## ASSIGNMENT 4

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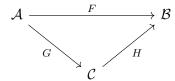
**Problem 1.** Show that any functor  $F: \mathcal{A} \to \mathcal{B}$  factors as  $F = H \circ G$  where G is surjective on objects and H is fully faithful and monic on objects.

Show that the surjective-on-objects functors are left orthogonal to the fully-faithful-and-monic-on-objects functors.

Solution. Define an intermediate category  $\mathcal{C}$  whose objects are  $\mathcal{C}_0 = F_0(\mathcal{A}_0) \subseteq \mathcal{B}_0$  and whose arrows are all the arrows in  $\mathcal{B}$  between these objects:

$$C(F(A), F(B)) := \mathcal{B}(F(A), F(B)).$$

This is a category because it inherits identities and composition from  $\mathcal{B}$ , in fact it's a subcategory of  $\mathcal{B}$  by construction. Now we'll define functors G and H such that the diagram



commutes in the following way. Since C is a subcategory of G the functor H will be an inclusion on objects and arrows; this is clearly a functor. It's vacuously injective on objects and fully-faithful by construction:

$$\mathcal{B}(H(F(A)), H(F(A'))) = \mathcal{B}(F(A), F(A')) =: \mathcal{C}(F(A), F(A'))$$

The functor G is just F with a restricted codomain so it's a functor because F is and it's surjective on objects because for any  $F(A) \in \mathcal{C}_0$  we have  $A \in \mathcal{C}_0$  and G(A) := F(A) by definition. The diagram commutes because for any  $A \in \mathcal{A}_0$  and  $f : A \to A'$  in  $\mathcal{A}_1$ 

$$H \circ G(A) = H(G(A)) = H(F(A)) = F(A)$$

and

$$H\circ G(f)=H(G(f))=H(F(f))=F(f).$$

For orthogonality suppose we have a commuting diagram

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{H} & \mathcal{C} \\
\downarrow F & & \downarrow G \\
\mathcal{B} & \xrightarrow{K} & \mathcal{D}
\end{array}$$

in **Cat** where F is surjective on objects and G is fully faithful and injective on objects. We'll show there exists a unique solution to the lifting problem. Define a candidate lift  $L: \mathcal{B} \to \mathcal{C}$  as follows. On objects, we take  $B \in \mathcal{B}_0$  and take any  $A \in \mathcal{A}_0$  in the fibre of F over B. Such an A exists because F is surjective on objects. Then we apply  $H_0$ :

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$$L_0(B) := H_0(A)$$
, where  $F_0(A) = B$ 

We need to check this is well-defined. Suppose A' is another object in the fibre of F above B, then

$$G_0(H_0(A')) = K_0(F_0(A')) = K_0(B) = K_0(F_0(A)) = G_0(H_0(A))$$

and since G is injective on objects we have

$$H_0(A') = H_0(A)$$

showing  $L_0$  is well-defined. Now for any arrow  $\beta: B \to B'$  in  $\mathcal{B}$  we can apply K to get an arrow  $K(\beta): K(B) \to K(B')$  in  $\mathcal{D}$ . Notice  $K_0(B) = G_0(L_0(B))$  and  $K_0(B') = G_0(L_0(B'))$  by commutativity of the original square and definition of  $L_0$ . Since G is fully faithful there exists a unique map

$$L_1(\beta): L_0(B) \to L_0(B').$$

Define  $L: \mathcal{B} \to \mathcal{C}$  by  $(L_0, L_1)$  on objects and arrows respectively. Identities and composition are preserved by the uniqueness in the definition of  $L_1$  coming from G being fully-faithful. This solves the lifting problem because for each  $A \in \mathcal{A}_0$ 

$$H_0(A) = L_0(F_0(A))$$

and for ever  $f:A\to A'$  in  $\mathcal A$  the square commutes so

$$K_1(F_1(f)) = G_1(H_1(f))$$

and by definition of  $L_1$  and the fact that G is fully faithful we must have

$$H_1(f) = L_1(F_1(f)).$$

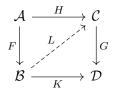
This shows  $H = L \circ F$ . On the other hand, for any  $B \in \mathcal{B}_0$  and  $A \in \mathcal{A}_0$  such that  $F_0(A) = B$  we have

$$G_0 \circ L_0(B) = G_0 \circ H_0(A) = K_0 \circ F_0(A) = K_0(B).$$

For  $\beta: B \to B'$  in  $\mathcal{B}_1$  we have that

$$G_1 \circ L_1(\beta) = K_1(\beta)$$

by construction so  $G \circ L = K$ . This shows L solves the lifting problem:



To show L is unique suppose L' also solved the lifting problem. Since F is surjective on objects and L and L' both solve the lifting problem we have that for each  $B \in \mathcal{B}_0$  there exists an  $A \in \mathcal{A}_0$  such that

$$(L')_0(B) = (L' \circ F)_0(A) = H_0(A) = (L \circ F)_0(A) = L_0(B).$$

Similarly we have that for any  $\beta \in \mathcal{B}_1$ 

$$G_1(L_1(\beta)) = K_1(\beta) = G_1((L')_1(\beta))$$

since L and L' both solve the lifting problem and since G is fully faithful we can conclude

$$L_1(\beta) = (L')_1(\beta).$$

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It follows that L = L' and the solution to the lifting problem is unique.

**Problem 2.** Let  $F, G : A \to B$  and let  $\alpha : F \implies G$  has components  $\alpha_X$  that are all invertible. Show that the family of inverses form a natural transformation  $G \implies F$ .

Solution. For any  $f: X \to Y$  in  $\mathcal{A}$  the naturality square

$$G(X) \xrightarrow{\alpha_X^{-1}} F(X)$$

$$G(f) \downarrow \qquad \qquad \downarrow^{F(f)}$$

$$G(Y) \xrightarrow{\alpha_Y^{-1}} F(Y)$$

commutes in  $\mathcal{B}$  because

$$F(f) \circ \alpha_X^{-1} = \alpha_Y^{-1} \circ \alpha_Y \circ F(f) \circ \alpha_X^{-1}$$
$$= \alpha_Y^{-1} \circ G(f) \circ \alpha_X \circ \alpha_X^{-1}$$
$$= \alpha_Y^{-1} \circ G(f).$$

This shows  $\alpha^{-1}: G \implies F$  is a natural transformation and since all its components are isomorphisms

$$\alpha_X \circ \alpha_X^{-1} = 1_{G(X)}$$
 ;  $\alpha_X^{-1} \circ \alpha_X = 1_{F(X)}$ 

it's a natural isomorphism.

**Problem 3.** Let  $\alpha: F \implies F'$  be a natural transformation between two parallel functors  $F, F': BG \to BH$  where G and H are groups and BG and BH are the categories with one object and arrows determined by G and H respectively. Describe  $\alpha$ .

Solution. The functors F and F' correspond to group homomorphisms  $\varphi, \varphi': G \to H$ . There's only one object in BG so  $\alpha$  only has on component  $h \in H$  such that for any  $g \in G$  the naturality square

$$\begin{array}{ccc}
* & \xrightarrow{h} & * \\
\varphi(g) \downarrow & & \downarrow \varphi'(g) \\
* & \xrightarrow{h} & *
\end{array}$$

commutes in BH. This says

$$h\varphi'(g) = \varphi(g)h$$

in H or equivalently that

$$\varphi(g) = h\varphi'(g)h^{-1}$$

We can think of conjugation by an element in H as an action on the set of group homomorphisms  $G \to H$  and in this context a natural transformation  $\alpha : F \implies F'$  witnesses that the group homomorphisms  $\varphi$  and  $\varphi'$  are conjugate.

**Problem 4.** Prove that  $-\times A: \mathbf{Set} \to \mathbf{Set}$  is a functor and the family of projections  $\pi_X: X \times A \to X$  induces a natural transformation  $(-\times A) \implies 1_{\mathbf{Set}}$ .

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Solution. It assigns a set X to its cartesian product  $X \times A$  and it assigns an arrow  $f: X \to Y$  to the function

$$f \times 1_A : X \times A \to Y \times A$$

defined by

$$f \times 1_A(x, a) := (f(x), a).$$

This assignment preserves identities because for every set X

$$1_X \times 1_A(x, a) = (1_X(x), a) = (x, a) = 1_{X \times A}(x, a)$$

and it preserves composition as for any composable  $f: X \to Y$  and  $g: Y \to Z$ 

$$(g \circ f) \times 1_A(x, a) = (g \circ f(x), a)$$
  
=  $(g(f(x)), a)$   
=  $g \times 1_A (f(x), a)$   
=  $((g \times 1_A) \circ (f \times 1_A)) (x, a)$ .

This shows  $-\times A$  is an endofunctor on **Set**. To see the projections induce a natural transformation to the identity functor we need to check that for every function  $X \to Y$  the naturality square

$$X \times A \xrightarrow{\pi_X} X$$

$$f \times 1_A \downarrow \qquad \qquad \downarrow f$$

$$Y \times A \xrightarrow{\pi_Y} Y$$

commutes. To see this we chase an element  $(x, a) \in X \times A$  through the diagram:

$$f \circ \pi_X(x, a) = f(\pi_X(x, a))$$

$$= f(x)$$

$$= \pi_Y(f(x), a))$$

$$= \pi_Y \circ (f \times 1_A)(x, a)$$

**Problem 5.** Let  $U: \mathcal{C} \to \mathcal{C}^{\to}$  denote the functor that sends objects in  $\mathcal{C}$  to their identity arrows and let  $D_0: \mathcal{C}^{\to} \to \mathcal{C}$  denote the domain functor that sends arrows in  $\mathcal{C}$  to their domains. Define a natural transformation  $U \circ D_0 \implies 1_{\mathcal{C}^{\to}}$ .

Solution. For every object  $f: X \to Y$  in  $\mathcal{C}^{\to}$  define the component

$$\alpha_f: U \circ D_0(f) \to f \quad ; \quad \alpha_f = (1_X, f)$$

in  $\mathcal{C}^{\to}$ . This makes sense because  $U \circ D_0(f) = 1_X$  and  $1_{\mathcal{C}^{\to}}(f) = f$  in  $\mathcal{C}$  and in  $\mathcal{C}$  we view this as a commuting square:

$$X = X \qquad \qquad \downarrow_f.$$

$$X \xrightarrow{f} Y$$

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For naturality we take an arbitrary arrow  $(h_0, h_1): f \to g$  in  $\mathcal{C}^{\to}$  where  $f: X \to Y$  and  $g: W \to Z$  are arrows in  $\mathcal{C}$  and we need to show the naturality square

$$U \circ D_0(f) \xrightarrow{\alpha_f} f$$

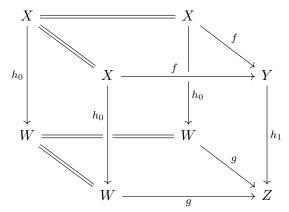
$$(U \circ D_0)(h_0, h_1) \downarrow \qquad \qquad \downarrow (h_0, h_1)$$

$$U \circ D_0(g) \xrightarrow{\alpha_g} g$$

commutes in  $\mathcal{C}$ . Note that the domain of  $h_0$  is the domain of X so

$$(U \circ D_0)(h_0, h_1) = U(h_0) = 1_{\partial_0(h_0)} = 1_X.$$

The naturality square above commutes in  $\mathcal{C}^{\rightarrow}$  whenever the cube



commutes in  $\mathcal{C}$ . The bottom, top, left, and back faces of the cube commute by the identity laws for composition in  $\mathcal{C}$ . The front and right faces of the cube commute by definition of  $(h_0, h_1)$  as an arrow in  $\mathcal{C}^{\rightarrow}$ . This shows the cube commutes and it follows that the naturality square commutes in  $\mathcal{C}^{\rightarrow}$ .