Chapter 3

The H$_2$-optimal control problem

In this chapter we present the solution of the H_2-optimal control problem. We consider the control system in Figure 1.2. Introducing the partition of G according to

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} \tag{3.1}$$

the closed-loop system

$$z = F(G, K)v \tag{3.2}$$

has the transfer function $F(G, K)$ given by

$$F(G, K) = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21} \tag{3.3}$$

The H_2-optimal control problem consists of finding a causal controller K which stabilizes the plant G and which minimizes the cost function

$$J_2(K) = \|F(G, K)\|_2^2 \tag{3.4}$$

where $\|F(G, K)\|_2$ is the H_2 norm, cf. Section 2.2.1.

The control problem is most conveniently solved in the time domain. We will assume that the plant G has the state-space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1v(t) + B_2u(t) \\ z(t) &= C_1x(t) + D_{12}u(t) \\ y(t) &= C_2x(t) + D_{21}v(t) \end{aligned} \tag{3.5}$$

Remark 3.1.
In the optimal and robust control literature, the state-space representation $G(s) = C(sI - A)^{-1}B + D$ is commonly written using the compact notation

$$G(s) := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3.6}$$

Hence the systems in (3.5) can also be written as

$$G(s) := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \tag{3.7}$$

where $D_{11} = 0$, $D_{22} = 0$.

In (3.5), the direct feedthrough from v to z has been assumed zero in order to obtain a finite H_2 norm for the closed-loop system. The direct feedthrough from u to y has been assumed zero because physical systems always have a zero gain at infinite frequency.

It is convenient to make the following assumptions on the system matrices.

Assumptions:

(A1) The pair (A, B_2) is stabilizable.

(A2) $D_{12}^T D_{12}$ is invertible.

(A3) $D_{12}^T C_1 = 0$.

(A4) The pair (C_1, A) has no unobservable modes on the imaginary axis.

(B1) The (C_2, A) is detectable.

(B2) $D_{21} D_{21}^T$ is invertible.

(B3) $D_{21} B_1^T = 0$.

(B4) The pair (A, B_1) has no uncontrallable modes on the imaginary axis.

The first set of assumptions (A1)–(A4) is related to the state feedback control problem, while the second set of assumptions (B1)–(B4) is related to the state estimation problem. Assumptions (A1) and (B1) are necessary for a stabilizing controller $u = Ky$ to exist. Assumptions (A2), (A3) and (B2), (B3) can be relaxed, but they are not very restrictive, and as we will see they imply convenient simplifications in the solution. Assumptions (A4) and (B4) are required for the Riccati equations which characterize the optimal controller to have stabilizing solutions. These assumptions can be relaxed, but the solution of the optimal control problem must then be characterized in an alternative way, for example in terms of linear matrix inequalities (LMIs).

It is appropriate to introduce the time-domain characterization in (2.45) of the cost (3.4), giving

$$J_2(K) = \sum_{k=1}^{m} [\int_0^\infty z(t)^T z(t) dt : v = e_k \delta(t)] \tag{3.8}$$

Notice, however, that the characterization (3.8) is introduced solely because the solution of the H_2-optimal control problem can be derived in a convenient way using (3.8). On the other hand, the *motivation* of the H_2-optimal is often more naturally stated in terms of the average frequency-domain characterization in (2.15) or the stochastic characterization in (2.46).

Remark 3.2.
In the linear quadratic optimal control literature, the cost function is usually defined as

$$J_{LQ}(K) = \sum_{k=1}^{m} \left[\int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt : v = e_k \delta(t) \right] \tag{3.9}$$

where Q is a symmetric positive semidefinite matrix and R is a symmetric positive definite matrix. It is easy to see that the cost functions (3.8) and (3.9) are equivalent, because the positive (semi)definite matrices Q and R can always be factorized as

$$Q = (Q^{1/2})^T Q^{1/2}, \;\; R = (R^{1/2})^T R^{1/2} \tag{3.10}$$

The factorizations in (3.10) are called Cholesky factorizations, cf. the MATLAB routine `chol`. Defining the matrices

$$C_1 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, \;\; D_{12} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \tag{3.11}$$

it follows that

$$z^T z = (C_1 x + D_{12} u)^T (C_1 x + D_{12} u) = x^T Q x + u^T R u \tag{3.12}$$

and the costs (3.8) and (3.9) are thus equivalent. Notice also that with C_1 and D_{12} defined by (3.11), assumption (A3) holds.

The solution of the H_2-optimal control problem can be done in two stages. In the first stage, an optimal state-feedback law is constructed. The second stage consists of finding an optimal state estimator.

3.1 The optimal state-feedback problem

The following result gives the optimal *state-feedback law* $\hat{u}(s) = K_x(s)\hat{x}(s)$ which minimizes the quadratic cost (3.8) or (3.9). The result is classical in optimal linear quadratic (LQ) control theory.

Theorem 3.1 H_2**-optimal state feedback control.**
Consider the system (3.5). Suppose that the assumptions (A1)–(A4) hold. Assume that the control signal $u(t)$ has access to the present and past values of the state, $x(\tau), \tau \leq t$. Then the cost (3.8) is minimized by the static state state-feedback controller

$$u(t) = K_{opt} x(t) \tag{3.13}$$

where

$$K_{opt} = -(D_{12}^T D_{12})^{-1} B_2^T S \tag{3.14}$$

and where S is the unique symmetric positive (semi)definite solution to the algebraic Riccati equation (ARE)

$$A^T S + SA - SB_2 (D_{12}^T D_{12})^{-1} B_2^T S + C_1^T C_1 = 0 \tag{3.15}$$

such that the matrix

$$A + B_2 K_{opt} \tag{3.16}$$

is stable, i.e. all its eigenvalues have negative real parts.

Moreover, the minimum value of the cost (3.8) achieved by the control law (3.13) is given by

$$\min_{K_x} J_2(K_x) = \mathrm{tr}(B_1^T S B_1) \tag{3.17}$$

Proof: By the theory of Riccati equations it follows from assumptions (A1) and (A4) that the algebraic Riccati equation (3.15) has a positive (semi)definite solution S such that the matrix in (3.16) is stable. Consider the integral in (3.8). It can be decomposed as

$$\int_0^\infty z(t)^T z(t) dt = \int_0^{0^+} z(t)^T z(t) dt + \int_{0^+}^\infty z(t)^T z(t) dt \tag{3.18}$$

where 0^+ denotes the time immediately after the impulse input at time $t = 0$. With $x(0) = 0$ and $v(t) = e_k \delta(t)$, we have formally

$$x(0^+) = \int_0^{0^+} e^{A(0^+ - \tau)}[B_1 e_k \delta(\tau) + B_2 u(\tau)] d\tau = B_1 e_k \tag{3.19}$$

It follows that the first integral in (3.18) is zero. Assuming that the algebraic Riccati equation (3.15) has a positive (semi)definite solution, the second integral in (3.18) can be expanded as

$$\begin{aligned} \int_{0^+}^\infty z(t)^T z(t) dt &= \int_{0^+}^\infty [C_1 x(t) + D_{12} u(t)]^T [C_1 x(t) + D_{12} u(t)] dt \\ &= \int_{0^+}^\infty \left[[C_1 x(t) + D_{12} u(t)]^T [C_1 x(t) + D_{12} u(t)] + \frac{d(x(t)^T S x(t))}{dt} \right] dt \\ &\quad - x(\infty)^T S x(\infty) + x(0^+)^T S x(0^+) \\ &= \int_{0^+}^\infty [u(t) - u^0(t)]^T D_{12}^T D_{12} [u(t) - u^0(t)] dt + x(0^+)^T S x(0^+) \end{aligned} \tag{3.20}$$

where

$$u^0(t) = K_{opt} x(t), \tag{3.21}$$

and $x(0^+)$ denotes the state immediately after the impulse input at time $t = 0$. We have also assumed that $x(\infty) = 0$ due to stability, and used the fact that S satisfies (3.15). Introducing (3.20) and (3.19), the cost (3.8) can be expressed as

$$\begin{aligned} J_2(K) &= \sum_{k=1}^m \left[\int_0^\infty z(t)^T z(t) dt : v = e_k \delta(t) \right] = \\ &= \sum_{k=1}^m \left[\int_0^\infty [u(t) - u^0(t)]^T D_{12}^T D_{12} [u(t) - u^0(t)] dt + x(0^+)^T S x(0^+) : v = e_k \delta(t) \right] \\ &= \sum_{k=1}^m \left[\int_0^\infty [u(t) - u^0(t)]^T D_{12}^T D_{12} [u(t) - u^0(t)] dt + e_k^T B_1^T S B_1 e_k : v = e_k \delta(t) \right] \end{aligned} \tag{3.22}$$

Here

$$\sum_{k=1}^{m} e_k^T B_1^T S B_1 e_k = \sum_{k=1}^{m} \mathrm{tr}(B_1^T S B_1 e_k e_k^T) = \mathrm{tr}(B_1^T S B_1 \sum_{k=1}^{m} e_k e_k^T) = \mathrm{tr}(B_1^T S B_1) \tag{3.23}$$

Hence,

$$J_2(K) = \sum_{k=1}^{m} \left[\int_0^\infty [u(t) - u^0(t)]^T D_{12}^T D_{12} [u(t) - u^0(t)] dt : v = e_k \delta(t) \right] + \mathrm{tr}(B_1^T S B_1) \tag{3.24}$$

Since the integrals in (3.24) are non-negative, and are equal to zero when $u(t) = u^0(t)$, it follows that the cost $J_2(K_x)$ is minimized by the static state feedback (3.13), or (3.21), and the minimum cost is given by (3.17). □

Problem 3.1. Verify the last step in (3.20).

Notice that the optimal state-feedback controller consists of a *static state feedback*, although no restrictions on the controller structure was imposed. Another point to notice about the optimal controller is that it does not depend on the disturbance v, since the controller is independent of the matrix B_1. This matrix enters only in the expression (3.17) for the minimum cost.

Remark 3.3.
By equation (3.20), the control law (3.21) is optimal for all initial states $x(0^+)$. This implies that the solution of the H_2 control problem can be interpreted in a worst-case setting as the controller which minimizes the worst-case cost

$$\begin{aligned} J_{2,worst}(K) &:= \max_{x(0)} \left\{ \|z\|_2^2 : x^T(0)x(0) \leq 1 \right\} \\ &= \max_{x(0)} \left\{ \int_0^\infty z^T(t) z(t) dt : x^T(0)x(0) \leq 1 \right\} \end{aligned} \tag{3.25}$$

The optimal state-feedback control result suggests how an optimal output feedback controller $\hat{u}(s) = K(s)\hat{y}(s)$ can be obtained. In particular, the integral in the expansion (3.22) cannot be made equal to zero if the whole state vector $x(t)$ is not available to the controller. In the output feedback problem, one should instead make this integral as small as possible by using an optimal state estimate $\hat{x}(t)$ instead. Therefore, we give next the solution to an H_2-optimal state estimation problem.

3.2 The optimal state-estimation problem

In this section we consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 v(t) \\ z(t) &= C_1 x(t) \\ y(t) &= C_2 x(t) + D_{21} v(t) \end{aligned} \tag{3.26}$$

Notice that in the estimation problem, we need not consider the input u. As u is a known input, its effect on the state is exactly known.

We consider stable causal *state estimators* F such that the state estimate $\hat{x}(t)$ is determined from the measured output y according to (in the Laplace-domain) $\hat{\hat{x}}(s) = F(s)y(s)$. In the H_2-optimal estimation problem, the problem is to construct an estimator such that the quadratic H_2-type cost

$$J_e(F) = \sum_{k=1}^{m} [\int_0^\infty [x(t) - \hat{x}(t)]^T C_1^T C_1 [x(t) - \hat{x}(t)] dt : v = e_k \delta(t)] \tag{3.27}$$

is minimized. The solution of the optimal state estimation problem is given as follows.

Theorem 3.2 ***H_2*-optimal state estimation.**
Consider the system (3.26). Suppose that the assumptions (B1)–(B4) hold. The stable causal state estimator which minimizes the cost (3.27) is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L[y(t) - C_2\hat{x}(t)] \tag{3.28}$$

where

$$L = PC_2^T(D_{21}D_{21}^T)^{-1} \tag{3.29}$$

and where P is the unique symmetric positive (semi)definite solution to the algebraic Riccati equation

$$AP + PA^T - PC_2^T(D_{21}D_{21}^T)^{-1}C_2P + B_1B_1^T = 0 \tag{3.30}$$

such that the matrix

$$A - LC_2 \tag{3.31}$$

is stable, i.e. all its eigenvalues have negative real parts.

Moreover, the minimum value of the cost (3.27) achieved by the estimator (3.28) is given by

$$\min_F J_e(F) = \mathrm{tr}(C_1PC_1^T) \tag{3.32}$$

Remark 3.4.
Notice that the optimal state estimator is independent of the matrix C_1 in the cost (3.27). The matrix enters only in the expression (3.32) for the minimum cost.

Remark 3.5.
The solution defined by (3.29) and (3.30) is the dual to the solution (3.14) and (3.15) of the optimal state-feedback problem in the sense that (3.14) and (3.15) reduce to (3.29), (3.30) by making the substitutions

$$\begin{aligned} A &\to A^T \\ C_1 &\to B_1^T \\ B_2 &\to C_2^T \\ D_{12} &\to D_{21}^T \end{aligned} \tag{3.33}$$

and the costs (3.17) and (3.32) are dual under the substitution $B_1 \to C_1^T$.

The estimator (3.28) is the celebrated *Kalman filter* due to Rudolf Kalman and R. S. Bucy (1960, 1961). Originally, it has been introduced in a stochastic framework for recursive state estimation in stochastic systems. Notice that the cost (3.27) equals the H_2 norm of the transfer function from the disturbance v to the estimation error $C_1(x - \hat{x})$. From what was mentioned previously in Section 2.2.1 it follows that if v is stochastic white noise, the cost (3.27) can be characterized as

$$J_e(F) = \lim_{t_f \to \infty} E[\frac{1}{t_f} \int_0^{t_f} [x(t) - \hat{x}(t)]^T C_1 C_1 [x(t) - \hat{x}(t)] dt] \tag{3.34}$$

Remark 3.6.
In the standard formulation of the stochastic estimation problem the system is defined as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + v_1(t) \\ z(t) &= C_1 x(t) \\ y(t) &= C_2 x(t) + v_2(t) \end{aligned} \tag{3.35}$$

where $v_1(t)$ and $v_2(t)$ are mutually independent white noise processes with covariance matrices R_1 and R_2, respectively. This formulation reduces to the one above by the identifications

$$\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} [\, B_1^T \quad D_{21}^T \,] \tag{3.36}$$

and

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} v \tag{3.37}$$

The optimal estimator result is more difficult to prove than the optimal state-feedback controller. We shall only provide a proof based on duality between the optimal state-feedback and estimation problems.

Proof of Theorem 3.2: Consider a state estimator $\hat{x}(s) = F(s)y$ with transfer function $F(s)$. Then we have in terms of operators,

$$C_1(x - \hat{x}) = [C_1 G_1 - C_1 F G_2] v \tag{3.38}$$

where (cf. (3.26))

$$G_1(s) = (sI - A)^{-1} B_1, \;\; G_2(s) = C_2(sI - A)^{-1} B_1 + D_{21} \tag{3.39}$$

The cost (3.27) can then be characterized as

$$J_e(F) = \|C_1 G_1 - C_1 F G_2\|_2^2 \tag{3.40}$$

But by the definition (2.15), the H_2 norm of a transfer function matrix is equal to the H_2 norm of its transpose. Hence

$$J_e(F) = \|G_1^T C_1^T - G_2^T F^T C_1^T\|_2^2 \tag{3.41}$$

Notice that

$$\begin{bmatrix} C_1 G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1} B_1 + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \tag{3.42}$$

Hence

$$[\, G_1(s)^T C_1^T \quad G_2(s)^T \,] = B_1^T (sI - A^T)^{-1} [\, C_1^T \quad C_2^T \,] + [\, 0 \quad D_{21}^T \,] \tag{3.43}$$

and the transposed system

$$r = [\, G_1^T C_1^T \quad G_2^T \,] \begin{bmatrix} \nu \\ \eta \end{bmatrix} \tag{3.44}$$

thus has the state-space representation

$$\begin{aligned} \dot{p}(t) &= A^T p(t) + C_1^T \nu(t) + C_2^T \eta(t) \\ r(t) &= B_1^T p(t) + D_{21}^T \eta(t) \end{aligned} \tag{3.45}$$

Using the control law

$$\eta = -F^T C_1^T \nu \tag{3.46}$$

the closed-loop system (3.44), (3.46) is described by

$$r = (G_1^T C_1^T - G_2^T F^T C_1^T)\nu \tag{3.47}$$

Hence the H_2-optimal estimation problem is equal to the dual control problem which consists of finding a controller (3.46) such that the H_2 norm of the closed-loop system is minimized. The dual control problem is, however, equivalent to a state-feedback control problem, because the fact that $C_1^T \nu$ is available to the controller means that the state $p(t)$ in (3.45) is also available. More specifically, the problem of finding a control law (3.46) which minimizes the H_2 norm of the closed-loop transfer function, equation (3.41), is equivalent to the problem of finding a state-feedback law

$$\eta(t) = -F_p^T p(t) \tag{3.48}$$

where $p(t)$ is available through equation (3.45) and knowledge of $C_1 \nu(t)$ and $\eta(t)$. But the problem of finding an optimal state-feedback controller for the dual system (3.45) is known, and from our previous result it is given by (3.14) and (3.15) under to the substitutions (3.33). This gives the optimal stationary state-feedback with

$$\eta(t) = -F_{opt}^T p(t), \;\; F_{opt}^T = (D_{21} D_{21}^T)^{-1} C_2 P \tag{3.49}$$

where P is given by the algebraic Riccati equation (3.30). In order to derive the optimal state estimator, notice that combining (3.49) and (3.45), the dual state-feedback controller (3.49) corresponds to a controller $\eta = -F^T C_1^T \nu$, (3.46), with state-space representation

$$\begin{aligned} \dot{p}(t) &= A^T p(t) + C_1^T \nu(t) - C_2^T F_{opt}^T p(t) \\ &= (A - F_{opt} C_2)^T p(t) + C_1^T \nu(t) \\ \eta(t) &= -F_{opt}^T p(t) \end{aligned} \tag{3.50}$$

Taking the negative of the transpose of (3.50) gives the state-space representation of $(F^T C_1^T)^T = C_1 F$, and thus the optimal estimator $\hat{x} = Fy$,

$$\begin{aligned} \dot{\hat{x}}(t) &= (A - F_{opt} C_2)\hat{x}(t) + F_{opt} y(t) \\ \hat{z}(t) &= C_1 \hat{x}(t) \end{aligned} \tag{3.51}$$

which is equal to (3.28). □

Remark 3.7.
As noted above, the H_2-optimal estimator is dual to an optimal state feedback problem. It follows that for all the properties of the latter there is a corresponding dual property of the former. In particular, it follows from Remark 3.3 that the optimal estimator can be interpreted as a worst-case estimator as follows. By Remark 3.3 the dual state-feedback problem gives the optimal controller (3.49) which minimizes the worst-case performance from $p(0) \in R^n$ to $r \in L_2[0,\infty)$. By duality of the optimal estimation and the optimal state feedback problem, it follows that the optimal estimator can be interpreted in a worst-case setting as the estimator which minimizes the worst-case estimation error

$$\begin{aligned} J_{e,worst}(K) &:= \max_v \left\{ \|x(t) - \hat{x}(t)\|^2 : \|v\|_2 \leq 1 \right\} \\ &= \max_v \left\{ [x(t) - \hat{x}(t)]^T [x(t) - \hat{x}(t)] : \int_{-\infty}^t v^T(\tau) v(\tau) d\tau \leq 1 \right\} \end{aligned} \tag{3.52}$$

This observation gives a nice deterministic interpretation of the Kalman filter as the filter which minimizes the pointwise worst-case estimation error. This property can be compared with the H_∞ estimator, which will be studied in section 4.2.

3.3 The optimal output feedback problem

We are now ready to present the solution to the H_2-optimal control problem for the system shown in Fig. 1.2. The solution will be based on the cost function expansion in equation (3.24). Recall that (3.24) can be written as

$$\begin{aligned} J_2(K) &= \sum_{k=1}^m \left[\int_0^\infty [u(t) - K_{opt}x(t)]^T D_{12}^T D_{12} [u(t) - K_{opt}x(t)] dt : v = e_k \delta(t) \right] \\ &\quad + \mathrm{tr}(B_1^T S B_1) \end{aligned} \tag{3.53}$$

In contrast to the state-feedback case, when only the output y is available to the controller the integrals cannot be made equal to zero. Instead, the cost (3.53) is equivalent to an H_2-optimal state estimation problem, because the best one can do is to compute an estimate $\hat{x}$ of the state, and to determine the control signal according to

$$u(t) = K_{opt}\hat{x}(t) \tag{3.54}$$

giving the cost

$$\begin{aligned} J_2(K) &= \sum_{k=1}^m \left[\int_0^\infty [\hat{x}(t) - x(t)]^T K_{opt}^T D_{12}^T D_{12} K_{opt} [\hat{x}(t) - x(t)] dt : v = e_k \delta(t) \right] \\ &\quad + \mathrm{tr}(B_1^T S B_1) \end{aligned} \tag{3.55}$$

Hence, it follows that the minimum of the quadratic cost (3.8) is achieved by the controller (3.54), where $\hat{x}$ is an H_2-optimal state estimate for the system (3.5). We summarize the result as follows.

Theorem 3.3 H_2**-optimal control.**
Consider the system (3.5). Suppose that the assumptions (A1)–(A4) and (B1)–(B4) hold. Then the controller $u = Ky$ which minimizes the H_2 cost (3.8) is given by the equations

$$\begin{aligned} \dot{\hat{x}}(t) &= (A + B_2 K_{opt})\hat{x}(t) + L[y(t) - C_2\hat{x}(t)] \\ u(t) &= K_{opt}\hat{x}(t) \end{aligned} \tag{3.56}$$

where

$$K_{opt} = -(D_{12}^T D_{12})^{-1} B_2^T S \tag{3.57}$$

and

$$L = P C_2^T (D_{21} D_{21}^T)^{-1} \tag{3.58}$$

and S and P are the symmetric positive (semi)definite solutions of the algebraic Riccati equations (3.15) and (3.30), respectively.

Moreover, the minimum value of the cost achieved by the controller (3.56) is

$$\min_K J_2(K) = \mathrm{tr}(D_{12} K_{opt} P K_{opt}^T D_{12}^T) + \mathrm{tr}(B_1^T S B_1) \tag{3.59}$$

Proof: First we show that the optimal controller (3.56) gives a stable closed loop. Introduce the estimation error $\tilde{x}(t) := x(t) - \hat{x}(t)$. Then the closed loop consisting of the system (3.5) and the controller (3.56) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A + B_2 K_{opt} & -B_2 K_{opt} \\ 0 & A - LC_2 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_1 - LD_{21} \end{bmatrix} v \tag{3.60}$$

Stability of the closed-loop follows from stability of the matrices $A + B_2K_{opt}$ and $A - LC_2$. The fact that the controller (3.56) minimizes the H_2 cost (3.8) follows from the expansion (3.53) of Theorem 3.1 and Theorem 3.2. □

The optimal controller (3.56) consists of an H_2-optimal state estimator, and an H_2-optimal state feedback of the estimated state. A particular feature of the solution is that the optimal estimator and state feedback can be calculated independently of each other. This feature of H_2-optimal controllers is called the *separation principle.*

Remark 3.8.
The optimal controller (3.56) is a classical result in linear optimal control, where one usually assumes that the disturbances are stochastic white noise processes with a Gaussian distribution (cf. Remark 3.6), and the cost is defined as

$$J_{LQG}(K) = \lim_{t_f \to \infty} E\left[\frac{1}{t_f}\int_0^{t_f} [x(t)^T Q x(t) + u(t)^T R u(t)] dt\right] \tag{3.61}$$

cf. Remark 3.1. This problem is known as the *Linear Quadratic Gaussian (LQG)* control problem.

3.4 Some comments on the application of $\mathbf{H}_2$ optimal control

The H_2-optimal (LQG) control problem solves a well-defined optimal control problem defined by the quadratic cost in (3.4), (3.8), or (3.61). Practical applications of the method require, however, that the cost be defined in a manner which corresponds to the control objectives. This is not always very straightforward to achieve, and some observations are therefore in order.

Here, we will only discuss two particular problems in H_2 controller design. The first one concerns the problem of applying the approach to the case with general deterministic inputs, not necessarily impulses. The second problem is concerned with the selection of weights in the quadratic cost (3.61).

$\mathbf{H}_2$-optimal control against general deterministic inputs

In the deterministic interpretation of the H_2 cost, equation (3.8), it is in many cases not very realistic to assume an impulse disturbance. For example, a step disturbance would often be a more realistic disturbance form. It is therefore important to generalize the control problem to other types of deterministic disturbance signals.

Assume that the system (3.5) is subject to an input v with a rational Laplace-transform. Such an input can be characterized in the time domain by a state-space equation

$$\begin{aligned} \dot{x}_v(t) &= A_v x_v(t), \;\; x(0) = b \\ v(t) &= C_v x_v(t) \end{aligned} \tag{3.62}$$

or

$$\begin{aligned} \dot{x}_v(t) &= A_v x_v(t) + bw(t), \;\; w(t) = \delta(t) \\ v(t) &= C_v x_v(t) \end{aligned} \tag{3.63}$$

where the input w is an impulse function. By augmenting the system (3.5) with the disturbance model (3.63), the problem has thus been taken to the standard form with an impulse input.

One problem with the above approach is that for some important disturbance types, such as step, ramp and sinusoidal disturbances, the disturbance model (3.63) is unstable. As the disturbance dynamics is not affected by the controller, the augmented plant consisting of (3.5) and (3.63) will not be stabilizable. It follows that the Riccati equation (3.15) does not have a stabilizable solution in these cases. This problem can be handled in the following way, which we will here demonstrate for the case with a step disturbance.

Assume that the disturbance v is a *step disturbance*, described by

$$v(t) = \begin{cases} 0, & t < 0 \\ v_{step}, & t \geq 0 \end{cases} \tag{3.64}$$

where v_{step} is a constant vector. The disturbance can be characterized by the model

$$\dot{v}(t) = v_{step}\delta(t) \tag{3.65}$$

Notice that this model is not stable. One approach would be to make the model stable, by using $\dot{v}(t) = -av(t) + v_{step}\delta(t)$ for some $a >> 0$. A more direct approach is, however, to eliminate the unstabilizable mode. This is achieved by differentiating the system equations (3.5), giving

$$\begin{aligned} \ddot{x}(t) &= A\dot{x}(t) + B_1\dot{v}(t) + B_2\dot{u}(t) \\ \dot{y}(t) &= C_2\dot{x}(t) + D_{21}\dot{v}(t) \end{aligned} \tag{3.66}$$

Notice that if the input u is weighted in the cost, a steady-state offset after the step disturbance will result. Therefore, it is more realistic to weight the input variation $\dot{u}(t)$ instead. The controlled output z is therefore redefined according to

$$z(t) = C_1 x(t) + D_{12}\dot{u}(t) \tag{3.67}$$

Introducing $z_x := C_1 x$, the system can be written as

$$\begin{aligned} \begin{bmatrix} \ddot{x}(t) \\ \dot{z}_x(t) \\ \dot{y}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 & 0 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ z_x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ D_{21} \end{bmatrix} \dot{v}(t) + \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix} \dot{u}(t) \\ z(t) &= \begin{bmatrix} 0 & I & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ z_x(t) \\ y(t) \end{bmatrix} + D_{12}\dot{u}(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ z_x(t) \\ y(t) \end{bmatrix} \end{aligned} \tag{3.68}$$

This characterization is in standard form with an impulse disturbance input $\dot{v}$ given by (3.65). Notice, however, that assumption (B2) associated with the measurement noise does not hold. In order to satisfy this assumption as well, we can add a noise term v_{meas} on the measured output,

$$y(t) = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ z_x(t) \\ y(t) \end{bmatrix} + v_{meas}(t) \tag{3.69}$$

As there is often a measurement noise present in practical situations, this modification seems to be realistic one.

Selection of weighting matrices in H_2-optimal control

The second design topic which we will discuss is the specification of the controlled output z, or equivalently, the choice of weighting matrices Q and R in the quadratic cost (3.9) or (3.61). For convenience, we will focus the discussion in this section on the stochastic problem formulation, and the associated cost (3.61).

It has been stated that the LQG problem is rather artificial, because the quadratic cost (3.61) seldom, if ever, corresponds to a physically relevant cost in practical applications. This may hold for the quadratic cost (3.61). However, it should be noted that it is often meaningful to consider the variances of the individual variables. Therefore, we introduce the individual costs

$$\begin{aligned} J_i(K) &= \lim_{t_f \to \infty} E\left[\frac{1}{t_f}\int_0^{t_f} x_i(t)^2 dt\right], \quad i = 1, \dots, n \\ J_{n+i}(K) &= \lim_{t_f \to \infty} E\left[\frac{1}{t_f}\int_0^{t_f} u_i(t)^2 dt\right], \quad i = 1, \dots, m \end{aligned} \tag{3.70}$$

Thus, the costs $J_i(K), i = 1, \dots, n$ denote the variances of the state variables, and $J_{n+i}(K)$ denote the variances of the inputs.

In contrast to the quadratic cost (3.61), the individual costs $J_i(K)$ are often physically relevant. If the states x_i represent quality variables (such as concentrations, basis weight etc), then the variance is a measure of the quality variations. The input variances, again, show how much control effort is required. It is therefore well motivated to control the system in such a way that all the individual costs (3.70) are as small as possible.

As all the costs cannot be made arbitrarily small simultaneously, the problem defined in this way is a *multiobjective optimization problem*, and it can be solved efficiently using techniques developed in multiobjective optimization. The set of optimal points in multiobjective optimization is characterized by the set Pareto-optimal, or noninferior, points.

Definition 3.1. (Pareto-optimality)
Consider the multiobjective control problem associated with the individual costs $J_i(K)$, $i = 1, \dots, n+m$ in equation (3.70). A stabilizing controller K_0 is a *Pareto-optimal*, or *noninferior*, controller if there exists no other stabilizing controller K such that

$$\begin{aligned} &J_i(K) \le J_i(K_0) \text{ for all } i = 1, \dots, n+m \text{ and} \\ &J_j(K) < J_j(K_0) \text{ for some } j \in \{1, \dots, n+m\} \end{aligned}$$

Thus, a Pareto-optimal controller is simply any controller such that no individual cost can be improved without making at least one other cost worse. It is clear that the search for a suitable controller should be made in the set of Pareto-optimal controllers. The multiobjective optimal control problem is connected to the H_2 or LQG optimal control problem as follows. It turns out that the set of Pareto-optimal controllers consists of precisely those controllers which minimize linear combinations of the individual costs, i.e.,

$$\begin{aligned} J_{LQG}(K) &= \sum_{i=1}^{n} q_i J_i(K) + \sum_{i=1}^{m} r_i J_{n+i}(K) \\ &= E\left[\frac{1}{t_f}\int_0^{t_f} [x(t)^T Q x(t) + u(t)^T R u(t)] dt\right] \end{aligned} \tag{3.71}$$

where

$$Q = \text{diag}\,(q_1, \dots, q_n)\,, \quad R = \text{diag}\,(r_1, \dots, r_m) \tag{3.72}$$

and q_i, r_i are positive.

The multiobjective control problem is thus reduced to a selection of the weights in the LQ cost in such a way that all the individual costs $J_i(K)$ have satisfactory values. Thus, although the LQ cost may not be physically well motivated as such, it is still relevant via the individual costs $J_i(K)$, which often can be motivated physically. There are systematic methods to find suitable weights (3.72) in such a way that the individual costs have satisfactory values. A particularly simple case is when one cost, $J_j(K)$ say, is the primary cost to be minimized, while the other costs should be restricted. In this case the problem can be formulated as a constrained minimization problem,

$$\text{Minimize } J_j(K) \tag{3.73}$$

subject to

$$J_i(K) \leq c^2, \;\; i = 1, \ldots, n+m, i \neq j \tag{3.74}$$

3.5 Literature

In addition to the books referenced in Chapter 1, there are some excellent classic texts on the optimal LQ and LQG problems. Standard texts on the subject are for example Anderson and Moore (1971, 1979), Åström (1970) and Kwakernaak and Sivan (1970). A newer text with a very extensive treatment of the theory is Grimble and Johnson (1988).

The minimax property of the Kalman filter, which was mentioned in Remark 3.7, has been observed by many people, see for example Krener (1980). The multiobjective approach to LQG design described in Section 3.4 has been discussed in Toivonen and Mäkilä (1989).

References

Anderson, B. D. O. and J. B. Moore (1971). *Linear Optimal Control.* Prentice-Hall, Englewood Cliffs, N.J.

Anderson, B. D. O., and J. B. Moore (1971). *Optimal Filtering.* Prentice-Hall, Englewood Cliffs, N.J.

Åström, K. J. (1970). *Introduction to Linear Stochastic Control Theory.* Academic Press, New York.

Grimble, M. J. and M. A. Johnson (1988). *Optimal Control and Stochastic Estimation.* Volumes 1 and 2. Wiley, Chichester.

Krener, A. J. (1980). Kalman-Bucy and minimax filtering. *IEEE Transactions on Automatic Control Theory* **AC-25**, 291–292.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems.* Wiley, New York.

Toivonen, H. T. and P. M. Mäkilä (1989). Computer-aided design procedure for multiobjective LQG control problems. *Int. J. Control* **49**, 655–666.