

CATEGORY THEORY COURSE NOTES

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INTRODUCTION

These are course notes from the winter 2024 semester Category Theory course at Dalhousie university taught by Dorette Pronk. We follow Robin's notes [?] with occasional reference to Emily Riehl's book [?]. She gave us the option to give presentations/guest lectures to replace exams in the course.

There are 'analytic' and 'synthetic' perspectives to category theory. An analytic point of view considers categories defined by their objects, arrows, and the structures they admit. This focuses more on the compositionality of relationships between the objects in a category. The synthetic point of view views categories as objects in a bigger categorical structure and studying them by the structure preserving relationships between them in that larger context. We begin with the analytic, peeking inside some familiar (concrete) categories and checking out the types of formal structures we can have when we have access to objects that are sets and have elements before defining generalized elements. After that we get more synthetic to talk about adjunctions and monads before finishing up with limits and colimits along with some relationships between the two perspectives.

1. 'ANALYTIC' CATEGORY THEORY

A category is an abstract setting where 'composition of functions' makes sense.

Definition 1. A category is a directed multigraph along with an algebra of paths.

The directed multigraph tells you what the 'objects' and 'arrows' are and the algebra of paths determines the composition of arrows.

For a category \mathcal{C} , the objects are denoted \mathcal{C}_0 and the arrows are denoted \mathcal{C}_1 . Let $d_0 : \mathcal{C}_1 \rightarrow \mathcal{C}_0$ denote the function that picks out the domain of an arrow and $d_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_0$ the function that picks out the codomain. The path algebra says: for any $f, g \in \mathcal{C}_1$, if $d_1(f) = d_0(g)$ then there exists a unique composite $g \circ f \in \mathcal{C}_1$ whose domain and codomain agree with those of f and g respectively. Associativity of the path algebra encodes associativity of arrow composition in the category \mathcal{C} . The constant paths in the path algebra correspond to identity arrows in the category.

Examples. Here are bunch of examples you can verify yourself!

- (a) The category of sets and functions is a category with function composition.
- (b) The category of groups and group homomorphisms is a category again with function composition.
- (c) Vaguely but more generally speaking the category of (algebraic thing) with (algebraic thing) homomorphisms is a category. There's a way to talk about these kinds of 'algebraic theories' more generally that we'll encounter later in the course!
- (d) Any directed multigraph can be made into a category by freely adding finite paths and identities.
- (e) The small categories \mathbb{K} and \mathbb{K}^* have as many objects and one less non-identity arrow. The first one is pictured below:

$$A \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}$$

- (f) Any set with a pre-order defines a category whose objects are the elements of the set and whose arrows are determined by the pre-order relation.

2. 'SYNTHETIC' CATEGORY THEORY