York University

Department of Electrical Engineering & Computer Science

MATH1090B. December 2019: Final Examination

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SOLUTIONS

Boolean Logic 1. (2 MARKS)

I. Pick *ONE* correct answer from among (a) and (b) below:
Correct answer = 1 point; incorrect = −1 point; no answer = 0 points.
The way we **apply** the Deduction Theorem is:
(a) In order to prove $A \vdash B$ it suffices to prove $\vdash A \to B$.
(b) **In order to prove $\vdash A \to B$ it suffices to prove $A \vdash B$.**

II. Pick *ONE* correct answer from among (i) and (ii) below:
Correct answer = 1 point; incorrect = −1 point; no answer = 0 points.
The Deduction Theorem **says**:
(i) **If $A \vdash B$, then $\vdash A \to B$.**
(ii) If $\vdash A \to B$, then $A \vdash B$.

Boolean Logic 2. (2 MARKS) Is the following a well formed formula? **Prove the correctness of your answer either using formula calculations or the recursive definition of formulas**.

$$((\bot))$$

Answer. NO. If it *were* a formula, then it would show up in a formula-calculation. **BUT**: In every step of such a calculation we may put left-right brackets **iff** we applied "glue". There is no glue here, so both sets of brackets are WRONG. So this string CANNOT appear in a formula calculation and thus is not a formula.

We may also answer like this:

We know from class that the number of left brackets of a *well formed formula* equals the complexity —i.e., the glue-count— of the formula. Here we we have 2 brackets but 0 complexity. Not a formula! □

Boolean Logic 3. (5 Marks) Prove **by Resolution**:

$$\vdash ((A \to B) \to A) \to A$$

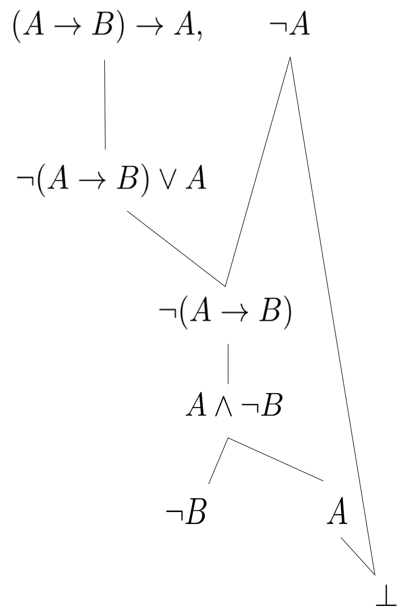
Caution: **0 Marks** gained **if any other technique is used. In particular, Post's theorem is NOT allowed.**

Proof. By DThm suffices to prove instead

$$(A \to B) \to A \vdash A$$

Before we apply Resolution we note (proof by contra) that it suffices to prove instead

$$(A \to B) \to A, \neg A \vdash \bot$$



$\square$

Boolean Logic 4. (4 MARKS) Is

$$A \to B, B \vdash A \tag{1}$$

a theorem-*schema*?

You have ONE of two options:

(a) You opt for "yes" *and* **produce** a *syntactic proof*.

(b) You opt for "no" *and use* the appropriate **tool** (must *name* and describe how the tool is used in general!) to show precisely why not.

Careful if you opt for (b): Since (1) is a schema, you *cannot* expect your justification to "work" for all A and B. **You *must* pick specific simple A and B instances**!

Opting for a choice, (a) or (b), ***without justification earns 0 points***.

Answer. NO. I show this by (Boolean) soundness:

If (1) is a theorem schema, then the following is a theorem

$$p \to q, q \vdash p \tag{2}$$

If (2) is correct, then (Boolean Soundness) we must also have

$$p \to q, q \models_{taut} p \tag{3}$$

But (3) fails: Just take a state s such that $s(p) = \mathbf{f}$ and $s(q) = \mathbf{t}$. So, (2) fails and thus (1) fails. $\square$

Predicate Logic 1. (3 MARKS) **True** or **False** and **WHY**: For any formula A, we have $\vdash (\exists \mathbf{x})(A \to A)$.

Proof. TRUE. Here is a proof:

$$\begin{array}{lll} (1) & A \to A & \langle \text{axiom} \rangle \\ (2) & (\exists \mathbf{x})(A \to A) & \langle (1) + \text{Dual Spec} \rangle \end{array}$$

□

Predicate Logic 2. (5 MARKS) Use an **Equational** proof to establish

$$\vdash (\exists \mathbf{x})(A \to (\forall \mathbf{x})A)$$

Caution! You may **NOT** assume that A has no free occurrences of $\mathbf{x}$.

Any <u>other</u> type of CORRECT proof gets 1 MARK.

Proof. We know that "$(\exists \mathbf{x})\ldots$" means "$\neg(\forall \mathbf{x})\neg\ldots$", thus we must prove

$$\vdash \neg(\forall \mathbf{x})\neg(A \to (\forall \mathbf{x})A)$$

$$\begin{array}{ll}
 & \neg(\forall \mathbf{x})\neg(A \to (\forall \mathbf{x})A) \\
\Longleftrightarrow & \langle \text{WL + Tautology; Denom: } \neg(\forall \mathbf{x})\mathbf{p} \rangle \\
 & \neg(\forall \mathbf{x})(A \wedge \neg(\forall \mathbf{x})A) \\
\Longleftrightarrow & \langle \text{WL (BL works too here) + "}\forall \text{ over } \wedge \text{ thm"; Denom: } \neg\mathbf{p} \rangle \\
 & \neg\Big((\forall \mathbf{x})A \wedge (\forall \mathbf{x})\neg(\forall \mathbf{x})A\Big) \\
\Longleftrightarrow & \langle \vdash Y \equiv (\forall \mathbf{x})Y \text{, if } x \text{ } \underline{\text{is not free in } Y} \text{ + BL; Denom: } \neg((\forall \mathbf{x})A \vee \mathbf{p}) \rangle^1 \\
 & \neg\Big((\forall \mathbf{x})A \wedge \neg(\forall \mathbf{x})A\Big) \\
\Longleftrightarrow & \langle \text{Tautology} \rangle \\
 & \neg(\forall \mathbf{x})A \vee (\forall \mathbf{x})A
\end{array}$$

Bingo!

□

Predicate Logic 3. (5 MARKS) **Assume** the "*Renaming the Bound Variable*" theorem for $\forall$ **proved**. (In fact, it **was** proved in class) and use an **Equational** proof —**not any other way**— to show: If $\mathbf{z}$ does not occur in $(\exists \mathbf{x})A$ as either free or bound, then

$$\vdash (\exists \mathbf{x})A \equiv (\exists \mathbf{z})\big(A[\mathbf{x} := \mathbf{z}]\big) \tag{1}$$

Proof. The dummy renaming proved in class assumes that $\mathbf{z}$ is fresh for $(\forall \mathbf{x})A$ and proves

$$\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{z})\big(A[\mathbf{x} := \mathbf{z}]\big) \tag{2}$$

We prove (1) as follows, mindful of "$(\exists \mathbf{x})\ldots$" means "$\neg(\forall \mathbf{x})\neg\ldots$".

$$\begin{array}{ll}
 & \neg(\forall \mathbf{x})\neg A \\
\Longleftrightarrow & \langle \text{BL + (2); Denom: } \neg\mathbf{p}\text{; } \textbf{Note}\text{: } \mathbf{z} \text{ being fresh for } (\exists \mathbf{x})A \text{ is so for } \neg(\forall \mathbf{x})\neg A \text{ too.} \rangle \\
 & \neg(\forall \mathbf{z})\neg\big(A[\mathbf{x} := \mathbf{z}]\big)
\end{array}$$

The above is the same as (1). □

Predicate Logic 4. (4 MARKS) For any A and B, prove $\vdash (\forall \mathbf{x})A \to (\exists \mathbf{x})(A \vee B)$.

Proof. Suffices to prove (DThm)

$$(\forall \mathbf{x})A \vdash (\exists \mathbf{x})(A \vee B)$$

$$\begin{array}{lll} (1) & (\forall \mathbf{x})A & \langle \text{hyp} \rangle \\ (2) & A & \langle (1) + \text{Spec} \rangle \\ (3) & A \vee B & \langle (2) + \text{Post} \rangle \\ (4) & (\exists \mathbf{x})(A \vee B) & \langle (3) + \text{Dual Spec} \rangle \end{array}$$

□

[1] Ping-pong and **Ax2**: $\vdash (\forall \mathbf{x})Y \to Y$ + **Ax3**: $\vdash Y \to (\forall \mathbf{x})Y$

Predicate Logic 5. (5 MARKS) Prove that

$$(\forall \mathbf{y})(\exists \mathbf{x})A \to (\exists \mathbf{x})(\forall \mathbf{y})A \tag{$*$}$$

is **not** a theorem.

Hint. Use 1st-order soundness and a *countermodel*.

Proof. According to the hint. Simplify first. If the above *IS* a theorem then so is

$$(\forall \mathbf{y})(\exists \mathbf{x})\phi(x, y) \to (\exists \mathbf{x})(\forall \mathbf{y})\phi(x, y) \tag{1}$$

for a predicate ϕ of arity two.

Interpret (1) over $\mathbb{N}$, that is, take $\mathscr{D} = (\mathbb{N}, I)$, where $\phi^{\mathscr{D}}(x, y)$ is the standard $y < x$ over $\mathbb{N}$. But then (1) translates as

$$(\forall \mathbf{y} \in \mathbb{N})(\exists \mathbf{x} \in \mathbb{N})y < x \to (\exists \mathbf{x} \in \mathbb{N})(\forall \mathbf{y} \in \mathbb{N})y < x \tag{2}$$

which is **false**, since lhs of $\to$ is $\mathbf{t}$ but rhs is $\mathbf{f}$.

Thus (1) is not a theorem hence nor is the general case $(*)$. $\square$

Predicate Logic 6. (5 MARKS) Show that (1) below is impossible in our logic:

$$A \to B \vdash A \to (\forall \mathbf{x})B, \text{ if } \mathbf{x} \text{ is not free in } A \tag{1}$$

Your method must use **ONLY ONE OF** the techniques listed below:

(a) **Semantic**: That is, select **specific** *very simple* formulae A and B and find a *countermodel* for the formula

$$(A \to B) \to (A \to (\forall \mathbf{x})B) \tag{2}$$

(b) **Formal (syntactic)**: That is, you assume that (1) IS available in our logic, and using it as a "lemma" you prove the known to us impossible "rule" $C \vdash (\forall \mathbf{x})C$, thus contradicting (1).

Proof. I give a proof with each methodology, but you only needed to give **ONE** methodology.

(a) **Semantic**: If (1) is correct, then so is (2) by the DThm. I will find a *countermodel* for a simple special case of (2). So, if the special case does not work [is not a theorem] nor is the general case. I take A to be $\top$ and B to be $\mathbf{x} = \mathbf{y}$. I work with the interpretation

$$\mathscr{D} = (\mathbb{N}, I) \tag{3}$$

where I take $x^{\mathscr{D}} = y^{\mathscr{D}} = 0$.

(2) translates as

$$\overbrace{(\top \to 0 = 0)}^{\mathbf{t}} \to \overbrace{(\top \to (\forall \mathbf{x} \in \mathbb{N})\mathbf{x} = 0)}^{\mathbf{f}} \tag{4}$$

The interpretation (4) being false, (2) is not a theorem (soundness) as we established (3) to be a countermodel for an *instance* of (2).

(b) **Syntactic**: If (1) is possible, then I will prove that so is $C \vdash (\forall \mathbf{x})C$ —for any C— which we know is **not** possible (so, neither is (1)).

1.	C	$\langle$hyp$\rangle$
2.	$\top \to C$	$\langle$1. + Post$\rangle$
3.	$\top \to (\forall \mathbf{x})C$	$\langle$2. + (1) above$\rangle$
4.	$(\forall \mathbf{x})C$	$\langle$3. + Post$\rangle$

$\square$

Predicate Logic 7. (5 MARKS) You must use the technique of the "auxiliary hypothesis metatheorem" in the proof that you are asked to write here. Any other proof (***IF correct***) will *MAX* at 1 MARK.

For any formulas A, B, and C show that

$$\vdash (\exists \mathbf{x})\big(A \lor B \to C\big) \to (\exists \mathbf{x})(A \to C) \land (\exists \mathbf{x})(B \to C)$$

Proof. By the DThm it suffices to prove

$$(\exists \mathbf{x})\big(A \lor B \to C\big) \vdash (\exists \mathbf{x})(A \to C) \land (\exists \mathbf{x})(B \to C)$$

(1)	$(\exists \mathbf{x})\big(A \lor B \to C\big)$	$\langle$hyp$\rangle$
(2)	$A[\mathbf{x} := \mathbf{z}] \lor B[\mathbf{x} := \mathbf{z}] \to C[\mathbf{x} := \mathbf{z}]$	$\langle$aux. hyp for (1); $\mathbf{z}$ fresh$\rangle$
(3)	$A[\mathbf{x} := \mathbf{z}] \to C[\mathbf{x} := \mathbf{z}]$	$\langle$(2) + Post$\rangle$
(4)	$B[\mathbf{x} := \mathbf{z}] \to C[\mathbf{x} := \mathbf{z}]$	$\langle$(2) + Post$\rangle$
(5)	$(\exists \mathbf{x})(A \to C)$	$\langle$(3) + Dual Spec$\rangle$
(6)	$(\exists \mathbf{x})(B \to C)$	$\langle$(4) + Dual Spec$\rangle$
(7)	$(\exists \mathbf{x})(A \to C) \land (\exists \mathbf{x})(B \to C)$	$\langle$(5, 6) + Post$\rangle$

In step (5) we used the fact that $A[\mathbf{x} := \mathbf{z}] \to C[\mathbf{x} := \mathbf{z}]$ is $(A \to C)[\mathbf{x} := \mathbf{z}]$. Similar comment for step (6). $\square$