Assignment One

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Question 1

a. $f(4n) - f(n) = \log_2 4n - \log_2 n = \log_2 4 = 2$

b. $f(4n)/f(n) = \sqrt{4n}/\sqrt{n} = 2$

c. $f(4n)/f(n) = 4n/n = 4$

d. $f(4n)/f(n) = (4n)^2/n^2 = 4^2$

e. $f(4n)/f(n) = (4n)^3/n^3 = 4^3$

f. $f(4n)/f(n) = 2^{4n}/2^n = 2^{3n} = (2^n)^3$

Question 2

a. $t(n) \in O(g(n)) \Leftrightarrow t(n) \le cg(n) \quad for\ all\ n \ge n_0,\ where\ c > 0.$

$$\Rightarrow g(n) \ge \left(\frac{1}{c}\right) t(n) \quad for\ all\ n \ge n_0,\ where\ c > 0.$$

$$\Rightarrow g(n) \ge c' t(n) \quad for\ all\ n \ge n_0, where\ c' = \frac{1}{c} > 0.$$

$$\Rightarrow g(n) \in \Omega(t(n))$$

b. We should show $\Theta(\alpha g(n)) \subseteq \Theta(g(n))$ and $\Theta(g(n)) \subseteq \Theta(\alpha g(n))$ to prove $\Theta(\alpha g(n)) = \Theta(g(n))$.

i. Let $f(n) \in \Theta(\alpha g(n))$, then show that $f(n) \in \Theta(g(n))$.

$f(n) \in \Theta(\alpha g(n))$

$\Rightarrow c_2 \alpha g(n) \le f(n) \le c_1 \alpha g(n) \quad for\ all\ n \ge n_0,\ where\ c_1 > 0, c_2 > 0.$

$\Rightarrow \quad c_2' g(n) \le f(n) \le c_1' g(n) \quad for\ all\ n \ge n_0,\ where\ c_1' = c_1 \alpha > 0, c_2' = c_2 \alpha > 0.$

$\Rightarrow\ f(n) \in \Theta\big(g(n)\big)$

Therefore $\Theta\big(\alpha g(n)\big) \subseteq \Theta\big(g(n)\big)$.

ii. Let $f(n) \in \Theta\big(g(n)\big)$, then show that $f(n) \in \Theta\big(\alpha g(n)\big)$.

$f(n) \in \Theta\big(g(n)\big)$

$\Rightarrow\ c_2 g(n) \le f(n) \le c_1 g(n) \quad for\ all\ n \ge n_0,\ where\ c_1 > 0, c_2 > 0.$

$\Rightarrow\ \frac{c_2}{\alpha}\alpha g(n) \le f(n) \le \frac{c_1}{\alpha}\alpha g(n) \quad for\ all\ n \ge n_0,\ where\ c_1 > 0, c_2 > 0.$

$\Rightarrow\quad c_2{}'\alpha g(n) \le f(n) \le c_1{}'\alpha g(n) \quad for\ all\ n \ge n_0,\ where\ c_1{}' = \frac{c_1}{\alpha} > 0, c_2{}' = \frac{c_2}{\alpha} > 0.$

$\Rightarrow\ f(n) \in \Theta\big(g(n)\big)$

Therefore $\Theta\big(g(n)\big) \subseteq \Theta\big(\alpha g(n)\big)$.

So $\Theta\big(\alpha g(n)\big) = \Theta\big(g(n)\big)$.

c. We should show $\Theta\big(g(n)\big) \subseteq O\big(g(n)\big) \cap \Omega\big(g(n)\big)$ and $O\big(g(n)\big) \cap \Omega\big(g(n)\big) \subseteq \Theta\big(g(n)\big)$ to prove $\Theta\big(g(n)\big) = O\big(g(n)\big) \cap \Omega\big(g(n)\big)$.

i. Let $f(n) \in \Theta\big(g(n)\big)$, then show that $f(n) \in O\big(g(n)\big) \cap \Omega\big(g(n)\big)$.

$f(n) \in \Theta\big(g(n)\big)$

$\Rightarrow\ c_2 g(n) \le f(n) \le c_1 g(n) \quad for\ all\ n \ge n_0,\ where\ c_1 > 0, c_2 > 0.$

$\Rightarrow f(n) \le c_1 g(n) \quad for\ all\ n \ge n_0,\ where\ c_1 > 0$

and

$f(n) \ge c_2 g(n) \quad for\ all\ n \ge n_0,\ where\ c_2 > 0.$

$\Rightarrow\ f(n) \in O\big(g(n)\big)$ and $f(n) \in \Omega\big(g(n)\big)$.

$\Rightarrow\ f(n) \in O\big(g(n)\big) \cap \Omega\big(g(n)\big)$.

Therefore $\Theta\big(g(n)\big) \subseteq O\big(g(n)\big) \cap \Omega\big(g(n)\big)$.

ii. Let $f(n) \in O\big(g(n)\big) \cap \Omega\big(g(n)\big)$, then show that $f(n) \in \Theta\big(g(n)\big)$.

$f(n) \in O(g(n)) \cap \Omega(g(n))$

$\Rightarrow\ f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.

$\Rightarrow f(n) \le c_1 g(n) \quad for\ all\ n \ge n_1,\ where\ c_1 > 0$

and

$f(n) \ge c_2 g(n) \quad for\ all\ n \ge n_2,\ where\ c_2 > 0.$

$\Rightarrow\ c_2 g(n) \le f(n) \le c_1 g(n) \quad for\ all\ n \ge n_0,\ where\ c_1 > 0, c_2 > 0, n_0 = \max(n_1, n_2).$

$\Rightarrow\ f(n) \in \Theta(g(n))$

Therefore $O(g(n)) \cap \Omega(g(n)) \subseteq \Theta(g(n))$.

So $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$.

d. This assertion is false.

There is a counterexample as followed.

$$t(n) = \begin{cases} n & if\ n\ is\ odd \\ n^2 & if\ n\ is\ even \end{cases} \qquad g(n) = \begin{cases} n^3 & if\ n\ is\ odd \\ n & if\ n\ is\ even \end{cases}$$

Question 3

a. $x(n) = 3x(n-1)$ for $n > 1,\ x(1) = 4$

$$\begin{aligned} x(n) &= 3x(n-1) \\ &= 3[3x(n-2)] = 3^2 x(n-2) \\ &= 3^2[3x(n-3)] = 3^3 x(n-3) \\ &= \ldots \\ &= 3^i x(n-i) \\ &= \ldots \\ &= 3^{n-1} x(1) = 4 \cdot 3^{n-1} \end{aligned}$$

b. $x(n) = x(n-1) + n$ for $n > 0,\ x(0) = 0$

$$\begin{aligned} x(n) &= x(n-1) + n \\ &= [x(n-2) + (n-1)] + n = x(n-2) + (n-1) + n \\ &= [x(n-3) + (n-2)] + (n-1) + n = x(n-3) + (n-2) + (n - \end{aligned}$$

$1) + n$

$$= \cdots$$

$$= x(n-i) + (n-i+1) + (n-i+2) + \cdots + n$$

$$= \ \ldots$$

$$= x(0) + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

c. $x(n) = x(n/2) + n$, for $n > 1$, $x(1) = 1$ (solve for $n = 2^k$)

$$x(n) = x(2^k) = x(2^{k-1}) + 2^k$$

$$= x(2^{k-2}) + 2^{k-1} + 2^k$$

$$= \cdots$$

$$= x(2^{k-i}) + 2^{k-i+1} + 2^{k-i+2} + \cdots + 2^k$$

$$= \cdots$$

$$= x(1) + 2^1 + 2^2 + \cdots + 2^k$$

$$= 2^{k+1} - 1 = 2n - 1$$