Week 4. By Deng Yufan.

Prob 1. Prove the following are not RL.

a. $A = \{0^n 1^m \mid n \le m\}$.

If $A$ is a RL, then:

Using pumping lemma. there is a $n$ satisfying the condition in lemma.

So $0^n 1^n \in A$. $\exists\, x = 0^p, y = 0^q, z = 0^{n-p-q} 1^n$ $(p+q \le n)$, $xy^* z \in A$.

But $xy^2 z = 0^{n+q} 1^n \notin A$.

b. $B = \{0^n 1 0^n \mid n \ge 1\}$.

If $B$ is a RL. then use the pumping lemma.

$|0^n 1 0^n| \ge n$, $0^n 1 0^n \in B$. then ~~there is~~ $\exists\, x = 0^p, y = 0^q, z = 0^{n-p-q} 1 0^n$. $xy^* z \in B$

But $xz = 0^{n-q} 1 0^n \notin B$.

c. $C = \{w^2 \mid w \in \{0,1\}^*\}$.

If $C$ is a RL. then use the pumping lemma.

$|0^n 1 0^n 1| \ge n$. $0^n 1 0^n 1 \in C$. then $\exists\, x = 0^p, y = 0^q, z = 0^{n-p-q} 1 0^n 1$. $xy^* z \in C$.

But $xz = 0^{n-q} 1 0^n 1 \notin C$. ~~since~~ otherwise the end of $w$ is 1. but $0^{n-q} 1 \ne 0^n 1$.

d. $D = \{w w^R \mid w \in \{0,1\}^*\}$.

If $D$ is a RL. then use the pumping lemma.

$|0^n 1 1 0^n| \ge n$. $0^n 1 1 0^n \in D$. then $\exists\, x = 0^p\ y = 0^q\ z = 0^{n-p-q} 1 1 0^n$. $xy^* z \in D$.

But $xz = 0^{n-q} 1 1 0^n \notin D$, since the only two 1 are not symmetrical.

Prob 2. a. If $L$ is a language, and $a$ is a symbol, let $L\backslash a \stackrel{\cdot}{=} \{w \mid wa \in L\}$.

Prove $L$ is RL $\Rightarrow$ $L\backslash a$ is RL.

We prove $L$ can be presented as RE $\Rightarrow$ $L\backslash a$ can be represented as RE by induction the structure of RE.

As base, we have $\phi\backslash a \stackrel{L}{=} (\phi)$ is RE. $\varepsilon\backslash a \stackrel{L}{=} (\phi)$ is RE. $b\backslash a \stackrel{L}{=} (\phi)$ $(b \neq a)$ is RE.

$a\backslash a \stackrel{L}{=} (\varepsilon)$ is RE.

As induction, if $E, F$ is RE, $L(E)\backslash a$, $L(F)\backslash a$ can be represented as RE $E'$ and $F'$, respectively, then:

$$L(E+F)\backslash a = \{w \mid wa \in L(E) \text{ or } wa \in L(F)\}$$

$$= \{w \mid wa \in L(E)\} \cup \{w \mid wa \in L(F)\}$$

$$= L(E'+F')$$

$$L(EF)\backslash a = \{w \mid wa \in L(EF)\}$$

$$= \{xy \mid x \in L(E),\ ya \in L(F)\} \cup \{w \mid wa \in L(E),\ \varepsilon \in L(F)\}.$$

so if $\varepsilon \in L(F)$, $L(EF)\backslash a = L(EF'+E')$, else $L(EF)\backslash a = L(EF')$.

$$L(E^*)\backslash a = \{w \mid wa \in \bigcup_{i=0}^{+\infty} L(E)^i\}$$

if $\exists w_0$, $w_0 a \in L(E)$, discussing $a$ is at the end of which $L(E)$, we have

$$LHS = \bigcup_{i=0}^{+\infty} \{w \mid w \in L(E)^i [L(E)\backslash a]\}.$$

$$= L(E^*E').$$

else $\forall w$, $wa \notin L(E)$, then $wa \notin L(E)^*$.

$$L(E^*)\backslash a = L(\phi).$$

So $E+F$, $EF$, $E^*$ is closed under $L(E)\backslash a$.

b. Let $a \backslash L = \{ w \mid aw \in L \}$. Prove $L$ is RL $\Rightarrow$ $a \backslash L$ is RL.

We provide a better method than (a).

Let $M = (Q, \Sigma, \delta, q_0, F)$ DFA. $L(M) = A$. We can assume $a \in \Sigma$, otherwise $a \backslash L = \phi$ is RL.

Let $M' = (Q, \Sigma, \delta', q_1, F)$ DFA. which $\delta'(q, a) = \delta(q, a)$.

and $q_1 = \delta(q_0, a)$.

We prove $aw \in L(M) \Leftrightarrow w \in L(M')$.

$\Rightarrow$: $aw \in L(M) \Rightarrow \hat{\delta}(q_0, aw) \in F \Rightarrow \hat{\delta}(\hat{\delta}(q_0, a), w) \in F$

$\Rightarrow \hat{\delta}'(q_1, w) \in F \Rightarrow w \in L(M')$.

$\Leftarrow$: $w \in L(M') \Rightarrow \hat{\delta}'(\delta'(q_0, a), w) \in F \Rightarrow \hat{\delta}(q_0, aw) \in F \Rightarrow aw \in L(M')$.

So $a \backslash L$ is RL.

Prob 3. Prove the RL is closed under following operations.

a. $\min(L) = \{ w \mid w \in L$, but no proper prefix of $w$ is in $L \}$.

Let DFA $M = (Q, \Sigma, \delta, q_0, F)$, $L(M) = L$.

Let NFA $M' = (Q, \Sigma, \delta', q_0, F)$. $\delta'(q, a) = \begin{cases} \{\delta(q, a)\} & (q \notin F) \\ \phi & (q \in F). \end{cases}$

We prove $L(M') = \min(L)$, or say $w \in L$, but no proper prefix of $w$ is in $L \Leftrightarrow w \in L(M')$.

$\Rightarrow$: $w \in L \Rightarrow \hat{\delta}(q_0, w) \in F$.

no proper prefix of $w$ is in $L \Rightarrow \hat{\delta}(q_0, xy) \notin F$, if $xy = w$, $y \neq \phi$.

By induction, we can easily prove $\hat{\delta}'(q_0, w) = \{ \hat{\delta}(q_0, w) \}$.

First $\hat{\delta}'(q_0, \varepsilon) = \{q_0\} = \{\hat{\delta}(q_0, \varepsilon)\}$.

if $\hat{\delta}'(q_0, x) = \{\hat{\delta}(q_0, x)\}$,

$\hat{\delta}'(q_0, xa) = \delta(\hat{\delta}(q_0, x), a)$

Since $x$ is proper prefix of $w$, $\hat{\delta}(q_0, x) \notin F$

So $\hat{\delta}'(q_0, xa) = \{\hat{\delta}(q_0, xa)\}$

so $\hat{\delta}'(q_0, w) \cap F \neq \phi$. $w \in L(M')$.

$\Leftarrow$: If $xy = w$, $y \neq \varepsilon$. $x \in L$. then $\delta(q_0, x) \in F$.

We first prove $\hat{\delta}'(q_0,w) \subseteq \{\hat{\delta}(q_0,w)\}$.

Induction the length of w. we have $\hat{\delta}'(q_0,\varepsilon) = \{q_0\} = \{\hat{\delta}(q_0,\varepsilon)\}$.

if $\hat{\delta}'(q_0,w) \subseteq \{\hat{\delta}(q_0,w)\}$, then $\hat{\delta}'(q_0,wa) = \bigcup_{q \in \hat{\delta}'(q_0,w)} \delta'(q,a)$.

if $\hat{\delta}'(q_0,w) = \phi$. then $\hat{\delta}'(q_0,wa) = \phi$. (1)

if $\hat{\delta}'(q_0,w) = \{\hat{\delta}(q_0,w)\}$, then $\hat{\delta}'(q_0,wa) = \delta'(\hat{\delta}(q_0,w),a)$.

if $\hat{\delta}(q_0,w) \in F$, then $\hat{\delta}'(q_0,wa) = \phi$. (2)

if $\hat{\delta}(q_0,w) \notin F$, then $\hat{\delta}'(q_0,wa) = \{\delta(\hat{\delta}(q_0,w),a)\} = \{\hat{\delta}(q_0,wa)\}$. (3)

So $\hat{\delta}'(q_0,wa) \subseteq \{\hat{\delta}(q_0,wa)\}$.

We prove $w \in L$, but no proper prefix of $w$ is in $L$ $\Leftrightarrow$ $w \in L(M')$. this ends the proof.

$\Rightarrow$: $w \in L \Rightarrow \hat{\delta}(q_0,w) \in F$.

no proper prefix of w is in L $\Rightarrow$ write w as $xay$ $(y \neq \varepsilon)$: $x \notin L$.

$\Rightarrow \hat{\delta}(q_0,x) \notin F$.

when it is only possible to be (3).

According to the proof above. we have $\hat{\delta}'(q_0,w) = \{\hat{\delta}(q_0,w)\}$, then $\hat{\delta}'(q_0,w) \cap F \neq \phi$.

$\Leftarrow$: $w \in L(M') \Rightarrow \hat{\delta}'(q_0,w) \cap F \neq \phi \Rightarrow \hat{\delta}'(q_0,w) = \{\hat{\delta}(q_0,w)\}$.

$\Rightarrow \hat{\delta}(q_0,w) \in F \Rightarrow w \in L$.

if $xy = w$. $y \neq \varepsilon$. $x \in L$. then $\hat{\delta}(q_0,x) \in F$.

if $\hat{\delta}'(q_0,x) = \phi$. $\hat{\delta}'(q_0,xa) = \phi$.

if $\hat{\delta}'(q_0,x) = \{\hat{\delta}(q_0,x)\}$. $\hat{\delta}'(q_0,xa) = \phi$.

So $\hat{\delta}'(q_0,xa) = \phi$. thus $\hat{\delta}'(q_0,xy) = \phi$. So $w \notin L(M')$. contradiction.

So no proper prefix of w is in L.

b. $max(L) = \{w \mid$ no nonempty string $x$ that $wx \in L\}$.

~~We prove $max(L) = C\,min(C_L)$. where $C$ is complement operation.~~

~~$min(C_L) = \{w \mid w \notin L$, but all proper prefix of $w$ is in $L\}$.~~

Consider DFA $M = (Q, \Sigma, \delta, q_0, F)$.

~~If for all $w \neq \varepsilon$~~ Let DFA $M' = (Q, \Sigma, \delta, q_0, F')$.

$F' = \{q \in F \mid \forall q' \in F, w \neq \varepsilon, \hat{\delta}(q, w) \neq q'\}$.

We proof $w \in L$, no nonempty string $x$ that $wx \in L \Leftrightarrow w \in L(M')$.

$\Rightarrow$: $w \in L \Rightarrow \hat{\delta}(q_0, w) \in F$. if $\hat{\delta}(q_0, w) \notin F'$. then $\exists q' \in F$, $w' \neq \varepsilon$, $\hat{\delta}(\hat{\delta}(q_0, w), w') = q'$.

then $\hat{\delta}(q_0, ww') = q' \in F$. so $ww' \in L$. contradiction. So $\hat{\delta}(q_0, w) \in F'$. $w \in L(M')$.

$\Leftarrow$: $w \in L(M') \Rightarrow \hat{\delta}(q_0, w) \in F' \subseteq F \Rightarrow w \in L$.

For any $x \neq \varepsilon$, $q' \in F$, $\hat{\delta}(q_0, wx) = \hat{\delta}(\hat{\delta}(q_0, w), x) \neq q'$.

So no nonempty string $x$ that $wx \in L$.

C. $init(L) = \{w \mid \exists x, wx \in L\}$.

Induction the structure of RE.

As basis. $init(L(\phi)) = L(\phi)$, $init(L(\varepsilon)) = L(\varepsilon)$. $init(L(a)) = L(\varepsilon + a)$.

If $E, F$ is RE. $init(L(E)) = L(E')$, $init(L(F)) = L(F')$.

~~$init(E+F) = E'$~~ $init(L(E+F)) = L(E' + F')$.

$init(L(EF)) = L(EF' + E')$. since we can discuss if $x$ takes up all $L(F)$.

$init(L(E^*)) = L(E^*E' + \varepsilon)$.

$$\text{since } init(L(E^*)) = \bigcup_{n=0}^{+\infty} init(L(E^n)) = \bigcup_{n=1}^{+\infty} init(L(E^{n-1}E)) \cup L(\varepsilon)$$

$$= \bigcup_{n=1}^{+\infty} L(E^{n-1}E' + (E^{n-1})') \cup L(\varepsilon).$$

$$= \bigcup_{n=1}^{+\infty} L(E^{n-1}E') \cup L(\varepsilon).$$

$$= L(E^*E' + \varepsilon).$$

Prob 4. Let $half(L)=\{w \mid \exists x, |x|=|w|, wx\in L\}$. Prove $L$ is RL $\Rightarrow half(L)$ is RL.

Let DFA $M=(Q,\Sigma,\delta,q_0,F)$. $L(M)=L$.

Let DFA $M'=(Q\times 2^Q, \Sigma, \delta', (q_0,F), F'=\{(q,s)\mid s\subseteq Q, q\in s\})$.

where $\delta'(q,s,a)=(\delta(q,a), \bigcup_{|c|=1}\bigcup_{\delta(q',c)\in s}\{q'\})$.

We first prove $\hat\delta'(q,s,w)=(\hat\delta(q,w),\bigcup_{|c|=|w|}\bigcup_{\hat\delta(q',c)\in s}\{q'\})$.

Induction the length of $w$. we have $\hat\delta'(q,s,\epsilon)=(q,s)=(\hat\delta(q,\epsilon),\bigcup_{|c|=0}\bigcup_{\hat\delta(q',c)\in s}\{q'\})$.

If $\hat\delta'(q,s,w)=(\hat\delta(q,w),\bigcup_{|c|=|w|}\bigcup_{\hat\delta(q',c)\in s}\{q'\})$. we prove $\hat\delta'(q,s,wa)=(\hat\delta(q,wa),\bigcup_{|c|=|w|+1}\bigcup_{\hat\delta(q',c)\in s}\{q'\})$.

we have $\hat\delta'(q,s,wa)=\delta'(\hat\delta'(q,s,w),a)$

$$=\delta'(\hat\delta(q,w),\bigcup_{|c|=|w|}\bigcup_{\hat\delta(q',c)\in s}\{q'\}, a)$$

$$=(\delta(\hat\delta(q,w),a),\bigcup_{|c|=1}\bigcup_{\delta(q',c)\in \bigcup_{|c'|=|w|}\bigcup_{\hat\delta(q'',c')\in s}\{q''\}}\{q'\})$$

$$=(\hat\delta(q,wa),\bigcup_{|c|=1}\bigcup_{|c'|=|w|}\bigcup_{\hat\delta(\delta(q',c),c')\in s}\{q'\})$$

$$=(\hat\delta(q,wa),\bigcup_{|c|=1}\bigcup_{|c'|=|w|}\bigcup_{\hat\delta(q',cc')\in s}\{q'\})$$

$$=(\hat\delta(q,wa),\bigcup_{|c|=|w|+1}\bigcup_{\hat\delta(q',c)\in s}\{q'\}).$$

Now we prove $L(M')=half(L)$. which is $\exists x. |x|=|w|. wx\in L \iff w\in L(M')$.

$\Rightarrow$: $wx\in L \Rightarrow \hat\delta(q_0,wx)\in F$. Let $q=\hat\delta(q_0,w)$, then $\hat\delta(q,x)\in F$.
Let $S=\bigcup_{|c|=|w|}\bigcup_{\hat\delta(q',c)\in F}\{q'\}$.

then $\hat\delta'(q_0,F,w)=(\hat\delta(q_0,w),\bigcup_{|c|=|w|}\bigcup_{\hat\delta(q',c)\in F}\{q'\})=(q,S)$.

Since $|x|=|w|$, $\hat\delta(q,x)\in F$. So consider $c=x$, $q'=q$. we have $q\in S$. then $(q,S)\in F'$.

So $\hat\delta'(q_0,F,w)\in F'$. $w\in L(M')$.

$\Leftarrow$: $w\in L(M') \Rightarrow \hat\delta'(q_0,F,w)\in F'$.

$\Rightarrow \hat\delta(q,w)\in\bigcup_{|c|=|w|}\bigcup_{\hat\delta(q',c)\in F}\{q'\}$.

So $\exists |c|=|w|$, $\hat\delta(\hat\delta(q,w),c)\in F$.

that is $\hat\delta(q,wc)\in F$, $wc\in L$.

Consider $x=c$, we have $|x|=|w|$, $wx\in L$.