CATEGORY THEORY COURSE NOTES

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Introduction

These are course notes from the winter 2024 semester Category Theory course at Dalhousie university taught by Dorette Pronk. We follow Robin's notes [?] with occasional reference to Emily Riehl's book [?]. She gave us the option to give presentations/guest lectures to replace exams in the course.

Some people talk about 'analytic' and 'synthetic' perspectives to studying categories. To my understanding, the analytic perspective is to 'analyze' categories but sudying their objects, arrows, and the structures/properties they have. This focuses more on the compositionality of relationships between the objects in a category. The synthetic perspective is more about viewing categories as the objects in a categorical context studying them in terms of the structure preserving maps into/out of them.

We begin with the analytic, peeking inside some familiar (concrete) categories and checking out the types of formal structures we can have while having access to familiar objects with underlying sets containing elements. We shake off the elements with some interesting examples and discuss generalizations for such situations. After that we get more synthetic to talk about adjunctions and monads before finishing up with limits and colimits and some relationships between the two perspectives.

A bit of History

This little summary is from Dorette's slides from her honours talk in Feb 2020. Category theory was developed for a couple of reasons. One was to contextualize the mathematical objects people cared about and study them in terms of their morphisms (Emmy Neother started this idea) and another was the idea of 'natural' operations that we could perform on such objects. For example, functions between groups should preserve the group structure in order to be consider morphisms of groups and the isomorphisms between finite dimensional vector spaces and their duals depending on a chosen basis are instances of these ideas showing up.

Category theory was particularly important in (re)developing the foundatinos of algebraic topology. Euler characteristic was the first known result of a 'topological invariant.' Riemann studied connectivity of complex varieties (zero sets of polynomial equations); for example disconnecting a sphere can be done with one circle and disconnecting a torus requires at least two. Möbius strips were a first instance of studying orientability of surfaces. Betti numbers were inspired by Riemann's connectivity of surfaces to quantify connectivity in higher dimensions using higher dimensional spheres and boundary relations. Poincare studies solutions to differential equations on algebraic varieties and found betti numbers played an important role in these questions; he introduced torsion coefficients to capture the monodromy. Nowadays we see Betti numbers as ranks of abelian groups that show up in the study of what we now know as (singular) hoomology.

Emmy Noether was one of the first people to suggest studying the algebraic complexes associated to spaces, specifically we should study the 'holes' in a complex directly as the equivalence classes of cycles in the complex modulo their boundaries. This also led to Alexandrov's theory of 'continuous decompositions' which is essentially the perspective of (the image of) a continuous map as a bundle of fibers above points in its codomain. Homology doesn't intertwine the group structure and the topological structure, it relates continuous maps with group homomorphisms.

1. An 'Analytic' Appraoch to Categories

A category is an abstract setting where 'composition of functions' makes sense.

Definition 1. A category is a directed multigraph along with an algebra of paths.

The directed multigraph tells you what the 'objects' and 'arrows' are and the algebra of paths determins the composition of arrows.

For a category $\mathcal{C}$, the objects are denoted $\mathcal{C}_0$ and the arrows are denoted $\mathcal{C}_1$. Let $d_0 : \mathcal{C}_1 \to \mathcal{C}_0$ denote the function that picks out the domain of an arrow and $d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ the function that picks out the codomain. The path algebra says: for any $f, g \in \mathcal{C}_1$, if $d_1(f) = d_0(g)$ then there exists a unique composite $g \circ f \in \mathcal{C}_1$ whose domain and codomain agree with those of f and g respectively. Associativity of the path algebra encodes associativity of arrow composition in the category $\mathcal{C}$. The constant paths in the path algebra correspond to identity arrows in the category.

Examples. Here are bunch of examples you can verify yourself!

(a) The category of sets and functions is a category with function composition.
(b) The category of groups and group homomorphisms is a category again with function composition.
(c) Vaguely but more generally speaking the category of (algebraic thing) with (algebraic thing) homomorphisms is a category. There's a way to talk about these kinds of 'algebraic theories' more generally that we'll encounter later in the course!
(d) Any directed multigraph can be made into a category by freely adding finite paths and identities.
(e) The small categories $\nVdash$ and $\nvDash$ have as many objects and one less non-identity arrow. The first one is pictured below:

$$\begin{tikzcd} A \arrow[loop right] \end{tikzcd}$$

(f) The category of topological spaces and continuous maps between them is a category.
(g) Any set with a pre-order defines a category whose objects are the elements of the set and whose arrows are determined by the pre-order relation.
(h) If $\mathcal{C}$ is a category and $X \in \mathcal{C}_0$ is an object, the *slice category over* X, $\mathcal{C}/X$, has arrows in $\mathcal{C}$ with codomain X as its objects and commuting triangles (over X) in $\mathcal{C}$ as its arrows. Composition and identities are inherited from $\mathcal{C}$.

1.1. January 12th.

I missed this class for a tennis tournament, we wrote down some definitions and talked about the path category associated to a directed (multi)graph. We talked about **Rel** and **Par** as further examples.

1.2. January 15th.

Definition 2. The *dual* of a category $\mathcal{C}$ is denoted $\mathcal{C}^{op}$. It has the same objects as $\mathcal{C}$ and the arrows are determined by those in $\mathcal{C}$ but are written in the opposite direction:

$$\mathcal{C}^{op}(A, B) = \mathcal{C}(B, A)$$

Examples. (1) Check that $\mathcal{C}^{op}$ is in fact a category.
(2) The category **Rel** is isomorphic to its opposite, $\mathbf{Rel}^{op}$.
(3) The category $\mathbf{Mat}(\mathbf{R})$ has natural numbers as objects and arrows $m \to n$ are $n \times m$-matrices with coefficients in the ring R. Composition is matrix multiplication and identites are the identity matrices.

Now that we have some examples of categories, let's look at some properties that arrows in a category may have.

Definition 3. An arrow $f : A \to B$ is a *monomorphism* if $g \circ f = h \circ f$ implies $g = h$. In pictures, if the diagram

$$A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$$

commutes, then $g = h$.

Examples. What are the monomorphisms in **Set**? **Grp**? **Rel**?

The dual notion of a monomorphism is an epimorphism. It can be defined as a monomorphism in the opposite category or that result can be proven from the following definition:

Definition 4. An arrow $f : A \to B$ is an *epimorphism* if $f \circ g = f \circ h \circ f$ implies $g = h$. In pictures, if the diagram

$$C \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B$$

commutes, then $g = h$.

Examples. What are the epimorphisms in **Set**? **Grp**? **Rel**?

Proposition 5. *The mono/epimorphisms in $\mathcal{C}$ are precisely the epi/monomorphisms in $\mathcal{C}^{op}$ respectively.*

Definition 6. An arrow with a left (post-compositional) inverse is called a *retraction* and an arrow with a right (pre-compositional) inverse is called a *section*.

Examples. An *inverse* of an arrow is a left and a right inverse. Show that such an arrow is necessarily unique.

Definition 7. An *isomorphism* is an arrow with a left and right inverse.

Examples. Find or construct a category with at least one arrow that's monic and epic but not an isomorphism.

Lemma 8. *If $g \circ f$ is monic then f is monic and if $g \circ f$ is epic then g is epic.*

Solution. These are dual statements, it suffices to prove one. □

Lemma 9. *The following are equivalent:*

- *(a) The map $f : A \to B$ is an isomorphism.*
- *(b) f is an epic section*
- *(c) f is a monic retraction*

Solution. □

Definition 10. A *subobject* is an equivalence class of monomorphisms with a common codomain. Two mono's with a common codomain represent the same subobject if they're isomorphic in the slice category over that coodmain.

Definition 11. An *idempotent* is an arrow $e : A \to A$ such that $e^2 = e$.

Dorette thinks about these as 'really nice subobjects.' We'll hear about why next class when we talk about partially defined maps a bit more.