Chapter 6

Structured uncertainties and structured singular value synthesis

In Chapter 5, the only assumption imposed on the uncertainties was their norm-boundedness. No structure was assumed on the uncertainty blocks. In many cases, however, more is known about the set of plant uncertainties, and the robust stability condition introduced in Chapter 5 may then lead to unnecessarily conservative designs. Therefore, various forms of structured uncertainties have also been studied. In this chapter, an important type of structured uncertainty will be considered, which gives rise to structured singular value synthesis.

Example 6.1.
Consider a system with one input and two outputs described by $y = Gu$, with

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tag{6.1}$$

where each of G_1 and G_2 are described by the uncertainty description (5.4), i.e.,

$$G_1 = G_{0,1} + W_{01}\Delta_1, \quad \|\Delta_1\|_\infty \le 1 \tag{6.2}$$

$$G_2 = G_{0,2} + W_{02}\Delta_2, \quad \|\Delta_2\|_\infty \le 1 \tag{6.3}$$

Hence the plant is described by

$$\begin{aligned} G &= \begin{bmatrix} G_{0,1} \\ G_{0,2} \end{bmatrix} + \begin{bmatrix} W_{01}\Delta_1 \\ W_{02}\Delta_2 \end{bmatrix} \\ &= \begin{bmatrix} G_{0,1} \\ G_{0,2} \end{bmatrix} + \begin{bmatrix} W_{01} & 0 \\ 0 & W_{02} \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} \\ &= G_0 + W_1 \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} W_2 \end{aligned} \tag{6.4}$$

where

$$G_0 = \begin{bmatrix} G_{0,1} \\ G_{0,2} \end{bmatrix} \tag{6.5}$$

$$W_1 = \begin{bmatrix} W_{01} & 0 \\ 0 & W_{02} \end{bmatrix}, \quad W_2 = \begin{bmatrix} I \\ I \end{bmatrix} \tag{6.6}$$

The uncertain plant characterization in (6.4) is of the same form as (5.11), but with the difference that the uncertainty is now restricted to the structure

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}, \quad \|\Delta_i\|_\infty \leq 1, \ \ i = 1, 2 \tag{6.7}$$

The uncertainty structure in (6.4) is a subset of the unstructured norm-bounded uncertainty studied in Chapter 5. Hence the robust stability conditions given in Chapter 5 are conservative if applied to the uncertain plant described by (6.4), (6.7). Therefore, special procedures for dealing with structured norm-bounded uncertainties have been developed.

A general structured norm-bounded uncertainty with norm bound equal to $\delta > 0$ is defined by the set

$$\mathbf{\Delta}_s(\delta) = \left\{ \Delta = \text{block diag}(\Delta_1, \ldots, \Delta_s), \ \ \Delta_i \in H_\infty^{r_i \times r_i}, \|\Delta_i\|_\infty \leq \delta \right\} \tag{6.8}$$

where $H_\infty^{r_i \times r_i}$ denotes the set of stable $r_i \times r_i$ transfer functions with bounded H_∞ norm. The uncertainty set $\mathbf{\Delta}_s(\delta)$ is thus characterized by the following set of parameters: the number of blocks s, the dimensions of the uncertainty blocks, r_i, and the 'uncertainty radius' δ, i.e., the maximum norm of the blocks Δ_i. Notice that the structure of $\Delta \in \mathbf{\Delta}_s(\delta)$ implies that $\|\Delta\|_\infty \leq \delta$ as well.

An uncertain plant with structured uncertainty is described by

$$G = G_0 + W_1 \Delta W_2, \ \ \Delta \in \mathbf{\Delta}_s(\delta) \tag{6.9}$$

or by the equivalent representations of Figures 5.2 and 5.3, where the uncertainty Δ is now assumed to belong to the set $\mathbf{\Delta}_s(\delta)$. For convenience, we formulate the robust stabilization problem for plants with structured uncertainties.

Robust stabilization problem for plants with structured uncertainties. *Consider the uncertain plant described by (6.9) and the uncertainty set (6.8). Find a controller such that the control system in Figure 5.1 (or equivalently, Figures 5.2 and 5.3) is stable for all uncertainties $\Delta \in \mathbf{\Delta}_s(\delta)$.*

In analogy with (5.17), the closed-loop plant F defined by (5.15) (Figure 5.3) is stable for all $\Delta \in \mathbf{\Delta}_s(\delta)$ if and only if

$$\det(I - F(j\omega)\Delta(j\omega)) \neq 0, \ \ \text{all } \omega \text{ and } \Delta \in \mathbf{\Delta}_s(\delta) \tag{6.10}$$

In contrast to the case with unstructured uncertainties, the condition (6.10) cannot be reduced to a simple condition on the H_∞ norm of the closed-loop transfer function F. Instead, based on (6.10) a quantity $\mu(F(j\omega))$ is defined as follows. Introduce the smallest value of the uncertainty magnitude δ such that there exists $\Delta \in \mathbf{\Delta}_s(\delta)$ which destabilizes the plant, i.e.,

$$\delta_{min}(F(j\omega)) = \min \left\{ \delta : \det(I - F(j\omega)\Delta(j\omega)) = 0 \text{ for some } \Delta \in \mathbf{\Delta}_s(\delta) \right\} \tag{6.11}$$

From the definition of $\delta_{min}(F(j\omega))$ it follows that the uncertain plant is stable for all structured uncertainties $\Delta \in \mathbf{\Delta}_s(\delta)$ with the uncertainty radius δ, if and only if

$$\delta < \delta_{min}(F(j\omega)), \text{ all } \omega \tag{6.12}$$

Thus, robust stability with respect to the set of structured uncertainties can be characterized by the quantity

$$\mu(F(j\omega)) = \delta_{min}(F(j\omega))^{-1} \tag{6.13}$$

so that the uncertain plant is stable for all $\Delta \in \mathbf{\Delta}_s(\delta)$ with the uncertainty radius δ, if and only if

$$\mu(F(j\omega)) < \delta^{-1}, \text{ all } \omega \tag{6.14}$$

The quantity $\mu(F)$ is called the *structured singular value* of F. It was introduced by John C. Doyle in 1982 (Doyle 1982) in order to provide a condition for robust stability with respect to structured uncertainties. Notice that the structured singular value depends not only on F itself but also on the structure of the uncertainty set $\mathbf{\Delta}_s(\delta)$ characterized by the number of block s and their dimensions r_i. The structured singular value can be considered as a kind of generalization of the maximum singular value $\bar{\sigma}$, and it reduces to $\bar{\sigma}$ in the case when there is only one uncertainty block; $\mu(F) = \bar{\sigma}(F)$, if $s = 1$.

For convenience, we summarize the stability condition for structured linear time-invariant (LTI) uncertainties below.

Theorem 6.1 Robust stability against structured LTI uncertainties. *Consider the system in Figure 5.2 or 5.3, where the uncertainty is assumed to be structured according to (6.8). The system is stable for all uncertainties $\Delta \in \mathbf{\Delta}_s(\delta)$ if and only if the nominal closed-loop transfer function $F = F(P, K)$ is stable and*

$$\sup_{\omega} \mu(F(j\omega)) < \delta^{-1} \tag{6.15}$$

The robust stabilization problem for plants with structured uncertainty thus consists of finding a stabilizing controller such that the structured singular value of the closed-loop systems satisfies the bound (6.15). The synthesis of controllers which are (sub)optimal with respect to a performance criterion expressed in terms of a structured singular value is called *structured singular value synthesis* or *μ-synthesis.*

The definition (6.11), (6.13) of the structured singular value does not provide much help for computing its value. The structured singular value turns out to be very hard to calculate numerically (cf. Toker and Özbay (1995)), and no efficient algorithm for its computation exists. Instead, a more tractable approach is to calculate an upper bound on μ which is easier to compute. In particular, define the set of diagonal matrices D with the following structure corresponding to the structure of $\mathbf{\Delta}_s(\delta)$,

$$\mathbf{D}_s = \left\{ D = \mathrm{diag}(d_1 I_{r_1}, \ldots, d_s I_{r_s}) \right\} \tag{6.16}$$

where I_r denotes the $r \times r$ identity matrix, and the d_i are complex-valued scalar constants. Then it can be shown that

$$\mu(F) \leq \bar{\sigma}(DFD^{-1}), \quad D \in \mathbf{D}_s \tag{6.17}$$

Thus an upper bound on the structured singular value is given by

$$\mu(F(j\omega)) \leq \inf_{D(j\omega) \in \mathbf{D}_s} \bar{\sigma}[D(j\omega)F(j\omega)D(j\omega)^{-1}] \tag{6.18}$$

Doyle (1982) has shown that if the number of blocks in the uncertainty structure (6.8) is less or equal three ($s \leq 3$), then the upper bound is strict, i.e.,

$$\mu(F(j\omega)) = \inf_{D(j\omega) \in \mathbf{D}_s} \bar{\sigma}[D(j\omega)F(j\omega)D(j\omega)^{-1}], \quad s \leq 3 \tag{6.19}$$

For $s > 3$, numerical studies indicate that the upper bound is nearly always within 15% larger than the structured singular value (Doyle 1982). These estimates are based on the difference between the upper bound and a lower bound on μ. Thus, the upper bound provides a useful and not overly conservative approximation of μ.

Let $D(j\omega)$ denote a stable transfer function with the structure in equation (6.16), i.e., $D(j\omega) \in \mathbf{D}_s$ for all ω,

$$D(j\omega) = \mathrm{diag}(d_1(j\omega)I_{r_1}, \ldots, d_s(j\omega)I_{r_s}) \tag{6.20}$$

where $d_i(\cdot)$ are stable scalar transfer functions. Then it follows from (6.17) and the definition of the H_∞ norm that

$$\sup_\omega \mu(F(j\omega)) \leq \sup_\omega \bar{\sigma}[D(j\omega)F(j\omega)D(j\omega)^{-1}] = \|DFD^{-1}\|_\infty, \quad D(j\omega) \in \mathbf{D}_s \tag{6.21}$$

In combination with the robust stability condition with respect to structured uncertainties it follows that the existence of a filter D with $D(j\omega) \in \mathbf{D}_s$, such that

$$\|DFD^{-1}\|_\infty < \delta^{-1} \tag{6.22}$$

implies that the transfer function F is robustly stable for the set (6.8) of structured uncertainties. On the other hand, the existence of $D(j\omega) \in \mathbf{D}_s$ such that (6.22) holds is not a necessary condition for robust stability unless $s \leq 3$.

Next consider the robust stabilization problem for plants with structured uncertainties. In view of the fact that (6.22) implies robust stability, a natural approach to design a controller which achieves robust stability for structured uncertainties involves the construction of both a controller K and a filter $D \in \mathbf{D}_s$ such that the closed-loop transfer function DFD^{-1} satisfies the H_∞-norm bound (6.22). The filter D is commonly called *scaling filter*, because it scales the components of the input and output vectors of the closed-loop systems. We can thus state on approximate robust stabilization problem for plants with structured uncertainties as follows.

DK-optimization problem for plants with structured uncertainties.
Consider the uncertain plant described by (6.9). Find a stabilizing controller K and a scaling filter $D(j\omega) \in \mathbf{D}_s$ such that the scaled closed-loop transfer function $DF(P,K)D^{-1}$ satisfies the H_∞-norm bound (6.22). If the norm bound (6.22) is achieved, then the closed loop is robustly stable with respect to the structured uncertainty set $\boldsymbol{\Delta}_s(\delta)$.

Remark 6.1.
Recall that $F(P,K)$ denotes the closed-loop transfer function of the plant P and the controller K in Figure 5.2, given by equation (5.15). Hence $DF(P,K)D^{-1}$ denotes the closed-loop transfer function from the signal v_D to the signal z_D of the systems described by

$$\begin{aligned} \begin{bmatrix} z_D \\ y \end{bmatrix} &= \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} D^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v_D \\ u \end{bmatrix} \\ u &= Ky \end{aligned} \tag{6.23}$$

Recalling that for the uncertainty description (6.9), the augmented plant P is defined by equation (5.12), we have

$$\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} D^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & DW_2 \\ W_1 D^{-1} & G_0 \end{bmatrix} \tag{6.24}$$

Unfortunately, there is no explicit method for the simultaneous construction of K and D such that (6.22) holds. Therefore, the problem is in practice solved in an iterative manner by alternately reducing the H_∞ cost in (6.22) with respect to K (with D fixed) and with respect to D (with K fixed). Such a procedure is known as *DK-iteration*, and it can be summarized as follows.

Algorithm (DK-iteration).

Step 0. Specify the maximum number of iterations, MAXIT and a tolerance level $\epsilon > 0$. Select a scaling filter $D \in \mathbf{D}_s$.

Step 1. (K-step). With D fixed, apply a γ-iteration procedure to find an H_∞-optimal controller K and $\gamma < \gamma_{inf} + \epsilon$ such that the H_∞-norm bound $\|DFD^{-1}\| < \gamma$ holds. If $\gamma < \delta^{-1}$, K is robustly stabilizing and the problem is solved. Otherwise, go to Step 2.

Step 2. (D-step). With K fixed, calculate a new scaling filter D by (approximately) minimizing $\bar{\sigma}[D(j\omega)F(j\omega)D(j\omega)^{-1}]$ with respect to $D(j\omega)$. If the procedure gives a filter such that $\bar{\sigma}[D(j\omega)F(j\omega)D(j\omega)^{-1}] < \delta^{-1}$, all ω, then the controller K is robustly stabilizing, and the problem is solved. Otherwise, go to Step 3.

Step 3. If the number of iterations is equal to MAXIT, no solution to the robust stabilization problem has been found. Otherwise, continue from Step 1.

Step 1 consists of a standard H_∞-optimal control problem, which can be solved by the procedure described in Chapter 4. Step 2 involves a minimization of $\bar{\sigma}[D(j\omega)F(j\omega)D(j\omega)^{-1}]$ with respect to $D(j\omega) \in \mathbf{D}_s$. This can be performed by numerical optimization at a number of frequencies $\{\omega_i\}$, and approximating a rational scaling filter D to the optimized $\{D(j\omega_i)\}$. Notice that the accuracy of the approximation will improve as the filter order is increased, and there is in general no upper bound on the order of the optimal scaling filter D. As the order of the optimal controller is equal to the order of the plant P augmented with the scaling filter D and its inverse D^{-1} (cf. equations (6.23) and (6.24)), it follows that the controller order can also be quite high. In fact, μ-optimal controller design is known in general to generate controllers of excessively high orders. In many cases the factors D and D^{-1} contribute with more states than the original plant P, and it is not uncommon to have μ-optimal controllers with orders exceeding two or three times the order of the plant. The design is therefore often followed by a subsequent controller reduction in order to obtain a low-order suboptimal controller.

In analogy with unstructured uncertainties, it is also of interest to study the case when the structured uncertainties are not restricted to be linear and time-invariant, but may be nonlinear and/or time-varying, and bounded in L_2-induced norm, cf. Remark 5.3. In contrast to unstructured uncertainties, where the necessary and sufficient condition for robust stability is equivalent for both uncertainty classes (cf. Remark 5.3), in the structured case the assumption of nonlinear or linear time-varying uncertainties gives rise to a more restrictive robust stability condition. More precisely, we have the following result, which is due to Shamma (1994).

Theorem 6.2 Robust stability to structured time-varying uncertainties. *Consider the system in Figure 5.2 or 5.3, where the uncertainty is assumed structured according to (6.8). The system is robustly stable with respect to the set of* **structured nonlinear and/or time-varying uncertainties** *with L_2-induced norm less than or equal to δ, $\|\Delta\| \leq \delta$, if and only if the nominal closed-loop transfer function $F = F(P, K)$ is stable, and there exists a* **real-valued, frequency-independent** *diagonal matrix with the structure in (6.16), $D \in \mathbf{D}_s$, D real, such that $\|DFD^{-1}\|_\infty < \delta^{-1}$.*

Notice that the theorem gives a sufficient and necessary robust stability condition. The condition is more restrictive (as it should) than the one in Theorem 6.1, because there is less freedom in the choice of D. The corresponding DK-optimization problem consists of finding a stabilizing K and a real-valued scaling $D \in \mathbf{D}_s$ such that (6.22) holds. Observe that as only a real-valued scaling matrix D is required, the DK-optimization procedure is significantly simplified.

Several years after the upper bound in (6.21) had been introduced, it was found that it gives in fact a sufficient and necessary stability condition with respect to structured *arbitrarily slowly time-varying linear uncertainties* with bounded L_2-induced norm. The following result is due to Poolla and Tikku (1995).

Theorem 6.3 Robust stability against slowly time-varying uncertainties. *Consider the system in Figure 5.2 or 5.3, where the uncertainty is assumed structured according to (6.8). The system is robustly stable with respect to the set of* **structured arbitrarily slowly time-varying linear uncertainties** *with L_2-induced norm less than or equal to δ, $\|\Delta\| \leq \delta$, if and only if the nominal closed-loop transfer function $F =$*

$F(P,K)$ is stable, and there exists a scaling filter $D(j\omega) \in \mathbf{D}_s$, such that $\|DFD^{-1}\|_\infty < \delta^{-1}$.

Theorem 6.3 states that the DK-optimization problem really solves a robust stabilization problem for the set of slowly time-varying uncertainties. This is not so bad, because it is probably more realistic to assume that uncertainties may vary slowly with time than to assume that they are time-invariant.

It is interesting to note that as it is known that equality in (6.21) does not always hold, it follows that there is a gap between robustness to linear time-invariant uncertainties and linear, arbitrarily slowly time-varying uncertanties.

Remark 6.2 *Generalizations.*
The structured uncertainty description can be generalized to cases where the block diagonal uncertainty (6.8) contains a number of repeated blocks, or some of the blocks are real-valued and frequency-independent. Repeated blocks are needed to describe some uncertainty structures, and real-valued uncertainties can be used to characterize parametric uncertainties. Procedures for dealing with these generalizations have been developed, but they will not be considered here.

Structured singular value synthesis has been implemented in the `MATLAB` toolbox *μ-Analysis and Synthesis Toolbox*. The toolbox contains various routines related to μ, such as upper and lower bounds, as well as an implementation of the DK-iteration procedure. The user has control over the controller order by fixing the order of the scaling filter D. The toolbox also has routines for dealing with repeated and real-valued uncertainty blocks, cf. Remark 6.2. For anybody who plans to use the toolbox it is highly recommended to reserve a sufficient amount of time and patience, and not least to read the toolbox manual in advance!

References

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