Problem 1: Weak selection

Consider the general two-strategy game

$$\begin{array}{c|cc} & A & B \\ A & a & b \\ B & c & d \end{array} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in a finite population of size N. Assume the population evolves according to an unstructured Moran process.

(a) Show that for weak selection, $w \ll 1$, the fixation probability of strategy A is given by

$$\rho_A \approx \frac{1}{N}\frac{1}{1-(\alpha N-\beta)w/6}$$

with $\alpha = a+2b-c-2d$ and $\beta = 2a+b+c-4d$.

You can use the following formulae:

(i) For small w, one can approximate $\prod_{i=1}^{k}(1-wx_i) \approx 1-w\sum_{i=1}^{k}x_i$.

(ii) For small w, it holds $\frac{1-wy}{1-wz} \approx 1-w(y-z)$.

(iii) $\sum_{k=1}^{N}\sum_{i=1}^{k} i = N(N+1)(N+2)/3!$.

(3 Points)

Now consider the specific game

$$\begin{array}{c|cc} & A & B \\ A & 20 & 2 \\ B & 17 & 1 \end{array} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(b) Decide for which N strategies A and B are evolutionarily stable in the limit of weak selection.

(1 Point)

(c) Compute for which N strategy A is risk dominant in the limit of weak selection.

(1 Point)

1a) $i :=$ no. of type A individuals.

$$\frac{g_i}{f_i} = \frac{1-w+wG_i}{1-w+wF_i} = \frac{1-w(1-G_i)}{1-w(1-F_i)} \approx 1-w(F_i-G_i) \text{ for } w \ll 1.$$

$$\rho_A = \frac{1}{1+\sum_{k=1}^{N-1}\prod_{i=1}^{k}\left(\frac{g_i}{f_i}\right)} \approx \frac{1}{1+\sum_{k=1}^{N-1}\prod_{i=1}^{k}\left[1-w(F_i-G_i)\right]} \approx \frac{1}{1+\sum_{k=1}^{N-1}\left[1-w\sum_{i=1}^{k}(F_i-G_i)\right]} \text{ for } w \ll 1.$$

$$F_i - G_i = \frac{(i-1)a+(N-i)b}{N-1} - \frac{ic+(N-i-1)d}{N-1} = \frac{ia-a+Nb-ib-ic-Nd+id+d}{N-1}$$

$$= \frac{i(a-b-c+d)+N(b-d)-a+d}{N-1}$$

$$= i\underbrace{\frac{(a-b-c+d)}{N-1}}_{x} + \underbrace{\frac{N(b-d)-a+d}{N-1}}_{y}$$

$$1+\sum_{k=1}^{N-1}\left[1-w\sum_{i=1}^{k}(F_i-G_i)\right] = 1+\sum_{k=1}^{N-1}1 - w\sum_{k=1}^{N-1}\sum_{i=1}^{k}(F_i-G_i)$$

$$= N - w\sum_{k=1}^{N-1}\sum_{i=1}^{k}(F_i-G_i)$$

$$= N - w\sum_{k=1}^{N-1}\sum_{i=1}^{k}(ix+y)$$

$$= N - w\left[x\sum_{k=1}^{N-1}\sum_{i=1}^{k} i + y\sum_{k=1}^{N-1}\sum_{i=1}^{k} 1\right]$$

$$= N - w\left[x\,\frac{(N-1)N(N+1)}{3!} + y\,\frac{N(N-1)}{2}\right]$$

$$= N - w\left[\frac{(a-b-c+d)}{N-1}\cdot\frac{(N-1)N(N+1)}{6} + \frac{N(b-d)-a+d}{N-1}\cdot\frac{N(N-1)}{2}\right]$$

$$= N - w\left[\frac{(a-b-c+d)N(N+1)}{6} + \frac{N(bN-dN)+N(-a+d)}{2}\right]$$

$$= N - \frac{wN}{6}\left[(a-b-c+d)(N+1) + 3bN - 3dN - 3a + 3d\right]$$

$$= N - \frac{wN}{6}\left(aN - bN - cN + dN + a - b - c + d + 3bN - 3dN - 3a + 3d\right)$$

$$= N - \frac{wN}{6}\left[N(a+2b-c-2d) - (2a+b+c-4d)\right] = N - \frac{wN}{6}(\alpha N - \beta)$$

$$\therefore \rho_A = \frac{1}{1+\sum_{k=1}^{N-1}\prod_{i=1}^{k}\left(\frac{g_i}{f_i}\right)} = \frac{1}{N - \frac{wN}{6}(\alpha N - \beta)} = \frac{1}{N}\,\frac{1}{1-\frac{w}{6}(\alpha N - \beta)}$$

1b) A is ESS_N if

i. $c(N-1) < b + a(N-2)$

ii. $d(N-2) + c(2N-1) < b(N+1) + a(2N-4)$

i. $17(N-1) < 2 + 20(N-2)$

$3N > 40 - 2 - 17$

$N > 7$

ii. $N - 2 + 34N - 17 < 2N + 2 + 40N - 80$

$7N > 59$

$N > 8.4$

$\therefore$ A is ESS_N for $N \geq 9$.

B is ESS_N if

i. $b(N-1) < c + d(N-2)$

ii. $a(N-2) + b(2N-1) < c(N+1) + d(2N-4)$

i. $2N - 2 < 17 + N - 2$

$N < 17$

ii. $20N - 40 + 4N - 2 < 17N + 17 + 2N - 4$

$5N < 55$

$N < 11$

$\therefore$ B is ESS_N for $2 \leq N \leq 10$.

1c) $(N-2)(a-d) > N(c-b)$

$(N-2)(20-1) > N(17-2)$

$19N - 38 > 15N$

$4N > 38$

$N > 9.5$

$\therefore$ A is risk dominant for $N \geq 10$.

Problem 2: Strong selection

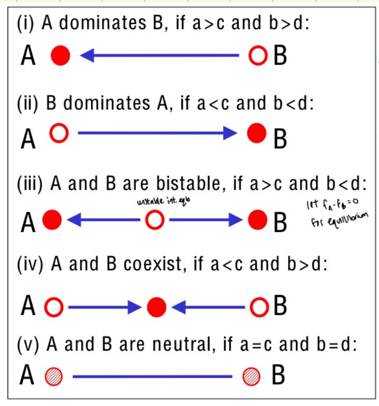
Consider the two-strategy game

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}.$$

(a) In an *infinite* population with replicator dynamics, decide for all games of this type whether strategies A and B are dominant, coexisting, or bi-stable, based on the two variables $\xi = a - c$ and $\zeta = d - b$. α **(1 Point)**

β

2a) Let $\alpha = a - c$, $\beta = d - b$.



$\rightarrow \alpha > 0, \beta < 0$

$\rightarrow \alpha < 0, \beta > 0$

$\rightarrow \alpha > 0, \beta > 0$

$\rightarrow \alpha < 0, \beta < 0$

$\rightarrow \alpha = 0, \beta = 0$

Made with Goodnotes

Now consider a population of *finite* size N that evolves according to a unstructured Moran process. Suppose the fitness of A and B are given respectively by

$$f_i = \frac{a(i-1)+b(N-i)}{N-1}$$
$$g_i = \frac{ci+d(N-i-1)}{N-1}.$$

Note that this corresponds to limit of strong selection, $w = 1$, as compared to the lecture.
We want to classify the evolutionary stability of A and B as a function of the population size N and the payoff values a, b, c, and d. To this end we analyze the difference in fitness $h_i = f_i - g_i$.

(b) Show that

$$h_i = \xi' \frac{i}{N-1} - \zeta' \frac{N-i}{N-1}$$

with

$$\overset{\alpha'}{\xi'} = \overset{\alpha}{\xi} - \frac{a-d}{N} \quad \text{and} \quad \underset{\beta'}{\zeta'} = \underset{\beta}{\zeta} + \frac{a-d}{N}.$$

What happens in the limit of large N? **(1 Point)**

(c) Show that for $\xi' > 0 > \zeta'$ strategy A is dominant. Derive a criterion for the dominance of B. **(1 Point)**

(d) Now suppose that $\xi', \zeta' > 0$. Show that if $\frac{1}{N-1} < \xi'/\zeta' < N-1$ it follows $h_1 < 0 < h_{N-1}$. Show that $h_1 < 0 < h_{N-1}$ is a criterion for bi-stability of A and B.
Similarly, show that if $\xi', \zeta' < 0$ and $\frac{1}{N-1} < \xi'/\zeta' < N-1$ it follows that $h_1 > 0 > h_{N-1}$. What does $h_1 > 0 > h_{N-1}$ imply for the evolutionary stability of A and B? **(1 Point)**

2b) Show that $h_i = \overbrace{\left[\underbrace{(a-c)}_{\alpha} - \frac{a-d}{N}\right]}^{\alpha'}\left(\frac{i}{N-1}\right) - \overbrace{\left[\underbrace{(d-b)}_{\beta} + \frac{a-d}{N}\right]}^{\beta'}\left(\frac{N-i}{N-1}\right)$

$$h_i = f_i - g_i = \frac{a(i-1) + b(N-i) - ci + d(N-i-1)}{N-1}$$

$$= \frac{ai - a - ci + b(N-i) - d(N-i) + d}{N-1}$$

$$= \frac{(a-c)i}{N-1} - \frac{(d-b)(N-i)}{N-1} - \frac{a-d}{N-1}$$

$$= \frac{i}{N-1}(a-c) - \frac{N-i}{N-1}(d-b) - \frac{a-d}{N-1}\left(\frac{i}{N-1} + \frac{N-i}{N-1}\right)$$

$$= \frac{i}{N-1}\left[(a-c) - \left(\frac{a-d}{N-1}\right)\right] - \frac{N-i}{N-1}\left[(d-b) - \left(\frac{a-d}{N-1}\right)\right]$$

$$\frac{a-d}{N-1} = \frac{a-d}{N} \cdot \frac{N}{N-1}$$
$$= \frac{a-d}{N} \cdot \frac{i+(N-i)}{N-1}$$
$$= \frac{a-d}{N}\left(\frac{i}{N-1} + \frac{N-i}{N-1}\right)$$

As $N \to \infty$,

$\frac{i}{N-1} \to 0$, $\frac{a-d}{N-1} \to 0$, $\frac{N-i}{N-1} \to 1$.

$h_i \to 0 \cdot (a-c) - 1 \cdot (d-b) = b-d$

2c) Let $\alpha = a-c$, $\beta = d-b$.

$$\alpha' = \alpha - \frac{a-d}{N}, \quad \beta' = \beta - \frac{a-d}{N}$$

Assume $\alpha' > 0 > \beta'$. A dominates if $a>c$ and $b>d$

finite real number

$$\alpha' > 0 \iff \alpha - \frac{a-d}{N} > 0 \iff a-c > \frac{a-d}{N} \qquad \therefore a-c>0 \text{ and } a>c.$$

$$\beta' < 0 \iff \beta - \frac{a-d}{N} < 0 \iff d-b < \frac{a-d}{N} \qquad \therefore d-b<0 \text{ and } d<b.$$

$$h_i = f_i - g_i = \underbrace{\alpha'\frac{i}{N-1}}_{>0} - \underbrace{\beta'\frac{N-i}{N-1}}_{<0} > 0$$

$\therefore$ for $\alpha' > 0 > \beta'$, A dominates B.

2d) $$\frac{|\alpha'|}{|\beta'|} < N-1 \iff \frac{|\alpha'|}{N-1} < |\beta'| \iff \frac{|\alpha'|}{N-1} - |\beta'| < 0$$

$$\frac{|\alpha'|}{|\beta'|} > \frac{1}{N-1} \iff |\alpha'| > \frac{|\beta'|}{N-1} \iff |\alpha'| - \frac{|\beta'|}{N-1} > 0$$

$$h_{N-1} = |\alpha'|\frac{N-1}{N-1} - |\beta'|\frac{N-(N-1)}{N-1} = |\alpha'| - \frac{|\beta'|}{N-1} > 0$$

$$h_1 = \frac{|\alpha'|}{N-1} - |\beta'|\frac{N-1}{N-1} = \frac{|\alpha'|}{N-1} - |\beta'| < 0$$

$\therefore$ for $\frac{1}{N-1} < \frac{\alpha'}{\beta'} < N-1$, $h_1 < 0 < h_{N-1}$ if $\alpha', \beta' > 0$.

$h_1 > 0 > h_{N-1}$ if $\alpha', \beta' < 0$.

For bi-stability, $\alpha > 0$ and $\beta > 0$. Since $h_1 < 0 < h_{N-1}$ if $\alpha, \beta > 0$, it is a criterion.
For $h_1 > 0 > h_{N-1}$ i.e. $\alpha < 0$ and $\beta < 0$, A and B coexist.