УНИВЕРСИТЕТ ИТМО

Факультет систем управления и робототехники

Nonlinear systems

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5. Feedback linearization

Motivation

Why and when do we need linearization ?

Motivation

Example…

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu$$

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$$\dot{x}_2 = -k_1 x_1 - (k_2 + b)x_2 \longrightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 - b \end{bmatrix}}_{\text{is Hurwtz}} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

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$$u = \frac{a}{c}[\sin(x_1+\delta) - \sin\delta] + \frac{1}{c}[-k_1x_1 - k_2x_2]$$

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Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

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$$u = \alpha(x) + \gamma^{-1}(x)v \Longrightarrow \dot{z} = Az + Bv$$

$$v = -Kz$$

Design K such that $(A - BK)$ is Hurwtz

$$u = \alpha(x) - \gamma^{-1}(x)KT(x)$$

Closed-loop system in the x-coordinates:

$$\dot{x} = f(x) + G(x)[\alpha(x) - \gamma^{-1}(x)KT(x)]$$

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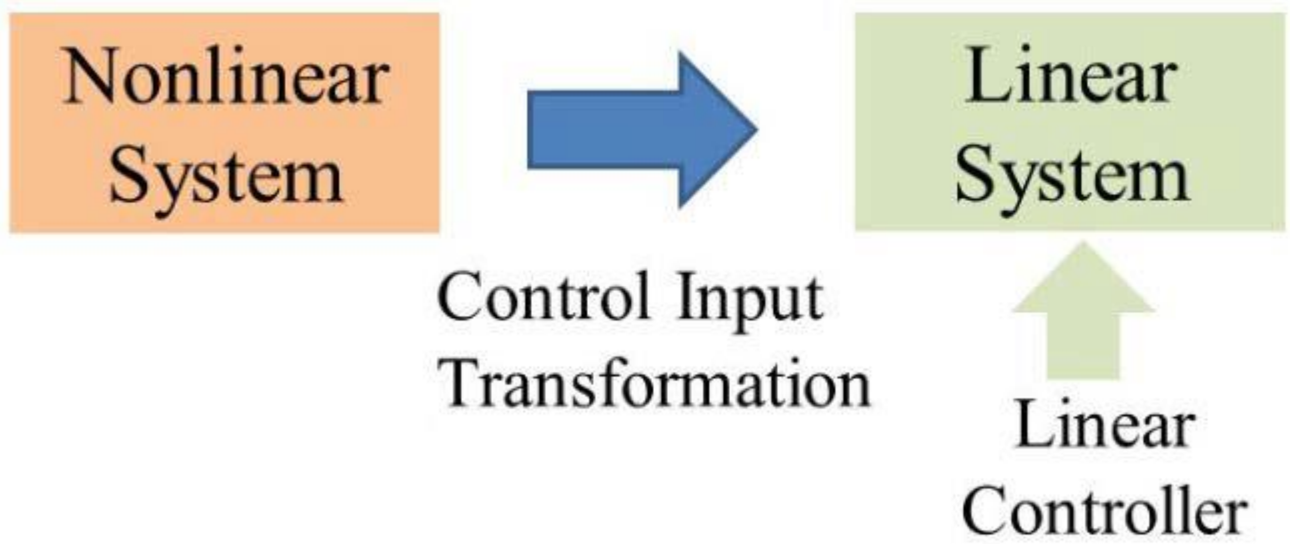

5. Feedback linearization

$$u = \alpha(x) - \gamma^{-1}(x) K T(x)$$

?

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In fact, we have $\hat{\alpha}, \hat{\gamma}, \hat{T}$

$$u = \hat{\alpha}(x) - \hat{\gamma}^{-1}(x)K\hat{T}(x)$$

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Closed-loop system:

$$\dot{z} = (A - BK)z + B\delta(z)$$

$$\delta = \gamma\left[\hat{\alpha} - \alpha + \gamma^{-1}KT - \hat{\gamma}^{-1}K\hat{T}\right]$$

$$u = \hat{\alpha}(x) - \hat{\gamma}^{-1}(x) K \hat{T}(x)$$

Closed-loop system:

$$\dot{z} = (A - BK)z + B\delta(z)$$

$$\delta = \gamma\left[\hat{\alpha} - \alpha + \gamma^{-1} KT - \hat{\gamma}^{-1} K \hat{T}\right]$$

where: $\hat{\alpha}$, $\hat{\gamma}$, $\hat{T}$ are nominal models of α, γ and T.

$$V(z) = z^T P z,$$

$$P(A - BK) + (A - BK)^T P = -I$$

If $\|\delta(z)\| \leq k\|z\|$ for all z, where

$$0 \leq k < \frac{1}{2\|PB\|}$$

then the origin is globally exponentially stable

- Example: pendulum equation

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT$$

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \qquad u = T - T_{ss} = T - \frac{a}{c}\sin\delta,$$

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$$u = \frac{1}{c}\{a[\sin(x_1 + \delta) - \sin\delta] - k_1x_1 - k_2x_2\}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix} \qquad \text{is Hurwith}$$

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$$T = u + \frac{a}{c}\sin\delta = \frac{1}{c}[a\sin(x_1+\delta) - k_1 x_1 - k_2 x_2]$$

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$$T = \frac{1}{\hat{c}}[\hat{a}\sin(x_1 + \delta) - k_1 x_1 - k_2 x_2]$$

$$\dot{x} = (A - BK)x + B\delta(x)$$

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1 + \delta) - \left(\frac{c - \hat{c}}{\hat{c}}\right)(k_1 x_1 + k_2 x_2)$$

- Example: pendulum equation

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1 + \delta_1) - \left(\frac{c - \hat{c}}{\hat{c}}\right)(k_1 x_1 + k_2 x_2)$$

$$|\delta(x)| \le k\|x\| + \varepsilon$$

$$k = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| + \left|\frac{c - \hat{c}}{\hat{c}}\right|\sqrt{k_1^2 + k_2^2}, \qquad \varepsilon = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| |\sin(x_1 + \delta_1)|$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \qquad PB = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}$$

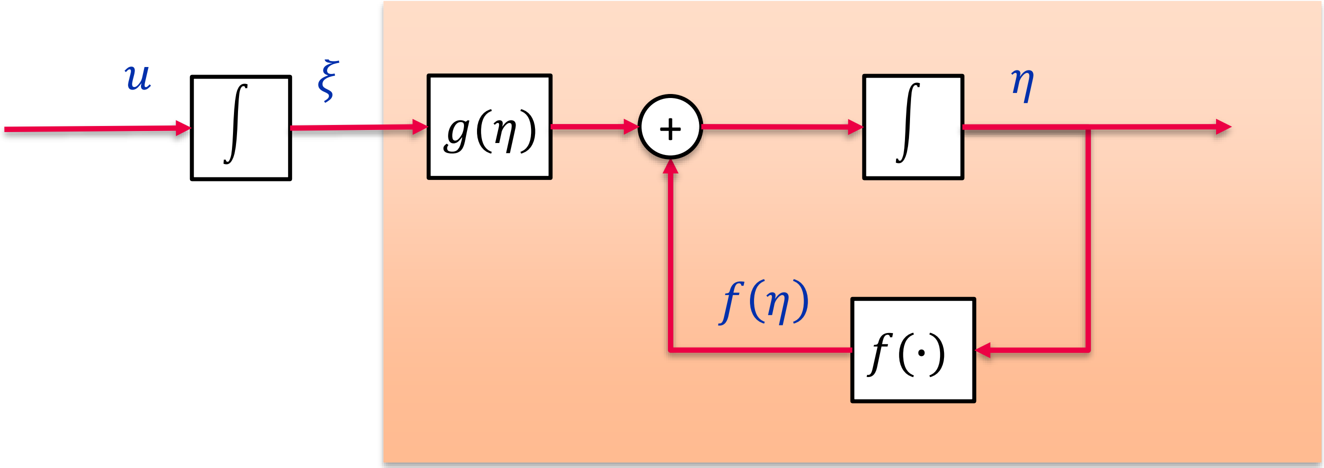
$$\sin\delta_1 = 0 \;\Rightarrow\; \varepsilon = 0$$

6. Backstepping

Consider the system

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$
$$\dot{\xi} = u$$

$\eta \in \mathbb{R}^n, \quad \xi, u \in \mathbb{R}$



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Stabilize the origin using state feedback

View ξ as "virtual" control input to

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$

Suppose there is $\xi = \phi(\eta)$ that stabilizes the origin of

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

With the Lyapunov function in the form

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \le -W(\eta), \quad \forall \eta \in \mathbb{D}$$

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Transform the original system by adding and subtracting $g(\eta)\phi(\eta)$

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$$\dot{\xi} = u$$

And we will introduce a replacement

$$z = \xi - \phi(\eta)$$

We get:

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z$$
$$\dot{z} = u - \dot{\phi}$$

$$\dot{\phi}(\eta) = \frac{\partial\phi(\eta)}{\partial\eta}\dot{\eta} \quad \Longrightarrow \quad \dot{\phi}(\eta) = \frac{\partial\phi(\eta)}{\partial\eta}[f(\eta) + g(\eta)\xi]$$

Taking $v = u - \dot{\phi}$ reduces the system to the cascade connection

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z$$
$$\dot{z} = v$$

now the first component has an asymptotically stable origin when the input is zero.

Consider the Lyapunov function

$$V_c(\eta, \xi) = V(\eta) + \frac{1}{2}z^2$$

Its derivative is equal to

$$\dot{V}_c = \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta}g(\eta)z + zv$$

$$\leq -W(\eta) + \frac{\partial V}{\partial \eta}g(\eta)z + zv$$

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Choosing
$$v = -\frac{\partial V}{\partial \eta}g(\eta) - kz \qquad k > 0$$

Yields
$$\dot{V}_c \leq -W(\eta) - kz^2$$

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$

$$\dot{\xi} = u$$

Substituting for u, v and $\dot{\phi}$ we obtain the state feedback control law:

$$v = u - \dot{\phi} \qquad z = \xi - \phi(\eta)$$

$$\dot{\phi}(\eta) = \frac{\partial\phi(\eta)}{\partial\eta}[f(\eta) + g(\eta)\xi]$$

$$v = -\frac{\partial V}{\partial\eta}g(\eta) - kz$$

$$u = \frac{\partial\phi(\eta)}{\partial\eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial\eta}g(\eta) - k[\xi - \phi(\eta)]$$

$$\dot{\eta} = f(\eta) + g(\eta)\xi \qquad (*)$$

$$\dot{\xi} = u$$

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \qquad (**)$$

$$u = \frac{\partial \phi(\eta)}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta} g(\eta) - k[\xi - \phi(\eta)] \qquad (***)$$

Lemma

Consider the system $(*)$. Let $\phi(\eta)$ be a stabilizing state feedback control for $(*)$ with $\phi(0) = 0$, and $V(\eta)$ be a Lyapunov function that satisfies $(**)$ with some definite function $W(\eta)$. Then, the state feedback control $(***)$ stabilizes the origin of $(*)$, with $V(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2$ as a Lyapunov function. Moreover, if all the assumptions hold globally and $V(\eta)$ is radially unbounded, the origin will be globally asymptotically stable.

Example 1

Consider the system:

$$\dot{x}_1 = x_1^2 - x_1^3 + \underline{x_2}$$
$$\dot{x}_2 = u$$

$$x_2 = \phi(x_1) = -x_1^2 - x_1 \quad \Longrightarrow \quad \dot{x}_1 = -x_1 - x_1^3$$

$$V(x_1) = \frac{1}{2}x_1^2 \quad \Longrightarrow \quad \dot{V} = -x_1^2 - x_1^4 \qquad \forall x_1 \in \mathbb{R}$$

Next, we will introduce a replacement:

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1^2 + x_1$$

$$\dot{x}_1 = -x_1 - x_1^3 + z_2$$
$$\dot{z}_2 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)$$

- Example 1

$$\dot{x}_1 = -x_1 - x_1^3 + z_2$$

$$\dot{z}_2 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)$$

$$V_c(x) = V(x) + \frac{1}{2} z^2$$

$$V(x) = \frac{1}{2} x_1^2$$

$$V_c(x) = \frac{1}{2} x_1^2 + \frac{1}{2} z_2^2$$

$$\dot{V}_c = x_1(-x_1 - x_1^3 + z_2) + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)]$$

$$\dot{V}_c = -x_1^2 - x_1^4 + z_2[x_1 + u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)]$$

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$$\boxed{u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2}$$

$$-z_2$$

- Example 1

$$\boxed{u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2}$$

$$z_2 = x_2 + x_1^2 + x_1$$

$$\dot{V}_c = -x_1^2 - x_1^4 - z_2^2$$

$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2$$

Example 1

For system:

$$\boxed{\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u \\ u = -x_1 - (1 + 2x_1)&(-x_1^3 + x_2 + x_1^2) - (x_2 + x_1^2 + x_1) \end{aligned}}$$

the origin $x = 0$ is globally asymptotically stable

Example 2

Consider the third-order system:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\end{aligned}$$

After one step of backstepping, we know that the second-order system:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\end{aligned}$$

with x_3 as input, can he globally stabilized by the control:

$$x_3 = -x_1 - (1 + 2x_1)(-x_1^3 + x_2 + x_1^2) - (x_2 + x_1^2 + x_1)$$

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$$x_3 = -x_1 - (1 + 2x_1)(-x_1^3 + x_2 + x_1^2) - (x_2 + x_1^2 + x_1) \stackrel{\text{def}}{=} \phi(x_1, x_2)$$

with $$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2 + x_1)^2$$

Example 2

By analogy, we introduce a replacement:

$$z_3 = x_3 - \phi(x_1, x_2)$$

It was

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$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = x_3 - \phi(x_1, x_2) + \phi(x_1, x_2)$$
$$\dot{x}_3 = u$$
$$\dot{x}_3 = \dot{z}_3 + \dot{\phi}(x_1, x_2)$$

We get:

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = z_3 + \phi(x_1, x_2)$$
$$\dot{z}_3 = u - \dot{\phi}(x_1, x_2)$$

Example 2

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = z_3 + \phi(x_1, x_2)$$
$$\dot{z}_3 = u - \dot{\phi}(x_1, x_2)$$

$$\dot{x}_1 = \boxed{x_1^2 - x_1^3 + x_2}$$
$$\dot{x}_2 = \boxed{z_3 + \phi}$$

$$\dot{z}_3 = u - \frac{\partial \phi}{\partial x_1}\boxed{(x_1^2 - x_1^3 + x_2)} - \frac{\partial \phi}{\partial x_2}\boxed{(z_3 + \phi)}$$

Example 2

Consider the Lyapunov function:

$$V_c = V + \frac{1}{2}z_3^2$$

Its derivative is equal to:

$$\dot{V}_c = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + z_3\dot{z}_3$$

$$\dot{V}_c = \frac{\partial V}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V}{\partial x_2}(z_3 + \phi) + z_3\left(u - \dot{\phi}\frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi)\right)$$

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Its derivative is equal to:

$$\dot{V}_c = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + z_3\dot{z}_3$$

$$\dot{V}_c = \frac{\partial V}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V}{\partial x_2}(z_3 + \phi) + z_3\left(u - \dot{\phi}\,\frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi)\right)$$

$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 + z_3\left(\frac{\partial V}{\partial x_2} + u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi)\right)$$

Example 2

We obtain:

$$u = -\frac{\partial V}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$$

and

for a square

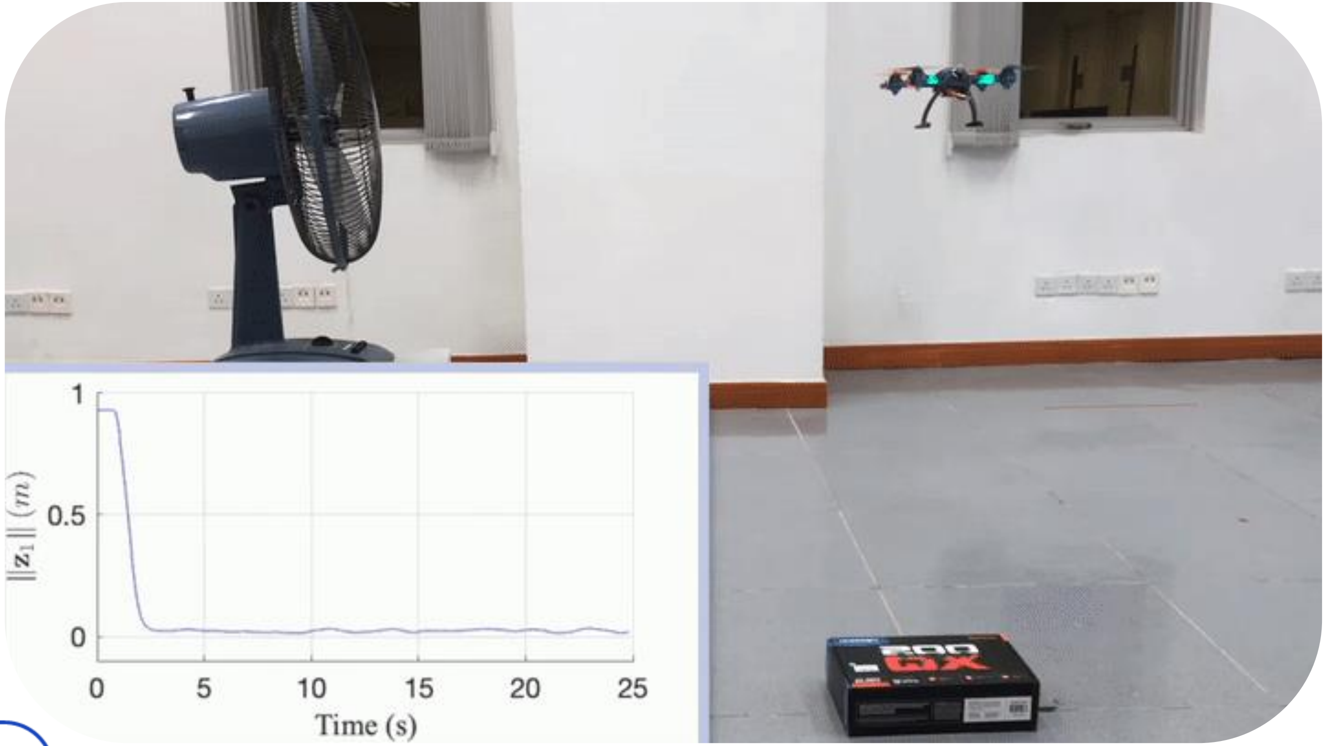
$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 + z_3^2$$

Generalize

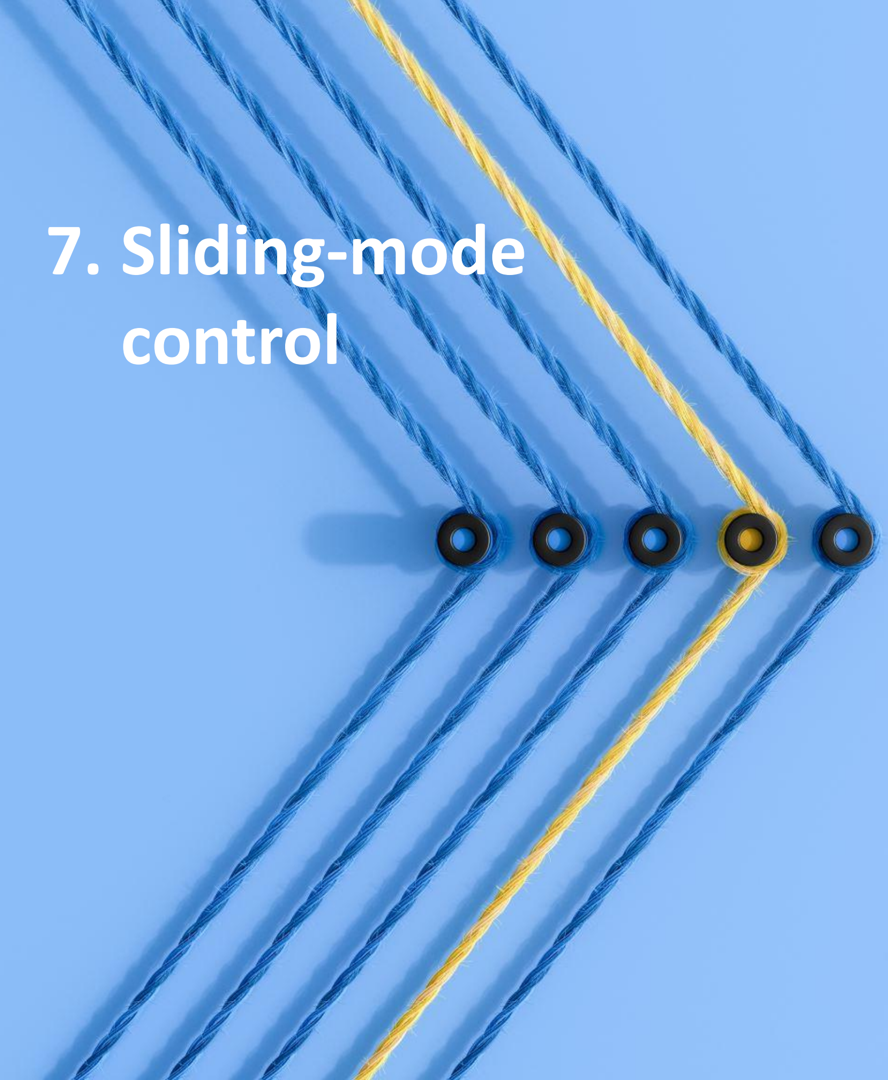
$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \cdots, z_{k-1}) + g_{k-1}(x, z_1, \cdots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \cdots, z_k) + g_k(x, z_1, \cdots, z_k)u
\end{aligned}
$$

$g_i(x, z_1, \cdots, z_i) \neq 0 \qquad$ for $1 \leq i \leq k$

Adaptive Backstepping Control of a Quadcopter



7. Sliding-mode control

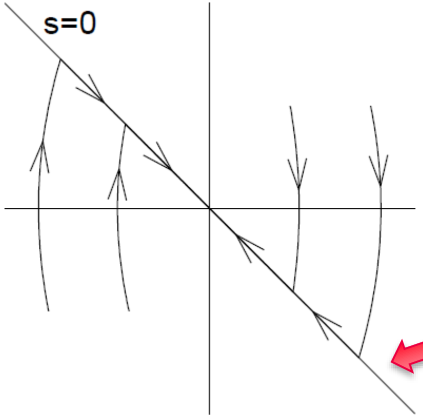


Consider the system :

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = h(x) + g(x)u \qquad g(x) \geq g_0 > 0$$

Sliding Manifold (Surface):



$$s = a_1 x_1 + x_2 = 0$$

$$s(t) \equiv 0 \implies \dot{x}_1 = -a_1 x_1$$

$$a_1 > 0 \implies \lim_{t\to\infty} x_1(t) = 0$$

$$\dot{s} = a_1\dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)u$$

Suppose: $$\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq \varrho(x)$$

With $V = \frac{1}{2}s^2$ as a Lyapunov function candidate we have:

$$\dot{V} = s\dot{s} = s[a_1 x_2 + h(x)] + g(x)su \leq \boxed{g(x)|s|\varrho(x) + g(x)su}$$

Taking $u = -\beta \operatorname{sgn}(s)$ where $\beta(x) \geq \boxed{\varrho(x) + \beta_0,\ \beta_0 > 0}$

$$s > 0, \quad u = -\beta(x)$$

$$\dot{V} \leq \boxed{g(x)|s|\varrho(x) - g(x)\beta(x)|s|}$$

$$\dot{V} \leq g(x)|s|\varrho(x) - \boxed{g(x)\boxed{(\varrho(x) + \beta_0)}|s|} = -g(x)\beta_0|s|$$

$$s < 0, \qquad u = \beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) + g(x)su \quad \Longrightarrow \quad \dot{V} \le g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

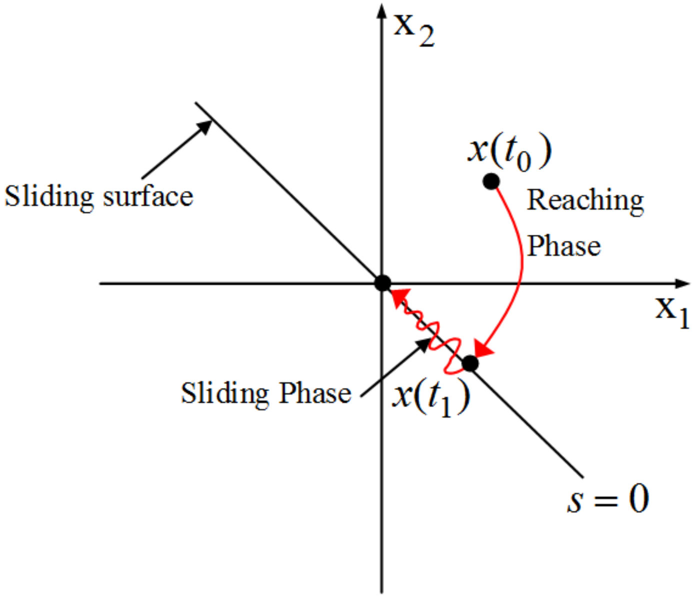
$$\dot{V} \le g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

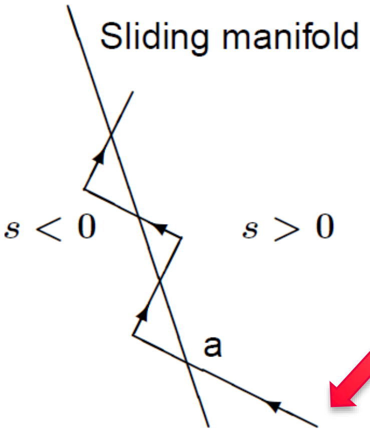
$$\text{sign}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

$$u = -\beta(x)\text{sign}(s)$$

$$\dot{V} \le -g(x)\beta_0|s| \le -g_0\beta_0|s|$$

$$\dot{V} \le -g_0\beta_0\sqrt{2V}$$





Chattering due to delay in control switching.

Solutions

1. divide the control into continuous and switching components

2. eliminate chattering is to replace the signum function by a high-slope saturation function

1. Divide the control into continuous and switching components

Nominal models

$$\hat{h}(x) \rightarrow h(x)$$

$$\hat{g}(x) \rightarrow g(x)$$

Taking: $u = \frac{[a_1 x_2 + \hat{h}(x)]}{\hat{g}(x)} + v$

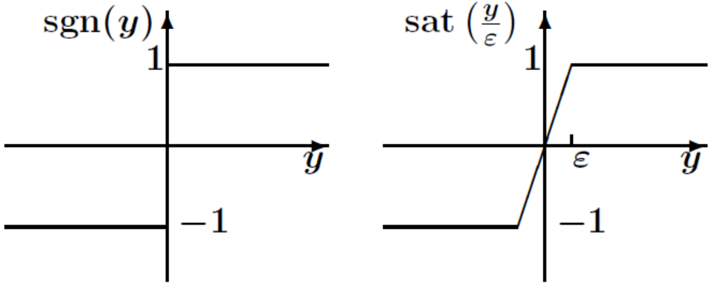
$$\dot{s} = a_1 \left[1 - \frac{g(x)}{\hat{g}(x)}\right] x_2 + h(x) - \frac{g(x)}{\hat{g}(x)} \hat{h}(x) + g(x) v \stackrel{\text{def}}{=} \delta(x) + g(x)$$

If: $\left|\frac{\delta(x)}{g(x)}\right| \leq \varrho(x) \rightarrow v = -\beta(x)\,\mathrm{sgn}(s)$

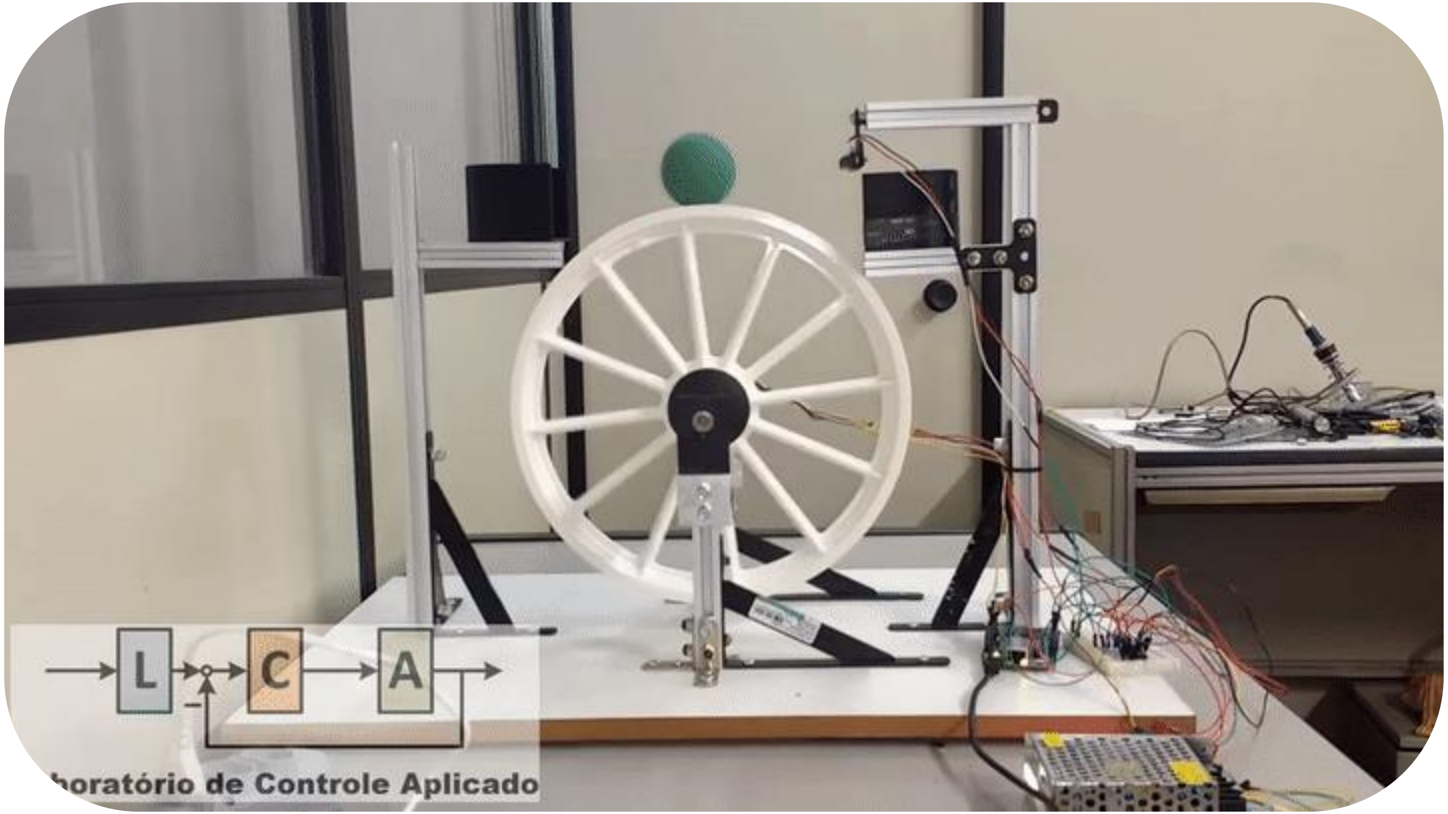
2. Eliminate chattering is to replace the signum function by a high-slope saturation function

$$u = -\beta(x)\text{sat}\left(\frac{s}{\varepsilon}\right)$$

$$\text{sat}(y) = \begin{cases} y, & if\,|y| \le 1 \\ \text{sgn}(y), & if\,|y| > 1 \end{cases}$$



Sliding-Mode Control of a Ball on Wheel System



Stabilization of a Ball on Sphere System Using Sliding Mode Control

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