ASSIGNMENT 3

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Problem 1. A functor that doesn't preserve mono's.

Solution. Consider the functor F from the walking arrow category $\mathbf{2}$

$$a \xrightarrow{\varphi} b$$

to the two-pronged fork category $\mathbf{2Frk}$

$$x \underset{h}{\overset{g}{\rightrightarrows}} y \xrightarrow{f} z$$

defined by picking out the handle of the fork.

$$F(\varphi) = f.$$

The map $\varphi \in \mathbf{2}$ is vacuously monic but f is not: $f \circ g = f \circ h$ are equal by definition but $g \neq h$. $\square$

Problem 2. A functor that doesn't reflect mono's.

Solution. The functor $F : \mathbf{2} \to \mathbf{2Frk}$ has a retraction $G : \mathbf{2Frk} \to \mathbf{2}$ defined by

$$G(g) = 1_a = G(h), \quad G(f) = \varphi$$

As mentioned above, $G(f) = \varphi$ is vacuously monic in $\mathbf{2}$ but f is not monic in $\mathbf{2Frk}$. $\square$

Problem 3. Faithful functors reflect mono's

Solution. Let $F : \mathcal{C} \to \mathcal{D}$ be faithful and suppose $F(f)$ is monic in $\mathcal{D}$. Further suppose $f \circ g = f \circ h$ in $\mathcal{C}$. Applying F to these composites we see

$$F(f) \circ F(g) = F(f \circ g) = F(f \circ h) = F(f) \circ F(h).$$

Since $F(f)$ is monic $F(g) = F(h)$ and since F is faithful we have $g = h$. This shows f is monic in $\mathcal{C}$ and concludes the proof. $\square$

Problem 4. If $g \circ f$ is monic then f is monic and if $g \circ f$ is epic then g is epic.

Solution. Epi's and mono's are dual so these two statements (along with their) are formally dual and it suffices to prove the first one.

Assume $g \circ f$ is monic and suppose $f \circ h = f \circ k$. Post-composing with g and omitting brackets for composition we see

$$g \circ f \circ h = g \circ f \circ k$$

Since $g \circ f$ is monic we get $h = k$. This implies f is monic. $\square$

Problem 5. Show that every functor preserves sections and retractions.

Solution. Sections and retractions are formally dual so it suffices to prove every functor preserves sections.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and s a section in $\mathcal{C}_1$, ie. there exists composable $r \in \mathcal{C}_1$ such that $r \circ s = 1$ is the identity on the domain of s. Functors preserve composition and identities so we have

$$F(r) \circ F(s) = F(r \circ s) = F(1) = 1.$$

This shows $F(s)$ is a section (of $F(r)$) in $\mathcal{D}$ and so F preserves sections. $\square$

Problem 6. Show that the unique functor $\mathcal{C} \to \mathbf{1}$ is faithful if and only if $\mathcal{C}$ is a preorder category.

Solution. Suppose the functor $\mathcal{C} \to \mathbf{1}$ is faithful. Then for each $A, B \in \mathcal{C}$ the induced map sending all arrows to the identity

$$\mathcal{C}(A, B) \to \{1\} \tag{$\star$}$$

is injective. More explicitly for any $f, g : A \to B$ in $\mathcal{C}$ we must have $f = g$. This implies ever hom-set of $\mathcal{C}$ has at most one arrow, ie. $\mathcal{C}$ is a preorder category.

On the other hand if $\mathcal{C}$ is a preorder category then it's hom-sets have at most one arrow. This means the map on hom-sets ($\star$) induced by the unique functor $\mathcal{C} \to \mathbf{1}$ is injective and therefore the functor is faithful. $\square$

Problem 7. Problem 1.3.ix in Riehl's book (Cat's in Context). The commutator subgroup, the automorphism group, and the center of a group are all groups we can assign to a group.

These are all functorial on the discrete category of groups in a trivial way. Is it still functorial on the category of groups with isomorphisms between them? On the category of groups with epimorphisms/monomorphisms between them? On the category of groups and all group homomorphisms.

Solution.

(a) Assigning each group $G \in \mathbf{Grp}_0$ its commutator subgroup $C(G)$ is functorial on all of $\mathbf{Grp}$. For $\varphi : G \to H$ we can define $C(\varphi)$ on the generators of $C(G)$ by applying φ

$$C(\varphi)(ghg^{-1}h^{-1}) := \varphi(ghg^{-1}h^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1}$$

The second equality comes from the fact that φ is a group homomorphism. Similarly $C(\varphi)$ is a group homomorphism because φ is. Identities are clearly preserved

$$C(1_G)(ghg^{-1}h^{-1}) = 1_G(ghg^{-1}h^{-1}) = ghg^{-1}h^{-1} = 1_{C(G)}(ghg^{-1}h^{-1})$$

and so is composition

$$\begin{aligned} C(\varphi \circ \psi)(ghg^{-1}h^{-1}) &= \varphi \circ \psi(ghg^{-1}h^{-1}) \\ &= (\varphi \circ \psi(g))(\varphi \circ \psi(h))(\varphi \circ \psi(g)^{-1})(\varphi \circ \psi(h)^{-1}) \\ &= C(\varphi)\left(\psi(g)\psi(h)\psi(g)^{-1}\psi(h)^{-1}\right) \\ &= C(\varphi) \circ C(\psi)(ghg^{-1}h^{-1}). \end{aligned}$$

(b) Assigning each group $G \in \mathbf{Grp}_0$ to its center $Z(G)$ is functorial on any subcategory of group whose arrows are epimorphisms, including the subcategory of isomorphisms.

Let $\varphi : G \to H$ be a group homomorphism. Then $g \in Z(G)$ whenver it commutes with all $h \in G$. Since φ is a group homomorphism it preserves this commutativity relation on the image: if $gh = hg$ for all $h \in G$ then

$$\varphi(g)\varphi(h) = \varphi(gh) = \varphi(hg) = \varphi(h)\varphi(g)$$

shows $\varphi(g) \in Z(\mathrm{im}\varphi) \supseteq Z(H)$. The epimorphisms in **Grp** are precisely the surjective group homomorphisms so

$$Z\varphi : Z(G) \to Z(H) \quad ; \qquad Z\varphi(g) = \varphi(g)$$

is well-defined whenever φ is an epimorphism. Identities are epi's and they're clearly preserved:

$$Z(1_G)(g) = 1_G(g) = g = 1_{Z(G)}$$

Epimorphisms are stable under composition and

$$Z(\varphi \circ \psi)(g) = \varphi \circ \psi(g) = \varphi\left(Z(\psi)(g)\right) = \left(Z(\varphi) \circ Z(\psi)\right)(g)$$

shows they're preserved. It follows that $Z(-)$ is functorial on any subcategory of groups whose arrows are all epimorphisms.

(c) The automorphism group assignment, Aut, extends to the subcategory of isomorphisms in **Grp**. If $\varphi : A \to B$ is a group isomorphism define $\mathrm{Aut}(\varphi)$ by conjugation with φ:

$$\mathrm{Aut}(\varphi) : \mathrm{Aut}(A) \to \mathrm{Aut}(B) \quad ; \qquad \mathrm{Aut}(\varphi)(\alpha) = \varphi^{-1} \circ \alpha \circ \varphi$$

This preserves identities: for all $A \in \mathbf{Grp_0}$

$$\mathrm{Aut}(1_A)(\alpha) = 1_A^{-1} \circ \alpha \circ 1_A = \alpha = 1_{\mathrm{Aut}(A)}(\alpha)$$

It also preserves composition: for all composable isomorphisms of groups φ and ψ

$$\begin{aligned}
\mathrm{Aut}(\varphi \circ \psi)(\alpha) &= (\varphi \circ \psi)^{-1} \circ \alpha \circ (\varphi \circ \psi) \\
&= \psi^{-1} \circ \varphi^{-1} \circ \alpha \circ \varphi \circ \psi \\
&= \psi^{-1} \circ \mathrm{Aut}(\varphi)(\alpha) \circ \psi \\
&= \mathrm{Aut}(\psi) \circ \mathrm{Aut}(\varphi)(\alpha).
\end{aligned}$$

The automorphism group assignment is not functorial on the subcategory of epimorphisms in **Grp** so it's not functorial on all of **Grp**.

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