ELEC-H-415 Finite-Difference Time-Domain

1 Introduction

The Finite-Difference Time-Domain (FDTD) method is widely used to numerically solve Maxwell's equations. In this project, our objective is to develop a 2-dimensional (2D) FDTD code. This code will then be applied to problems theoretically tackled during the lectures. By performing numerical experiments, you are asked to think by yourselves about the concepts of wave propagation.

To help you in doing the project, a reference is provided on the Virtual University : *Understanding the Finite-Difference Time-Domain Method*, John B. Schneider, 2015, that can be also freely downloaded from the web. Chapters 3 and 8 from this reference explain the ingredients required to fulfill the minimal goals of this project (see below). But it also allows you to go further.

2 The 1D FDTD algorithm

Electromagnetics is governed by the well-known Maxwell's equations, including two curl equations which read, in a source-free region :

$$\begin{cases} \dfrac{\partial \vec{E}}{\partial t} = \dfrac{1}{\varepsilon} \vec{\nabla} \times \vec{H} \\ \dfrac{\partial \vec{H}}{\partial t} = -\dfrac{1}{\mu} \vec{\nabla} \times \vec{E} \end{cases} \tag{1}$$

where, conventionally, the $\vec{H}$ field is used in place of the magnetic field $\vec{B} = \mu \vec{H}$

The aim of the Finite-Difference Time-Domain is to numerically solve Maxwell's equations in complex media by approximating the derivatives in (1) with finite differences.

Let us recall the basic formula for the first derivative of a function f at x_0 :

$$f'(x_0) = \frac{f(x_0 + \frac{\delta}{2}) - f(x_0 - \frac{\delta}{2})}{\delta} + \mathcal{O}(\delta^2) \tag{2}$$

so that any derivative can be estimated by a finite difference :

$$f'(x_0) \simeq \frac{f(x_0 + \frac{\delta}{2}) - f(x_0 - \frac{\delta}{2})}{\delta} \tag{3}$$

for δ values tending towards zero.

2.1 The Yee Algorithm

The Yee algorithm is made of 4 steps :

1. Replace the derivatives in (1) with finite differences by discretizing space and time.
2. Evaluate the magnetic field one time-step into the future.
3. Evaluate the electric field one time-step into the future.
4. Repeat the previous two steps.

Let us consider a one-dimensional space (the x coordinate) and a z-polarized electric field $\vec{E}$ $= E_z\vec{1}_z$. Eqs (1) read

$$\begin{cases} \dfrac{\partial E_z}{\partial t} = \dfrac{1}{\varepsilon}\dfrac{\partial H_y}{\partial x} \\ \dfrac{\partial H_y}{\partial t} = \dfrac{1}{\mu}\dfrac{\partial E_z}{\partial x} \end{cases} \tag{4}$$

To apply step 1 in (4), it is required to discretize space and time.

$$\begin{aligned} E_z(x,t) = E_z(a\Delta x, b\Delta t) = E_z^{a,b} \\ H_y(x,t) = H_y(a\Delta x, b\Delta t) = H_y^{a,b} \end{aligned} \tag{5}$$

where Δx is the spatial step and Δt the time step. To discretize (4), the fields are calculated at different space-time locations as described in Fig. 1 and Fig. 2.

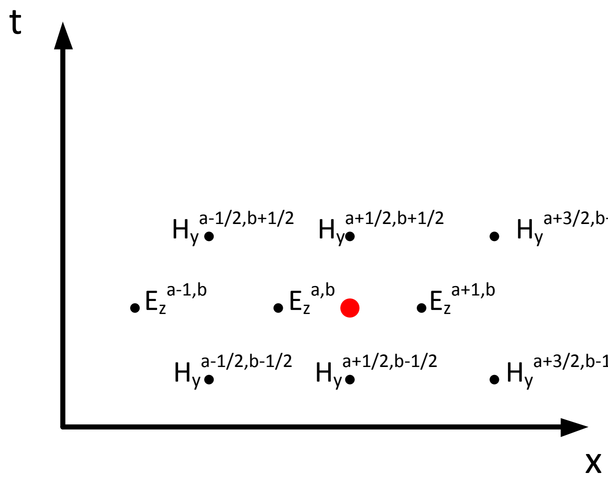

FIGURE 1 – Discretization and space-time field locations. We can see the $\frac{1}{2}$ offsets between E_z and H_y, which are used to discretize at the red dot.

Let us calculate the second of (4) at the $(a + 1/2, b)$ location (the red dot in Fig.1).

$$\mu \left.\frac{\partial H_y}{\partial t}\right|_{(a+1/2)\Delta x, b\Delta t} = \left.\frac{\partial E_z}{\partial x}\right|_{(a+1/2)\Delta x, b\Delta t} \tag{6}$$

This yields

$$\mu \frac{H_y^{a+1/2,b+1/2} - H_y^{a+1/2,b-1/2}}{\Delta t} = \frac{E_z^{a+1,b} - E_z^{a,b}}{\Delta x} \tag{7}$$

This can be rewritten as an update equation for H_y :

$$\boxed{H_y^{a+1/2,b+1/2} = H_y^{a+1/2,b-1/2} + \frac{\Delta t}{\mu \Delta x}(E_z^{a+1,b} - E_z^{a,b}) \tag{8}}$$

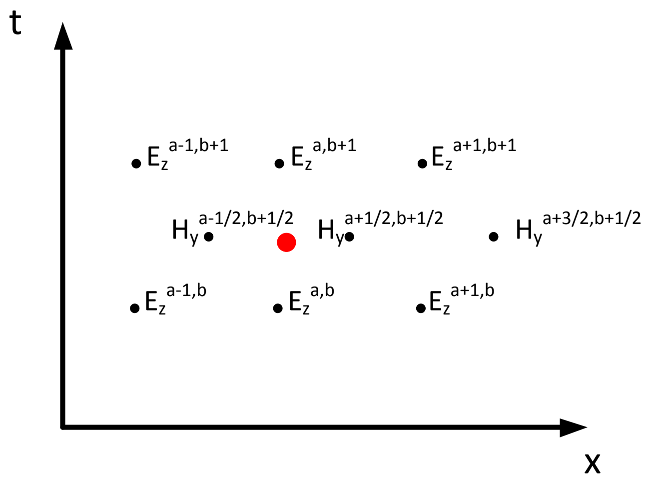

FIGURE 2 – Discretization and space-time field locations. We can see the $\frac{1}{2}$ offsets between E_z and H_y, which are used to discretize at the red dot.

Let us apply the same method for the first of (4) at the $(a, b + 1/2)$ location (the red dot in Fig.2). :

$$\varepsilon \left.\frac{\partial E_z}{\partial t}\right|_{a\Delta x, (b+1/2)\Delta t} = \left.\frac{\partial H_y}{\partial x}\right|_{a\Delta x, (b+1/2)\Delta t} \tag{9}$$

This yields

$$\varepsilon \frac{E_z^{a,b+1} - E_z^{a,b}}{\Delta t} = \frac{H_y^{a+1/2,b+1/2} - H_y^{a-1/2,b+1/2}}{\Delta x} \tag{10}$$

that can be rewritten as an update equation for E_z :

$$\boxed{E_z^{a,b+1} = E_z^{a,b} + \frac{\Delta t}{\varepsilon \Delta x}(H_y^{a+1/2,b+1/2} - H_y^{a-1/2,b+1/2}) \tag{11}}$$

Equations (8) and (11) only depend on the past and can be used to update E_z and H_y over time. Classically, the coefficients in these equations are rewritten in terms of "how far the energy can propagate in free-space in one time step" : $c\Delta t$. By introducing $\varepsilon = \varepsilon_r \varepsilon_0$, $\mu = \mu_r \mu_0$ and the Courant number $S_c = c\Delta t / \Delta x$, we have :

$$\begin{aligned} \frac{1}{\varepsilon}\frac{\Delta t}{\Delta x} &= \frac{1}{\varepsilon}\frac{\sqrt{\varepsilon_0 \mu_0}\ c\Delta t}{\Delta x} = \frac{1}{\varepsilon_r}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{c\Delta t}{\Delta x} &&= \frac{Z_0}{\varepsilon_r} S_c \\ \frac{1}{\mu}\frac{\Delta t}{\Delta x} &= \frac{1}{\mu}\frac{\sqrt{\varepsilon_0 \mu_0}\ c\Delta t}{\Delta x} = \frac{1}{\mu_r}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{c\Delta t}{\Delta x} &&= \frac{1}{Z_0 \mu_r} S_c \end{aligned} \tag{12}$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the free-space impedance.

2.2 Stability

To obtain an efficient spatial discretization, a rule-of-thumb is to choose

$$\Delta x \sim \frac{\lambda}{10} \tag{13}$$

where λ is the smallest wavelength according to the simulated geometry and maximal frequency.

To ensure stable results in FDTD simulations, the time step size Δt has to be restricted. Physically, it is understandable that the energy cannot propagate more than a spatial step during a time step, and in 1D :

$$\Delta t \leq \frac{\Delta x}{c} \tag{14}$$

where c is the speed of light in free-space. In 2D, this condition is more severe :

$$\Delta t \leq \frac{\Delta x}{c\sqrt{2}} \tag{15}$$

3 Project and simulations

The goals of this project are (i) to develop your own code to numerically solve Maxwell's equations and (ii) to visualize propagation to help you in the course understanding.

The project general philosophy is : define and analyze your own set of simulations that illustrate the theoretical lectures, or other phenomena of electromagnetics. A basic set of simulations is here proposed. They are not mandatory. Do not limit yourself to these simulations.

In two words : be creative !

3.1 Minimal requirements

The minimal requirements you have to fulfill to succeed in this project are :

— Derive theoretically a 2D FDTD scheme in an inhomogeneous dielectric medium with losses. Assume that the electric field is polarized normal to the propagation plane, i.e. TM_z case (E_z, H_x, H_y).
— Implement this scheme with Matlab, assuming that $E_z = 0$ at the boundaries (what is the consequence of this assumption ?).
— Implement an arbitrary wave source either by hardwiring E_z at a given point of the grid, or by adding an additional source J_z .

3.2 Content of the report

The final report should contain :

— The theoretical derivation of your FDTD model : from Maxwell to Matlab.
— The full commented Matlab code.
— The validation procedure.

The report should not content the simulation results. These will be discussed during the oral presentation of the project.

3.3 Examples of simulations

3.3.1 SAR Study

— Calculate the power absorbed by the human head due a cell phone (model the cell phone by a point source and the head by a cylinder).

— Some manufacturers have proposed to place a metallic plate in the back of phones to reduce absorbed power. Is this technique efficient ?

3.3.2 Knife-edge model

— Analyze the coverage area of a base station located behind a large building by using a knife-edge geometry (replace the building by a knife-edge model).
— Plot the field power in terms of $\nu = h\sqrt{\frac{2}{\lambda}(\frac{1}{d_1} + \frac{1}{d_2})}$.

3.3.3 Body Area Networks

— By modeling the body by a single cylinder (or three cylinders with the arms), compare propagation around the body at different frequencies.
— Compare the received power by assuming that the receive antenna has the same gain whatever the frequency.

3.3.4 Fast Fading

Analyze fast fading in a room (modeled as a closed metallic box, to increase reflections off the walls).

TABLE 1 – Parameters

Parameter	Value
Frequency	$f = 2.45$ GHz
Wavelength	c/f
Spatial Step	$\delta = \lambda/10$
Room size	$500\delta \times 500\delta$
Source location	$(250, 250)\delta$
Time steps	4500

- Run the simulation using the parameters in Table 1.
- Study and discuss the electric field in zone 1 between time steps 150 and 250.
- Study and discuss the electric field in zone 2 between time steps 450 and 550.
- Study and discuss the electric field in zone 3 between time steps 650 and 750.
- Study and discuss the electric field in zone 4 between time steps 3000 and 4500.
- In a second step, place a metallic plate in front of the source and analyse the distribution of the electric field between time steps 3000 and 4500. Analyze and compare the result.

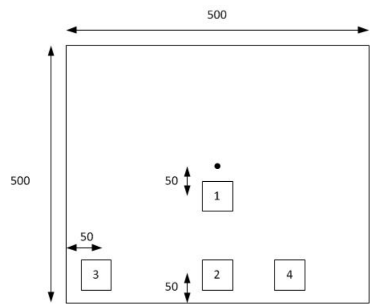

FIGURE 3 – Room scheme, the local areas have sizes of 30x30.

3.4 Dielectric slab

- Simulate propagation inside a dielectric slab (as a first approximation of an optical fibre).
- Compare with the theory of the dielectric slab.