Chapter 5

Robust stability and the $\mathbf{H}_\infty$ norm

An important application of the H_∞ control problem arises when studying robustness against model uncertainties. It turns out that the condition that a control system is robustly stable in spite of a certain kind of model uncertainties can be expressed quantitatively in terms of an H_∞ norm bound which the control system should satisfy. The connection of the H_∞ norm and robust stability will be described in this chapter.

We will consider a plant

$$y = Gu \tag{5.1}$$

where it is assumed that G can be characterized as

$$G = G_0 + \Delta_A \tag{5.2}$$

Here, G_0 is the *nominal plant* model, which is assumed to be linear and finite dimensional. The *additive uncertainty* Δ_A is unknown, but it is assumed that an upper bound on its magnitude can be estimated as a function of frequency. More precisely, for a SISO system we can specify a filter $W(j\omega)$ such that

$$|\Delta_A(j\omega)| \leq |W(j\omega)|, \text{ all } \omega \qquad \text{(SISO)} \tag{5.3}$$

The uncertainty bound $|W(j\omega)|$ characterizes the assumed maximum model uncertainty at various frequencies, and it can often be estimated from experimental data. In particular, if the plant model has been determined by frequency response experiments, the maximum error of the frequency response can often directly be estimated from the available experimental frequency response data.

For convenience, the characterization (5.2), (5.3) of an uncertain plant is usually stated in an equivalent form obtained by defining Δ such that $\Delta_A = W\Delta$. Then (5.2), (5.3) is equivalent to

$$G = G_0 + W\Delta, \ |\Delta(j\omega)| \leq 1, \text{ all } \omega \qquad \text{(SISO)} \tag{5.4}$$

Here, the uncertainty block Δ is assumed to be uniformly bounded at all frequencies, and the varying amount of plant uncertainty at various frequencies is completely incorporated into the filter W. The filter W is therefore commonly called *uncertainty weighting filter* or simply the *uncertainty weight.*

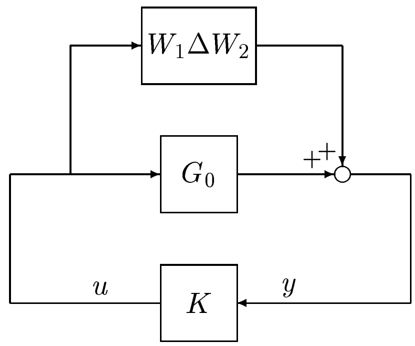

Figure 5.1: Control of uncertain system.

For *MIMO systems*, the norm-bounded uncertainty description is generalized by assuming that the uncertainty block Δ_A is bounded in its induced matrix norm $\|\Delta(j\omega)\| = \bar{\sigma}(\Delta(j\omega))$ according to

$$\bar{\sigma}(\Delta_A(j\omega)) \leq |W(j\omega)|, \text{ all } \omega \qquad \text{(MIMO)} \tag{5.5}$$

where W is a scalar weighting filter. The characterization (5.4) is then replaced by

$$G = G_0 + W\Delta, \ \ \bar{\sigma}(\Delta(j\omega)) \leq 1, \ \text{ all } \omega \qquad \text{(MIMO)} \tag{5.6}$$

or, equivalently (recalling the definition of the H_∞ norm)

$$G = G_0 + W\Delta, \ \ \|\Delta\|_\infty \leq 1 \qquad \text{(MIMO)} \tag{5.7}$$

Although the uncertain plant in (5.4) or (5.7) has been formulated for the additive uncertainty (5.2), it is straightforward to extend the uncertainty description to other forms of norm-bounded uncertainties. An important uncertainty type is the *output multiplicative uncertainty*,

$$G = (I + \Delta_M)G_0 \tag{5.8}$$

Assuming that

$$\bar{\sigma}(\Delta_M(j\omega)) \leq |W(j\omega)|, \text{ all } \omega \tag{5.9}$$

the uncertain plant can be written in the equivalent form

$$\begin{aligned} G &= (I + \Delta W)G_0 \\ &= G_0 + \Delta W G_0, \ \ \|\Delta\|_\infty \leq 1 \end{aligned} \tag{5.10}$$

which is of the same general form as (5.7), with uncertainty weight WG_0 and with the uncertainty and weight in reversed order. *Input multiplicative uncertainties* can be characterized in a similar way.

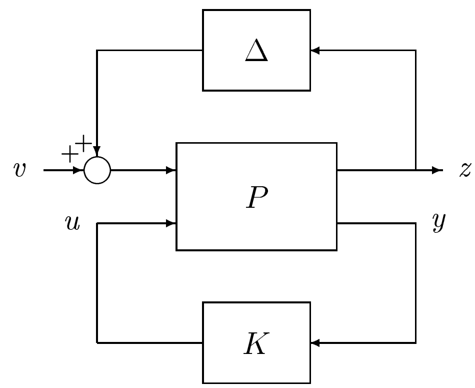

Figure 5.2: Standard representation of uncertain control system.

A general description of an uncertain system with a norm-bounded uncertainty is given by

$$G = G_0 + W_1 \Delta W_2, \quad \|\Delta\|_\infty \leq 1 \tag{5.11}$$

See Figure 5.1.

For the control system in Figure 5.1, it is important to study whether the closed loop is stable for all possible G, i.e., for all norm-bounded Δ such that $\|\Delta\|_\infty \leq 1$. In this case the control system is said to be *robustly stable* and the controller K is said to be *robustly stabilizing*. The problem of finding such a robustly stabilizing controller is stated as follows.

Robust stabilization problem.
Consider the uncertain plant described by (5.11). Find a controller such that the control system in Figure 5.1 is stable for all uncertainties Δ which satisfy the norm bound $\|\Delta\|_\infty \leq 1$.

The robust stabilization problem can be stated in terms of an equivalent H_∞-optimal control problem. The result will be based on the observation that the control system in Figure 5.1 can be written in the equivalent form in Figure 5.2, where P is an augmented plant defined as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0 & W_2 \\ W_1 & G_0 \end{bmatrix} \tag{5.12}$$

To see the equivalence of the control systems in Figures 5.1 and 5.2, it is sufficient to show that the block diagrams give the same transfer function from u to y. Notice that by Figure 5.2 and equation (5.12) we have

$$\begin{aligned} y &= [P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}]u \\ &= [G_0 + W_1\Delta(I - 0 \cdot \Delta)^{-1}W_2]u \\ &= [G_0 + W_1\Delta W_2]u \end{aligned} \tag{5.13}$$

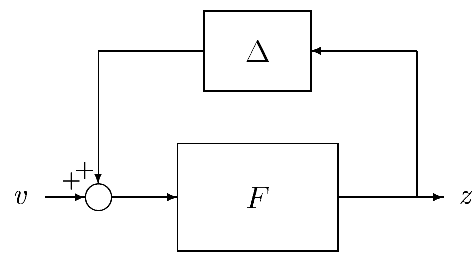

Figure 5.3: Representation of uncertain system.

which is the same as in Figure 5.1.

In order to obtain a qualitative appreciation of the robust stability problem, notice that the uncertainty block Δ appears in a feedback loop around the closed-loop plant, and classical stability analysis can be applied to provide stability conditions for this feedback loop. More precisely, the control system in Figure 5.2 can be written in the equivalent form shown in Figure 5.3,

$$z = (I - F\Delta)^{-1} F v, \quad F = F(P, K) \tag{5.14}$$

where $F = F(P, K)$ denotes the closed-loop transfer from v to z for the nominal system (with $\Delta = 0$) in Figure 5.2, i.e.,

$$F(P, K) = P_{11} + P_{12}(I - KP_{22})^{-1} K P_{21} \tag{5.15}$$

According to classical control theory of SISO systems, the feedback loop in Figure 5.3 is stable if $F(P, K)$ is stable and the loop transfer function $F\Delta$ has magnitude less than one; $|F(j\omega)\Delta(j\omega)| < 1$, all ω (implying that $|F(j\omega_c)\Delta(j\omega_c)| < 1$ at the cross-over frequency ω_c, where $F\Delta$ has phase $-\pi$). This result is known as the *small gain theorem*. By the assumption that Δ is norm-bounded, such that $|\Delta(j\omega)| \leq 1$, robust stability is thus guaranteed if the closed-loop transfer function F satisfies $|F(j\omega)| < 1$, all ω, or equivalently, the H_∞-norm bound $\|F\|_\infty < 1$ holds. It turns out that the assumption on the uncertainty block Δ, $\|\Delta\|_\infty \leq 1$, leaves sufficient freedom so that a norm-bounded Δ which actually destabilizes the system can always be found if $|F(j\omega)| \geq 1$, for some ω, i.e., if $\|F\|_\infty \geq 1$. Hence, $\|F\|_\infty < 1$ is both a sufficient and necessary condition for robust stability.

For MIMO systems, we can use the following (simplified) qualitative arguments. Notice that the transfer function in (5.14) can be written

$$(I - F\Delta)^{-1} F = \frac{1}{\det(I - F\Delta)} \mathrm{adj}(I - F\Delta) \cdot F \tag{5.16}$$

Assuming that F and Δ are both stable, the stability of the operator in (5.16) is determined by the zeros of $\det(I - F\Delta)$. Since the system in Figure 5.3 is stable for sufficiently small Δ, it follows by interpolation that if there exists a norm-bounded Δ such that the operator in (5.16) is unstable, i.e., it has a right-half-plane pole, then there also exists a norm-bounded Δ such that the operator in (5.16) has a pole on the stability boundary, i.e., on the imaginary axis, implying $\det(I - F(j\omega)\Delta(j\omega)) = 0$ for some ω. Hence the robust stability condition is equivalent to the condition that

$$\det(I - F(j\omega)\Delta(j\omega)) \neq 0, \ \text{all } \omega \text{ and } \Delta \text{ such that } \|\Delta\|_\infty \leq 1 \tag{5.17}$$

Some analysis shows that the condition (5.17) holds if and only if $\bar{\sigma}(F(j\omega)) < 1$, all ω, or equivalently, $\|F\|_\infty < 1$.

The above results are summarized in the following robust stability condition.

Theorem 5.1 Condition for robust stability.
Consider the system in Figure 5.2 or 5.3. The system is stable for all uncertainties which satisfy the norm bound $\|\Delta\|_\infty \leq 1$ if and only if the nominal closed-loop transfer function $F = F(P, K)$ is stable and

$$\|F\|_\infty < 1 \tag{5.18}$$

Remark 5.1.
As mentioned above, the 'if'-part,

$$\|F\|_\infty < 1 \quad \Rightarrow \quad \text{robust stability}$$

follows from the classical small gain theorem, because the H_∞-norm bound guarantees $|F(j\omega)\Delta(j\omega)| < 1$ for SISO systems and $\bar{\sigma}(F(j\omega)\Delta(j\omega)) < 1$ for MIMO systems. By contrast, the 'only if'-part,

$$\text{robust stability} \quad \Rightarrow \quad \|F\|_\infty < 1$$

is newer, and was proved only in the 1980's. The result follows basically from the fact that the uncertainty can be chosen to have arbitrary phase. Hence, if $\|F\|_\infty \geq 1$ it follows that a norm-bounded uncertainty $\|\Delta\|_\infty \leq 1$ can always be found which destabilizes the system.

Remark 5.2.
The robust stability condition in (5.18) is formulated for norm-bounded uncertainties with a norm bound which is normalized so that $\|\Delta\|_\infty \leq 1$. Any other magnitude of the norm bound is assumed to be incorporated as a corresponding factor in the weight filters W_1 and W_2. Sometimes it is, however, convenient to formulate the robustness criterion explicitly for norm-bounded uncertainties which satisfy the general norm bound $\|\Delta\|_\infty \leq \delta$. It is easy to see that the corresponding robust stability condition is then

$$\|F\|_\infty < \delta^{-1} \tag{5.19}$$

It may for example be of interest to determine the largest norm-bounded uncertainty that can be allowed such that it is still possible to find a controller which gives robust stability. The *maximally robustly stabilizing controller* can be found by minimizing $\|F\|_\infty$ (by γ-iteration). The controller is then robustly stabilizing for all Δ such that $\|\Delta\|_\infty < \|F\|_\infty^{-1}$.

Remark 5.3. *Robust stability against nonlinear and/or time-varying uncertainties.* Although the robust stability result has above been formulated for linear, time-invariant (LTI) uncertainties, it can easily be generalized to **non-linear**, and possibly **time-varying uncertainties**. Suppose that the uncertainty is not restricted to be linear, but is only assumed to satisfy the norm bound

$$\|\Delta\| \leq 1 \tag{5.20}$$

where $\|\Delta\|$ denotes the L_2-induced norm

$$\|\Delta\| = \sup \left\{ \frac{\|\Delta v\|_2}{\|v\|_2} \; : \; v \neq 0 \right\} \tag{5.21}$$

Notice that this definition of the L_2-induced norm coincides with the definition of the H_∞ norm form for linear time-invariant systems (equations (2.51), (2.61)), but now Δ may be nonlinear and/or time-varying. Then it can be shown that the system in Figure 5.2 or 5.3 is stable for all, possibly nonlinear and/or time-varying uncertainties Δ which satisfy the L_2-induced norm bound (5.20) if and only if the nominal closed-loop transfer function $F = F(P, K)$ is stable and satisfies the H_∞-norm bound (5.18), i.e., $\|F\|_\infty < 1$.

Hence the condition for robust stability against nonlinear, time-varying uncertainties is the same as the condition for robust stability against linear time-invariant uncertainties. This result may appear counter-intuitive, as one would expect that the first, more general, uncertainty class would correspond to a more restrictive condition on the closed loop. The result is, however, a consequence of the fact that the nominal system is linear and time-invariant, and then the worst, destabilizing uncertainty is also linear and time-invariant.

By Remark 5.3 the robust stability condition provides a quantitative way of treating *plant nonlinearities as uncertainties about a nominal linear plant model.* Such an approach is of course only feasible for not too strongly nonlinear plants, where the norm bound of the uncertainty which captures the nonlinearities is not excessive.

The significance of the robust stability condition (5.18) is that it gives a quantitative characterization of robust stability in terms of the H_∞ norm. By the procedure in Chapter 4 we know how to find a controller which achieves the norm bound (5.18), and which is robustly stabilizing. Before the introduction of H_∞ control theory the solution of the robust stabilization problem was not known.

Although theoretically important, the robust stabilization problem alone is perhaps of limited practical interest. Firstly, many processes, for example in chemical process control systems, are known to be open-loop stable. The maximally robustly stabilizing control strategy is then to operate the plant in open loop, because any feedback controller gives a reduction in robust stability for an open-loop stable plant. Secondly, even for an unstable uncertain plant, a controller which is only designed for robust stability will in general be conservative. In practical controller design, the robust stability condition (5.18) is useful as a robustness bound used in combination with other performance related design criteria. This will be discussed in Chapter 7.

Notes and references

The formulation of robust stability in terms of the H_∞ norm is due to a number of researchers, see for example Doyle and Stein (1981), Chen and Desoer (1982), Kimura (1984), Vidyasagar (1985), and Vidyasagar and Kimura (1986). Highly readable accounts of the robust stability issue are found in the books by Vidyasagar (1985) and McFarlane and Glover (1990).

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