CATEGORY THEORY COURSE NOTES

DENI SALJA

Introduction

These are course notes from the winter 2024 semester Category Theory course at Dalhousie university taught by Dorette Pronk. We follow Robin's notes [?] with occasional reference to Emily Riehl's book [?]. She gave us options to do seminar-style presentations or guest lectures to replace exams.

Categories are a lot of things to a lot of people. They are places where mathematical objects are studied by structure preserving morphisms between them that satisfy some composition laws. Like the mathematical objects they so often contextualize, categories can be studied in terms of structure preserving morphisms between them in a 'higher' categorical context! We'll get there eventually.

We begin with some familiar examples of (concrete) categories and check out some of the formal structures and properties they have, but first. . .

A bit of History

This little summary is from Dorette's slides from her honours talk in Feb 2020. Category theory was developed for a couple of reasons. One was to contextualize the mathematical objects people cared about and study them in terms of their morphisms (Emmy Neother started this idea) and another was the idea of 'natural' operations that we could perform on such objects. For example, functions between groups should preserve the group structure in order to be consider morphisms of groups and the isomorphisms between finite dimensional vector spaces and their duals depending on a chosen basis are instances of these ideas showing up.

Category theory was particularly important in (re)developing the foundatinos of algebraic topology. Euler characteristic was the first known result of a 'topological invariant.' Riemann studied connectivity of complex varieties (zero sets of polynomial equations); for example disconnecting a sphere can be done with one circle and disconnecting a torus requires at least two. Möbius strips were a first instance of studying orientability of surfaces. Betti numbers were inspired by Riemann's connectivity of surfaces to quantify connectivity in higher dimensions using higher dimensional spheres and boundary relations. Poincare studies solutions to differential equations on algebraic varieties and found betti numbers played an important role in these questions; he introduced torsion coefficients to capture the monodromy. Nowadays we see Betti numbers as ranks of abelian groups that show up in the study of what we now know as (singular) hoomology.

Emmy Noether was the first to suggest studying the algebraic complexes associated to spaces, specifically we should study the 'holes' in a complex directly as the equivalence classes of cycles in the complex modulo their boundaries. This also led to Alexandrov's theory of 'continuous decompositions' which is essentially the perspective of (the image of) a continuous map as a bundle of fibers above points in its codomain. Homology doesn't intertwine the group structure and the topological structure, it relates continuous maps with group homomorphisms. I'll ask for a link to her slides, anyway category theory will help us contextualize relationships between mathematical objects and state and prove more precise statements about them.

1. Jan 8th

A category is an abstract setting where *composition* makes sense.

Definition 1. A category is a directed multigraph, $(\mathcal{C}_0, \mathcal{C}_1)$ along with an associative and unital algebra of paths.

The directed multigraph tells you what the 'objects' and 'arrows' are and the algebra of paths says arrows whose domain and codomain agree can be composed. The objects of $\mathcal{C}$ are denoted $\mathcal{C}_0$ and the arrows are denoted $\mathcal{C}_1$. Let $d_0 : \mathcal{C}_1 \to \mathcal{C}_0$ denote the function that picks out the domain of an arrow and $d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ the function that picks out the codomain. The path algebra says that for any $f, g \in \mathcal{C}_1$, if $d_1(f) = d_0(g)$ then there exists a unique composite $g \circ f \in \mathcal{C}_1$ whose domain and codomain agree with those of f and g respectively:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Associativity of the path algebra encodes associativity of arrow composition in the category $\mathcal{C}$.

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

The constant paths in the path algebra correspond to identity arrows, $1_A : A \to A$, and satisfy an identity law,

$$1_B \circ f = f = f \circ 1_A$$

for any $f : A \to B$.

Examples. Here are a bunch of examples you can verify yourself. To show something is a category you need to check there are well-defined objects, arrows, composition, identities, and that identity and associativity laws for composition hold.

(a) The category of sets and functions is a category with function composition.
(b) The category of groups and group homomorphisms is a category again with function composition.
(c) Vaguely but more generally speaking the category of (algebraic thing) with (algebraic thing) homomorphisms is a category. There's a way to talk about these kinds of 'algebraic theories' more generally that we'll encounter later in the course!
(d) Any directed multigraph can be made into a category by freely adding finite paths and identities.
(e) The small categories $\nVdash$ and $\nvDash$ have as many objects and one less non-identity arrow. The first one is pictured below:

$$\begin{tikzcd} A \arrow[loop right] \end{tikzcd}$$

(f) The category of topological spaces and continuous maps between them is a category.
(g) Any set with a pre-order defines a category whose objects are the elements of the set and whose arrows are determined by the pre-order relation.
(h) If $\mathcal{C}$ is a category and $X \in \mathcal{C}_0$ is an object, the *slice category over* X, $\mathcal{C}/X$, has arrows in $\mathcal{C}$ with codomain X as its objects and commuting triangles (over X) in $\mathcal{C}$ as its arrows. Composition and identities are inherited from $\mathcal{C}$.

1.1. **January 12th.** I missed this class for a tennis tournament, we wrote down some definitions and talked about the path category associated to a directed (multi)graph. We talked about **Rel** and **Par** as further examples.

1.2. **January 15th.**

Definition 2. The *dual* of a category $\mathcal{C}$ is denoted $\mathcal{C}^{op}$. It has the same objects as $\mathcal{C}$ and the arrows are determined by those in $\mathcal{C}$ but are written in the opposite direction:

$$\mathcal{C}^{op}(A, B) = \mathcal{C}(B, A)$$

Examples.
(1) Check that $\mathcal{C}^{op}$ is in fact a category.
(2) The category **Rel** is isomorphic to its opposite, $\mathbf{Rel}^{op}$.
(3) The category $\mathbf{Mat}(\mathbf{R})$ has natural numbers as objects and arrows $m \to n$ are $n \times m$-matrices with coefficients in the ring R. Composition is matrix multiplication and identites are the identity matrices.

Now that we have some examples of categories, let's look at some properties that arrows in a category may have.

Definition 3. An arrow $f : A \to B$ is a *monomorphism* if $g \circ f = h \circ f$ implies $g = h$. In pictures, if the diagram

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$$

commutes, then $g = h$.

Examples. What are the monomorphisms in **Set**? **Grp**? **Rel**?

The dual notion of a monomorphism is an epimorphism. It can be defined as a monomorphism in the opposite category or that result can be proven from the following definition:

Definition 4. An arrow $f : A \to B$ is an *epimorphism* if $f \circ g = f \circ h \circ f$ implies $g = h$. In pictures, if the diagram

$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

commutes, then $g = h$.

Examples. What are the epimorphisms in **Set**? **Grp**? **Rel**?

Proposition 5. *The mono/epimorphisms in $\mathcal{C}$ are precisely the epi/monomorphisms in $\mathcal{C}^{op}$ respectively.*

Definition 6. An arrow with a left (post-compositional) inverse is called a *retraction* and an arrow with a right (pre-compositional) inverse is called a *section*.

Examples. An *inverse* of an arrow is a left and a right inverse. Show that such an arrow is necessarily unique.

Definition 7. An *isomorphism* is an arrow with a left and right inverse.

Examples. Find or construct a category with at least one arrow that's monic and epic but not an isomorphism.

Lemma 8. *If $g \circ f$ is monic then f is monic and if $g \circ f$ is epic then g is epic.*

Solution. These are dual statements, it suffices to prove one. □

Lemma 9. *The following are equivalent:*

(a) *The map $f : A \to B$ is an isomorphism.*
(b) *f is an epic section*
(c) *f is a monic retraction*

Solution. □

Definition 10. A *subobject* is an equivalence class of monomorphisms with a common codomain. Two mono's with a common codomain represent the same subobject if they're isomorphic in the slice category over that coodmain.

Definition 11. An *idempotent* is an arrow $e : A \to A$ such that $e^2 = e$.

Dorette thinks about these as 'really nice subobjects.' We'll hear about why next class when we talk about partially defined maps a bit more.

1.3. **January 17th.**

Examples. Let $r : A \to B$ be a retraction of $s : B \to A$, a section. Then $r \circ s = 1$ implies $(s \circ r)^2 = s \circ r$ is idempotent.

Not every subobject gives an idempotent because there may not be a projection onto that subobject; equivalently: not every projection has a global section. Idempotents that can be given this way are called *split idempotents*. In **Set** for example, every idempotent $e : X \to X$ splits, via its image, as a retraction followed by a section:

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, two heads, "r_e"'] & & X \\ & \mathrm{im}e \arrow[ur, hook, "s_e"'] & \end{tikzcd}$$

where r_e is just e with a restricted codomain and s_e is subset inclusion.

Proposition 12. *Idempotent splittings are unique in the sense that for any two splittings $e = s \circ r$ and $e = r' \circ s'$ of an idempotent e there exists a unique isomorphism $h : \mathrm{dom}s \to \mathrm{dom}s'$ such that $h \circ r = r'$ and $s' \circ h = s$.*

Solution. Define $h = r' \circ s$ and check that everything works. The inverse is $h^{-1} = r \circ s'$. □

Corollary 13. *Split idempotents correspond to subobjects.*

Definition 14. The universal way to split idempotents in a category $\mathcal{C}$ is called the 'Karoubi envelope.' This is a new category whose objects are the idempotents in $\mathcal{C}$ and whose arrows $f : e_1 \to e_2$ are given by arrows $f : \mathrm{dom}e_1 \to \mathrm{dom}e_2$ in $\mathcal{C}$ such that $e_2 \circ f \circ e_2 = f$ so the diagram

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[d, "f"'] \arrow[dr, "f"] & A \arrow[d, "f"] \\ B \arrow[r, "e_2"'] & B \end{tikzcd}$$

Composition is given by stacking commuting squares

$$
\begin{tikzcd}
A \arrow[r, "e_1"] \arrow[d, "f"'] \arrow[dr, "f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_2"'] \arrow[d, "g"'] \arrow[dr, "g"] & B \arrow[d, "g"] \\
C \arrow[r, "e_3"'] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "e_1"] \arrow[d, "g\circ f"'] \arrow[dr, "g\circ f"] & A \arrow[d, "g\circ f"] \\
C \arrow[r, "e_3"'] & C
\end{tikzcd}
$$

and the identity arrow on e is e.

$$
\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e"'] \arrow[dr, "e"] & A \arrow[d, "e"] \\
A \arrow[r, "e"'] & A
\end{tikzcd}
$$

Examples. For the category of open subsets of $\mathbb{R}^n$ and the smooth functions between them, the Karoubi envelope is the category of (smooth) manifolds. This relies on the tubular neighborhood theorem for manifolds (that you can embed them in $\mathbb{R}^n$ and thicken them with an interval to get an open subset of $\mathbb{R}^n$).
In a similar fashion, vector bundles form the karoubi envolope of the category of trivial bundles.

The original category $\mathcal{C}$ embeds into the Karoubi envelope:

$$\mathcal{C} \to Kar(\mathcal{C})$$

by sending each object to its identity map and arrows to the obvious squares. Notice this assignment preserves the 'category structure,' namely identities and composition.

Split epimorphisms are exmaples of arrows that factor in a universal way as a composite of arrows from two special classes of arrows. A factorization system in a category describes when every arrow factors as a composite of two arrows from two special classes of arrows. An example of a factorization system in **Set** is the epi-mono factorization which comes from noticing that every function $f : A \to B$ factors as an epimorphism (given by restricting its codomain to the image) followed by a monomorphism (given by including the image as a subset).

Definition 15. Let $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}_1$ be two classes of arrows that are closed under composition and both containing the identity arrows. Then $\mathcal{C}$ has an $\mathcal{E}$-$\mathcal{M}$ factorization system if every arrow in $\mathcal{C}$ factors as a composite of an arrow in $\mathcal{E}$ followed by an arorw in $\mathcal{M}$ and this factorization is unique up to a unique map between the intermediate objects. We write arrows in $\mathcal{E}$ with two heads and arrows in $\mathcal{M}$ with tails, the universal property looks like:

$$
\begin{tikzcd}
& X \arrow[dr] \arrow[dd, bend right, "\exists!", "\cong"'] & \\
A \arrow[ur] \arrow[rr, "f"] \arrow[dr] & & B \\
& Y \arrow[ur] &
\end{tikzcd}
$$

1.4. **January 9th.** Today we start with the notion of orthogonality (of arrows in a category).

Definition 16. Let $f : A \to B$ and $g : C \to D$ be two arrows in a category $\mathcal{C}$. We say 'f is *left orthogonal* to g' or 'g is *right orthogonal* to f,' and we write $f \perp g$ if every commutative diagram

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "h_2"'] & D \end{tikzcd}$$

has a unique filler

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "h_2"'] \arrow[ur, dashed, "k"] & D \end{tikzcd}$$

making each of the triangles commute.

Remark. If $f \perp g$ in $\mathcal{C}$ then $g^{op} \perp f^{op}$ in $\mathcal{C}^{op}$.

Examples.

(a) Every epimorphism is left orthogonal to every monomorphism in **Set**.

(b) For $f : \{0, 1\} \to \{*\}$ the right orthogonal arrows are functions with singleton fibers. This is required to ensure uniqueness of the filler since $h_1 : \{0, 1\} \to C$ is able to separate points in $\mathcal{C}$ and having singleton fibers is another way to say the function is injective.

We say f is *weakly* orthogonal to g if the filler, k, in Definition 16 exists but is not unique.

Examples.

(a) For $\emptyset \to \{*\}$ (in **Set**), the weakly right orthogonal arrows are the surjective functions. The (strictly) right orthogonal arrows are the bijections. To see why, ask yourself 'what do the fibers of such a map have to be?'

(b) In **Top**, the weakly right orthogonal maps to the inclusion $\{0\} \hookrightarrow [0, 1]$ are those that have the path lifting property with respect to that inclusion. This class includes the covering maps but it also contains more!

(c)

1.5. **January 22nd.** We have seen that in **Set** all epi's are left orthogonal to all mono's. In **Set**, mono's are also weakly left orthogonal to epi's.

Definition 17. Let $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}_1$ be two classes of arrows in $\mathcal{C}$. We say $\mathcal{E}$ is left orthogonal to $\mathcal{M}$ if every arrow in $\mathcal{E}$ is left orthogonal to every arrow in $\mathcal{M}$.

What if we want to add stuff into $\mathcal{E}$ and $\mathcal{M}$?

Definition 18. The class of arrows that are right orthogonal to all arrows in $\mathcal{E}$ is denoted $\mathcal{E}_\perp$. The class of arrows that are left orthogonal to all arrows in $\mathcal{M}$ are denoted ${}_\perp\mathcal{M}$.

Proposition 19. *For any class of arrows $\mathcal{E}$ we have that $\mathcal{E}_\perp$ contains all iso's, ${}_\perp(\mathcal{E}_\perp) \supseteq \mathcal{E}$.*

Solution. □

Proposition 20. *The class of arrows left orthogonal to the class of isomorphisms is all of the arrows: ${}_\perp Iso = \mathcal{C}_1$.*

Proposition 21. *Claim—* $({}_\perp(\mathcal{E}_\perp))_\perp = \mathcal{E}_\perp$

Solution. Suppose $A \subseteq A'$, then $A_\perp \supseteq A'_\perp$ and ${}_\perp A \supseteq \perp A'$ so that

$$A \subseteq_\perp (A_\perp).$$

Applying $\perp$ once more on the right we get

$$A_\perp \supseteq ({}_\perp(A_\perp))_\perp$$

and also we have $(\perp (A_\perp))_\perp \supseteq A_\perp$. This shows both subset inclusions and the equality follows. □

Proposition 22. *For any class of arrows A there are two maximal classes of arrows A_1 and A_2 such that $(A_1) \perp = A_2$ and ${}_\perp(A_2) = A_1$ and $A_1 \supseteq A$.*

Solution. Let $A_1 =_\perp (A_\perp)$ and let $A_2 = A_\perp$. □

Lemma 23. *Let $A \subseteq \mathcal{C}_1$ be a class of arrows.*

(1) Isomorphisms are right (and left) orthogonal to A.

(2) If $x \in A \cap A_\perp$ then x is an iso.

(3) $A_\perp$ and ${}_\perp A$ are closed under composition.

(4) If $h \circ g \in A_\perp$ and h is monic then $g \in A_\perp$. Is the dual statement: if $g \circ h \in {}_\perp A$ and g is epic then $h \in {}_\perp A$?

(5) If $h \circ g \in A_\perp$ and $h \in A_\perp$ then $g \in A_\perp$. Write down the dual statement.

Solution. Exercise in diagram chasing and unraveling definitions. □

Definition 24. A factorization system in a category $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of classes of arrows $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}_1$ such that $\mathcal{E}$ and $\mathcal{M}$ contain all iso's, $\mathcal{E}$ and $\mathcal{M}$ are closed under composition, and every map f factors as $f = g \circ h$ where $h \in \mathcal{E}$ and $g \in \mathcal{M}$ uniquely up to a unique isomorphism (between the mediating object(s) in the middle).

1.6. **January 24th.**

Proposition 25. *If $(\mathcal{E}, \mathcal{M})$ is a factorization system in $\mathcal{C}$ then*

(a) $\mathcal{E}$ is left orthogonal to cM

(b) Factorizations extend to commutative squares.

Solution. Consider the lifting problem:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, two heads, "e"'] & A' \arrow[d, tail, "m"] \\
B \arrow[r, "b"'] & B'
\end{tikzcd}$$

where the two headed arrow is in $\mathcal{E}$ and the arrow with a tail is in $\mathcal{M}$. We factor both the top and bottom arrows and obtain a filler by uniqueness of factorizations:

$$\begin{tikzcd}
A \arrow[r, two heads, "e_a"] \arrow[d, two heads, "e"'] & E_A \arrow[r, tail, "m_a"] & A' \arrow[d, tail, "m"] \\
B \arrow[r, two heads, "e_b"'] & E_B \arrow[u, dashed, "k"] \arrow[r, tail, "m_b"] & B'
\end{tikzcd}$$

Notice $\ell = m_a \circ k \circ e_b$ solve the lifting problem. To see this solution is unique suppose there's another solution ℓ' and factor it

$$\begin{tikzcd}
\bullet \arrow[rr, two heads, "e_a"] \arrow[dd, two heads, "e"'] & & \bullet \arrow[r, tail, "m_a"] & \bullet \arrow[dd, tail, "m"] \\
& \bullet \arrow[ur, dashed, "\theta"] \arrow[urr, tail, "m_{\ell'}"] & & \\
\bullet \arrow[rr, two heads, "e_b"'] \arrow[ur, two heads, "e_{\ell'}"] \arrow[uurrr, "\ell'"] \arrow[uurrr, bend right, "\ell"'] & & \bullet \arrow[r, tail, "m_b"'] \arrow[ul, dashed, "\theta'"] & \bullet
\end{tikzcd}$$

By uniqueness we have $\theta \circ \theta' = k$ and consequently $\ell = \ell'$:

$$\begin{aligned}
\ell &= m_a \circ k \circ e_b \\
&= m_a \circ \theta \circ \theta' \circ e_b \\
&= m_{\ell'} \circ \theta \circ e_b \\
&- m_{\ell'} \circ e_{\ell'} \\
&= \ell'.
\end{aligned}$$

It follows that the lift ℓ is unique.

Factor the vertical arrows in a commuting square, squish it into a (rectangular) lifting problem that can be uniquely solved.

□

Definition 26. Given a class $\mathcal{E}$ of maps we say that an arrow f in in $\mathcal{C}$ has a maximal $\mathcal{E}$-factorization if it can be factors as $f = f' \circ e$ with $e \in \mathcal{E}$ such that for all commutative squares there exists a unique filler as in the diagram below:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dd, "g'"'] & \bullet \arrow[d, "e"] \\
& \bullet \arrow[d, "f"] \\
\bullet \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd} .$$

We say $\mathcal{C}$ has a maximal $\mathcal{E}$-factorization if every arrow in $\mathcal{C}$ has a maximal $\mathcal{E}$-factorization.

The dual concept is a *maximal $\mathcal{M}$-cofactorization system.*

Proposition 27. *The following are equivalent:*

(1) *There are two classes $\mathcal{E}$ and $\mathcal{M}$ such that*
    $\mathcal{E}$ is left orthogonal to $\mathcal{M}$
    $\mathcal{E}$ is closed under (post-)composition with iso's.
    $\mathcal{M}$ is closed under (pre-)composition with iso's.
    Each map f can be factored as $f = m \circ e$ for $e \in \mathcal{E}$ and $m \in \mathcal{M}$.
(2) *$(\mathcal{E}, \mathcal{M})$ is a pair of maximal orthogonal classes of maps such that every map in $\mathcal{C}$ factors as $m \circ e$ for some $m \in cM$ and $e \in cE$.*
(3) *$\mathcal{E}$ is a class of maps containing all iso's and closed under composition such that each map $f \in \mathcal{C}$ can be factored as a maximal $\mathcal{E}$-factorization*

(4) $\mathcal{M}$ is a class of maps closed under composotion and containing all iso's such that every map factors as a maximal $\mathcal{M}$-factorization. (dual to the previous one)

(5) $(\mathcal{E}, \mathcal{M})$ is a factorization system.

Solution. Next class. □

1.7. **January 26th.** Today we prove the previous lemma. Dorette wrote the proof down beforehand so I'm not going to live-TeX all of it, I'll just give sketches/hints.Before that, a correction from a couple classes back:

Consider the lifting problem:

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & X \arrow[d] \\ \{*\} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

The square can only commute if the top horizontal arrow agrees on the fiber of g above the point picked out by the bottom arrow. A lift such as the dashed line above then exists whenever g only has empty or singleton fibers above each point. In this case weak orthogonality implies orthogonality.

(Sketch) of last Proposition. The third and fourth statements are duals so we only deal with the third. We show a chain of implications in the order they're stated skipped the fourth by duality.

For (1) implies (2) we only need to show maximality of either $\mathcal{E}$ or $\mathcal{M}$. We have one of the containments in the assumption so show the other one. For this you pick $g \in \mathcal{E}$ or $\mathcal{M}$, factor it, and find a lift to show one of the maps is a section/retraction (depending on which one you started proving). Use that unique lift to find a lift for another square with both sides of it given by the same factorization of g. This gives a way to write g factoring through an iso and shows g satisfies the required maximality property

For (2) $\implies$ (3), 'closed under composition' and 'containing all iso's' was proved in a previous class so we only need to show that the factorization is maximal. Play two factorizations against each other to set up a lifting problem with a unique solution. We ran into some issues with this proof so we're leaving it as an exercise.

For (3) $\implies$ (5), suppose you have $\mathcal{E}$ and define $\mathcal{M}$ to be all the maps whose maximal factorization, $f = f' \circ e$ has e being an isomorphism. Check all the axioms for a factorization system are defined using the maximality property of the factorization $\mathcal{E}$-factorizations.

For (5) $\implies$ (1), suppose you have a factorization system. We only need to show $\mathcal{E}$ is left orthogonal to $\mathcal{M}$. Set up the lifting problem, factor the horizontal arrows and use closure under composition of $\mathcal{E}$ and $\mathcal{M}$ along with uniqueness of factorizations to solve the lifing problem uniquely.

□

1.8. **Jan 29.** Snow day and Dorette was sick so no class.

1.9. **Jan 31st.** One last thing about factorization systems:

Lemma 28. *If $(\mathcal{E}, \mathcal{M})$ is a factorization system for $\mathcal{C}$ then*

(1) If $g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ then $f \in \mathcal{M}$
(2) If $g \circ f \in \mathcal{M}$ and $f \in \mathcal{E}$ then $g \in \mathcal{E}$
(3) $\mathcal{M}$ contains all sections iff all $\mathcal{E}$ maps that are sections are iso's.
(4) $\mathcal{E}$ contains all retractions iff all $\mathcal{M}$ maps that are retractions are iso's.

Solution. □

Examples. Isomorphisms with all the other arrows in any category (in either order) forms a factorization system. Epimorphisms and monomorphisms in **Set**, **Grp**, **Top**, form factorization systems and in the category of rings localizations and conservative morphisms form another important one in algebraic geometry.

Definition 29. A functor $F : \mathcal{C} \to \mathcal{D}$ assigns objects to objects and arrows to arrows

$$F_0 : \mathcal{C}_0 \to \mathcal{D}_0, \quad F_1 : \mathcal{C}_1 \to \mathcal{D}_1$$

such that domains, codomains, identities, and composition are all preserved.

Examples. We can pick out objects, arrows, isomorphisms, in a category by looking at functors into out from $\nVdash$,$\nvDash$, and the 'walking isomorphism' category (with 2 objects and 2 non-identity arrows that mutually inverse).

1.10. **Feb 7.** We're wrapping up a proof from the factorization system business. Namely that given a maximal $\mathcal{E}$-factorization for every arrow we can define $\mathcal{M}$ to be the arrows whose maximal $\mathcal{E}$-factorizations have an isomorphism as the $\mathcal{E}$-factor in the factorizaiton.

To see we can $\mathcal{E}$-$\mathcal{M}$ factorizations notice we can factor maximal $\mathcal{E}$-factorizations, $f = f' \circ e$, twice by factoring the non-$\mathcal{E}$ factor, $f' = f'' \circ e'$. Then show e' is an isomorphism by leveraging the modified lifting property in the square:

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "e"'] & \bullet \arrow[d, "e"] \\
\bullet \arrow[r, dashed] \arrow[d, "e'"'] & \bullet \arrow[d, "f'"] \\
\bullet \arrow[r, "f''"'] \arrow[ur, dashed, "k"] & \bullet
\end{tikzcd}$$

Now show $k \circ e' = 1$ and $e' \circ k = 1$ by chasing diagrams/equations. That concludes the proof.

Example. Let $\varphi : A \to B$ be a ring homomorphism between unital commutative rings. We say φ *inverts a inA* if $\varphi(a)$ is invertible in B.

We say φ is *conservative* if it doesn't create new inverses, ie. if φ inversts a then a is invertible in A. If $S \subseteq A$ has no zero-divisor (and multiplicatively closed) then there exists a ring $S^{-1}A$ and ring homomorphism

$$\ell : A \to S^{-1}A$$

called the *localization of* S satisfying the universal property that for any $\varphi : A \to B$ that inverts all the elements of S there exists a unique map $S^{-1}A \to B$ factoring φ through ℓ:

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\ell"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

From here we can see that every ring homomorphism factors as a localization followed by a conservative map: given $\varphi : A \to B$ take S to be the set of all $a \in A$ such that φ inverts A; this set has no zero-divisors and the unique map induced by the localization in the diagram above doesn't create new inverses because everyone who could've been inverted already was. As a result we get a factorization system of localizations and coservative ring homomorphisms.

Remark 30. The previous factorization system example works for rings. A similar localization construction works for categories but we can't do quite the same thing to make a factorization system; we need to take a colimit of all the factorizations.

Definition 31. A **subcategory** $\mathcal{A}$ of a category $\mathcal{B}$ consists of a collection of objects and arrows in $\mathcal{B}$ that contains all identities and is closed under composition.

Note that if $f : A \to A'$ is monic in $\mathcal{A}$ then it doesn't need to be monic in $\mathcal{B}$ necessarily, as $\mathcal{B}$ could have more arrows than $\mathcal{A}$ that mess things up. However, if $g : B \to B'$ is monic in $\mathcal{B}$ and contained in $\mathcal{A}$ then it's still monic in $\mathcal{A}$ because it satisfied a stronger condition in $\mathcal{B}$.

In general, when we look at functors $F : \mathcal{A} to \mathcal{B}$ we often ask whether they *preserve* or *reflect* certain properties.

Definition 32. A functor $F : \mathcal{A} \to \mathcal{B}$ **preserves** property P if $F(f)$ has property P whenever f has property P.

It **reflects** property P if f has property P whenever $F(f)$ has property P.

Examples. All functor preserve isomorphisms. Localizations wouldn't be very interesting if all functors reflected isomorphisms. An explicit counter example is any functor out of the walking arrow category that picks out an isomorphism, pick your favourite one.

Monomorphisms don't necessarily get preserved or reflected by arbitrary functors. Counterexamples will be in the next assignment.

Let's look at some properties of functors. Monic, epic, and iso are all standard properties of arrows in a category. Similar properties exist for functors but there are some subtle variations.

Definition 33. A functor $F : \mathcal{A} \to \mathcal{B}$ is **full** if all of the induced functions on hom-sets

$$\mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$$

are surjective.

Definition 34. A functor $F : \mathcal{A} \to \mathcal{B}$ is **faithful** if all of the induced functions on hom-sets

$$\mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$$

are injective.

Definition 35. A functor $F : \mathcal{A} \to \mathcal{B}$ is **essentially surjective** if for all $B \in \mathcal{B}$ there exists an isomorphism $FA \cong B$ for some $A \in \mathcal{A}$.

1.11. Feb 9.

Definition 36. A functor F is **replete** if its image is closed under isomorphism. That is, whenever $B \cong FA$ there exists a A' such that $B = FA'$.

Remark 37. A **contravariant functor** $\mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{op} \to \mathcal{D}$; it flips the direction of arrows.

Examples.

(1) Functors $\mathbb{B}G \to \mathbb{B}H$ correspond to group homomorphisms $G \to H$.

(2) The assignment $\mathbb{B}(-) : \mathbf{Grp} \to \mathbf{Cat}$ is a functor sending each group G to its *delooping* $\mathbb{B}G$. As mentioned, group homomorphisms get mapped to functors between the groupoids.

(3) If P and Q are preorders regarded as categories, the functors between them are precisely the order-preserving functions between them.

(4) Representing a preorder as a category is a functor from the cateogory of pre-ordered sets with order preserving functions to **Cat**.

(5) For any category $\mathcal{C}$ there's a unique functor
$$! : \mathcal{C} \to \mathbb{1}$$
For this reason we call $\mathbb{1}$ the **terminal category**.

(6) Similarly there's a unique functor to any category $\mathcal{C}$ from the empty category $\emptyset$
$$\emptyset \to \mathcal{C}$$
which gives us reason to call $\emptyset$ the **initial category**.

(7) Consider the power-set functor
$$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$$
that sends functions $f : A \to B$ to the functions $\mathcal{P}(f)$ defined on $U \subseteq A$ to its image in B
$$\mathcal{P}(f)(U) = f(U) \subseteq B.$$
It's a routine exercise to check this preserves identities and composition.

(8) Consider the other power-set functor
$$\mathcal{P}^* : \mathbf{Set}^{op} \to \mathbf{Set}$$
defined on functions $f : A \to B$ by the pre-image: $\mathcal{P}^*(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ defined on $V \subseteq B$ by
$$\mathcal{P}^*(f)(V) = f^{-1}(V).$$

(9) Consider yet another power-set functor
$$\mathcal{P} : \mathbf{Rel} \to \mathbf{Set}$$
defined by sending sets to their power sets and relations between them to their images: $R \subseteq A \times B$
$$\mathcal{P}(R)(U) = \{b \in B \mid \exists a \in A.aRb\}$$

(10) Remember the arrow category $\mathcal{C}^{\mathbf{2}}$. The reason we use this notation is because arrows in $\mathcal{C}$ correspond precsiely to functors

$$\mathbf{2} \to \mathcal{C}$$

where $\mathbf{2}$ is the walking arrow category.

(11) There's a functor

$$\mathcal{C} \to \mathcal{C}^{\mathbf{2}}$$

which sends objects to their identity arrows and arrows between them to the obvious commuting squares and there are two different functors

$$\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$$

given by projecting onto the domain and codomain components respectively.

There are several factorization systems on **Cat** and for that reason there are several ways of talking about 'the image' of a functor. For instance, we'll talk about the 'replete image' of a functor in these terms later.

Proposition 38. *Let $\mathcal{E} \subseteq \mathbf{Cat}_1$ denote the functors that are full and bijective on objects. The right class $\mathcal{M} \subseteq \mathbf{Cat}_1$ consists of the faithful functors. This is a factorization system on* **Cat**.

Solution. Solving the lifting problem is left as an exercise. Dorette shows the factorization. For $F : \mathcal{C} \to \mathcal{D}$ we factor F through the localization of $\mathcal{C}$ by the congruence relation

$$f \sim g \iff F(f) = F(g).$$

Another way to define this is by taking hom-sets to be the images of hom-sets in $\mathcal{C}$ under F. The identification above happens implicitly then and we need to check that the composition is well-defined on representatives (in $\mathcal{C}$) of maps in the image of F. Functoriality ensures the composition (and identities) are in fact well-defined. □

1.12. **February 12th.** Finishing up the last proposition and then looking at a parallel example of localization in **Set** factoring maps in **Set** through their images. This is another way to think of the isomorphism theorems

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & A/\sim_f \arrow[ur, tail] & \end{tikzcd}$$

This idea generalizes to categories; namely the localization of a category. The equivalence relations defined on the hom-sets need to 'play nicely' with the composition operation. In particular we need the equivalence relation to be a **congruence** on the hom-sets: if $f \sim g$ then

$$f \circ \ell \sim g \circ \ell, \quad k \circ f \sim k \circ g$$

for any composable k and ℓ. Now the definition of a localization makes sense:

Definition 39. The **localization** of $\mathcal{C}$ by a *congruence* on hom-sets is the category whose objects are those of $\mathcal{C}$ and whose arrows are equivalcence classes in the hom-set quotients.

Here are some other examples of facrization systems on **Cat**. Some of these are from Robin's notes, some are from Andre Joyal's 'catlab' blog. Feel free to google that, there's lots of cool stuff there.

Examples. (1) Let $\mathcal{E}$ denote the essentially surjective functors and let $\mathcal{M}$ be the replete, fully faithful, and injective on objects.

(2) Let $\mathcal{E}$ be the functors that are surjective on objects and let $\mathcal{M}$ be the functors that are fully faithful and monic on objects. For $F : \mathcal{A} \to \mathcal{B}$ a functor, the intermediate category for this factorization has objects given by

$$F_0(\mathcal{A}_0) \cup \{B \in \mathcal{B}_0 | \ \exists A \in \mathcal{A}_0 . B \cong F(A)\}$$

We needed the entire essential image of F in $\mathcal{B}$ for the objects, then take the of subcategory $\mathcal{C}$ that makes the inclusion fully faithful.This is called the **replete image** of F.

Now let's talk about functors in more than one variable.

$$F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$$

where $\mathcal{A} \times \mathcal{B}$ has objects $\mathcal{A}_0 \times \mathcal{B}_0$ and arrows $\mathcal{A}_1 \times \mathcal{B}_1$ with composition and identities defined coordinate-wise.

Notice that for each $A \in \mathcal{A}_0$ we get a functor

$$F(A, -) : \mathcal{B} \to \mathcal{C}.$$

Similarly for each $B \in \mathcal{B}_0$ we get a functor

$$F(-, B) : \mathcal{A} \to \mathcal{C}.$$

Functoriality in each coordinate is not enough to give functoriality on the product, we also need that they play nicely together. More precisely we need that $F_A(B) = F_B(A)$ agree on objects and that for $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$ we require

$$F_{A'}(g) \circ F_{B'}(f) = F_{B'}(f) \circ F_A(g)$$

to ensure

$$F(1_{A'}, g) \circ F(f, 1_B) = F(f, g) = F(f, 1_{B'}) \circ F(1_A, g)$$

is well-defined.

Definition 40. Natural transformations are structure-preserving maps between functors $F, G : \mathcal{A} \to \mathcal{B}$. A natural transformation $\alpha : F \implies G$ is a family of arrows

$$(\alpha_A : F(A) \to G(A))_{A \in \mathcal{A}_0} \subseteq \mathcal{B}_1$$

such that the following *naturality square* commutes in $\mathcal{B}$: for all $f : A \to A'$ in $\mathcal{A}$,

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(A') \arrow[r, "\alpha_{A'}"'] & G(A') \end{tikzcd}$$

Proposition 41. *Natural transformations $\alpha : F \implies G$ can also be defined indexed by the families of arrows*

$$(\alpha_f : F(A) \to G(A'))_{f \in \mathcal{A}_1}$$

such that a similar condition recovering naturality commutes.

1.13. **Feb 16.** A natural transformation $\alpha : F \implies G$ between functors $F, G : \mathcal{A} \to \mathcal{B}$ is a family of maps $\alpha_X : F(X) \to G(X)$ satisfying the following naturality condition: for all $f : X \to Y$ in $\mathcal{A}$

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd} .$$

We call the maps α_X the **components** of the natural transformation α. **Vertical composition** of natural transformations

$$F \xRightarrow{\alpha} G \xRightarrow{\beta} L$$

is given by composing their components in $\mathcal{B}$. This defines a new natural transformation because pasting together naturality squares in $\mathcal{B}$ gives new naturality squares:

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[rr, bend left, "(\gamma\circ\alpha)_X"] & GX \arrow[r, "\gamma_X"] \arrow[d, "Gf"] & HX \arrow[d, "Lf"] \\ FY \arrow[r, "\alpha_Y"'] \arrow[rr, bend right, "(\gamma\circ\alpha)_Y"'] & GY \arrow[r, "\gamma_Y"] & HY \end{tikzcd} .$$

The identity transformation on a functor F consists of identity components $1_F = (1_{F(X)})_{X \in \mathcal{A}_0}$. This makes the collection of functors from $\mathcal{A}$ to $\mathcal{B}$, $\mathbf{Cat}(\mathcal{A}, \mathcal{B})$, into a category. We know how to compose functors in $\mathbf{Cat}$

$$\mathbf{Cat}(\mathcal{B}, \mathcal{C}) \times \mathbf{Cat}(\mathcal{A}, \mathcal{B}) \to \mathbf{Cat}(\mathcal{A}, \mathcal{B}).$$

and if these homs are categories we could expect the composition of functors above to extend to a functor on the natural transformations as well. This **horizontal** composition can be defined in two ways. To see where the two possible definitions come from suppose $\alpha : F \implies G$ and $\beta : H \implies K$ are natural transformations where $F, G \in \mathbf{Cat}(\mathcal{A}, \mathcal{B})$ and $H, K \in \mathbf{Cat}(\mathcal{B}, \mathcal{C})$. Functors preserve commuting diagrams so applying H and K to the naturality squares of α in $\mathcal{B}$ gives new commuting squares

$$\begin{tikzcd} HFX \arrow[r, "H\alpha_X"] \arrow[d, "HFf"'] & HGX \arrow[d, "HGf"] \\ HFY \arrow[r, "H\alpha_Y"'] & HGY \end{tikzcd} \quad ; \quad \begin{tikzcd} KFX \arrow[r, "K\alpha_X"] \arrow[d, "KFf"'] & KGX \arrow[d, "KGf"] \\ KFY \arrow[r, "K\alpha_Y"'] & KGY \end{tikzcd}$$

which describe new natural transformations $H \circ \alpha : H \circ F \implies H \circ G$ and $k \circ \alpha : K \circ F \implies K \circ G$. This is called 'whiskering α by H and K' and the '$\circ$' notation is used for whiskering on this side because we're applying H to the data of α. On the other side, naturality of β gives families of commuting squares

$$\begin{tikzcd} HFX \arrow[r, "\beta_{FX}"] \arrow[d, "HFf"'] & KFX \arrow[d, "KFf"] \\ HFY \arrow[r, "\beta_{FY}"'] & KFY \end{tikzcd} \quad ; \quad \begin{tikzcd} HGX \arrow[r, "\beta_{GX}"] \arrow[d, "HGf"'] & KGX \arrow[d, "KGf"] \\ HGY \arrow[r, "\beta_{GY}"'] & KGY \end{tikzcd} ,$$

in $\mathcal{C}$ which describe natural transformations $\beta_F : H \circ F \implies K \circ F$ and $\beta_G : H \circ G \implies K \circ G$. This is similarly called 'whiskering β by F and G'. Whiskering α and β by H, K and F, G respectively gives us four natural transformations which compose to give two possible definitions for horizontal composition $\beta \circ \alpha : H \circ F \implies K \circ G$:

$$\begin{tikzcd}
H \circ F \arrow[r, Rightarrow, "H\circ\alpha"] \arrow[d, Rightarrow, "\beta_F"'] & H \circ G \arrow[d, Rightarrow, "\beta_G"] \\
K \circ F \arrow[r, Rightarrow, "K\circ\alpha"'] & K \circ G
\end{tikzcd}$$

To see these definitions agree we need to check their components are equal, namely that the square

$$\begin{tikzcd}
H \circ F(X) \arrow[r, "H(\alpha_X)"] \arrow[d, "\beta_{F(X)}"'] & H \circ G(X) \arrow[d, "\beta_{G(X)}"] \\
K \circ F(X) \arrow[r, "K(\alpha_X)"'] & K \circ G(X)
\end{tikzcd}$$

commutes fpr each $X \in \mathcal{A}_0$. All these squares commute because they are precisely the naturality squares of β associated to the components of α. If natural transformations are new to you and you need it to look exactly like the definition just redraw the square with the β's horizontally instead of vertically.

1.14. **Yoneda's Lemma.** Categories are contexts for mathematical objects that allow us to study them in terms of the morphisms between them. For example, sets are determined by their elements and one way to study a set, S, is to view it as an object in the category **Set**, and then look at the functions

$$\{*\} \to S$$

corresponding to the elements of S. This is a special example where it suffices to look at one kind of arrow in the category but it doesn't always work, for example in the category of abelian groups (and group homomorphisms) **Ab** the only such group homomorphism is 0.

When every map can be described in terms of[1] maps out of the terminal object we can call these maps **elements**. If we can't do that we can always consider all the maps with a common codomain; these are called **generalized elements** (of their codomain). Think:

$$(X \to Y \in \mathcal{C}) \quad \rightsquigarrow \quad \text{"An } X\text{-shaped thing in } Y\text{"}$$

The generalized elements of an object $Y \in \mathcal{C}_0$ are encoded in the contravariant hom-functor

$$\mathcal{C}(-, Y) : \mathcal{C}^{op} \to \mathbf{Set}.$$

Some more notation/language:

- Functors $F : \mathcal{C}^{op} \to \mathbf{Set}$ are called *presheaves*
- *Local sections* of F at $X \in \mathcal{C}_0$ are the elements of the set $F(X)$.

[1]I'm being vague, we haven't talked about limits/colimits yet. The category **Set** is completely determined by coproducts of its terminal object. I think this is called being well-copowered and being well-powered is the dual concept (with initial object and products)

- The (contravariant) hom-functors are special presheaves that represent the objects of $\mathcal{C}$ in the much bigger category of presheaves on $\mathcal{C}$ and people sometimes call them (the) *representable functors* [2].

The reason we want to view objects in $\mathcal{C}$ in the presheaf category $\mathbf{Cat}(\mathcal{C}^{op}, \mathbf{Set})$ is because the presheaf category has nice properties that we'll talk about in the future, probably after (co)limits and we do this by assigning each object in $\mathcal{C}$ to its representable presheaf. It's straightforward to check this is a functor:

$$\mathcal{C} \to \mathbf{Cat}(\mathcal{C}^{op}, \mathbf{Set}) \quad ; \qquad \left(\begin{array}{c} X \\ \big\downarrow f \\ Y \end{array} \right) \mapsto \left(\begin{array}{c} \mathcal{C}(-, X) \\ \big\downarrow f \circ (-) \\ \mathcal{C}(-, Y) \end{array} \right)$$

In the lanuage mentioned above, Yoneda's lemma says the natural transformations from a representable presheaf (on X) to another presheaf F naturally correspond to the local sections of F at X.

Theorem 42. *For any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ and every object $X \in \mathcal{C}_0$, there exists an isomorphism*

$$\mathbf{Nat}(\mathcal{C}(-, X), F) \xrightarrow[\cong]{\varphi_X} F(X)$$

that's natural in X in the sense that for all $g : Y \to X$ in $\mathcal{C}$ the naturality square

$$\begin{tikzcd} \mathbf{Nat}(\mathcal{C}(-, X), F) \arrow[r, "\varphi_X"] \arrow[d, "(=)\cdot(g \circ (-))"'] & F(X) \arrow[d, "F(g)"] \\ \mathbf{Nat}(\mathcal{C}(-, Y), F) \arrow[r, "\varphi_Y"'] & F(Y) \end{tikzcd}$$

commutes in $\mathbf{Set}$ [3].

Proof. For each $X \in \mathcal{C}_0$ define φ_X on an arbitrary natural transformation $\alpha : \mathcal{C}(-, X) \implies F$ by evaluating its X-component $\alpha_X : \mathcal{C}(X, X) \to F(X)$ at the identity map $1_X \in \mathcal{C}(X, X)$:

$$\varphi_X(\alpha) := \alpha_X(1_X)$$

On the other hand, for each $x \in F(X)$ define a natural transformation $\psi_{X,x} : \mathcal{C}(-, X) \implies F$ by evaluating maps in $\mathcal{C}$ at F and then evaluating the resulting functions at $x \in F(X)$. That is, for every map $g : Y \to X$ in $\mathcal{C}_1$ define the component

$$(\psi_{X,x})_Y : \mathcal{C}(Y, X) \to F(Y) \quad ; \qquad (\psi_{X,x})_Y(g) := F(g)(x)$$

To see this is a natural transformation let $h : Y \to Z$ be a map in $\mathcal{C}$ and check that the following naturality square commutes by diagram chasing.

[2]This is a bit abusive: Representable functors are really any functors that are isomorphic to the hom-functors and that isomorphism is called a representation. One thinks of the object in $\mathcal{C}$ as representing the functor in that case. The hom-functors are canonical representables since the isomorphism in question is the identity

[3]The '$\cdot$' on the left vertical arrow denotes composition of natural transformations written in applicative order (right to left).

$$\begin{tikzcd}
\mathcal{C}(Z,X) \arrow[r, "(\psi_{X,x})_Z"] \arrow[d, "(-)\circ h"'] & F(Z) \arrow[d, "F(h)"] \\
\mathcal{C}(Y,X) \arrow[r, "(\psi_{X,x})_Y"'] & F(Y)
\end{tikzcd}$$

We claim the induced function

$$F(X) \xrightarrow{\psi_X} \mathbf{Nat}(\mathcal{C}(-,X),F) \qquad ; \psi_X(x) := \psi_{X,x}$$

is the inverse of φ_X. For each $x \in F(X)$

$$\begin{aligned}
(\varphi_X \circ \psi_X)(x) &= \varphi_X(\psi_{X,x}) \\
&= (\psi_{X,x})_X(1_X) \\
&= F(1_X)(x) \\
&= 1_{F(X)}(x)
\end{aligned}$$

shows it's a one-sided inverse. For each natural transformation $\alpha : \mathcal{C}(-,X) \implies F$ we get another natural transformation

$$\begin{aligned}
(\psi_X \circ \varphi_X)(\alpha) &= \psi_X(\alpha_X(1_X)) \\
&= \psi_{X,\alpha_X(1_X)}.
\end{aligned}$$

Evaluating each Y-compoenent at an arbitrary $g : Y \to X$

$$\begin{aligned}
(\psi_{X,\alpha_X(1_X)})_Y(g) &= F(g)(\alpha_X(1_X)) \\
&= F(g) \circ \alpha_X(1_X) \\
&= \alpha_Y \circ (1_X \circ g) \\
&= \alpha_Y \circ (g)
\end{aligned}$$

shows $(\psi_X \circ \varphi_X)(\alpha) = \alpha$ and so ψ_X is a two-sided inverse of φ_X. The second to last equality used naturality of α.

The naturality of this correspondence is straightforward but tedious so we leave it as an exercise (and potential bonus homework problem?) in type-checking and diagram chasing. Note that for $g : Y \to X$ the function

$$\mathbf{Nat}(\mathcal{C}(-,X),F) \xrightarrow{(=)\cdot(g\circ(-))} \mathbf{Nat}(\mathcal{C}(-,Y),F)$$

from the naturality diagram in the statement of the theorem is defined on a natural transformation $\alpha : \mathcal{C}(-,X) \implies F$ by composing natural transformations $\alpha \cdot (g \circ (-))$. For $W \in \mathcal{C}_0$ the component of this composite is

$$\mathcal{C}(W,Y) \xrightarrow{g\circ(-)} \mathcal{C}(W,X) \xrightarrow{\alpha_W} F(W")$$

$\square$

Yoneda's lemma gives a few important corollaries.

Corollary 43. *The functor*

$$y : \mathcal{C} \to \mathbf{Cat}(\mathcal{C}^{op}, \mathbf{Set}) \quad ; \quad X \mapsto \mathcal{C}(-, X)$$

is fully faithful.

Solution. Yoneda's lemma gives us an isomorphism

$$\mathbf{Nat}(\mathcal{C}(-, X), \mathcal{C}(-, Y)) \cong \mathcal{C}(X, Y)$$

for all objects $X, Y \in \mathcal{C}_0$. The left hand side is the collection of maps between $y(X) = \mathcal{C}(-, X)$ and $y(Y) = \mathcal{C}(-, Y)$ in the presheaf category and the isomorphism takes place in **Set** which means it's a bijection. Injectivity of this map tells us y is faithful and surjectivity tells us it's full. □

The previous corollary tells us that the functor y embeds $\mathcal{C}$ into its presheaf category and that's why y is called the *Yoneda embedding.* The next corollary can gives us a tool for proving two objects in a category are isomorphic.

Corollary 44. *Representing objects $X \in \mathcal{C}_0$ for the representable functors $\mathcal{C}(-, X)$ are unique.*

Solution. Fully faithful functors reflect isomorphisms so any isomorphism $y(X) \cong y(X')$ must come from an isomorphism $X \cong X'$. □

The last corollary may not make sense to us at this point because we haven't really talked about comma categories but it's important because it relates the concept of representability with universal constructions (namely colimits, which we'll get to).

Corollary 45. *A presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable if and only if the comma category (y/Δ_F) has a terminal object where Δ_F is the constant functor on F and y is the Yoneda embedding.*

Solution. □

The following example can be thought of as foreshadowing and the details will make a lot more sense after we talk about limits.

Example. Groups can be defined in any category with (finite) products. Representable functors preserve products. Yoneda's lemma and its corollaries tell us that an object G is a group object in $\mathcal{C}$ if and only if the representable functor $\mathcal{C}(-, G)$ is a group object in the presheaf category $\mathbf{Cat}(\mathcal{C}^{op}, \mathbf{Set})$. This is really important in modern algebraic geometry among other places and generalizes to other kinds of 'algebraic/essentially algebraic' objects (not just groups) because the representable functors actually preserve all limits.