# Chapter 4

# The $\mathbf{H}_\infty$-optimal control problem

In this chapter we present a state-space solution to the $H_\infty$-optimal control problem. As in Chapter 3, we consider the control system in Figure 1.2. The performance measure to be minimized is now taken as the $H_\infty$-norm of the closed-loop transfer function, i.e., we consider the cost

$$J_\infty(K) = \|F(G,K)\|_\infty \tag{4.1}$$

where $F(G,K)$ is the transfer function of the closed-loop system,

$$z = F(G,K)v$$

and is given by equation (3.3),

$$F(G,K) = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}$$

The control problem is most conveniently solved in the time domain using a state-space based approach. We will assume that the plant $G$ has the state-space representation

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1v(t) + B_2u(t), \;\; x(0) = 0 \\
z(t) &= C_1x(t) + D_{12}u(t) \\
y(t) &= C_2x(t) + D_{21}v(t)
\end{aligned} \tag{4.2}$$

Notice that the only inputs to the system relevant in the following analysis are $v$ and $u$. Therefore a zero initial state is assumed.

We will make the same assumptions on the system matrices as we made in the context of the $H_2$-optimal control problem. For convenience, we repeat the assumptions here:

**Assumptions:**

**(A1)** The pair $(A, B_2)$ is stabilizable.

**(A2)** $D_{12}^TD_{12}$ is invertible.

**(A3)** $D_{12}^TC_1 = 0$.

**(A4)** The pair $(C_1, A)$ has no unobservable modes on the imaginary axis.

**(B1)** The $(C_2, A)$ is detectable.

**(B2)** $D_{21}D_{21}^T$ is invertible.

**(B3)** $D_{21}B_1^T = 0$.

**(B4)** The pair $(A, B_1)$ has no uncontrallable modes on the imaginary axis.

As was the case in the $H_2$-optimal problem, assumptions (A2)–(A4) and (B2)–(B4) can be relaxed, but they imply convenient simplifications in the solution, and they are not very restrictive.

The direct minimization of the cost $J_\infty(K)$ turns out to be a very hard problem, and it is therefore not feasible to tackle it directly. Instead, it is much easier to construct conditions which state whether there exists a stabilizing controller which achieves the $H_\infty$-norm bound

$$J_\infty(K) < \gamma \tag{4.3}$$

for a given $\gamma > 0$. In that case, the conditions also provide a specific controller which achieves the bound (4.3). One can then use the conditions for checking the achievability of (4.3) for various values of $\gamma$, and in this way determine the minimum of $J_\infty(K)$ to any degree of accuracy. Such a procedure is called *'$\gamma$-iteration'*.

**Remark 4.1.**
As the synthesis of $H_\infty$-optimal controllers is almost exclusively performed via the inequality (4.3), the problem of finding a controller which achieves the $H_\infty$-norm bound (4.3) is commonly called the "$H_\infty$-optimal control problem". It would, however, be more accurate to refer to this problem as a suboptimal $H_\infty$ problem, or an $H_\infty$ performance bound problem.

In order to derive conditions for checking whether there exists a controller which achieves the bound (4.3), recall that the $H_\infty$ performance measure can be characterized in terms of the worst-case gain in terms of $L_2$-norm, i.e. (cf. equation (2.61)),

$$J_\infty(K) = \sup \left\{ \frac{\|z\|_2}{\|v\|_2} : v \neq 0 \right\} \tag{4.4}$$

The performance bound (4.3) is thus equivalent to

$$\frac{\|z\|_2}{\|v\|_2} < \gamma, \quad \text{all } v \neq 0 \tag{4.5}$$

or

$$L(v, u) = \|z\|_2^2 - \gamma^2 \|v\|_2^2 < 0, \quad \text{all } v \neq 0 \tag{4.6}$$

As the $H_\infty$-optimal controller which achieves the bound (4.6) will be derived in the time domain, it is convenient to give an explicit time-domain expression of the inequality (4.6). By Parseval's theorem and the time-domain expression of the $L_2$-norm, equation (2.5), we have that (4.6) is equivalent to

$$L(v, u) = \int_0^\infty [z(t)^T z(t) - \gamma^2 v(t)^T v(t)] dt < 0, \quad \text{all } v \neq 0 \tag{4.7}$$

The problem of finding a controller $u = Ky$ which satisfies the inequality (4.7) for all $v \neq 0$ can now be stated in terms of a max-min problem as

$$\max_{v \neq 0}\{\min_{u=Ky} L(v,u)\} < 0 \tag{4.8}$$

The problem has thus been stated in the form of a particular *dynamic game* problem. Here the first 'player' $v$ tries to make the cost $L(u,v)$ as large as possible, while the second 'player' attempts to guarantee that $L(v,u) < 0$ regardless of the action of $v$. As the system is linear and the cost $L(v,u)$ is quadratic, the associated dynamic game problem is called a *linear quadratic game problem*, in analogy with linear quadratic control.

The solution of the $H_\infty$-optimal control problem via the game problem defined by (4.7) has a similar structure as the solution of the $H_2$ problem described in Chapter 3. Thus, the solution is obtained in two stages: the first stage consists of an $H_\infty$-optimal state-feedback control problem and an associated variable tranformation, and the second stage consists of an $H_\infty$-optimal estimation problem.

## 4.1 The $\mathbf{H_\infty}$-optimal state-feedback problem

The following result gives the optimal *state-feedback law* $u(s) = K_x(s)x(s)$ which achieves the inequality $J_\infty(K_x) < \gamma$, provided such a controller exists.

**Theorem 4.1** $H_\infty$**-optimal state feedback control.**
*Consider the system (4.2). Suppose that the assumptions (A1)–(A4) hold. Assume that the control signal* $u(t)$ *has access to the present and past values of the state,* $x(\tau), \tau \leq t$. *Then there exists a state-feedback controller such that* $J_\infty(K) < \gamma$, *i.e., the inequality (4.7) holds for all* $v \neq 0$, *if and only if there exists a positive (semi)definite solution to the algebraic Riccati equation*

$$A^T X + XA - XB_2(D_{12}^T D_{12})^{-1}B_2^T X + \gamma^{-2}XB_1B_1^T X + C_1^T C_1 = 0 \tag{4.9}$$

*such that the matrix*

$$A - B_2(D_{12}^T D_{12})^{-1}B_2^T X + \gamma^{-2}B_1B_1^T X \tag{4.10}$$

*is stable, i.e. all its eigenvalues have negative real parts.*

*Moreover, when these conditions are satisfied, a controller which achieves the bound (4.7) is given by the static state-feedback controller*

$$u(t) = K_\infty x(t) \tag{4.11}$$

*where*

$$K_\infty = -(D_{12}^T D_{12})^{-1}B_2^T X \tag{4.12}$$

A partial proof of the result will be given along the same lines as the $H_2$-optimal case.

*Proof of $H_\infty$-optimal state-feedback controller (partial):* Assume that the algebraic Riccati equation (4.9) has a positive (semi)definite solution $X$ such that the matrix in (4.10) is stable. Then we have, in analogy with (3.20), the expansion

$$
\begin{aligned}
\int_0^\infty [z(t)^T z(t) \; - \; \gamma^2 v(t)^T v(t)]dt &= \int_0^\infty \left[[C_1 x(t) + D_{12}u(t)]^T [C_1 x(t) + D_{12}u(t)] - \gamma^2 v(t)^T v(t)\right] dt \\
&= \int_0^\infty \Big[[C_1 x(t) + D_{12}u(t)]^T [C_1 x(t) + D_{12}u(t)] - \gamma^2 v(t)^T v(t) + \frac{d(x(t)^T X x(t))}{dt}\Big] dt \\
&\quad - x(\infty)^T X x(\infty) + x(0)^T X x(0) \\
&= \int_0^\infty \Big[[u(t) - u^0(t)]^T D_{12}^T D_{12} [u(t) - u^0(t)] \\
&\qquad - \gamma^2 [v(t) - v^0(t)]^T [v(t) - v^0(t)]\Big] dt \qquad (4.13)
\end{aligned}
$$

where

$$
\begin{aligned}
u^0(t) &= -(D_{12}^T D_{12})^{-1} B_2^T X x(t) \\
&= K_\infty x(t) \qquad (4.14) \\
v^0(t) &= \gamma^{-2} B_1^T X x(t) \qquad (4.15)
\end{aligned}
$$

and where we have introduced $x(0) = 0$ (by assumption) and $x(\infty) = 0$ (by stability).

The expansion (4.13) shows that using the feedback (4.11), $u(t) = u^0(t)$, the inequality $L(v,u) \leq 0$ holds for all $v$. With this feedback, the maximum value for $L(v,u)$ is equal to zero, and is achieved if and only if $v(t) = v^0(t)$ for all $t$. But as $x(0) = 0$, this implies that $x(t) = 0$, and hence $v(t) = 0$ for all $t \geq 0$. Hence it follows that the inequality (4.7) holds for all $v \neq 0$.

The proof can be completed by showing that if a state-feedback controller which achieves the bound (4.7) exists, then the algebraic Riccati equation (4.9) has a symmetric positive definite or semidefinite solution such that (4.10) is stable. One way to show this part of the result is to consider a finite-horizon control problem, and take the limit as the final time goes to infinity. We omit this part of the proof, which is more technical. □

It is seen that the $H_\infty$-optimal state-feedback law is obtained in a very similar manner as the $H_2$-optimal controller. In order to derive the $H_\infty$-optimal output feedback controller we make use of the expansion (4.13), which transforms the problem to an equivalent $H_\infty$-optimal estimation problem.

Referring to the expansion (4.13), define the signals

$$
\begin{aligned}
\tilde{z}(t) &= (D_{12}^T D_{12})^{1/2} [u(t) - u^0(t)] \\
&= (D_{12}^T D_{12})^{1/2} u(t) + (D_{12}^T D_{12})^{-1/2} B_2^T X x(t) \qquad (4.16) \\
\tilde{v}(t) &= v(t) - v^0(t) \\
&= v(t) - \gamma^{-2} B_1^T X x(t) \qquad (4.17)
\end{aligned}
$$

Then we have

$$
L(v,u) = \int_0^\infty [z(t)^T z(t) - \gamma^2 v(t)^T v(t)]dt = \int_0^\infty [\tilde{z}(t)^T \tilde{z}(t) - \gamma^2 \tilde{v}(t)^T \tilde{v}(t)]dt \qquad (4.18)
$$

or

$$L(v,u) = \|z\|_2^2 - \gamma^2\|v\|_2^2 = \|\tilde{z}\|_2^2 - \gamma^2\|\tilde{v}\|_2^2 \tag{4.19}$$

Then the inequality (4.7) holds for all $v \neq 0$ if and only if

$$\tilde{L}(\tilde{v},u) = \|\tilde{z}\|_2^2 - \gamma^2\|\tilde{v}\|_2^2 < 0, \quad \text{all } \tilde{v} \neq 0 \tag{4.20}$$

The problem of finding a controller which achieves the bound (4.7) can thus be stated in terms of the signals $\tilde{v}$ and $\tilde{z}$ and using the equivalent bound (4.20). Introducing the signals $\tilde{v}$ and $\tilde{z}$ into the system equations (4.2) gives

$$\begin{aligned} \dot{x}(t) &= (A + \gamma^{-2}B_1B_1^TX)x(t) + B_1\tilde{v}(t) + B_2u(t), \;\; x(0) = 0 \\ \tilde{z}(t) &= (D_{12}^TD_{12})^{-1/2}B_2^TXx(t) + (D_{12}^TD_{12})^{1/2}u(t) \\ y(t) &= (C_2 + \gamma^{-2}D_{21}B_1^TX)x(t) + D_{21}\tilde{v}(t) \end{aligned} \tag{4.21}$$

or,

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \tilde{B}_1\tilde{v}(t) + \tilde{B}_2u(t), \;\; x(0) = 0 \\ \tilde{z}(t) &= \tilde{C}_1x(t) + \tilde{D}_{12}u(t) \\ y(t) &= \tilde{C}_2x(t) + \tilde{D}_{21}\tilde{v}(t) \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \tilde{A} &= A + \gamma^{-2}B_1B_1^TX \\ \tilde{B}_1 &= B_1 \\ \tilde{B}_2 &= B_2 \\ \tilde{C}_1 &= (D_{12}^TD_{12})^{-1/2}B_2^TX \\ \tilde{C}_2 &= C_2 + \gamma^{-2}D_{21}B_1^TX \\ \tilde{D}_{12} &= (D_{12}^TD_{12})^{1/2} \\ \tilde{D}_{21} &= D_{21} \end{aligned} \tag{4.23}$$

Notice that in the state-feedback case the output $\tilde{z}$ can be made equal to zero by the controller (4.11), $u(t) = K_\infty x(t)$. In the output-feedback case the best one can do is to base the controller on an estimate of $x(t)$, or rather of the output $\tilde{z}(t)$. We are thus led to study an $H_\infty$-optimal estimation problem.

## 4.2 The $H_\infty$-optimal estimation problem

In this section we study the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1v(t), \;\; x(0) = 0 \\ z(t) &= C_1x(t) \\ y(t) &= C_2x(t) + D_{21}v(t) \end{aligned} \tag{4.24}$$

We consider stable causal estimators $F$ of the output $z$ based on the measured output $y$ such that $\hat{z}(s) = F(s)y(s)$. In the $H_\infty$-optimal estimation problem, we define the $H_\infty$ norm of the transfer function from the disturbance $v$ to the estimation error $z - \hat{z}$,

$$J_{e,\infty}(F) = \sup \left\{ \frac{\|z - \hat{z}\|_2}{\|v\|_2} : v \neq 0 \right\} \tag{4.25}$$

In analogy with the state-feedback problem, we consider conditions for the existence of an estimator which achieves the bound

$$J_{e,\infty}(F) < \gamma \tag{4.26}$$

or, equivalently,

$$L_e(v, \hat{z}) = \|z - \hat{z}\|_2^2 - \gamma^2 \|v\|_2^2 < 0, \quad \text{all } v \neq 0 \tag{4.27}$$

The solution of the $H_\infty$-optimal estimation problem can be characterized as follows.

**Theorem 4.2 $H_\infty$-optimal estimator.**
*Consider the system (4.24). Suppose that the assumptions (B1)–(B4) hold. There exists a stable estimator $F$ which achieves the $H_\infty$-norm bound $J_{e,\infty}(F) < \gamma$ if and only if there exists a symmetric positive definite or semidefinite solution $Y$ to the algebraic Riccati equation*

$$AY + YA^T - YC_2^T(D_{21}D_{21}^T)^{-1}C_2Y + \gamma^{-2}YC_1^TC_1Y + B_1B_1^T = 0 \tag{4.28}$$

*such that the matrix*

$$A - YC_2^T(D_{21}D_{21}^T)^{-1}C_2 + \gamma^{-2}YC_1^TC_1 \tag{4.29}$$

*is stable, i.e. all its eigenvalues have negative real parts.*

*Moreover, when these conditions are satisfied, an estimator which achieves the bound (4.26) is given by*

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + L_\infty[y(t) - C_2\hat{x}(t)], \ \ \hat{x}(0) = 0 \\ \hat{z}(t) &= C_1\hat{x}(t) \end{aligned} \tag{4.30}$$

*where*

$$L_\infty = YC_2^T(D_{21}D_{21}^T)^{-1} \tag{4.31}$$

Notice that the $H_\infty$-optimal estimator has a similar structure as the $H_2$-optimal estimator given in Section 3.2. In particular, the estimator has the same order as the system. In contrast to the $H_2$-optimal estimator, the $H_\infty$-optimal estimator also depends on the output $z$ which is estimated, cf. the dependence of (4.28) on the matrix $C_1$.

**Remark 4.2.**
In analogy with the $H_2$-optimal estimator, the solution defined by (4.31) and (4.28) is the

dual to the solution (4.12) and (4.9) of the optimal state-feedback problem in the sense that (4.12) and (4.9) are taken to (4.31), (4.28) by making the substitutions in (3.33).

*Proof of $H_\infty$-optimal estimator (partial):* The $H_\infty$-optimal estimator can be proved via a dual problem in a similar way as the $H_2$-optimal estimator. Using the fact that the $H_\infty$-norm of a transfer function matrix equals the norm of the transpose,

$$\|G\|_\infty = \|G^T\|_\infty \tag{4.32}$$

(because $G$ and $G^T$ have the same singular values, cf. (2.74)), it can be shown in analogy with Section 3.2 that the $H_\infty$-optimal estimation problem is equivalent to an $H_\infty$-optimal state-feedback problem for a transposed system. More precisely, note that the estimation error is given by

$$z - \hat{z} = [G_1 - FG_2]v \tag{4.33}$$

where

$$G_1 = C_1(sI - A)^{-1}B_1, \;\; G_2 = C_2(sI - A)^{-1}B_1 + D_{21} \tag{4.34}$$

Hence, by the definition in (4.25),

$$J_{e,\infty}(F) = \|G_1 - FG_2\|_\infty = \|G_1^T - G_2^T F^T\|_\infty \tag{4.35}$$

Here, the latter expression is the $H_\infty$ norm of the closed loop of the dual system

$$r = [\, G_1^T \quad G_2^T \,] \begin{bmatrix} \nu \\ \eta \end{bmatrix} \tag{4.36}$$

and the control law

$$\eta = -F^T \nu, \tag{4.37}$$

because the closed loop is given by

$$r = (G_1^T - G_2^T F^T)\nu \tag{4.38}$$

The $H_\infty$-optimal estimation problem has thus been transformed to an $H_\infty$-optimal control problem. In analogy with the $H_2$-optimal estimation problem, we observe that the dual system (4.36) has the state-space representation

$$\begin{aligned} \dot{p}(t) &= A^T p(t) + C_1^T \nu(t) + C_2^T \eta(t), \;\; p(0) = 0 \\ r(t) &= B_1^T p(t) + D_{21}^T \eta(t) \end{aligned} \tag{4.39}$$

The optimal dual controller (4.37) which achieves the performance bound on $J_{e,\infty}(K)$ can be derived as an optimal state-feedback controller for the dual system (4.39) due to the fact that the state $p$ is completely known from the inputs $\nu$ and $\eta$. Taking the transpose of the optimal dual controller gives the optimal state etimator (4.30). The derivation is completely analogous with the $H_2$ problem, and we omit the details. □

## 4.3 The $H_\infty$-optimal output feedback problem

In this section, the state-feedback result of Section 4.1 and the estimator presented in Section 4.2 are combined to solve the $H_\infty$-optimal control problem for the system shown in Figure 1.2.

Recall that the $H_\infty$-norm bound (4.3), or the inequality (4.7), holds for the system (4.2) if and only if the inequality (4.20) holds for the system (4.21). Whereas the output $\tilde{z}$ could be made identically equal to zero by the state-feedback controller (4.11), the best one can do when only the output $y$ is available to the controller is to compute an estimate $\hat{\tilde{z}}$ of the output, and to determine the control signal such that $\hat{\tilde{z}} = 0$. More precisely, from Theorem 4.2 we know that an $H_\infty$-optimal estimate of the output $\tilde{z}$ of the transformed system (4.22) can be written as

$$\hat{\tilde{z}}(t) = \tilde{C}_1 \hat{x}(t) + \tilde{D}_{12} u(t) \tag{4.40}$$

As $\tilde{D}_{12} = D_{12}^T D_{12}$ is invertible, we can apply the control law

$$\begin{aligned} u(t) &= -\tilde{D}_{12}^{-1} \tilde{C}_1 \hat{x}(t) \\ &= -(D_{12}^T D_{12})^{-1} B_2^T X \hat{x}(t) \\ &= K_\infty \hat{x}(t) \end{aligned} \tag{4.41}$$

making $\hat{\tilde{z}}(t) = 0$. With this controller, the output of the system (4.22) is given by $\tilde{z} = \tilde{C}_1(x - \hat{x})$, and the cost $\tilde{L}(\tilde{v}, u)$ in equation (4.20) is

$$\begin{aligned} \tilde{L}(\tilde{v}, u) &= \|\tilde{C}_1(x - \hat{x})\|_2^2 - \gamma^2 \|\tilde{v}\|_2^2 \\ &= \|z - \hat{\tilde{z}}\|_2^2 - \gamma^2 \|\tilde{v}\|_2^2 \end{aligned} \tag{4.42}$$

Hence, the inequality (4.20), $\tilde{L}(\tilde{v}, u) < 0$ holds for all $\tilde{v} \neq 0$ if and only if the estimate $\hat{\tilde{z}}$ satisfies the bound

$$L_e(\tilde{v}, \hat{\tilde{z}}) = \|z - \hat{\tilde{z}}\|_2^2 - \gamma^2 \|\tilde{v}\|_2^2 < 0, \quad \text{all } \tilde{v} \neq 0 \tag{4.43}$$

But from Section 4.2 we know how to construct an estimator which achieves the bound (4.43) for the system (4.22), provided such an estimator exists. Thus we have the following characterization of the $H_\infty$-optimal controller.

**Theorem 4.3** $H_\infty$**-optimal control problem.**
*Consider the system (4.2). Suppose that the assumptions (A1)–(A4) and (B1)–(B4) hold. Then there exists a controller $u = Ky$ which achieves the $H_\infty$-norm bound (4.3) $J_\infty(K) < \gamma$, or equivalently, the inequality (4.7), if and only if the following conditions are satisfied:*

*(a) There exists a symmetric positive definite or semidefinite solution $X$ to the Riccati equation (4.9) such that the matrix in (4.10) is stable.*

*(b) There exists a symmetric positive definite or semidefinite solution $Z$ to the algebraic Riccati equation associated with the system (4.22) and the estimation performance bound (4.43),*

$$\tilde{A}Z + Z\tilde{A}^T - Z\tilde{C}_2^T(\tilde{D}_{21}\tilde{D}_{21}^T)^{-1}\tilde{C}_2 Z + \gamma^{-2}Z\tilde{C}_1^T\tilde{C}_1 Z + \tilde{B}_1\tilde{B}_1^T = 0 \tag{4.44}$$

*such that the matrix*

$$\tilde{A} - Z\tilde{C}_2^T(\tilde{D}_{21}\tilde{D}_{21}^T)^{-1}\tilde{C}_2 + \gamma^{-2}Z\tilde{C}_1^T\tilde{C}_1 \tag{4.45}$$

*is stable, i.e. all its eigenvalues have negative real parts.*

*Moreover, when these conditions are satisfied, a controller which achieves the performance bound (4.3) is given by*

$$\begin{aligned}\dot{\hat{x}}(t) &= \tilde{A}\hat{x}(t) + \tilde{B}_2 u(t) + L_Z[y(t) - \tilde{C}_2\hat{x}(t)]\\ u(t) &= K_\infty\hat{x}(t)\end{aligned} \tag{4.46}$$

*where*

$$K_\infty = -(D_{12}^T D_{12})^{-1}B_2^T X \tag{4.47}$$

*and*

$$L_Z = Z\tilde{C}_2^T(\tilde{D}_{21}\tilde{D}_{21}^T)^{-1} \tag{4.48}$$

*Proof:* By Theorem 4.1, condition (a) is necessary and sufficient for the existence of a state-feedback controller which achieves the $H_\infty$-norm bound. Hence it is a necessary condition for the existence of an output feedback controller which achieves the norm bound. The fact that (b) is a sufficient condition follows from the construction of the controller, and by showing that the resulting closed-loop system is stable.

Necessity of (b) can be shown by assuming that there exists a controller $u = Ky$ which achieves the $H_\infty$-norm bound. Then (4.20) holds for the closed-loop system. Form the estimator

$$\begin{aligned}\dot{\hat{x}}(t) &= \tilde{A}\hat{x}(t) + \tilde{B}_2\left[u(t) - (Ky)(t)\right]\\ \hat{\tilde{z}}(t) &= \tilde{C}_1\hat{x}(t) + \tilde{D}_{12}\left[u(t) - (Ky)(t)\right]\end{aligned} \tag{4.49}$$

For the input $u = Ky$, we have $\hat{\tilde{z}} \equiv 0$, and it follows that the estimator achieves the performance bound (4.43) since (4.20) holds. But as the control input $u(t)$ is exactly known, and its effect on the system (4.22) and the estimator (4.49) is equivalent, it follows that the above estimator achieves the performance bound for the system (4.22) for any input, or equivalently, the condition (b) is satisfied. The estimator constructed in this way is not necessarily stable. It can be shown that a stable estimator which achieves the performance bound can also be constructed, but the proof is omitted. □

Notice that introducing the control law and the matrices (4.23) into the equation for $\hat{x}$, the controller (4.46) can be written as

$$\begin{aligned}\dot{\hat{x}}(t) &= [A + \gamma^{-2}B_1B_1^T X - B_2(D_{12}^T D_{12})^{-1}B_2^T X - L_Z(C_2 + \gamma^{-2}D_{21}B_1^T X)]\hat{x}(t) + L_Z y(t)\\ u(t) &= -(D_{12}^T D_{12})^{-1}B_2^T X\hat{x}(t)\end{aligned} \tag{4.50}$$

The $H_\infty$-optimal controller (4.46) has a similar structure as the $H_2$-optimal controller presented in Section 3.3, and it consists of an $H_\infty$-optimal estimator, and an $H_\infty$-optimal state-feedback of the state of the optimal estimator. However, a notable difference from the $H_2$-optimal control problem is that the estimator part of the controller now depends on the optimal state-feedback controller via the transformed system (4.22). Hence the separation principle which was valid in the $H_2$-optimal control problem, does *not* hold in the $H_\infty$-optimal control problem.

However, in spite of the fact that the separation principle does not hold, it turns out that it is not necessary to explicitly solve the estimator Riccati equation (4.44) associated with the transformed system (4.22), but it is sufficient to solve the estimator Riccati equation (4.28) associated with the *untransformed* system (4.24). This is due to a remarkable connection between the solutions $X$, $Y$ and $Z$ of (4.9), (4.28) and (4.44), which we present in the following theorem.

**Theorem 4.4 Connection between $X$, $Y$ and $Z$.**
*Suppose that the Riccati equation (4.9) has a symmetric positive (semi)definite solution $X$. Then the Riccati equation (4.44) has a symmetric positive (semi)definite solution $Z$ if and only if:*

*(a) there exists symmetric positive (semi)definite solution $Y$ to (4.28), and*

*(b) $\rho(XY) < \gamma^2$, where the $\rho(XY)$ denotes the maximum eigenvalue of $XY$.*

*Moreover, when the above conditions hold, the solution $Z$ of (4.44) is given by*

$$Z = Y(I - \gamma^{-2}XY)^{-1} \tag{4.51}$$

*Proof:* ($\Rightarrow$) Assume first that the Riccati equation (4.44) has a symmetric positive (semi)definite solution $Z$. It is straightforward to verify that the matrix $Y = Z(I + \gamma^{-2}XZ)^{-1}$ obtained by solving (4.51) for $Y$ is symmetric and positive semidefinite, and it satisfies the Riccati equation (4.28). Moreover, we have from Lemma 4.1 below,

$$\rho(XY) = \rho(XZ(I + \gamma^{-2}XZ)^{-1}) = \gamma^2\rho(XZ(\gamma^2 I + XZ)^{-1}) < \gamma^2$$

Hence (a) and (b) hold.

($\Leftarrow$) In order to show the converse, assume that (a) and (b) hold. Then it is straightforward to verify that the matrix $Z$ defined by (4.51) is symmetric and positive semidefinite, and it satisfies the Riccati equation (4.44). □

In the proof of Theorem 4.4 we have used the following result.

**Lemma 4.1**
*Let $X$ and $Z$ be symmetric, positive semidefinite matrices. Then*

$$\rho(XZ(\gamma^2 I + XZ)^{-1}) = \frac{\rho(XZ)}{\gamma^2 + \rho(XZ)} < 1 \tag{4.52}$$

Combining Theorems 4.3 and 4.4, the solution of the $H_\infty$-optimal control problem can be summarized in the following theorem.

**Theorem 4.5** **$H_\infty$-optimal controller.**
*Consider the system (4.2). Suppose that the assumptions (A1)–(A4) and (B1)–(B4) hold. Then there exists a stabilizing controller $u = Ky$ which achieves the $H_\infty$-norm bound (4.3) $J_\infty(K) < \gamma$, or equivalently, (4.7), if and only if the following conditions are satisfied:*

*(a) There exists a symmetric positive definite or semidefinite solution $X$ to the Riccati equation (4.9) such that the matrix in (4.10) is stable.*

*(b) There exists a symmetric positive definite or semidefinite solution $Y$ to the Riccati equation (4.28) such that the matrix in (4.29) is stable.*

*(c) $\rho(XY) < \gamma^2$.*

*Moreover, when these conditions are satisfied, a controller which achieves the performance bound (4.3) is given by (4.46), or (4.50), where $L_Z$ is given by (4.48) and $Z$ is given by (4.51).*

**Remark 4.3.**
When the conditions of Theorem 4.5 are satisfied, there are obviously many controllers which achieve the $H_\infty$-norm bound $J_\infty(K) < \gamma$. The $H_\infty$-optimal controller (4.50) is called the *central controller*, because in a certain sense it is in the center of the set of all controllers which achieve the norm bound. It has also the property that it maximizes a certain entropy function. Notice that in the limit as $\gamma \to \infty$, the central $H_\infty$-optimal controller approaches the $H_2$-optimal controller (cf. the Riccati equations (3.15), (4.9) and (3.30), (4.28), respectively).

**Remark 4.4.**
The solution of the $H_\infty$-optimal control problem shows that when there exists a controller which achieves the $H_\infty$-norm bound, it can always be achieved with a controller of order $n = \dim(x)$. In fact, it can be shown that there is also a controller of order $n-1$ which achieves the bound. There are, however, at present no systematic methods to find such a controller of order $n-1$.

**Remark 4.5.**
The controller (4.50) was derived by starting from the optimal state-feedback controller and a variable transformation associated with the expansion (4.13), which reduces the problem to an optimal estimation problem. There is an analogous dual procedure, which starts from the optimal state estimator, and a variable which results in a state-feedback problem. This approach produces an alternative state-space realization of the central controller, which is different from the realization (4.50).

So far we have only constructed a controller which achieves the $H_\infty$ bound (4.3) for a given performance level $\gamma > 0$. In order to make the closed-loop $H_\infty$ norm as small as possible one must iterate on $\gamma$, i.e. check whether there exists a controller which achieves the performance bound (4.3), $J_\infty(K) < \gamma$, for various values of $\gamma$ until the minimum achievable $H_\infty$ norm has been determined to a sufficient degree of accuracy. This solution procedure is called $\gamma$-iteration. The minimum achievable $H_\infty$ norm is commonly denoted $\gamma_{inf}$, because it is the infimum (cf. the definition of infimum or greatest lower bound in Example 2.1) of all $\gamma$ such that the bound (4.3) is achievable by any controller,

$$\gamma_{inf} = \inf \{ \|F(G,K)\|_\infty : u = Ky, \ \ K \text{ stabilizing } \} \tag{4.53}$$

Suppose we want to find a controller which achieves a closed-loop $H_\infty$ norm which exceeds the minimum achievable norm by a tolerance level $\delta > 0$, i.e., $\|F(G,K)\|_\infty \leq \gamma_{inf} + \delta$. Then a simple iterative bisection method based on the controller synthesis method described above is given as follows.

**Algorithm ($\gamma$-iteration).**

*Step 0.* Select $\underline{\gamma}$ and $\bar{\gamma}$ such that $\underline{\gamma} < \gamma_{inf}$ and $\bar{\gamma} > \gamma_{inf}$.

*Step 1.* Set $\gamma = (\underline{\gamma} + \bar{\gamma})/2$.

*Step 2.* Check whether there exists a controller which achieves the bound (4.3), i.e., whether conditions (a)–(c) of Theorem 4.5 for the existence of an $H_\infty$-optimal controller are satisfied. If the conditions are satisfied, set $\bar{\gamma} = \gamma$ and go to step 3. If the conditions are not satisfied, set $\underline{\gamma} = \gamma$ and go to step 3.

*Step 3.* If $\bar{\gamma} - \underline{\gamma} > \delta$, continue from step 1. If $\bar{\gamma} - \underline{\gamma} \leq \delta$, the $H_\infty$-optimal controller which achieves the bound (4.3) with $\gamma = \bar{\gamma}$ satisfies $\|F(G,K)\|_\infty \leq \gamma_{inf} + \delta$.

Routines which implement the synthesis equations and the $\gamma$-iteration procedure for $H_\infty$-optimal control problem are `hinfopt` in `MATLAB`s *Robust Control Toolbox* and `hinfsyn` in the *$\mu$-Analysis and Synthesis Toolbox.* The routines solve the more general case where the orthogonality assumptions (A3) and (B3) need not hold, i.e., we may have $D_{12}^T C_1 \neq 0$ and $D_{21} B_1^T \neq 0$.

## 4.4 Historical notes

In order to put the methods presented in this chapter into perspective, a few historical notes are in order.

While linear quadratic ($H_2$) optimal control was applied successfully during the 1960's and 1970's, mainly to aerospace problems, its failure to explicitly address robustness issues was critized by a number of workers. As robustness is most naturally treated in a frequency domain framework, these researchers continued to develop frequency-domain methods during that time.

The $H_\infty$ control problem, and its connection to robustness, which will be discussed in Chapter 5, was introduced in the seminal work by George Zames in the late 1970's (Zames 1981). First the theory was presented entirely in the frequency domain, and the computation of $H_\infty$-optimal controllers was based on analytic function theory and operator-theoretic methods. These methods were quite complicated, and they gave only a limited insight into the structure of the solutions. For example, before the introduction of the state-space solution in 1988 it was not known what order a controller was required to have to achieve an $H_\infty$-norm bound. Also, the common folklore regarding the difficulty of $H_\infty$ control goes back to this period. An excellent exposition of theses methods is given in the book by Francis (1987).

The connection between the $H_\infty$ problem and linear quadratic game theory was first observed by Petersen (1987), and the stationary state feedback $H_\infty$ problem was then presented in Khargonekar, Petersen and Rotea (1988) and Zhou and Khargonekar (1988). The general state-space solution to the $H_\infty$-optimal control problem was first given by Glover and Doyle (1988), and developed in full in the classic paper by Doyle, Glover, Khargonekar and Francis (1989), and the preliminary conference version in Doyle *et al.* (1988).

The state-space solution revolutionized the practical numerical computation of $H_\infty$-optimal controllers. Whereas the calculation of an $H_\infty$-optimal controller before 1988 had been an extremely demanding task, the state-space solution has approximately the same complexity as the standard linear quadratic control problem. However, the original papers (Doyle *et al.* 1988, 1989) did not do much to increase the general understanding of $H_\infty$ control, and it is fair to say that even not many researchers in the field can completely read the original papers by Doyle *et al.* (1988, 1989). In fact, the paper by Doyle *et al.* (1989) has been called 'the most important unreadable paper in the history of control science'. Since then, increased understanding of the problem has made it possible to present the $H_\infty$-optimal controller along the significantly simpler lines as has been done in this chapter. This approach has been developed gradually by many people during the 1990's.

Somewhat ironically, the $H_\infty$-optimal control problem, which was originally introduced in the frequency domain to deal with the robustness issues which the then prevailing time-domain linear quadratic control methods failed to address, is today solved by algorithms which have a close resemblance to the procedures used to solve precisely the very same linear quadratic control problems.

Linear quadratic game theory, with which the state-space $H_\infty$ solution is closely related, was largely developed in the 1960's. The standard reference in this field is the book by Başar and Olsder (1982). After the connection between problem and dynamic game theory was first realized, it became clear that much of the required theory was already available in the game-theory literature. In particular, the optimal state-feedback result (Theorem 4.1) was fully developed in classical linear-quadratic game theory. A thorough presentation of the game-theory approach to $H_\infty$ control can be found in the book by Başar and Bernhard (1991).

The solution of the $H_\infty$-optimal estimation problem has been given in the (hard-to-read) paper by Nagpal and Khargonekar (1991). An alternative, and simpler, derivation is given by Banavar and Speyer (1991).

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