ASSIGNMENT 4

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Problem 1. Show that any functor $F : \mathcal{A} \to \mathcal{B}$ factors as $F = H \circ G$ where G is surjective on objects and H is fully faithful and monic on objects.

Show that the surjective-on-objects functors are left orthogonal to the fully-faithful-and-monic-on-objects functors.

Solution. Define an intermediate category $\mathcal{C}$ whose objects are $\mathcal{C}_0 = F_0(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and whose arrows are all the arrows in $\mathcal{B}$ between these objects:

$$\mathcal{C}(F(A), F(B)) := \mathcal{B}(F(A), F(B)).$$

This is a category because it inherits identities and composition from $\mathcal{B}$, in fact it's a subcategory of $\mathcal{B}$ by construction. Now we'll define functors G and H such that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{B} \\ & \mathcal{C} \arrow[ur, "H"'] & \end{tikzcd}$$

commutes in the following way. Since $\mathcal{C}$ is a subcategory of G the functor H will be an inclusion on objects and arrows; this is clearly a functor. It's vacuously injective on objects and fully-faithful by construction:

$$\mathcal{B}(H(F(A)), H(F(A'))) = \mathcal{B}(F(A), F(A')) =: \mathcal{C}(F(A), F(A'))$$

The functor G is just F with a restricted codomain so it's a functor because F is and it's surjective on objects because for any $F(A) \in \mathcal{C}_0$ we have $A \in \mathcal{C}_0$ and $G(A) := F(A)$ by definition. The diagram commutes because for any $A \in \mathcal{A}_0$ and $f : A \to A'$ in $\mathcal{A}_1$

$$H \circ G(A) = H(G(A)) = H(F(A)) = F(A)$$

and

$$H \circ G(f) = H(G(f)) = H(F(f)) = F(f).$$

For orthogonality suppose we have a commmuting diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{B} \arrow[r, "K"'] & \mathcal{D} \end{tikzcd}$$

in **Cat** where F is surjective on objects and G is fully faithful and injective on objects. We'll show there exists a unique solution to the lifting problem. Define a candidate lift $L : \mathcal{B} \to \mathcal{C}$ as follows. On objects, we take $B \in \mathcal{B}_0$ and take any $A \in \mathcal{A}_0$ in the fibre of F over B. Such an A exists because F is surjective on objects. Then we apply H_0:

$$L_0(B) := H_0(A), \quad \text{where} \quad F_0(A) = B$$

We need to check this is well-defined. Suppose A' is another object in the fibre of F above B, then

$$G_0(H_0(A')) = K_0(F_0(A')) = K_0(B) = K_0(F_0(A)) = G_0(H_0(A))$$

and since G is injective on objects we have

$$H_0(A') = H_0(A)$$

showing L_0 is well-defined. Now for any arrow $\beta : B \to B'$ in $\mathcal{B}$ we can apply K to get an arrow $K(\beta) : K(B) \to K(B')$ in $\mathcal{D}$. Notice $K_0(B) = G_0(L_0(B))$ and $K_0(B') = G_0(L_0(B'))$ by commutativity of the original square and definition of L_0. Since G is fully faithful there exists a unique map

$$L_1(\beta) : L_0(B) \to L_0(B').$$

Define $L : \mathcal{B} \to \mathcal{C}$ by (L_0, L_1) on objects and arrows respectively. Identities and composition are preserved by the uniqueness in the definition of L_1 coming from G being fully-faithful. This solves the lifting problem because for each $A \in \mathcal{A}_0$

$$H_0(A) = L_0(F_0(A))$$

and for ever $f : A \to A'$ in $\mathcal{A}$ the square commutes so

$$K_1(F_1(f)) = G_1(H_1(f))$$

and by definition of L_1 and the fact that G is fully faithful we must have

$$H_1(f) = L_1(F_1(f)).$$

This shows $H = L \circ F$. On the other hand, for any $B \in \mathcal{B}_0$ and $A \in \mathcal{A}_0$ such that $F_0(A) = B$ we have

$$G_0 \circ L_0(B) = G_0 \circ H_0(A) = K_0 \circ F_0(A) = K_0(B).$$

For $\beta : B \to B'$ in $\mathcal{B}_1$ we have that

$$G_1 \circ L_1(\beta) = K_1(\beta)$$

by construction so $G \circ L = K$. This shows L solves the lifting problem:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "G"] \\
\mathcal{B} \arrow[r, "K"'] \arrow[ur, dashed, "L"] & \mathcal{D}
\end{tikzcd}$$

To show L is unique suppose L' also solved the lifting problem. Since F is surjective on objects and L and L' both solve the lifting problem we have that for each $B \in \mathcal{B}_0$ there exists an $A \in \mathcal{A}_0$ such that

$$(L')_0(B) = (L' \circ F)_0(A) = H_0(A) = (L \circ F)_0(A) = L_0(B).$$

Similarly we have that for any $\beta \in \mathcal{B}_1$

$$G_1(L_1(\beta)) = K_1(\beta) = G_1((L')_1(\beta))$$

since L and L' both solve the lifting problem and since G is fully faithful we can conclude

$$L_1(\beta) = (L')_1(\beta).$$

It follows that $L = L'$ and the solution to the lifting problem is unique. $\square$

Problem 2. Let $F, G : \mathcal{A} \to \mathcal{B}$ and let $\alpha : F \implies G$ has components α_X that are all invertible. Show that the family of inverses form a natural transformation $G \implies F$.

Solution. For any $f : X \to Y$ in $\mathcal{A}$ the naturality square

$$\begin{tikzcd} G(X) \arrow[r, "\alpha_X^{-1}"] \arrow[d, "G(f)"'] & F(X) \arrow[d, "F(f)"] \\ G(Y) \arrow[r, "\alpha_Y^{-1}"'] & F(Y) \end{tikzcd}$$

commutes in $\mathcal{B}$ because

$$\begin{aligned} F(f) \circ \alpha_X^{-1} &= \alpha_Y^{-1} \circ \alpha_Y \circ F(f) \circ \alpha_X^{-1} \\ &= \alpha_Y^{-1} \circ G(f) \circ \alpha_X \circ \alpha_X^{-1} \\ &= \alpha_Y^{-1} \circ G(f). \end{aligned}$$

This shows $\alpha^{-1} : G \implies F$ is a natural transformation and since all its components are isomorphisms

$$\alpha_X \circ \alpha_X^{-1} = 1_{G(X)} \quad ; \quad \alpha_X^{-1} \circ \alpha_X = 1_{F(X)}$$

it's a natural isomorphism. $\square$

Problem 3. Let $\alpha : F \implies F'$ be a natural transformation between two parallel functors $F, F' : BG \to BH$ where G and H are groups and BG and BH are the categories with one object and arrows determined by G and H respectively. Describe α.

Solution. The functors F and F' correspond to group homomorphisms $\varphi, \varphi' : G \to H$. There's only one object in BG so α only has on component $h \in H$ such that for any $g \in G$ the naturality square

$$\begin{tikzcd} * \arrow[r, "h"] \arrow[d, "\varphi(g)"'] & * \arrow[d, "\varphi'(g)"] \\ * \arrow[r, "h"'] & * \end{tikzcd}$$

commutes in BH. This says

$$h\varphi'(g) = \varphi(g)h$$

in H or equivalently that

$$\varphi(g) = h\varphi'(g)h^{-1}$$

We can think of conjugation by an element in H as an action on the set of group homomorphisms $G \to H$ and in this context a natural transformation $\alpha : F \implies F'$ witnesses that the group homomorphisms φ and φ' are conjugate. $\square$

Problem 4. Prove that $- \times A : \mathbf{Set} \to \mathbf{Set}$ is a functor and the family of projections $\pi_X : X \times A \to X$ induces a natural transformation $(- \times A) \implies 1_{\mathbf{Set}}$.

Solution. It assigns a set X to its cartesian product $X \times A$ and it assigns an arrow $f : X \to Y$ to the function

$$f \times 1_A : X \times A \to Y \times A$$

defined by

$$f \times 1_A(x, a) := (f(x), a).$$

This assignment preserves identities because for every set X

$$1_X \times 1_A(x, a) = (1_X(x), a) = (x, a) = 1_{X \times A}(x, a)$$

and it preserves composition as for any composable $f : X \to Y$ and $g : Y \to Z$

$$\begin{aligned}(g \circ f) \times 1_A(x, a) &= (g \circ f(x), a) \\ &= (g(f(x)), a) \\ &= g \times 1_A\,(f(x), a) \\ &= ((g \times 1_A) \circ (f \times 1_A))\,(x, a).\end{aligned}$$

This shows $- \times A$ is an endofunctor on **Set**. To see the projections induce a natural transformation to the identity functor we need to check that for every function $X \to Y$ the naturality square

$$\begin{tikzcd} X \times A \arrow[r, "\pi_X"] \arrow[d, "f \times 1_A"'] & X \arrow[d, "f"] \\ Y \times A \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

commutes. To see this we chase an element $(x, a) \in X \times A$ through the diagram:

$$\begin{aligned}f \circ \pi_X(x, a) &= f(\pi_X(x, a)) \\ &= f(x) \\ &= \pi_Y(f(x), a)) \\ &= \pi_Y \circ (f \times 1_A)(x, a)\end{aligned}$$

□

Problem 5. Let $U : \mathcal{C} \to \mathcal{C}^{\to}$ denote the functor that sends objects in $\mathcal{C}$ to their identity arrows and let $D_0 : \mathcal{C}^{\to} \to \mathcal{C}$ denote the domain functor that sends arrows in $\mathcal{C}$ to their domains. Define a natural transformation $U \circ D_0 \implies 1_{\mathcal{C}^{\to}}$.

Solution. For every object $f : X \to Y$ in $\mathcal{C}^{\to}$ define the component

$$\alpha_f : U \circ D_0(f) \to f \quad ; \quad \alpha_f = (1_X, f)$$

in $\mathcal{C}^{\to}$. This makes sense becuase $U \circ D_0(f) = 1_X$ and $1_{\mathcal{C}^{\to}}(f) = f$ in $\mathcal{C}$ and in $\mathcal{C}$ we view this as a commuting square:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

For naturality we take an arbitrary arrow $(h_0, h_1) : f \to g$ in $\mathcal{C}^{\to}$ where $f : X \to Y$ and $g : W \to Z$ are arrows in $\mathcal{C}$ and we need to show the naturality square

$$\begin{tikzcd}
U \circ D_0(f) \arrow[r, "\alpha_f"] \arrow[d, "(U\circ D_0)(h_0,h_1)"'] & f \arrow[d, "(h_0,h_1)"] \\
U \circ D_0(g) \arrow[r, "\alpha_g"'] & g
\end{tikzcd}$$

commutes in $\mathcal{C}$. Note that the domain of h_0 is the domain of X so

$$(U \circ D_0)(h_0, h_1) = U(h_0) = 1_{\partial_0(h_0)} = 1_X.$$

The naturality square above commutes in $\mathcal{C}^{\to}$ whenever the cube

$$\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, equal] \arrow[dd, "h_0"'] & & X \arrow[dr, "f"] \arrow[dd, "h_0" near end] & \\
& X \arrow[rr, "f" near start] \arrow[dd, "h_0"' near start] & & Y \arrow[dd, "h_1"] \\
W \arrow[rr, equal] \arrow[dr, equal] & & W \arrow[dr, "g"] & \\
& W \arrow[rr, "g"'] & & Z
\end{tikzcd}$$

commutes in $\mathcal{C}$. The bottom, top, left, and back faces of the cube commute by the identity laws for composition in $\mathcal{C}$. The front and right faces of the cube commute by definition of (h_0, h_1) as an arrow in $\mathcal{C}^{\to}$. This shows the cube commutes and it follows that the naturality square commutes in $\mathcal{C}^{\to}$. $\square$