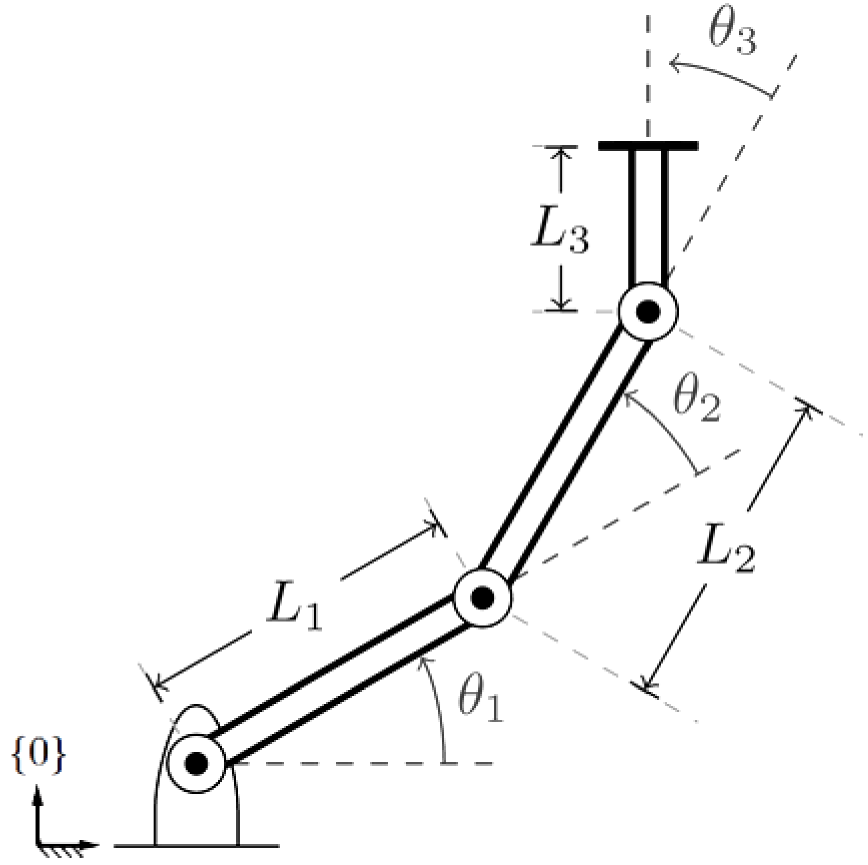
Прямая кинематика многозвенных манипуляторов

Forward kinematics for a 3R planar open chain

(x,y) – Позиция конца последнего звена

ϕ – угол поворота последнего звена

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + L_3 \cos(\theta_1+\theta_2+\theta_3),\\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + L_3 \sin(\theta_1+\theta_2+\theta_3),\\
\phi &= \theta_1+\theta_2+\theta_3.
\end{aligned}
$$



Denavit-Hartenberg (D-H) representation

$T_{04} = T_{01}T_{12}T_{23}T_{34},$ Product of homogeneous transformation matrices

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In general:

$$T_{0,n+1}(\theta) = T_{01}(\theta_1)\cdots T_{n-1,n}(\theta_n)T_{n,n+1}$$

Exponential coordinate representation of rotation

$\dot{p} = \hat{\omega} \times p.$ Solution:

$\dot{p} = [\hat{\omega}]p$ $\quad p(\theta) = e^{[\hat{\omega}]\theta} p(0).$

$$
\begin{aligned}
e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots \\
&= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2
\end{aligned}
$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3).$$

Rodrigues' formula for rotations

$R' = e^{[\hat{\omega}]\theta} R = \mathrm{Rot}(\hat{\omega}, \theta) R$ in a fixed-frame

$R'' = R e^{[\hat{\omega}]\theta} = R\,\mathrm{Rot}(\hat{\omega}, \theta)$ in a body-frame $\quad R'' = Re^{[\hat{\omega}_2]\theta_2} \neq R' = e^{[\hat{\omega}_2]\theta_2} R.$

Exponential coordinates of a homogeneous transformation

$$\begin{array}{rl} \exp: & [\mathcal{S}]\theta \in se(3) \rightarrow T \in SE(3), \\ \log: & T \in SE(3) \rightarrow [\mathcal{S}]\theta \in se(3). \end{array}$$

$$\begin{aligned} e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2 \frac{\theta^2}{2!} + [\mathcal{S}]^3 \frac{\theta^3}{3!} + \cdots \\ &= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \qquad G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2 \frac{\theta^3}{3!} + \cdots \end{aligned}$$

$$\begin{aligned} G(\theta) &= I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2 \frac{\theta^3}{3!} + \cdots \\ &= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots \right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots \right)[\omega]^2 \\ &= I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2. \end{aligned}$$

If $\|\omega\| = 1$

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}$$

If $\omega = 0$ *and* $\|v\| = 1$, *then*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$$

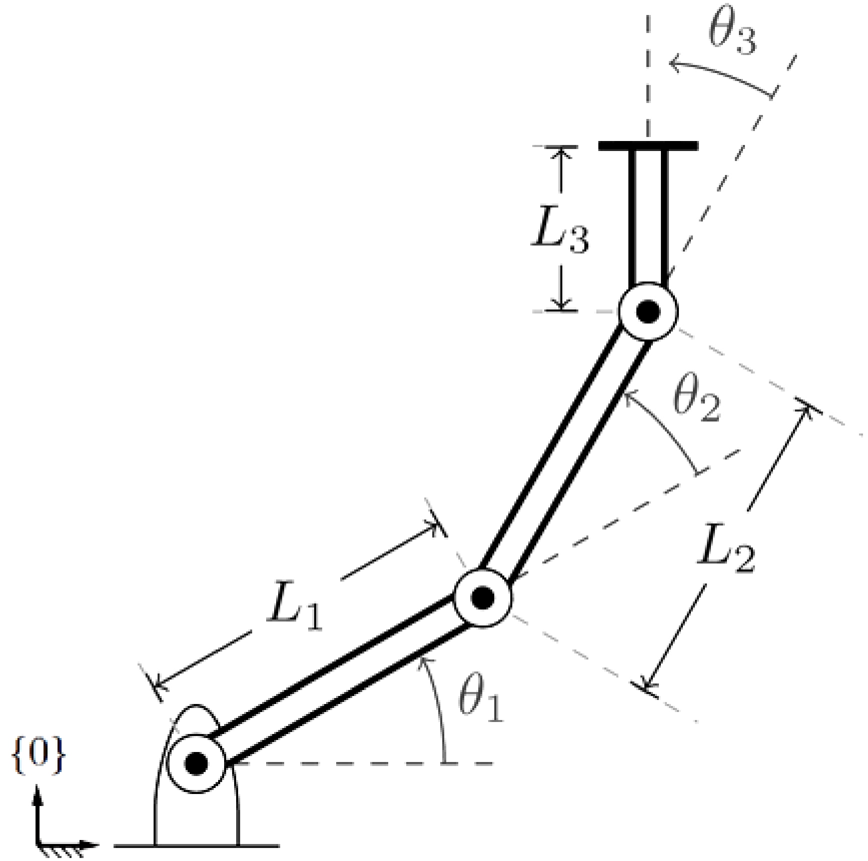
Exponential approach

М задает начальное положение, при котором углы поворота равны нулю

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

The screw axis corresponding to rotating about joint 3 expressed in the {0} frame

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}.$$



Product of exponentials (PoE) formula

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = \theta_2 = 0).$$

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0),$$

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M,$$

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Expressed in se(3) matrix form

Space form of the PoE formula:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M,$$

Body form of the PoE formula

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n},$$

where each $[\mathcal{B}_i]$ is given by $M^{-1}[\mathcal{S}_i]M$

$$\mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_i, \; i = 1, \ldots, n.$$

The matrix identity $e^{M^{-1}PM} = M^{-1}e^{P}M$

as $Me^{M^{-1}PM} = e^{P}M$

$$\begin{aligned}
T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots Me^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} \cdots Me^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= Me^{M^{-1}[\mathcal{S}_1]M\theta_1} \cdots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n},
\end{aligned}$$

Examples. 3R planar open chain

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,0,1)$	$(0,-L_1,0)$
3	$(0,0,1)$	$(0,-(L_1+L_2),0)$

i	ω_i	v_i
1	1	$(0,0)$
2	1	$(0,-L_1)$
3	1	$(0,-(L_1+L_2))$

$$M = \begin{bmatrix} 1 & 0 & L_1+L_2+L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$