CAT THEORY - A5

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Problem 1. *Let $\mathcal{C}$ be a category and let $A \in \mathcal{C}_0$ be an object.*

1 *Define the the slice category, $A/\mathcal{C}$, of $\mathcal{C}$ under A.*
2 *Show the $\{*\}/\mathbf{Set} \cong \mathbf{Set}_*$.*
3 *Define the slice category, $\mathcal{C}/A$, of $\mathcal{C}$ over A.*
4 *Show that $\mathbf{Set}/\{0,1\}$ is isomorphic to the category of partitioned sets of index 2 and partition preserving functions between them.*

Solution.

1 The under-category $A/\mathcal{C}$ has arrows of $\mathcal{C}$ with domain A as its objects and commuting triangles in $\mathcal{C}$ as arrows. Identities and composition are inherited from $\mathcal{C}$ as the diagrams

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f"] & \\ B \arrow[rr, equal] & & B \end{tikzcd} \quad ; \quad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ B \arrow[r, "k"'] \arrow[rr, bend right, "\ell \circ k"'] & C \arrow[r, "\ell"'] & D \end{tikzcd}$$

commute. The identity laws in $A/\mathcal{C}$ follow from the identity laws in $\mathcal{C}$ via the commuting diagram:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "f"] \arrow[dr, "g"] & \\ B \arrow[r, equal] \arrow[rr, bend right, "\ell"'] & B \arrow[r, "\ell"'] & C \end{tikzcd} \quad ; \quad \begin{tikzcd} & A \arrow[dl, "h"'] \arrow[d, "f"] \arrow[dr, "f"] & \\ A \arrow[r, "k"'] \arrow[rr, bend right, "k"'] & B \arrow[r, equal] & B \end{tikzcd}$$

Associativity follows from associativity in $\mathcal{C}$ via the commuting diagram:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] \arrow[drr, "k"] & & \\ B \arrow[r, "b"'] \arrow[rr, bend right, "c \circ b"'] & C \arrow[r, "c"'] \arrow[rr, bend right, "d \circ c"'] & D \arrow[r, "d"'] & E \end{tikzcd}$$

2 Define $F : \{*\}/\mathbf{Set} \to \mathbf{Set}_*$ on objects and arrows simultaneously by

$$F\left(\begin{tikzcd} & \{*\} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}\right) = (A, a(*)) \xrightarrow{f} (B, b(*)) \ . \tag{1}$$

This is well defined because $a(*) \in A,\ b(*) \in B$, and $f \circ a = b$ implies $f(a(*)) = b(*)$. Composition and identities in both categories are determined by the composition and identities from **Set** so it's clear that F preserves them and is a functor.

On the other hand define $G : \mathbf{Set}_* \to \{*\}/\mathbf{Set}$ on objects and arrows by

$$G\Big((A, a) \xrightarrow{f} (B, b) \Big) = \begin{tikzcd} & \{*\} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd} \tag{2}$$

where $\alpha(*) = a$ and $\beta(*) = b$. This is well defined because maps between pointed sets preserve the basepoints

$$f \circ \alpha(*) = f(a) = b = \beta(*).$$

Composition and identities are similarly preserved because they're inherited from **Set** in both categories so G is a functor. To see $G \circ F = 1_{\{*\}/\mathbf{Set}}$ we apply G to the diagram on the right hand side of equation (1) above and notice

$$G\Big((A, a(*)) \xrightarrow{f} (B, b(*)) \Big) = \begin{tikzcd} & \{*\} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

by definition of F. Similarly we check $F \circ G = 1_{\mathbf{Set}_*}$ by applying F to the diagram on the right hand side of equation (2) and noticing

$$F\left(\begin{tikzcd} & \{*\} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}\right) = (A, a) \xrightarrow{f} (B, b) \tag{1}$$

since $\alpha(*) = a$ and $\beta(*) = b$ by definition of G.

3 The slice category $\mathcal{C}/A$ has arrows in $\mathcal{C}$ with codomain $A \in \mathcal{C}_0$ as objects and commuting triangles in $\mathcal{C}$ as arrows. Identities and composition are inherited from $\mathcal{C}$ in the same fashion as the under-category $A/\mathcal{C}$ and the associatiity and identitiy laws hold similarly (just reverse the non-horizontal arrows in the diagrams above from part 1.)

4 Let $\mathbf{Set}_2$ denote the category of partitioned sets of index 2 with partition preserving functions between them. Any index 2 partition S can be uniquely identified with the disjoint union $S = S_0 \coprod S_1$ in **Set** and a partition preserving function $g : S \to T$ can be identified with unique map induced by the universal property of the coproduct in **Set**:

$$
\begin{tikzcd}
S_0 \arrow[r, "g_0"] \arrow[d, "i_0"'] & T_0 \arrow[d, "j_0"] \\
S_0 \coprod S_1 \arrow[r, dashed, "g"] & T_0 \coprod T_1 \\
S_1 \arrow[u, "i_1"] \arrow[r, "g_1"'] & T_1 \arrow[u, "j_1"']
\end{tikzcd}
$$

With this notation we define $F : \mathbf{Set}/\{0,1\} \to \mathbf{Set}_2$ by

$$
(\star) \qquad F\left(
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\
& \{0,1\} &
\end{tikzcd}
\right) =
\begin{tikzcd}
A_0 \arrow[r, "f|_{A_0}"] \arrow[d, "i_0"'] & B_0 \arrow[d, "j_0"] \\
A_0 \coprod A_1 \arrow[r, dashed, "f"] & B_0 \coprod B_1 \\
A_1 \arrow[u, "i_1"] \arrow[r, "f|_{A_1}"'] & B_1 \arrow[u, "j_1"']
\end{tikzcd}
$$

where $A_i = \alpha^{-1}(i)$ and $B_i = \beta^{-1}(i)$ for $i \in \{0,1\}$ are the fibers of α and β. Commutativity of the triangle on the left hand side implies

$$
f(A_0) \subseteq B_0 \qquad f(A_1) \subseteq B_1
$$

so that the diagram on the right hand side is well-defined. Functoriality follows immediately from the universal property of coproducts in **Set**.

On the other hand, define $G : \mathbf{Set}_2 \to \mathbf{Set}/\{0,1\}$ by

$$
(\star\star) \qquad G\left(
\begin{tikzcd}
S_0 \arrow[r, "g_0"] \arrow[d, "i_0"'] & T_0 \arrow[d, "j_0"] \\
S_0 \coprod S_1 \arrow[r, dashed, "g"] & T_0 \coprod T_1 \\
S_1 \arrow[u, "i_1"] \arrow[r, "g_1"'] & T_1 \arrow[u, "j_1"']
\end{tikzcd}
\right) =
\begin{tikzcd}
S \arrow[rr, "g"] \arrow[dr, "s"'] & & T \arrow[dl, "t"] \\
& \{0,1\} &
\end{tikzcd}
$$

where $S = S_0 \coprod S_1$ and $T = T_0 \coprod T_1$ are the sets being partitioned and s and t map S_i and T_i to $i \in \{0,1\}$ respectively. These can be described with the universal property of coproducts

$$
\begin{tikzcd}
S_0 \arrow[r, "!_0"] \arrow[d, "i_0"'] & \{0\} \arrow[d, "\subseteq"] & T_0 \arrow[l, "!_0"'] \arrow[d, "j_0"] \\
S_0 \coprod S_1 \arrow[r, dashed, "s"] & \{0,1\} & T_0 \coprod T_1 \arrow[l, dashed, "t"'] \\
S_1 \arrow[r, "!_1"'] \arrow[u, "i_1"] & \{1\} \arrow[u, "\subseteq"'] & T_1 \arrow[l, "!_1"] \arrow[u, "j_1"']
\end{tikzcd}
$$

so this is well-defined and functorial by the universal property of coproducts in **Set**. To see $G \circ F = 1_{\mathbf{Set}_2}$ we apply G to the diagram on the right hand side of equation ($\star$) and notice

$$
G\left(
\begin{tikzcd}
A_0 \arrow[r, "f|_{A_0}"] \arrow[d, "i_0"'] & B_0 \arrow[d, "j_0"] \\
A_0 \coprod A_1 \arrow[r, dashed, "f"] & B_0 \coprod B_1 \\
A_1 \arrow[r, "f|_{A_1}"'] \arrow[u, "i_1"] & B_1 \arrow[u, "j_1"']
\end{tikzcd}
\right)
=
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\
& \{0,1\} &
\end{tikzcd}
$$

because $\alpha|_{A_i} = !_i$ by definition of $A_i = \alpha^{-1}(i)$ for $i \in \{0,1\}$ respectively. To see $F \circ G = 1_{\mathbf{Set}/\{0,1\}}$ apply F to the diagram on the right hand side of equation ($\star\star$) and notice

$$
F\left(
\begin{tikzcd}
S \arrow[rr, "g"] \arrow[dr, "s"'] & & T \arrow[dl, "t"] \\
& \{0,1\} &
\end{tikzcd}
\right)
=
\begin{tikzcd}
S_0 \arrow[r, "g_0"] \arrow[d, "i_0"'] & T_0 \arrow[d, "j_0"] \\
S_0 \coprod A_1 \arrow[r, dashed, "g"] & T_0 \coprod B_1 \\
S_1 \arrow[r, "g_1"'] \arrow[u, "i_1"] & T_1 \arrow[u, "j_1"']
\end{tikzcd}
$$

because $g|_{S_i} = g_i$ by definition of g.

$\square$

Problem 2. *Show* $1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ *is representable and talk about the endomorphisms on* $1_{\mathbf{Set}}$.

Solution. The identity functor on **Set** is representable by a singleton:

$$y(\{*\}) = \mathbf{Set}^{op}(-, \{*\}) = \mathbf{Set}(\{*\}, -) \cong 1_{\mathbf{Set}}$$

because elements of a set A naturally correspond to functions functions $\{*\} \to A$. That is for $f : A \to A'$ the diagram

$$
\begin{tikzcd}
\mathbf{Set}(\{*\}, A) \arrow[r, "\varepsilon_A"] \arrow[d, "f\circ(-)"'] & A \arrow[d, "f"] \\
\mathbf{Set}(\{*\}, A') \arrow[r, "\varepsilon_{A'}"'] & A'
\end{tikzcd}
$$

commutes where $\varepsilon_A(\alpha) = \alpha(*)$ is defined by evaluation for each $\alpha : \{*\} \to A$. We saw this function and its inverse in problem 1 so the natural isomorphism of hom-sets is clear.

Yoneda's lemma implies there's only one endomorphism on $1_{\mathbf{Set}}$ in the functor category $[\mathbf{Set}, \mathbf{Set}]$.

$$\mathrm{Nat}(1_{\mathbf{Set}}, \mathbf{Set}) \cong \mathrm{Nat}(y(\{*\}, 1_{\mathbf{Set}})) \cong 1_{\mathbf{Set}}(\{*\}) = \{*\},$$

namely the identity natural transformation. $\square$

Problem 3. *Let $f : X \to Y$ be a function between sets. View $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are posetal categories under subset inclusion. Show there exists an adjoint triple*

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[rr, bend left=60, "\exists_f"] \arrow[rr, bend right=60, "\forall_f"'] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathcal{P}(Y) \arrow[ll, "f^*"]
\end{tikzcd}$$

Solution. For $S \subseteq X$ define

$$\exists_f(S) = \{y \in Y \mid \exists x \in S.f(x) = y\} \quad ; \quad \forall_f(S) = \{y \in Y \mid \forall x \in X.(f(x) = y) \implies x \in S\}.$$

If $S \subseteq S' \subseteq X$ then $y \in \exists_f(S)$ means $x \in S \subseteq S'$ such that $f(x) = y$ so $y \in \exists_f(S')$. Similarly for any $y \in \forall_f(S)$ we have that $f(x) = y$ implies $x \in S \subseteq S'$ so $y \in \forall_f(S')$. This implies

$$\exists_f(S) \subseteq \exists_f(S') \quad ; \quad \forall_f(S) \subseteq \forall_f(S')$$

On the other hand, for $T \subseteq Y$ we define

$$f^*(T) = \{x \in X \mid f(x) \in T\}$$

and notice for $T \subseteq T'$ we have $f(x) \in T$ implies $f(x) \in T'$ so $f^*(T) \subseteq f^*(T')$. Functoriality for all three of these assignments follows from the fact that $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are posetal categories.

To show these are adjunctions it suffices to show the unit and counit exist in each instance because naturality and the triangle identities will immediately follow from the posetal context we're in. We start with $\exists_f \dashv f^*$. For $S \subseteq X$

$$\begin{aligned}
f^* \circ \exists_f(S) &= \{x \in X \mid f(x) \in \exists_f(S)\} \\
&= \{x \in X \mid \exists x' \in S.f(x') = f(x)\}
\end{aligned}$$

and for any $s \in S$ we have $f(s) = f(s)$ so $S \subseteq f^* \circ \exists_f(S)$ shows the unit of adjunction exists. For $T \subseteq Y$ we unpack the definition

$$\begin{aligned}
\exists_f \circ f^*(T) &= \{y \in Y \mid \exists x \in f^*(T).f(x) = y\} \\
&= \{y \in Y \mid \exists y' \in T.f(x) = y' \text{ and } f(x) = y\} \\
&= \{y \in Y \mid \exists y' \in T.y' = y\} \\
&= T
\end{aligned}$$

to see that the counit is an isomorphism (equality because we're in a posetal category). For the other adjunction $f^* \dashv \forall_f$ we have

$$\begin{aligned}
\forall_f \circ f^*(T) &= \{y \in Y \mid \forall x \in X.(f(x) = y) \implies x \in f^*(T)\} \\
&= \{y \in Y \mid \forall x \in X.(f(x) = y) \implies f(x) \in T\}.
\end{aligned}$$

In particular for any $t \in T$ and $x \in X$, if $f(x) = t$ then $t \in T$ so $T \subseteq \forall_f \circ f^*(T)$ defines the unit of adjunction. On the other hand

$$\begin{aligned} f^* \circ \forall_f(S) &= \{x \in X \mid f(x) \in \forall_f(S)\} \\ &= \{x \in X \mid \forall x' \in X.(f(x') = f(x)) \implies x' \in S\} \end{aligned}$$

and in particular the condition becomes tautological for any $x \in f^* \circ \forall_f(S)$ so we have $x \in S$ showing the counit, $f^* \circ \forall_f(S) \subseteq S$, exists. $\square$