - \psi} \\ &= \frac{1}{\phi - \psi}(\phi^n(\frac{1}{\phi} + 1) - \psi^n(\frac{1}{\psi} + 1)) \end{aligned}$$

Using the fact that:

$$\frac{1}{\phi}+1=(\frac{2}{1+\sqrt{5}}+1)\cdot\frac{1+\sqrt{5}}{1+\sqrt{5}}=\frac{1+\sqrt{5}}{2}=\phi$$
$$\frac{1}{\psi}+1=(\frac{2}{1-\sqrt{5}}+1)\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{1-\sqrt{5}}{2}=\psi$$

We get that

$$F_{n+1}=\frac{\phi^{n+1}-\psi^{n+1}}{\phi-\psi}$$

Q. E. D

Part B

We will prove the following with induction: If $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_i \subseteq B_i$ for $i = 1, 2, \ldots, n$, then

$$A_1 \cup A_2 \cup \ldots \cup A_n \subseteq B_1 \cup B_2 \cup \ldots \cup B_n$$

Proof

Basis. For $n = 1$, $A_1 \subseteq B_1$

Hypothesis. Assume $A_1 \cup A_2 \cup \ldots \cup A_n \subseteq B_1 \cup B_2 \cup \ldots \cup B_n$.

Inductive step. Prove $A_1, A_2, \ldots, A_n, A_{n+1} \subseteq B_1, B_2, \ldots, B_n, B_{n+1}$

From the condition, $A_i \subseteq B_i$, we know that

$$A_{n+1} \subseteq B_{n+1}$$

and we have assumed that

$$A_1 \cup A_2 \cup \ldots \cup A_n \subseteq B_1 \cup B_2 \cup \ldots \cup B_n$$

therefore

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1} \subseteq B_1 \cup B_2 \cup \ldots \cup B_n \cup B_{n+1}$$

Q. E. D

Question 5

The number of ways that 20 candidates can be organized into a committee of 5 members is:

$$\binom{20}{5} = \frac{20!}{5!(20-5)!}$$

The number of ways a committee can be formed with n of the three candidates elected by the person on it is:

$$\binom{3}{n} \cdot \binom{17}{5-n}$$

The number of ways a committee can be formed with at least 1 of the three candidates elected by the person on it is:

$$\sum_{n=1}^{3} \binom{3}{n} \cdot \binom{17}{5-n}$$

This means that the probability of a committee being formed with at least 1 of the three candidates elected by the person on it is:

$$= \frac{\text{\# of committees of interest}}{\text{\# of possible committees}}$$

$$= \frac{\sum_{n=1}^{3} \binom{3}{n} \cdot \binom{17}{5-n}}{\binom{20}{5}}$$

$$\boxed{= \frac{137}{228}}$$

Question 6

Assume we want to obtain the correct answer. We can achieve this by doing the following:

Ask either person what the other person would say instead of what they would say. This would mean that the person A, who always tells the truth would say what the liar, person B says and the person B who says the lies would continue to lie about what person A says. This would result in obtaining the wrong answer both times. Therefore, by contradiction we can conclude that the other answer would be the truth.

Proof.

Person A: Always tells the truth

Person B: Always Lies

Hypothesis: Assume the person will say the correct answer. Then by contradiction, we can see that this is incorrect, so the other person will give the incorrect answer.

Solution: Ask either person what the other person would say. E.g. Person A says what person B would say; Person B says what person A would say

Contradiction: Person B, the liar, will lie about what person A would say as they always lie. Person A will lie because Person B lies. Hence we will obtain the incorrect answer either way.

We can see that if we choose the opposite option to the answer provided, we will obtain the correct solution.

Q. E. D

Question 7

The tree shown displays the following preorder and postorder traversals:

preorder: o v e f g t l s d n x a y u m k p j i c w b z h q r

postorder: t s l g f e n u m y a k x d v j z b h w c r q i p o