The second homework
Stability of nonlinear systems

Task 1. Consider the following nonlinear system:

1. Variant.
$$\begin{cases} \dot{x} = -x - x^3 - 2y, \\ \dot{y} = x - y; \end{cases}$$

2. Variant.
$$\begin{cases} \dot{x} = -3x + y, \\ \dot{y} = -x - 2y - \frac{y}{|y|+1}; \end{cases}$$

3. Variant.
$$\begin{cases} \dot{x} = -3x + y, \\ \dot{y} = x - 4y - \arctan 2y; \end{cases}$$

4. Variant.
$$\begin{cases} \dot{x} = -x - 3y, \\ \dot{y} = -x - 4y - \operatorname{sat} 2y, \end{cases} \quad \text{where } \operatorname{sat} 2y = \begin{cases} 2y, & \text{while } |y| < 1 \\ 2 \operatorname{sign} y, & \text{while } |y| \geqslant 1; \end{cases}$$

5. Variant.
$$\begin{cases} \dot{x} = -x - 2y, \\ \dot{y} = 4x - 2y - \frac{y}{|y|+3}; \end{cases}$$

6. Variant.
$$\begin{cases} \dot{x} = -2x - x^5 + y, \\ \dot{y} = -x - y; \end{cases}$$

7. Variant.
$$\begin{cases} \dot{x} = -x + y - \frac{2x}{|x|+4}, \\ \dot{y} = -x - 4y; \end{cases}$$

8. Variant.
$$\begin{cases} \dot{x} = -4x + 2y - \arctan x, \\ \dot{y} = -2x - 2y; \end{cases}$$

9. Variant.
$$\begin{cases} \dot{x} = -3x + y - \operatorname{sat} x, \\ \dot{y} = -x - 2y, \end{cases} \quad \text{where } \operatorname{sat} x = \begin{cases} x, & \text{while } |x| < 1 \\ \operatorname{sign} x, & \text{while } |x| \geqslant 1 \end{cases}$$

10. Variant.

$$\begin{cases} \dot{x} = -x + 2y - \frac{x}{|x|+2}, \\ \dot{y} = -2x - 2y; \end{cases}$$

- find all the equilibrium points;
- linearize the system around the equilibrium point and analyze the stability of linearized system;
- design the Lyapunov function and prove global stability of nonlinear system;
- draw the plots of linearized and nonlinear systems.

Task 2. Consider the following nonlinear system

$$\dot{x} = Ax + b\xi, \quad \sigma = c^* x, \tag{1}$$

$$\xi = \varphi(\sigma, t), \tag{2}$$

where

1. Variant.

$$A = \begin{bmatrix} -1 & 2 \\ -3 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \sin\sigma;$$

2. Variant.

$$A = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = -\frac{1}{1+e^{-\sigma}} + \frac{1}{2};$$

3. Variant.

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \operatorname{th}\sigma = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}};$$

4. Variant.

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \frac{\sigma}{|\sigma| + 2};$$

5. Variant.

$$A = \begin{bmatrix} -4 & 2 \\ -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \arctan\sigma;$$

6. Variant.

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi = \sin 0.5\sigma;$$

7. Variant.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \frac{1}{1 + \mathrm{e}^{-\sigma}} - \frac{1}{2};$$

8. Variant.

$$A = \begin{bmatrix} 1 & -2 \\ 4 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \operatorname{th} 0.5\sigma = \frac{e^{0.5\sigma} - e^{-0.5\sigma}}{e^{0.5\sigma} + e^{-0.5\sigma}};$$

9. Variant.

$$A = \begin{bmatrix} -1 & -2 \\ 4 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi = \frac{\sigma}{|\sigma| + 3};$$

10. Variant.

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi = \arctan 2\sigma;$$

Prove the exponential stability of the system (1), (2) using the circle criterion.

Theorem 1 (Circle criterion). *Let the following assumptions be fulfilled:*

- *function $\varphi(\sigma, t)$ in (2) saitisfies the following inequality for $\sigma \neq 0$:*

$$\mu_1 \leqslant \varphi(\sigma, t)/\sigma \leqslant \mu_2, \quad (\forall t \in (0, \infty));$$

- *there exists an eigenvalue $\lambda_i(A)$ of matrix A in (1): $\operatorname{Re} \lambda_i(A) \neq 0$;*
- *there exists $\mu_0 \in [\mu_1, \mu_2]$, for which the linear system (1), $\xi = \mu_0 \sigma$ is asymptotically stable;*
- *the "frequency inequality" is fulfilled:*

$$\operatorname{Re}\{[1 + \mu_1 W(i\omega)][1 + \mu_2 W(i\omega)]^*\} > 0, \quad (\omega \in [-\infty, +\infty]),$$

where the sign $*$ means the complex conjugare and function $W(i\omega)$ is defined by

$$W(i\omega) = c^*(A - i\omega I)^{-1} b.$$

Then nonlinear system (1), (2) *is globally exponentially stable: there exist constants* $c > 0$, $\varepsilon > 0$ *such that, for all soultions* $x(t)$ *of the system* (1), (2) *and for all* $t \geqslant t_0$ *the following inequality* $|x(t)| \leqslant c|x(t_0)|\mathrm{e}^{-\varepsilon(\mathrm{t}-\mathrm{t}_0)}$ *is fulfilled.*

Task 3. Consider the following nonlinear system:

$$\dot{x} = Ax + b\xi, \quad \sigma = c^*x, \tag{3}$$

$$\xi = \varphi(\sigma), \tag{4}$$

where

1. Variant.

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi = \arctan 2\sigma;$$

2. Variant.

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \frac{\sigma^3}{3};$$

3. Variant.

$$A = \begin{bmatrix} -1 & -3 \\ -1 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi = \operatorname{sat} 2\sigma = \begin{cases} 2\sigma, & \text{while } |\sigma| < 1 \\ 2\operatorname{sign}\sigma, & \text{while } |\sigma| \geqslant 1 \end{cases};$$

4. Variant.

$$A = \begin{bmatrix} -1 & -2 \\ 4 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi = \frac{\sigma}{|\sigma| + 3};$$

5. Variant.

$$A = \begin{bmatrix} 1 & -2 \\ 4 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \operatorname{th} 0.5\sigma = \frac{e^{0.5\sigma} - e^{-0.5\sigma}}{e^{0.5\sigma} + e^{-0.5\sigma}};$$

6. Variant.

$$A = \begin{bmatrix} -4 & 2 \\ -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \arctan \sigma;$$

7. Variant.

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \frac{\sigma^5}{5};$$

8. Variant.

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \operatorname{sat} \sigma = \begin{cases} \sigma, & \text{while } |\sigma| < 1 \\ \operatorname{sign} \sigma, & \text{while } |\sigma| \geqslant 1 \end{cases};$$

9. Variant.

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \frac{\sigma}{|\sigma| + 2};$$

10. Variant.

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi = \operatorname{th} \sigma = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}};$$

Prove the asympotical stability of the nonlinear system (3), (4) using the Popov criterion.

Theorem 2 (Popov criterion)**.** *Let the following assumptions be fulfilled:*

- *function $\varphi(\sigma)$ in (4) satisfies the following inequality for $\sigma \neq 0$:*

$$0 \leqslant \varphi(\sigma)/\sigma \leqslant \mu_0;$$

- *matrix A in system (3) is Hurwitz (stable);*
- *there exists ν such that for all $\omega \in [-\infty, +\infty]$ the "frequency inequality" is fulfilled:*

$$\mu_0^{-1} + \operatorname{Re}[(1 + i\omega\nu)W(i\omega)] > 0,$$

where $W(i\omega)$ is deifned by

$$W(i\omega) = c^*(A - i\omega I)^{-1} b.$$

Then nonlinear system (3), (4) is globally asymptotically stable.