

Chapter 8

Design issues

In the previous chapters we have considered solutions to a number of *controller synthesis problems*. These define solutions to well-defined optimal and robust control problems, once the problem has been stated. However, in *controller design*, the controller synthesis is only the final stage. Even more important for a good design is to state the problem properly. In particular, the design process involves the selection of the cost function and the controlled output z used in the controller synthesis step. The design parameters include relative weights between outputs and inputs in the cost, frequency-dependent weightings, realistic magnitudes and structures for the plant uncertainties Δ , etc.

Only in special instances the primary control objective can be stated directly in terms of a single cost function. This is for example the case in some quality control problems, where the disturbances can be modelled as stochastic processes and where the control objective is to minimize the stationary variances of quality variables. The control problem can then be stated directly as an H_2 -optimal (minimum variance) problem, cf. Section 3.4. If the process model is uncertain, the problem can be stated as an H_2 robust performance problem. Often, however, the control objectives are stated more loosely, typically requiring fast response, small overshoot, and sufficient robustness. It is then necessary to translate the control objectives into a quantitative controller synthesis problem. In this chapter we consider how the H_∞ optimal controller synthesis methods can be applied to achieve good controller designs in such situations.

8.1 General considerations

In this chapter we consider the general control system in Figure 8.1. Here y and u are the output and control signals, respectively, d and n are process and measurement disturbances, and r is a reference signal. Notice that the output y does not correspond directly to the measured output defined in previous chapters. For convenience, we use lower-case letters, although it is understood in the discussion that follows that the signals are defined in the Laplace domain, i.e., $y(s)$, $u(s)$ etc. Some manipulations give

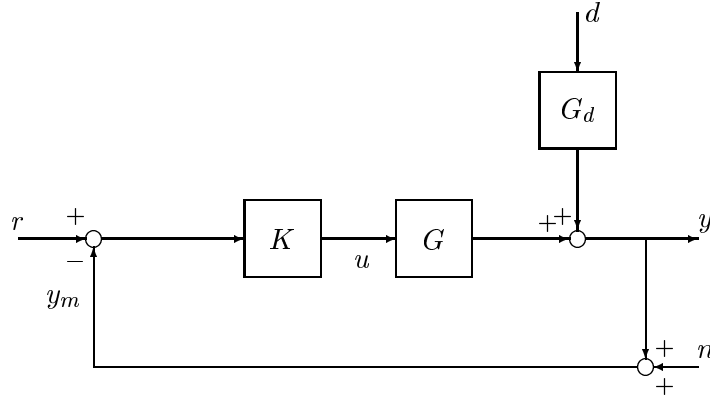


Figure 8.1: A general control system.

for the closed-loop system

$$y = (I + GK)^{-1}GKr + (I + GK)^{-1}G_d d - (I + GK)^{-1}GKn \quad (8.1)$$

Introducing the *sensitivity function*

$$S = (I + GK)^{-1} \quad (8.2)$$

and the *complementary sensitivity function*

$$T = (I + GK)^{-1}GK, \quad (8.3)$$

equation (8.1) can be written in the form

$$y = Tr + SG_d d - Tn \quad (8.4)$$

Notice that S and T are related according to

$$S + T = I \quad (8.5)$$

It is also of interest to study the control signal $u = K(r - y_m)$. From Figure 8.1 we have that u is given by

$$u = KSr - KSG_d d - KSn \quad (8.6)$$

The purpose of the control system in Figure 8.1 is to make the output y follow the reference signal r , i.e., to make the error signal

$$e = y - r \quad (8.7)$$

small. For the closed-loop system we have

$$e = -Sr + SG_d d - Tn \quad (8.8)$$

Remark 8.1.

The sensitivity function S gives the sensitivity reduction in the output which is achieved by feedback. This is seen by considering the open loop with no feedback in Figure 8.1,

$$y = GKr + G_d d \quad (8.9)$$

The response from r and d to y with feedback, equation (8.1), is obtained by multiplying (8.9) by S . Originally, the term sensitivity function derives from the fact that it equals the relative sensitivity dT/T of the closed-loop transfer function T (cf. equation (8.4)) to the relative plant model error dG/G , i.e., at a given frequency ω we have

$$\frac{dT/T}{dG/G} = S \quad (8.10)$$

which can be derived in a straightforward way by differentiating (8.3) with respect to G .

From (8.8) we see that good reference signal tracking, i.e., a small e , is achieved if the sensitivity function S is small. On the other hand, suppression of measurement noise n requires the complementary sensitivity function T to be small. However, due to the relation (8.5) both S and T cannot be small simultaneously. Hence there is necessarily a trade-off between reference signal tracking and sensitivity to measurement noise. Fortunately, the frequency contents of the tracking signal r and the noise signal n are usually concentrated to different frequency ranges: r consists typically of low-frequency components, whereas the noise n is important at higher frequencies. Therefore, controllers with both good (low-frequency) tracking properties and (high-frequency) noise suppression can be designed by making $S(j\omega)$ small at low frequencies and $T(j\omega)$ small at higher frequencies.

Besides performance, the sensitivity function is also associated with *robustness*. Notice that in the scalar case, the distance from a point on the Nyquist plot of the loop transfer function $G(j\omega)K(j\omega)$ to the point $(-1, 0)$ is $|1 + G(j\omega)K(j\omega)| = 1/|S(j\omega)|$. Hence the distance of the Nyquist plot to the point $(-1, 0)$ is greater than $1/M_S$ if and only if

$$|S(j\omega)| < M_S, \text{ all } \omega \quad (8.11)$$

Thus, classical stability margins impose a bound on $\|S\|_\infty$.

Robustness considerations do not only lead to a bound on the sensitivity function, however, but it turns out that the complementary sensitivity function T is also associated with robustness. In order to see this, assume that G is an uncertain plant model, such that the real plant is described by the nominal model G and a *multiplicative uncertainty*,

$$G_\Delta = (1 + \Delta_M)G \quad (8.12)$$

where Δ_M is the relative uncertainty, which is assumed bounded according to

$$|\Delta_M(j\omega)| \leq l(\omega) \quad (8.13)$$

but is otherwise unknown. Then we have by simple manipulations,

$$1 + G_\Delta K = 1 + (1 + \Delta_M)GK = (1 + GK)(1 + \Delta_M T) \quad (8.14)$$

The expression in (8.14) is the denominator of the closed-loop transfer function, and for stability, $1 + G_\Delta K \neq 0$ must hold. By (8.14), this is true for all Δ_M such that (8.13) holds if and only if

$$|T(j\omega)| < 1/l(\omega) \quad (8.15)$$

holds for all ω . In this way, robustness considerations impose a bound on the complementary sensitivity function.

The conditions imposed on S and T by performance and robustness requirements can be used directly for controller design. In particular, in various *loop-shaping* procedures, the controller is designed by shaping the magnitudes $|S(j\omega)|$ and $|T(j\omega)|$ as functions of frequency in such a way that performance and robustness criteria are met. There are classical loop-shaping procedures which have been developed for SISO systems. As will be described later, loop shaping can be formulated in terms of an H_∞ -optimal control problem. This provides a convenient loop-shaping procedure for both single-input single output and multivariable systems.

The controller design is, however, complicated by the fact that the magnitudes of the closed-loop transfer functions cannot be shaped freely. This is due to the fact that there are fundamental limitations on closed-loop transfer functions even for exactly known systems, which restrict the way in which the closed-loop transfer functions can be shaped by any stabilizing controller. These fundamental restrictions will be presented briefly in the next section.

8.2 Fundamental performance limitations

In controller design it is important to be aware of some fundamental performance constraints which apply to any stable control system. Thus, for the control system in Figure 8.1 it is for example theoretically not possible to achieve an arbitrary disturbance attenuation level at all frequencies *even* if the plant dynamics were exactly known. Instead, there are some limits on the achievable performance.

The performance limitations are related to some fundamental properties which any stable transfer function must satisfy. In particular, we have the following relationship for a minimum phase transfer function.

Theorem 8.1 Bode's gain-phase relationship.

Let $G(s)$ be a proper, stable transfer function with no RHP zeros and no time delays, and normalized so that $G(0) > 0$. Then, there is a unique relationship between the magnitude $|G(j\omega)|$ and the phase $\arg(G(j\omega))$ given by

$$\arg(G(j\omega_0)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \log |G(j\omega)|}{d \log \omega} \log \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right| \cdot \frac{d\omega}{\omega} \quad (8.16)$$

It turns out that equation (8.16) is the minimum possible phase lag that any system with a given magnitude response $|G(j\omega)|$ can have. Systems with no RHP zeros or time delays are therefore referred to as *minimum phase* systems. By contrast, *nonminimum*

phase systems, which do have RHP zeros and/or time delays, necessarily have more phase lag. Thus, the realization of a linear system with a given magnitude response necessarily introduces a certain amount of phase shift.

Remark 8.2.

Notice that the factor $\log[|\omega + \omega_0|/|\omega - \omega_0|]$ is infinite at $\omega = \omega_0$ and finite otherwise. Hence $\arg(G(j\omega_0))$ is primarily determined by the local slope of the logarithm of the magnitude, $d \log |G(j\omega)|/d \log \omega$ at $\omega = \omega_0$. Using the fact that $\int_{-\infty}^{\infty} \log[|\omega + \omega_0|/|\omega - \omega_0|] \cdot d\omega/\omega = \pi^2/2$ we have the following approximative formula for minimum phase systems,

$$\arg(G(j\omega_0)) \approx \frac{\pi}{2} \left(\frac{d \log |G(j\omega)|}{d \log \omega} \right)_{\omega=\omega_0} \quad [\text{ rad }] \quad (8.17)$$

Hence, a slope of -20 dB/decade in the gain at the vicinity of ω_0 , i.e., $d \log |G(j\omega)|/d \log \omega = -1$, implies that the phase angle is approximately $-\pi/2$ rad/sec, a slope of -40 dB/decade implies that the phase angle is approximately $-\pi$ rad/sec, etc. This leads to the conventional design wisdom that to ensure closed-loop stability, the slope of the loop transfer function $|G(j\omega)K(j\omega)|$ should be in the range -20 to -30 dB/decade at the gain cross-over point (where $|G(j\omega)K(j\omega)| = 1$), since this would imply a phase lag less than -180° .

The gain-phase relationship of Theorem 8.1 and related results have important implications for the achievable control system performance. This is due to the fact that the relationship holds for the *loop transfer function*

$$L = GK \quad (8.18)$$

Note that a small value of the sensitivity function S implies that the loop transfer function is large. On the other hand we recall from classical SISO control theory that for stability, $|L(j\omega_c)| < 1$ should hold at the phase crossover frequency ω_c , at which the phase lag is -180° , $\arg(L(j\omega_c)) = -180^\circ$. The relationship (8.16) again, applied to the loop transfer function $L(j\omega)$ gives the minimum phase for any given $|L(j\omega)|$. Hence it follows that there are limitations imposed on the function $|L(j\omega)|$, and thus on $|S(j\omega)|$, which should hold if the closed-loop system is required to be stable.

As discussed above, it is natural to require good reference signal tracking and disturbance rejection at low frequencies. Hence $S(j\omega)$ should be small for small ω . On the other hand, we know that all physical dynamical systems attenuate high frequencies, so that $G(j\omega) \rightarrow 0$ as $\omega \rightarrow \infty$. Hence $S(j\omega) \rightarrow I$ as $\omega \rightarrow \infty$.

The fact that $S(j\omega) \approx I$ should hold at high frequencies can also be motivated by robustness considerations. Recall the multiplicative uncertainty described by equation (8.12) and the associated stability condition (8.15). In practice, it is impossible to determine the phase shift or the *relative magnitude* of $G(j\omega)$ accurately at high frequencies. Therefore, at high frequencies we have $G(j\omega) \rightarrow 0$, and, due to large uncertainty in the phase and relative magnitude, the bound (8.13) on the relative uncertainty will be large, $l(\omega) \gg 1$. By (8.15), robust stability then imposes the condition $|T(j\omega)| < 1$. Hence, by (8.5), $S(j\omega) \approx I$ at high frequencies.

From the above discussion it follows that the sensitivity function S cannot be made small at all frequencies simultaneously. The best one can do is to make $S(j\omega)$ small in

a given frequency band. However, it may happen that as we try to make $S(j\omega)$ small in some frequency band, its value at some other frequencies will necessarily increase. It is interesting that the achievable average performance can be expressed quantitatively. In particular, we have the following classical result for SISO systems.

Theorem 8.2 Bode's Sensitivity Integral.

Suppose that $L(s)$ has

- no RHP zeros,
- at least two more poles than zeros, and
- N_p RHP-poles p_i .

Then for closed-loop stability the sensitivity function S must satisfy

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi \cdot \sum_{i=1}^{N_p} \text{Re}(p_i) \quad (8.19)$$

where $\text{Re}(p_i)$ denotes the real part of p_i .

For a stable plant $N_p = 0$ and the relation (8.19) reduces to

$$\int_0^\infty \log |S(j\omega)| d\omega = 0 \quad (8.20)$$

Thus we have that in terms of the logarithm of $|S|$, the area of sensitivity reduction, where $\log |S|$ is negative, i.e. $|S| < 1$, is *equal* to the area of sensitivity increase, where $\log |S|$ is positive, i.e. $|S| > 1$. The benefits and cost of feedback are hence balanced exactly.

Remark 8.3.

For MIMO systems conditions corresponding to (8.19) have been developed as well. Then S is a matrix, and the relation corresponding to (8.19) takes the form

$$\int_0^\infty \log |\det S(j\omega)| d\omega = \pi \cdot \sum_{i=1}^{N_p} \text{Re}(p_i) \quad (8.21)$$

By (8.19), the achievable performance is degraded by RHP (unstable) poles (because $\text{Re}(p_i) > 0$). There is also a corresponding sensitivity integral relation for nonminimum phase loop transfer functions with RHP zeros. In this case too, the performance is degraded (area of sensitivity increase, where $\log |S|$ is positive, is increased).

Recalling the interpretation of S (Remark 8.1), the sensitivity integral (8.19) tells us that in a way, the most we can achieve by control is to concentrate the control effort to a frequency band, such as $0 \leq \omega \leq \omega_1$, whereas the net effect taken over the whole frequency axis is zero, or even worse. Fortunately, for minimum phase systems the situation is not as bad as Theorem 8.2 might indicate. As the frequency range $\omega_1 < \omega < \infty$ which contributes to the positive part of (8.19) is infinite, $|S(j\omega)|$ may still be made arbitrarily close to one at all ω . More precisely, we have the following result.

Theorem 8.3 Achievable performance for minimum-phase systems.

Suppose that $G(s)$ has no right-half plane zeros. Let $\omega_1 > 0$. Then for any $\epsilon > 0$, $\delta > 0$ there exists a stabilizing K such that

$$|S(j\omega)| < \epsilon, \quad \text{all } |\omega| \leq \omega_1 \quad (8.22)$$

and

$$|S(j\omega)| < 1 + \delta, \quad \text{all } \omega \quad (8.23)$$

It should be observed that Theorem 8.3 applies to the case when the only requirement on $S(j\omega)$ is that it should be small in some frequency range. However, it is also often required that the loop transfer function should fall off sufficiently at high frequencies, for example $|L(j\omega)| < 1/\omega^m$ for some $m > 0$. In such cases it can be shown that (8.19) implies that if $|S|$ is made small in a frequency range $\omega \in [0, \omega_1]$, then $|S|$ must necessarily become large at some other frequencies.

The above results have dealt with minimum-phase systems. For *nonminimum phase systems* the situation is different. For such systems, the attempt to make the sensitivity function small in a frequency band necessarily results in large $S(j\omega)$ at some other frequencies. We have the following result.

Theorem 8.4 Performance limitation in nonminimum phase systems.

Define

$$M = \max_{|\omega| \leq \omega_1} |S(j\omega)| \quad (8.24)$$

and suppose that $G(s)$ has at least one right-half plane zero, i.e., $G(s_0) = 0$ for some s_0 with $\text{Re}(s_0) > 0$. Then there exists a positive number $c > 0$, which depends only on ω_1 and G , such that for every stabilizing K we have

$$M|S(j\omega)|^c \geq 1 \quad (8.25)$$

for some $\omega > \omega_1$.

The result states that as M tends to zero for a nonminimum phase system, $|S(j\omega)|$ increases without limit at some other frequency or frequencies. For this reason nonminimum phase systems are often difficult to control. This phenomenon is known in the literature by the metaphor *waterbed effect* (for obvious reasons). Although this term is usually reserved for the performance limitation valid for nonminimum phase systems, some authors use this term for the classical integral formula (8.19) as well.

8.3 Application of H_∞ -optimal control to loop shaping

We have seen that the control objectives can often be stated in terms of the magnitudes of various closed-loop transfer functions, such as the sensitivity function $S(j\omega)$ and the complementary sensitivity function $T(j\omega)$. We shall briefly sketch how these types

of problems can be solved in an efficient way using H_∞ -optimal controller synthesis methods.

Consider the control system in Figure 8.1. Recall that the error signal $e = y - r$ is given by (cf. equation (8.8))

$$e = -Sr + SG_d d - Tn \quad (8.26)$$

and the control signal u is given by (cf. equation (8.6))

$$u = KSr - KSG_d d - KSn \quad (8.27)$$

For good control performance in terms of reference signal tracking and attenuation of process disturbances the sensitivity function $S(j\omega)$ is required to be small in magnitude, whereas attenuation of measurement noise requires the complementary sensitivity function $T(j\omega)$ to be small at higher frequencies. It is also recalled that robustness considerations set bounds on both S and T , cf. equations (8.11) and (8.15). In practice, it is also often well motivated to bound the control signal, which implies a bound on the magnitude on KS , cf. equation (8.27).

The design specifications normally imply that S , T and KS should be small in different frequency bands. In particular, S should be small at low frequencies whereas T and KS should be small at higher frequencies. This can be achieved by defining the "stacked" transfer function matrix

$$N = \begin{bmatrix} W_S S \\ W_T T \\ W_u KS \end{bmatrix} \quad (8.28)$$

where W_S , W_T and W_u are weighting filters which give appropriate relative weights to the three transfer functions involved. The design specifications can be met by selecting the weights properly and minimizing a cost involving N . Such a control problem is called a *mixed sensitivity* problem, due to the "mixed sensitivity" nature of the specifications. Previously, it was common practice to base the design on the quadratic H_2 cost

$$\begin{aligned} J_2(N) = \|N\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} W_S S \\ W_T T \\ W_u KS \end{bmatrix}^T (-j\omega) \begin{bmatrix} W_S S \\ W_T T \\ W_u KS \end{bmatrix} (j\omega) d\omega \\ &= \|W_S S\|_2^2 + \|W_T T\|_2^2 + \|W_u KS\|_2^2 \end{aligned} \quad (8.29)$$

This approach has the property that even though the quadratic cost is small, there is no guarantee that S , T and KS are small at all frequencies in the frequency bands of interest because only the integral, i.e., an average value, is minimized.

In order to guarantee specified bounds on $S(j\omega)$, $T(j\omega)$ and $K(j\omega)S(j\omega)$ at all frequencies, it is more efficient to base the design on an H_∞ cost instead. Consider for simplicity the SISO case. The arguments generalize to the multivariable case in

a straightforward way by replacing the absolute values $|S(j\omega)|$, etc, by the maximum singular values $\bar{\sigma}(S(j\omega))$, etc. Then the bound

$$\|N\|_\infty < \gamma \quad (8.30)$$

implies

$$\|W_S S\|_\infty < \gamma, \quad \|W_T T\|_\infty < \gamma, \quad \|W_u K S\|_\infty < \gamma \quad (8.31)$$

or,

$$|W_S(j\omega)S(j\omega)| < \gamma, \quad (8.32)$$

$$|W_T(j\omega)T(j\omega)| < \gamma, \quad (8.33)$$

$$|W_u(j\omega)K(j\omega)S(j\omega)| < \gamma, \quad \text{all } \omega. \quad (8.34)$$

The bounds (8.32)–(8.34) can be written directly as bounds on $|S(j\omega)|$, $|T(j\omega)|$ and $|K(j\omega)S(j\omega)|$,

$$|S(j\omega)| < \gamma/|W_S(j\omega)|, \quad (8.35)$$

$$|T(j\omega)| < \gamma/|W_T(j\omega)|, \quad (8.36)$$

$$|K(j\omega)S(j\omega)| < \gamma/|W_u(j\omega)|, \quad \text{all } \omega. \quad (8.37)$$

In many cases, the frequency bands where the different transfer functions should be constrained are non-overlapping. This implies that the associated weight dominates in that frequency interval. For example, at low frequencies a small sensitivity function is required for good performance, whereas the other transfer functions are of less importance. Then, $|W_S(j\omega)| \gg |W_T(j\omega)|$, and $|W_S(j\omega)| \gg |W_u(j\omega)|$ for $|\omega| < \omega_1$ and some ω_1 . Then we have $\bar{\sigma}(N(j\omega)) \approx |W_S(j\omega)S(j\omega)|$. Moreover, H_∞ -optimal controllers have the property that, if γ is not too much larger than γ_{inf} , the closed-loop transfer function is close to the H_∞ norm bound at least at low frequencies, i.e., $\bar{\sigma}(N(j\omega)) \approx \gamma$. It follows that $|W_S(j\omega)S(j\omega)| \approx \gamma$, or

$$|S(j\omega)| \approx \gamma/|W_S(j\omega)|, \quad |\omega| < \omega_1 \quad (8.38)$$

Thus, the weight filter W_S determines the shape of the magnitude of the sensitivity function in the low-frequency band. Similar relations hold for the other transfer functions in their respective frequency bands. By the approximate relation (8.38), H_∞ -optimal controller design can be applied in *loop-shaping* techniques, in which the magnitudes $|S(j\omega)|$, etc, can be shaped by the choice of the weight filters in a very straightforward way.

In order to solve the H_∞ control problem which achieves the bound (8.30) on the transfer function matrix N , it is necessary to define a state-space model which has the closed-loop transfer function matrix N . This can be achieved by observing that in Figure 8.1, the signal y_m is given by

$$y_m = Tr + SG_d d + Sn \quad (8.39)$$

Hence we have for the control system in Figure 8.1 (setting $d = 0, r = 0$),

$$\begin{bmatrix} y_m \\ y \\ u \end{bmatrix} = \begin{bmatrix} S \\ -T \\ -KS \end{bmatrix} n \quad (8.40)$$

and a state-space representation with the closed-loop transfer function matrix N can thus be constructed in a straightforward way.

Weight filters.

By (8.35)–(8.37), the weight filters should be selected large on frequency ranges where it is important to constrain the magnitudes of the associated closed-loop transfer functions, and small at other frequencies.

The weight filters which are used to achieve the required closed-loop behaviour are typically low-pass, high-pass, or band-pass filters. A simple first-order weight filter often used in control system design is provided by the following observation. Suppose we want to construct a weight filter $W(j\omega)$ such that

- $1/|W(j\omega)| = A$ at small ω ,
- $1/|W(j\omega)| = B$ at large ω ,
- cross-over from low-frequency value to high-frequency value approximately at ω_0 .

Here, we have either $A > 1$ and $B < 1$ (high-pass filter) or $A < 1$ and $B > 1$ (low-pass filter). A first-order filter which achieves this is given by

$$W(s) = \frac{1}{B} \frac{s + B\omega_0}{s + A\omega_0} \quad (8.41)$$

The crossing from low-frequency to high-frequency value can be made steeper by selecting an n th order filter

$$W_n(s) = \frac{1}{B} \frac{(s + B^{1/n}\omega_0)^n}{(s + A^{1/n}\omega_0)^n} \quad (8.42)$$

Band-pass filters can be constructed by forming appropriate products of low-pass and high-pass filters, whose passbands overlap.

Elimination of steady-state offsets.

A natural requirement in many applications is that the controller should eliminate steady-state offsets in the error signal e . This implies that $S(0) = 0$ should hold (cf. equation (8.26)), which can be achieved by letting the weight filter W_s have the factor $1/s$. This is in accordance with the well-known fact that elimination of steady-state offsets requires the controller to have an integrator. In practical computations, it is important to notice that in the state-space representation of the system in Figure 8.1 augmented with the weight filters, the state corresponding to the unstable factor $1/s$ will be non-observable from the measured output available to the controller, and therefore, the controller synthesis methods fail to find a solution. For this reason, the factor $1/s$ is in practical computations replaced by the factor $1/(s + \epsilon)$, where $\epsilon > 0$ is a small constant.

Simple rules for weight selection.

The selection of proper weights in H_∞ based controller design can be quite tedious, and it may require much iteration. It would therefore be very useful to have general rules of thumb to assist the tuning process, cf. the rules used in the tuning of PID controllers (Ziegler-Nichols etc). Various tuning rules of this form have been proposed in the literature.

Here we will present a set of general rules due to Lennartson and Kristiansson (1999). For simplicity, we focus on the single-input single-output case. Extensions to MIMO systems can be found in the literature. In the approach proposed by Lennartson and Kristiansson, various criteria are specified for various frequency ranges. The frequency regions are defined with respect to the frequency ω_π , at which the plant phase equals -180° ; $\arg(G(j\omega_\pi)) = -\pi$. In this way three frequency regions are defined: a medium frequency (MF) range close to ω_π , a low frequency (LF) range, and a high frequency (HF) range. Usually the width of the MF range is taken to be one decade, so that we have the frequency regions

$$\begin{aligned} \text{LF range:} \quad & \omega < 0.3\omega_\pi \\ \text{MF range:} \quad & 0.3\omega_\pi < \omega < 3\omega_\pi \end{aligned} \tag{8.43}$$

$$\text{HF range:} \quad \omega > 3\omega_\pi \tag{8.44}$$

Different specifications for the closed-loop system are given for the various frequency ranges. In the *low frequency range*, the main objective is to make the closed-loop gain from the process disturbance d to the tracking error $e = y - r$ in Figure 8.1 small, and to achieve elimination of steady-state offsets. Assuming $G_d = G$, i.e., the disturbance d enters at the input, the closed-loop transfer function from d to e is SG (cf. equation (8.26)). Thus, the low-frequency cost is defined as

$$J_{LF} = \|W_{LF}SG\|_\infty \tag{8.45}$$

where $W_{LF}(s) = \frac{1}{s+\epsilon}$, for some small positive ϵ (cf. the discussion above).

In the *medium frequency range*, it is important to ensure acceptable stability margins. Recall that $\|S\|_\infty$ is the inverse of the smallest distance of the Nyquist plot of the loop-transfer function from the point $(-1, 0)$. It is therefore well motivated to introduce the bound (8.11), i.e.,

$$\|S\|_\infty < M_S \tag{8.46}$$

It has been suggested that appropriate values for the bound M_S are in the range of 1.4 to 2.0. A good compromise is $M_S = 1.7$. The complementary sensitivity function T is also related to robustness, cf. equation (8.15), and it should therefore be bounded as well. Thus, it is motivated to introduce the bound

$$\|T\|_\infty < M_T \tag{8.47}$$

where $M_T < M_S$. It has been suggested that an appropriate choice is $M_T = M_S/1.3$.

The bound (8.47) is called the 'peak M value' in classical controller design. There it is motivated by the fact that a peak in $|T(j\omega)|$ gives rise to oscillations at the

corresponding frequency after a step change in the setpoint. In addition to the bounds (8.46) and (8.47) a third medium frequency bound has been proposed in order to achieve a higher amplitude margin, cf. Lennartson and Kristiansson (1999).

Finally, in the *high frequency range* the most important requirement is to keep the control signal limited. This implies that the magnitude of the closed-loop transfer function KS should be kept small, cf. equation (8.27). Hence, the high-frequency cost is defined as

$$J_{HF} = \|W_{HF}KS\|_{\infty} \quad (8.48)$$

where $W_{HF}(s)$ is a high-pass filter whose passband is in the HF range. Making J_{HF} small will ensure that the high-frequency robustness condition (8.15) holds as well.

The controller design consists of finding a controller which makes the low and high frequency costs J_{LF} and J_{HF} as small as possible, subject to the condition that the medium frequency bounds (8.46) and (8.47) are satisfied. This may be achieved by searching for appropriate values for the weight filters in the stacked transfer function matrix (8.28) (using $W_S = W_{LF}G$ in order to capture the cost J_{LF}) and solving the associated H_{∞} control problem which achieves a performance bound of the form (8.30). In this procedure there is still a trade-off in the relative magnitudes of the low and high frequency costs. Therefore, a compromise which makes both the low and high frequency costs J_{LF} and J_{HF} sufficiently small should be selected.

8.4 Notes and references

Readable, elementary treatments of the material in this chapter can be found in Doyle *et al.* (1992), Maciejowski (1989) and Skogestad and Postlethwaite (1996).

The classical performance limitation results are due to Bode (1945). The classical results have been generalized by a number of researchers in the 1980's and 1990's in various directions, such as multivariable and discrete systems. An excellent exposition of performance limitations in control is given in the book by Seron *et al.* (1997).

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