CAT THEORY - A7

Problem 1. *Let $\mathcal{C}$ be a category and let F and G be presheaves on $\mathcal{C}$. Describe the presheaf F^G and show it satisfies the universal property.*

Solution. It should be the right adjoint to the product functor $-\times G$. Yoneda's lemma and cartesian closure say that for any object A we should have

$$F^G(A) \cong \mathrm{Hom}(y(A), F^G) \cong \mathrm{Hom}(y(A) \times G, F).$$

For any arrow $f : A \to B$ we have a natural transformation

$$y(f) : y(A) \implies y(B)$$

whose components are given by post-composition with f:

$$y(f)_X : \mathcal{C}(X, A) \to \mathcal{C}(X, B); g \mapsto f \circ g$$

This induces a natural transformation

$$y(f) \times 1_G : y(A) \times G \implies y(B) \times G$$

whose components are

$$(y(f) \times 1_G)_X : \mathcal{C}(X, A) \times G(X) \to \mathcal{C}(X, B) \times G(X); (g, a) \mapsto (f \circ g, a).$$

Now F^G is a presheaf so for $f : A \to B$ in $\mathcal{C}$ we define

$$F^G(f) : F^G(B) \to F^G(A)$$

to be the map

$$\mathrm{Hom}(y(B) \times G, F) \to \mathrm{Hom}(y(A) \times G, F)$$

defined on a transformation $\beta : y(B) \times G \implies F$ as

$$F^G(f)(\beta) = \beta \circ (y(f) \times 1_G).$$

Identities are preserved because for any $\alpha \in \mathrm{Hom}(y(A) \times G, F)$

$$\begin{aligned}
(F^G(1_A)(\alpha) &:= \alpha \circ (y(1_A) \times 1_G) \\
&= \alpha \circ (1_{y(A)} \times 1_G) \\
&= \alpha \circ 1_{y(A)\times 1_G} \\
&= \alpha \\
&= 1_{F^G(A)}(\alpha)
\end{aligned}$$

and composition is preserved because for any $f : A \to B$ and $f' : B \to C$ in $\mathcal{C}$ and $\gamma : F^G(C)$ we have

$$\begin{aligned}
F^G(f\circ f')(\gamma) &= \gamma\circ(y(f'\circ f)\times 1_G)\\
&= \gamma\circ((y(f')\circ y(f))\times 1_G)\\
&= \gamma\circ\big((y(f')\times 1_G)\circ(y(f)\times 1_G)\big)\\
&= \big(\gamma\circ(y(f')\times 1_G)\big)\circ(y(f)\times 1_G)\\
&= F^G(f)\big(\gamma\circ(y(f')\times 1_G)\big)\\
&= \big(F^G(f)\circ F^G(f')\big)(\gamma).
\end{aligned}$$

These calculations use the definition and functoriality of the Yoneda embedding, $y:\mathcal{C}\to[\mathcal{C}^{op},\mathbf{Set}]$, along with the product functor $(-)\times G$ on $[\mathcal{C}^{op},\mathbf{Set}]$. The natural isomorphism for the rest of the adjunction follows from the density lemma for the yoneda embedding and the fact that representable functors preserve limits. In particular presheaves are colimits of representables. Let $H\cong\operatorname{colim} y(X_\alpha)$ be an arbitrary presheaf and notice and compute

$$\begin{aligned}
\operatorname{Hom}(H,F^G) &\cong \operatorname{Hom}(\operatorname{colim} y(X_\alpha),F^G)\\
&\cong \lim\operatorname{Hom}(y(X_\alpha),F^G)\\
&\cong \lim\operatorname{Hom}(y(X_\alpha)\times G,F)\\
&\cong \operatorname{Hom}(\operatorname{colim} y(X_\alpha)\times G,F)\\
&\cong \operatorname{Hom}(H\times G,F).
\end{aligned}$$

You asked for the universal property of the adjunction, so here's a sketch. For any presheaves F and G define

$$\eta_F : F \implies (F\times G)^G$$

with components

$$(\eta_F)_X : F(X)\to(F\times G)^G(X)$$

on an element $a\in F(X)$ to be the natural transformation

$$(\eta_F)_X(a) : y(X)\times G \implies F\times G$$

with components

$$((\eta_F)_X(a))_Y : \mathcal{C}(Y,X)\times G(Y)\to F(Y)\times G(Y)$$

defined by

$$((\eta_F)_X(a))_Y\,(f,b) = (F(f)(a),b).$$

Naturality follows from functoriality of F after unpacking the mountain of definitions. I checked it on the board, I'm not TeX-ing it. Given a natural transformation $\alpha : F \implies H^G$ there's a natural transformation

$$\alpha^{\#} : F\times G \implies G$$

whose components are defined on $(b,b')\in F(B)\times G(B)$ in terms of α:

$$(\alpha^{\#})_B(b,b') = (\alpha_B)(b)(1_B,b').$$

Naturality of $\alpha^{\#}$ follows from naturality of α. Now

$$(\alpha^{\#})^G : (F \times G)^G \implies H^G$$

and

$$(\alpha^{\#})^G \circ \eta_F = \alpha.$$

This is another gross calculation I wrote out on the board that I'm not going to typeset; it unpacks the definitions above by evaluating nested components of natural transformations and appeals to naturality of α at the end and the definition of H^G on arrows. Uniqueness is forced by α, its aforemented naturality, and the definition of H^G on arrows in $\mathcal{C}^{op}$. I can send you a photo of the board as some justification if you want but I'll spare you the time and effort of parsing my definitions. $\square$

Problem 2. *Show that for any adjunction the following are equivalent*

(*i*) $\eta_{G \circ F(X)}$ *is an iso*
(*ii*) $G \circ F(\eta_X)$ *is an iso*
(*iii*) $\eta_{G(Y)}$ *is an iso*

Solution. The triangle identities give

$$G(\varepsilon_{F(X)}) \circ G \circ F(\eta_X) = 1_{G(F(X))} = G(\varepsilon_{F(X)}) \circ \eta_{G \circ F(X)}$$

so (i) and (ii) are equivalent because isomorphisms satisfy the two-for-three property. It's clear that (iii) implies (i) but the proof I had in mind for the other direction doesn't actually work. $\square$

Problem 3. *Show that retractions are stable under pullback. Show that sections aren't necessarily stable under pullback.*

Solution. Let $r : B \to A$ be a retraction and let $f : X \to A$ be any map in a category $\mathcal{C}$. The result follows from the universal property of the following pullback diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed] \arrow[dd, bend right, "1_X"'] & A \arrow[d, "s"] \arrow[dd, bend left, "1_A"] \\
P \arrow[r] \arrow[d, "r^* f"] & B \arrow[d, "r"] \\
X \arrow[r, "f"'] & A
\end{tikzcd}$$

Sections aren't stable under pullback in general; for example in **Set** take the inclusion $\{0\} \to \{0, 1\}$. It's a section of the unique map $\{0, 1\} \to \{0\}$ but its pullback along the inclusion $\{1\} \to \{0, 1\}$ is the unique map $\emptyset \to \{1\}$ which can't be a section because there are no smooth functions into the emptyset. $\square$

Problem 4. *Describe the induced monads and comonads for the following adjoint pairs of functors:*

(*i*) *The free-forget adjunction between sets and monoids.*
(*ii*) *The free-forget adjunction between graphs and categories.*

Solution. (*i*) Let $T = G \circ F$ be the associated monad where G us the forgetful functor and F is the free functor. For every set X the set $T(X)$ can be thought of as the set of finite non-empty lists (instead of words) with terms (instead of letters) in X. This makes the notation less cumbersome; computer scientists call this the 'list monad.' The unit

$$\eta : 1_{\mathbf{Set}} \implies T$$

is the unit of the adjunction; it includes a set into the lists of length 1:

$$\eta_X : X \to T(X); x \mapsto [x]$$

Multiplication

$$\mu : T^2 \implies T$$

has components

$$\mu_X : T(T(X)) \to T(X)$$

defined by concatenating the lists within a finite list of finite lists:

$$[[x_{0;0}, \ldots, x_{0;i_0}], \ldots, [x_{n;0}, \ldots, x_{n;i_n}]] \mapsto [x_{0;0}, \ldots, x_{n;i_n}].$$

The identity laws hold because

$$\mu_X([[x_0, \ldots, x_n]] = [x_0, \ldots, x_n] = \mu_X([[x_0], \ldots, [x_n]])$$

and multiplication is associative because concatenating lists is associative.

The comonad for the adjunction, $L = F \circ G$, is an endofunctor on the category of monoids. For a monoid M, $L(M)$ is the monoid of non-empty finite lists in M. The counit

$$\varepsilon : L \implies 1$$

is the counit of the adjunction. It's given by evaluating the list using the multiplication from M:

$$\varepsilon_M : L(M) \to M; [m_0, \ldots, m_n] \mapsto m_0 \ldots m_n$$

The comultiplication

$$\nu : L \implies L^2$$

takes a non-empty list in M and gives a list of lists of length 1 in M:

$$\nu_M : L(M) \to L^2(M); [m_0, \ldots, m_n] \mapsto [[m_0], \ldots, [m_n]]$$

The counit laws follow from the fact that concatenating length one lists in $L(M)$ is the same as evaluating length one lists in M and then listing them in $L(M)$:

$$\varepsilon_{L(M)}([[m_0], \ldots, [m_n]]) = [m_0] * \cdots * [m_n] = [m_0, \ldots, m_n] = L(\varepsilon_M)([[m_0], \ldots, [m_n]])$$

Coassociativity follows from

$$\nu_{L(M)}([[m_0], \ldots, [m_n]]) = [[[m_0], \ldots, [m_n]]] = L(\nu_M)([[m_0], \ldots, [m_n]])$$

where the first equality adds new brackets on the inside of the outer-most brackets while the second one adds brackets on the outside.

(ii) Let T be the monad of the free-forget adjunction. For any graph G the vertices of $T(G)$ are those of G and the edges of $T(G)$ are finite paths in G. We can write these finite paths as lists:

$$[\gamma_0, \ldots, \gamma_1] \in T(G)_1$$

The unit is the usual unit of adjunction. For a graph G,

$$\eta_G : G \to T(G)$$

is the identity on vertices and includes edges in G to length-one paths in $T(G)$. The multiplication

$$\mu : T^2 \implies T$$

has graph homomorphism components

$$\mu_G : T^2 G \to TG$$

given by identities on vertices and concatenating paths:

$$\mu_G([[\gamma_{0,0}, \ldots, \gamma_{0,i_0}], \ldots, [\gamma_{n,0}\gamma_{n,i_n}]]) := [\gamma_{0,0}, \ldots, \gamma_{n,i_n}]$$

The similarity with the list-monad in the previous part is obvious and the identity and associativity laws follow from identical calculations replacing all the elements '$x \in X$' with edges '$\gamma \in G_1$.'

The comonad, T', takes a (small) category $\mathcal{C}$ to the path category of its underlying graph, $T'(C)$. It inherits its counit from the counit of adjunction

$$\varepsilon_{\mathcal{C}} : T'(\mathcal{C}) \to \mathcal{C}$$

which is the identity on objects and sends finite paths from the underlying graph of $\mathcal{C}$ to their composites in $\mathcal{C}$. The comultiplication

$$\nu : T' \implies (T')^2$$

has component functors

$$\nu_{\mathcal{C}} : T'(\mathcal{C}) \to (T')^2(\mathcal{C})$$

that are identities on objects and send finite paths from the underlying graph of $\mathcal{C}$ to paths of length-one paths:

$$[\gamma_0, \ldots, \gamma_n] \mapsto [[\gamma_0], \ldots, [\gamma_n]]$$

The rest follows similarly to the comonad for monoids.

□