CAT THEORY - A6

Problem 1. *Consider the diagram*

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[rr, bend left] & B \arrow[r] \arrow[d] & C \arrow[d] \\
A' \arrow[r] \arrow[rr, bend right] & B' \arrow[r] & C'
\end{tikzcd}
$$

and prove:

(i) If the two inner squares are pullbacks then the outer square is a pullback.
(ii) If the outer square and the right square are pullbacks then the left square is a pullback.

Solution. $\square$

Problem 2. *Show that a category with pullbacks and products has equalizers as follows: given arrows $f, g : A \to B$ take the pullback indicated below where $\Delta = (1_B, 1_B)$ is the diagonal map*

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, "e"'] & B \arrow[d, "\Delta"] \\
A \arrow[r, "{(f,g)}"'] & B \times B
\end{tikzcd}
$$

Show that $e : E \to A$ is the equalizer of f and g.

Solution. $\square$

Let **Ab** denote the category of abelian groups and group homomorphisms. The kernel of an abelian group homomorphism $f : A \to B$ is the set

$$\ker f = \{a \in A \mid f(a) = 0\}$$

and the cokernel is the quotient

$$\operatorname{coker} f = B/\mathrm{im} f = \{b + f(A) \mid b \in B\}.$$

Problem 3. *Show that every equalizer in* **Ab** *is a kernel and that every coequalizer is a cokernel.*

Solution. $\square$

Problem 4. *Let $\mathcal{C}$ be a small category. Show that the category of presheaves $[\mathcal{C}^{op}, \mathbf{Set}]$ is complete and cocomplete.*

Hint: It suffices to show that products (indexed over an arbitrary set I) exist and the equalizer of an arbitrary pair of natural transformations $\alpha, \beta : F \implies G$ exists.

Bonus: Can we say the same about the category of abelian presheaves, $[\mathcal{C}^{op}, \mathbf{Ab}]$? Justify with a sentence or two, you don't need to do all the work above again.

Solution. □

Problem 5. *Let $F : \mathcal{C}^{op} \to \mathbf{Set}$ be a presheaf and let $\{*\}/\mathbf{Set}_*$ denote the slice category of sets under the singleton $\{*\}$ as in the previous homework assignment. Let*

$$\{*\}/\mathbf{Set}_* \xrightarrow{\pi} \mathbf{Set}$$

denote the functor that forgets distinguished points:

$$\pi\left(\begin{tikzcd} & \{*\} \arrow[dl, "x"'] \arrow[dr, "y"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}\right) = \left(X \xrightarrow{f} Y \right)$$

Compute the (strict) [1] *pullback of the diagram*

$$\begin{tikzcd} \mathbf{El}F \arrow[r, "p_2"] \arrow[d, "p_1"'] & \{*\}/\mathbf{Set} \arrow[d, "\pi"] \\ \mathcal{C}^{op} \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd}$$

You don't need to show π is a functor. Just define the category $\mathbf{El}F$ along with the projection functors, p_1 and p_2, and show it satisfies the universal property of the (strict) pullback in $\mathbf{Cat}$.[2]

Solution. □

[1] **Cat** is a 2-category so there are 'weaker' notions of pullback (pseudo and (op)lax) that we haven't discussed in this class where cells commute up to natural isomorphism or natural transformation. We haven't mentioned these in this course. Strict means you should treat **Cat** as a 1-category and use the definition of limit we've seen in class.

[2] $\mathbf{El}F$ is called the 'category of elements' associated to F