

Type Theory workbook

Denis Mazzucato

May 2019

Workbook for exercises given in the Type Theory (Maietti and Sambin) course at Padua in 2019

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Assunzioni

Per semplificare la scrittura degli esercizi utilizzo alcune semplificazioni, l'unico esercizio svolto per intero senza omettere alcun tipo di passaggio è la preservazione dell'uguaglianza tra programmi (1)

- Il tipo delle variabili dentro il contesto deve essere specificato solo dove è necessario. Ovvero quando la variabile è inserita per la prima volta dentro al contesto oppure durante l'utilizzo della regola *var*) che richiede di abbinare il giudizio con la variabile dentro al contesto. In qualche raro caso se compare una Γ al posto del contesto è per problemi di spazio e significa che il contesto non è mutato dal passo precedente.
- Se devo derivare un giudizio del tipo $a \in A \ [\Gamma, a \in A, \Delta]$ e il contesto $\Gamma, a \in A, \Delta$ è *semplice* allora concludo che riesco a derivarlo. Solo in casi dove il contesto non è banale continuo con la derivazione.
- Se il contesto è *banale* allora posso indebolirlo senza dover utilizzare le regole di indebolimento.
- Posso concludere con i tre punti verticali quando la stessa parte della dimostrazione è già stata svolta in un altro ramo differente nello stesso albero.
- Quando mi riferisco ad una *Label* la trovo definita sempre sopra il punto in cui mi trovo, per problemi di spazio alcune volte ometto di ripetere la formula e la rimpiazzo con dei puntini.

1 Equality preservation among programs

$$\mathbf{pf} \in Id(B, f(a), f(b)) \ [w \in Id(A, a, b)]$$

Date le seguenti assunzioni:

$$\pi_1) \ A \ type \ [\Gamma]$$

$$\pi_2) \ a \in A \ [\Gamma]$$

$$\pi_3) \ b \in A \ [\Gamma]$$

$$\pi_4) \ B \ type \ [\Gamma]$$

$$\pi_5) \ f(x) \in B \ [\Gamma, x \in A]$$

$$\begin{array}{c}
 \frac{\pi_5}{f(x) \in B \ [w, x \in A]} \text{I-Id)} \\
 \frac{}{id(f(x)) \in Id(B, f(x), f(x)) \ [w, x \in A]} \text{(B)} \\
 \\
 \frac{\pi_1 \quad \frac{\frac{A \ type \ [w, x]}{w, x, y \in A \ cont} \text{F-c)}}{x \in A \ [w, x \in A, y]} \text{var)} \quad \frac{\pi_1 \quad \frac{A \ type \ [w, x]}{w, x, y \in A \ cont} \text{F-c)}}{y \in A \ [w, x, y \in A]} \text{var)} \quad \frac{}{Id(A, x, y) \ type \ [w, x, y]} \text{F-Id)} \\
 \frac{}{Id(A, x, y) \ type \ [w, x, y]} \text{(D)} \\
 \\
 \frac{\frac{Id(A, x, y) \ type \ [w, x, y]}{w, x, y, z \in Id(A, x, y) \ cont} \text{F-c)}}{y \in A \ [w, x, y \in A, z]} \text{var)} \quad \frac{\pi_5 \quad \frac{Id(A, x, y) \ type \ [w, x, y]}{w, x, y, z \in Id(A, x, y) \ cont} \text{F-c)}}{f(x) \in B \ [w, x \in A]} \quad \frac{}{f(x) \in B \ [w, x, y, z]} \text{ind-ty)} \\
 \frac{}{f(y) \in B \ [w, x, y, z]} \text{sub-ter)} \\
 \frac{}{f(y) \in B \ [w, x, y, z]} \text{(E)} \\
 \\
 \frac{\pi_4 \quad \frac{\pi_5 \quad \frac{Id(A, x, y) \ type \ [w, x, y]}{w, x, y, z \in Id(A, x, y) \ cont} \text{F-c)}}{f(x) \in B \ [w, x \in A]} \quad \frac{}{f(x) \in B \ [w, x, y, z]} \text{ind-ty)} \quad \frac{}{f(y) \in B \ [w, x, y, z]} \text{(E)} \\
 \frac{}{Id(B, f(x), f(y)) \ type \ [w, x \in A, y \in A, z \in Id(A, x, y)]} \text{F-Id)} \\
 \frac{}{Id(B, f(x), f(y)) \ type \ [w, x \in A, y \in A, z \in Id(A, x, y)]} \text{(A)} \\
 \\
 \frac{\pi_1 \quad \pi_2 \quad \pi_3}{A \ type \ [\] \quad a \in A \ [\] \quad b \in A \ [\]} \\
 \frac{}{Id(A, a, b) \ type \ [\]} \text{F-c)} \\
 \frac{}{w \in Id(A, a, b) \ cont} \text{var)} \\
 \frac{\pi_2 \quad \pi_3}{a \in A \ [w] \quad b \in A \ [w]} \quad \frac{}{w \in Id(A, a, b) \ [w \in Id(A, a, b)]} \text{var)} \quad \frac{}{\dots} \text{(B)} \\
 \frac{}{El_{Id}(w, (x).id(f(x))) \in Id(B, f(a), f(b)) \ [w \in Id(A, a, b)]} \text{E-Id)}
 \end{array}$$

2 Simmetry

$$\mathbf{pf} \in Id(A, b, a) [w \in Id(A, a, b)]$$

Date le seguenti assunzioni: $\pi_1) A \text{ type } [\Gamma]$ $\pi_2) a \in A [\Gamma]$ $\pi_3) b \in A [\Gamma]$

$$Id(A, y, x) \text{ type } [w, x \in A, y \in A, z \in Id(A, a, b)]$$

$$(A)$$

$$\frac{\begin{array}{c} (A) \\ \dots \end{array} \quad \begin{array}{c} \pi_2 \\ a \in A [w] \end{array} \quad \begin{array}{c} \pi_3 \\ b \in A [w] \end{array} \quad w \in Id(A, a, b) [w] \quad \frac{x \in A [w, x] \quad id(x) \in Id(A, x, x) [w, x \in A]}{\text{I-Id)}}}{El_{Id}(w, id) \in Id(A, b, a) [w \in Id(A, a, b)]} \text{E-Id)}$$

3 Transitivity, Path Induction

$$\mathbf{pf} \in Id_p(A, e, g) [w_1 \in Id_p(A, e, f), w_2 \in Id_p(A, f, g)]$$

Date le seguenti assunzioni: $\pi_1) A \text{ type } [\Gamma]$ $\pi_2) e \in A [\Gamma]$ $\pi_3) f \in A [\Gamma]$ $\pi_4) g \in A [\Gamma]$

$$Id_p(A, e, y) \text{ type } [w_1, w_2, y \in A, z \in Id_p(A, f, y)]$$

$$(A)$$

$$\frac{\begin{array}{c} (A) \\ \dots \end{array} \quad \begin{array}{c} \pi_3 \\ f \in A [w_1, w_2] \end{array} \quad \begin{array}{c} \pi_6 \\ g \in A [w_1, w_2] \end{array} \quad w_2 \in Id_p(A, f, g) [w_1, w_2] \quad w_1 \in Id_p(A, e, f) [w_1, w_2]}{El_{Id_p}(w_2, w_1) \in Id_p(A, e, g) [w_1 \in Id_p(A, e, f), w_2 \in Id_p(A, f, g)]} \text{E-Id}_p)$$

4 Transitivity, Martin-Löf's

$$\mathbf{pf} \in Id(A, a, c) \ [w_1 \in Id(A, a, b), w_2 \in Id(A, b, c)]$$

Date le seguenti assunzioni:

$$\pi_1) \ A \text{ type } [\Gamma]$$

$$\pi_2) \ a \in A \ [\Gamma]$$

$$\pi_3) \ b \in A \ [\Gamma]$$

$$\pi_4) \ c \in A \ [\Gamma]$$

$$\frac{w \in Id(A, a, x) [w_1, w_2, x, w \in Id(A, a, x)]}{\lambda w. w \in Id(A, a, x) \rightarrow Id(A, a, x) [w_1, w_2, x \in A]} \text{I-}\rightarrow)$$

(B)

$$\frac{\frac{\frac{\pi_1}{A \text{ type } [\Gamma]} \quad \frac{\pi_2}{a \in A \ [\Gamma]} \quad x \in A \ [w_1, w_2, x \in A, y]}{Id(A, a, x) \text{ type } [w_1, w_2, x, y]} \quad \frac{\frac{\pi_1}{A \text{ type } [\Gamma]} \quad \frac{\pi_2}{a \in A \ [\Gamma]} \quad y \in A \ [w_1, w_2, x, y \in A]}{Id(A, a, y) \text{ type } [w_1, w_2, x, y]} \text{F-to)}}{Id(A, a, x) \rightarrow Id(A, a, y) \text{ type } [w_1, w_2, x \in A, y \in A, z \in Id(A, x, y)]} \text{(A)}$$

$$\frac{w_1 \in Id(A, a, b) \ [w_1, w_2] \quad \frac{\frac{(A) \quad \dots \quad b \in A \ [w_1, w_2] \quad c \in A \ [w_1, w_2] \quad w_2 \in Id(A, b, c) \ [w_1, w_2] \quad (B) \quad \dots}{El_{Id}(w_2, \lambda w. w) \in Id(A, a, b) \rightarrow Id(A, a, c) \ [w_1, w_2]} \text{E-Id)}}{Ap(El_{Id}(w_2, \lambda w. w), w_1) \in Id(A, a, c) \ [w_1 \in Id(A, a, b), w_2 \in Id(A, b, c)]} \text{E-}\rightarrow)$$

5 Propositional Equality among sum operators (Basic) [Nat]

$$\mathbf{pf} \in Id(Nat, x' +_1 0, x' +_2 0) [x' \in Nat]$$

Per questo esercizio uguaglianze composizionali e *definizionali* sono svolte *in place*, questi passaggi saranno indicati da linee tratteggiate. Date le seguenti definizioni:

- $x' +_1 y' \equiv El_{Nat}(y', x', (x, z).succ(z))$
- $x' +_2 y' \equiv El_{Nat}(x', y', (x, z).succ(z))$

La dimostrazione è basata sulla proposizione $x +_2 0 = 0$ e la si prova induttivamente. Se vale nel caso base, quindi applicata a 0 ($0 +_2 0 = 0$), e se vale nel passo induttivo, quindi applicata al successore, allora vale per un qualsiasi x . Quando la applichiamo al passo successivo ipotizziamo di avere una prova che vale al passo precedente.

Le sostituzioni composizionali e definizionali sono specificate alla fine, con l'identificativo associato per identificarle facilmente durante la prova. Possiamo eseguire queste sostituzioni formalmente utilizzando le regole di sostituzione e conversione.

$$id(succ(\hat{x})) \in Id(Nat, succ(\hat{x}), succ(\hat{x})) [x', x, z, \hat{x} \in Nat] \quad (D)$$

$$z \in Id(Nat, x, x +_2 0) [x', x, z \in Id(Nat, x, x +_2 0)] \quad (C)$$

$$Id(Nat, succ(\hat{x}), succ(\hat{y})) \text{ type } [x', x, z, \hat{x} \in Nat, \hat{y} \in Nat, \hat{z} \in Id(Nat, \hat{x}, \hat{y})] \quad (B)$$

$$\frac{\begin{array}{c} (B) \\ \dots \quad x \in Nat [x', x, z] \quad x +_2 0 \in Nat [x', x, z] \quad \dots \quad (C) \quad (D) \\ \dots \end{array}}{\frac{El_{Id}(z, (\hat{x}).id(succ(\hat{x}))) \in Id(Nat, succ(x), succ(x +_2 0)) [x', x \in Nat, z \in Id(Nat, x, x +_2 0)]}{El_{Id}(z, (\hat{x}).id(succ(\hat{x}))) \in Id(Nat, succ(x), succ(x) +_2 0) [x', x \in Nat, z \in Id(Nat, x, x +_2 0)]} \text{ E-Id} \quad (3)} \quad (A)$$

$$\frac{x' \in Nat [x' \in Nat] \quad Id(Nat, x', x' +_2 0) \text{ type } [x'] \quad \frac{id(0) \in Id(Nat, 0, 0) [x']}{id(0) \in Id(Nat, 0, 0 +_2 0) [x']} \quad (2) \quad (A)}{\frac{El_{Nat}(x', id(0), (x, z).El_{Id}(z, (\hat{x}).id(succ(\hat{x})))) \in Id(Nat, x', x' +_2 0) [x' \in Nat]}{El_{Nat}(x', id(0), (x, z).El_{Id}(z, (\hat{x}).id(succ(\hat{x})))) \in Id(Nat, x' +_1 0, x' +_2 0) [x' \in Nat]} \text{ Nat-e} \quad (1)}$$

$$(1) \quad x' +_1 0 \equiv El_{Nat}(0, x', (x, z).succ(z)) = x'$$

$$(2) \quad 0 +_2 0 \equiv El_{Nat}(0, 0, (x, z).succ(z)) = 0$$

$$(3) \quad succ(x) +_2 0 \equiv El_{Nat}(succ(x), 0, (x, z).succ(z)) = succ(El_{Nat}(x, 0, (x, z).succ(z))) \equiv succ(x +_2 0)$$

6 Axiom of choice

$$(\forall x \in A)(\exists y \in B(x))C(x, y) \rightarrow (\exists f \in A \rightarrow B)(\forall x \in A)C(x, Ap(f, x)) \text{ true}$$

Per questo esercizio sono state svolte alcune semplificazioni:

- Nessun tipo è stato provato essere derivabile in quanto essendo derviabili

$$A \text{ type } [\Gamma], B(x) \text{ type } [\Gamma, x \in A] \text{ and } C(x, y) \text{ type } [\Gamma, x \in A, y \in B(x)]$$

saranno derivabili pure loro combinazioni tra somme indiciate e prodotti dipendenti

- Per non inquinare eccessivamente lo spazio delle variabili utilizzo queste convenzioni:

$$x \in A, y \in B(x), f \in \Pi_{x \in A} B(x) \text{ and } z \in \Pi_{x \in A} \Sigma_{y \in B(x)} C(x, y)$$

- La precedente assunzione può creare confusione nello *scoping* delle variabili, per questo definisco una metrica di priorità per identificare univocamente lo *scope* di una variabile:

1. *Abstraction* (la più forte)
2. *Indexed sum type* oppure *Dependent product type*
3. *Context* (il più debole)

La dimostrazione è basata nell'interpretazione intuizionistica delle costanti logiche con le sostituzioni $\forall = \Pi$ e $\exists = \Sigma$ (*propositions-as-sets* - *Curry-Howard*). Iniziamo supponendo di avere una prova della prima parte $(\forall x)(\exists y)C(x, y)$, significa che abbiamo un metodo che quando applicato ad x tiene una prova di $(\exists y)C(x, y)$. Prendiamo f come metodo che dato una arbitraria x assegna la prima componente. Quindi sia $C(x, f(x))$ che segua con la seconda componente. Abbiamo così (ri)composto l'operatore.

$$\begin{array}{c}
 \text{E-II)} \frac{x \in A [z, x] \quad z \in \Pi_{x \in A} \Sigma_{y \in B(x)} C(x, y) [z, x]}{\frac{Ap(z, x) \in \Sigma_{y \in B(x)} C(x, y) [z, x]}{\pi_1(Ap(z, x)) \in B(x) [z, x]} \pi_1)} \quad \frac{x \in A [z, x] \quad z \in \Pi_{x \in A} \Sigma_{y \in B(x)} C(x, y) [z, x]}{\frac{Ap(z, x) \in \Sigma_{y \in B(x)} C(x, y) [z, x]}{\pi_2(Ap(z, x)) \in C(x, \pi_1(Ap(z, x))) [z, x]} \pi_2)} \text{E-II)} \\
 \text{(A)} \qquad \qquad \qquad \text{(C)} \\
 \text{(A)} \\
 \frac{\frac{x \in A [z, x] \quad \pi_1(Ap(z, x)) \in B(x) [z, x]}{Ap(\lambda x. \pi_1(Ap(z, x)), x) = \pi_1(Ap(z, x)) \in B(x) [z, x]} \beta\text{C-II)} \quad \text{(C)}}{C(x, Ap(\lambda x. \pi_1(Ap(z, x)), x)) = C(x, \pi_1(Ap(z, x))) \text{ type } [z, x]} \text{sub)} \\
 \frac{\pi_2(Ap(z, x)) \in C(x, \pi_1(Ap(z, x))) [z, x]}{\pi_2(Ap(z, x)) \in C(x, Ap(\lambda x. \pi_1(Ap(z, x)), x)) [z, x \in A]} \text{conv)} \\
 \frac{\pi_2(Ap(z, x)) \in C(x, Ap(\lambda x. \pi_1(Ap(z, x)), x)) [z, x \in A]}{\lambda x. \pi_2(Ap(z, x)) \in \Pi_{x \in A} C(x, Ap(\lambda x. \pi_1(Ap(z, x)), x)) [z]} \text{I-II)} \\
 \text{(B)} \\
 \text{(A)} \\
 \frac{\pi_1(Ap(z, x)) \in B(x) [z, x \in A]}{\lambda x. \pi_1(Ap(z, x)) \in \Pi_{x \in A} B(x) [z]} \text{I-II)} \quad \frac{\lambda x. \pi_2(Ap(z, x)) \in \Pi_{x \in A} C(x, Ap(\lambda x. \pi_1(Ap(z, x)), x)) [z]}{\langle \lambda x. \pi_1(Ap(z, x)), \lambda x. \pi_2(Ap(z, x)) \rangle \in \Sigma_{f \in \Pi_{x \in A} B(x)} \Pi_{x \in A} C(x, Ap(f, x)) [z \in \Pi_{x \in A} \Sigma_{y \in B(x)} C(x, y)]} \text{I-}\Sigma) \\
 \frac{\langle \lambda x. \pi_1(Ap(z, x)), \lambda x. \pi_2(Ap(z, x)) \rangle \in \Sigma_{f \in \Pi_{x \in A} B(x)} \Pi_{x \in A} C(x, Ap(f, x)) [z]}{\lambda z. \langle \lambda x. \pi_1(Ap(z, x)), \lambda x. \pi_2(Ap(z, x)) \rangle \in \Pi_{z \in \Pi_{x \in A} \Sigma_{y \in B(x)} C(x, y)} \Sigma_{f \in \Pi_{x \in A} B(x)} \Pi_{x \in A} C(x, Ap(f, x)) [z]} \text{I-II)}
 \end{array}$$

7 Bool terms inequality

$$\mathbf{pf} \in Id(Bool, true, false) \rightarrow N_0 [] \quad \equiv \quad \neg(true = false)$$

Date le seguenti definizioni:

$$\begin{aligned} f &\equiv El_{Bool}(z, (x).\hat{N}_1, (x).\hat{N}_0) \in U_0 [z \in Bool] \\ p_f &\equiv El_{Id}(w, (x).id(f(x))) \in Id(U_0, f(true), f(false)) [w \in Id(Bool, true, false)] \\ k &\equiv El_{Id}(w, (x).\langle \lambda y.y, \lambda y.y \rangle) \in T(x) \leftrightarrow T(y) [x \in U_0, y \in U_0, w \in Id(U_0, x, y)] \end{aligned}$$

La dimostrazione è basata sull'utilizzo degli universi, altrimenti non potremmo mai distinguere termini diversi all'interno della nostra teoria dei tipi. La funzione f propaga elementi diversi all'interno di universi distinti per poterne poi effettuare l'uguaglianza. La funzione k invece crea l'*isomorfismo* necessario per distinguere \hat{N}_1 da \hat{N}_0 .

$$\frac{\frac{\frac{\frac{\frac{El_{Id}(El_{Id}(w, (x).id(El_{Bool}(x, (x).\hat{N}_1, (x).\hat{N}_0))), (x).\langle \lambda y.y, \lambda y.y \rangle) \in T(\hat{N}_1) \leftrightarrow T(\hat{N}_0) [w]}{El_{Id}(El_{Id}(w, (x).id(f(x))), (x).\langle \lambda y.y, \lambda y.y \rangle) \in T(\hat{N}_1) \leftrightarrow T(\hat{N}_0) [w]} \quad \frac{El_{Id}(p_f(w), (x).\langle \lambda y.y, \lambda y.y \rangle) \in T(\hat{N}_1) \leftrightarrow T(\hat{N}_0) [w]}{k(\hat{N}_1, \hat{N}_0, p_f(w)) \in N_1 \leftrightarrow N_0 [w]} \quad \frac{\star \in N_1 [w]}{\pi_1(k(\hat{N}_1, \hat{N}_0, p_f(w))) \in N_1 \rightarrow N_0 [w]} \quad \frac{Ap(\pi_1(k(\hat{N}_1, \hat{N}_0, p_f(w))), \star) \in N_0 [w \in Id(Bool, true, false)]}{\lambda w.Ap(\pi_1(k(\hat{N}_1, \hat{N}_0, p_f(w))), \star) \in Id(Bool, true, false) \rightarrow N_0 []} \quad \text{E}_1-\times \quad \text{E}-\rightarrow \quad \text{I}-\rightarrow}{\lambda w.Ap(\pi_1(k(\hat{N}_1, \hat{N}_0, p_f(w))), \star) \in Id(Bool, true, false) \rightarrow N_0 []} \quad \text{I}-\rightarrow$$

$$\frac{\frac{y \in T(x') [\Gamma, y \in T(x')]}{\lambda y.y \in T(x') \rightarrow T(x') [\Gamma]} \quad \text{I}-\rightarrow \quad \frac{y \in T(x') [\Gamma, y \in T(x')]}{\lambda y.y \in T(x') \rightarrow T(x') [\Gamma]} \quad \text{I}-\rightarrow}{\langle \lambda y.y, \lambda y.y \rangle \in T(x') \leftrightarrow T(x') [x, y, w, x' \in U_0]} \quad \text{I}-\times \quad \text{(B)}$$

$$\frac{\frac{x' \in U_0 [x, y, w, x' \in U_0, y', z]}{T(x') \text{ type } [x, y, w, x', y', z]} \quad \text{E-Un}_0 \quad \frac{y' \in U_0 [x, y, w, x', y' \in U_0, z]}{T(y') \text{ type } [x, y, w, x', y', z]} \quad \text{E-Un}_0}{T(x') \leftrightarrow T(y') \text{ type } [x, y, w, x' \in U_0, y' \in U_0, z \in Id(U_0, x', y')]} \quad \text{F}-\times \quad \text{(A)}$$

$$\frac{\begin{array}{c} \text{(A)} \\ \dots \quad x \in U_0 [x, y, w] \quad y \in U_0 [x, y, w] \quad w \in Id(U_0, x, y) [x, y, w] \quad \dots \quad \text{(B)} \end{array}}{El_{Id}(w, (x).\langle \lambda y.y, \lambda y.y \rangle) \in T(x) \leftrightarrow T(y) [x \in U_0, y \in U_0, w \in Id(U_0, x, y)]} \quad \text{E-Id} \quad \text{(k)}$$

$$\frac{z \in Bool [z \in Bool] \quad \hat{N}_1 \in U_0 [z, x \in Bool] \quad \hat{N}_0 \in U_0 [z, x \in Bool]}{El_{Bool}(z, (x).\hat{N}_1, (x).\hat{N}_0) \in U_0 [z \in Bool]} \quad \text{(f)}$$

$$\frac{\text{(Equality preservation among programs 1)}}{El_{Id}(w, (x).id(f(x))) \in Id(U_0, f(true), f(false)) [w \in Id(Bool, true, false)]} \quad \text{(p}_f\text{)}$$

8 Nat terms inequality

$$\mathbf{pf} \in Id(Nat, 0, 1) \rightarrow N_0 [] \quad \equiv \quad \neg(0 = 1)$$

Date le seguenti definizioni:

$$\begin{aligned} f &\equiv El_{Nat}(m, \hat{N}_1, (x, z). \hat{N}_0) \in U_0 [m \in Nat] \\ p_f &\equiv El_{Id}(w, (x). id(f(x))) \in Id(U_0, f(0), f(1)) [w \in Id(Nat, 0, 1)] \\ k &\equiv El_{Id}(w, (x). \langle \lambda y. y, \lambda y. y \rangle) \in T(x) \leftrightarrow T(y) [x \in U_0, y \in U_0, w \in Id(U_0, x, y)] \end{aligned}$$

In maniera simile alla prova 7, la dimostrazione si basa sull'utilizzo degli universi. La funzione f propaga elementi diversi all'interno di universi distinti per poterne poi effettuare l'uguaglianza, al posto di utilizzare l'eliminatore dei *Bool* lo utilizzo sui *Nat* propagando \hat{N}_1 se ho come input 0 altrimenti \hat{N}_0 . La funzione k invece crea l'*isomorfismo* necessario per distinguere \hat{N}_1 da \hat{N}_0 .

$$\frac{\frac{\frac{\vdots}{\lambda w. Ap(\pi_1(k(\hat{N}_1, \hat{N}_0, p_f(w))), \star) \in Id(Nat, 0, 1) \rightarrow N_0 []} \text{I} \rightarrow)}{m \in Nat [m \in Nat] \quad U_0 \text{ type } [m] \quad \hat{N}_1 \in U_0 [m] \quad \hat{N}_0 \in U_0 [m, x \in Nat, z \in U_0]} \frac{}{El_{Nat}(m, \hat{N}_1, (x, z). \hat{N}_0) \in U_0 [m \in Nat]} (f)$$

9 Peano Axioms

Dimostrare quali assiomi di Peano continuano a valere dentro la nostra teoria dei tipi:

- Ax1.** $\vdash 0 \in Nat$
- Ax2.** $\vdash \forall x. x = x$ (eq. is reflexive)
- Ax3.** $\vdash \forall x. \forall y. x = y \rightarrow y = x$ (eq. is symmetric)
- Ax4.** $\vdash \forall x. \forall y. \forall z. x = y \wedge y = z \rightarrow x = z$ (eq. is transitive)
- Ax5.** $x \in Nat, x = y \vdash y \in Nat$ (Nat are closed under eq.)
- Ax6.** $x \in Nat \vdash succ(x) \in Nat$
- Ax7.** $\vdash \forall x. \forall y. x = y \leftrightarrow succ(x) = succ(y)$ (succ is injective)
- Ax8.** $\vdash \forall x. \neg(0 = succ(x))$
- Ax9.** $D \text{ set}, 0 \in D, \forall x. succ(x) \in D \vdash D \text{ true}$ (axiom of induction)

Per essere derivabili significa che siano derivabili i seguenti giudizi:

- Ax1.** $0 \in Nat$
- Ax2.** $\mathbf{pf} \in \Pi_{x \in Nat} Id(Nat, x, x)$
- Ax3.** $\mathbf{pf} \in \Pi_{x \in Nat} \Pi_{y \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, x)$
- Ax4.** $\mathbf{pf} \in \Pi_{x \in Nat} \Pi_{y \in Nat} \Pi_{z \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, z) \rightarrow Id(Nat, x, z)$
- Ax5.** $b \in Nat [a \in Nat, w \in Id(Nat, a, b)]$
- Ax6.** $succ(x) \in Nat [x \in Nat]$
- Ax7.** $\mathbf{pf} \in \Pi_{x \in Nat} \Pi_{y \in Nat} Id(Nat, x, y) \leftrightarrow Id(Nat, succ(x), succ(y))$
- Ax8.** $\mathbf{pf} \in \Pi_{x \in Nat} Id(Nat, 0, succ(x)) \rightarrow N_0$
- Ax9.** $\pi_1) D \text{ type } [], \pi_2) 0 \in D [], \pi_3) succ(x) \in D [x \in Nat]$
 $\mathbf{pf} \in D [x \in Nat]$

$$\begin{array}{c}
\frac{[] \text{ cont}}{0 \in Nat []} \text{Nat-I}_1) \\
(\mathbf{Ax1})
\end{array}
\quad
\begin{array}{c}
\frac{[] \text{ cont}}{Nat \text{ type } []} \text{Nat-F)} \\
\frac{x \in Nat \text{ cont}}{x \in Nat [x \in Nat]} \text{F-c)} \\
\frac{x \in Nat [x \in Nat]}{id(x) \in Id(Nat, x, x) [x \in Nat]} \text{var)} \\
\frac{id(x) \in Id(Nat, x, x) [x \in Nat]}{\lambda x.id(x) \in \Pi_{x \in Nat} Id(Nat, x, x) []} \text{I-Id)} \\
\text{I-}\rightarrow) \\
(\mathbf{Ax2})
\end{array}$$

$$\begin{array}{c}
(\text{Simmetry 2}) \\
\frac{El_{Id}(w, id) \in Id(Nat, y, x) [x \in Nat, y \in Nat, w \in Id(Nat, x, y)]}{\lambda w.El_{Id}(w, id) \in Id(Nat, x, y) \rightarrow Id(Nat, y, x) [x \in Nat, y \in Nat]} \text{E-Id)} \\
\text{I-}\rightarrow) \\
\frac{\lambda w.El_{Id}(w, id) \in Id(Nat, x, y) \rightarrow Id(Nat, y, x) [x \in Nat, y \in Nat]}{\lambda y.\lambda w.El_{Id}(w, id) \in \Pi_{y \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, x) [x \in Nat]} \text{I-II)} \\
\text{I-II)} \\
\frac{\lambda y.\lambda w.El_{Id}(w, id) \in \Pi_{y \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, x) [x \in Nat]}{\lambda x.\lambda y.\lambda w.El_{Id}(w, id) \in \Pi_{x \in Nat} \Pi_{y \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, x) []} \text{I-II)} \\
(\mathbf{Ax3})
\end{array}$$

$$\begin{array}{c}
(\text{Transitivity, Martin-Löf's 4}) \\
\frac{Ap(El_{Id}(w_2, \lambda w.w), w_1) \in Id(Nat, x, z) [x \in Nat, y \in Nat, z \in Nat, w_1 \in Id(Nat, x, y), w_2 \in Id(Nat, y, z)]}{\lambda w_2.Ap(El_{Id}(w_2, \lambda w.w), w_1) \in Id(Nat, y, z) \rightarrow Id(Nat, x, z) [\Gamma, w_1 \in Id(Nat, x, y)]} \text{E-}\rightarrow) \\
\text{I-II)} \\
\frac{\lambda w_2.Ap(El_{Id}(w_2, \lambda w.w), w_1) \in Id(Nat, y, z) \rightarrow Id(Nat, x, z) [\Gamma, w_1 \in Id(Nat, x, y)]}{\lambda w_1.\lambda w_2.Ap(El_{Id}(w_2, \lambda w.w), w_1) \in Id(Nat, x, y) \rightarrow Id(Nat, y, z) \rightarrow Id(Nat, x, z) [\Gamma, z \in Nat]} \text{I-II)} \\
\text{I-II)} \\
\frac{\lambda w_1.\lambda w_2.Ap(El_{Id}(w_2, \lambda w.w), w_1) \in Id(Nat, x, y) \rightarrow Id(Nat, y, z) \rightarrow Id(Nat, x, z) [\Gamma, z \in Nat]}{\lambda z.\lambda w_1.\lambda w_2.Ap(El_{Id}(w_2, \lambda w.w), w_1) \in \Pi_{z \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, z) \rightarrow Id(Nat, x, z) [\Gamma, y \in Nat]} \text{I-II)} \\
\text{I-II)} \\
\frac{\lambda y.\lambda z.\lambda w_1.\lambda w_2.Ap(El_{Id}(\dots), w_1) \in \Pi_{y \in Nat} \Pi_{z \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, z) \rightarrow Id(Nat, x, z) [x \in Nat]}{\lambda x.\lambda y.\lambda z.\lambda w_1.\lambda w_2.Ap(El_{Id}(\dots), w_1) \in \Pi_{x \in Nat} \Pi_{y \in Nat} \Pi_{z \in Nat} Id(Nat, x, y) \rightarrow Id(Nat, y, z) \rightarrow Id(Nat, x, z) []} \text{I-II)} \\
(\mathbf{Ax4})
\end{array}$$

$$\begin{array}{c}
\frac{[] \text{ cont}}{\vdots} \\
\frac{a \in Nat [a \in Nat, w \in Id(Nat, a, b)] \quad a = b \in Nat [a \in Nat, w \in Id(Nat, a, b)]}{b \in Nat [a \in Nat, w \in Id(Nat, a, b)]} \quad (1) \\
(\mathbf{Ax5})
\end{array}$$

(1) sse $nf(a) \equiv nf(b)$ allora con la prova w posso concludere con l'uguaglianza computazionale

$$\begin{array}{c}
\frac{[] \text{ cont}}{Nat \text{ type } []} \text{Nat-F)} \\
\frac{x \in Nat \text{ cont}}{x \in Nat [x \in Nat]} \text{F-c)} \\
\frac{x \in Nat [x \in Nat]}{succ(x) \in Nat [x \in Nat]} \text{var)} \\
\text{Nat-I}_2) \\
(\mathbf{Ax6})
\end{array}$$

$$\begin{array}{c}
\frac{[] \text{ cont} \quad \vdots \quad x' \in \text{Nat} [\Gamma]}{El_{\text{Nat}}(x', 0, (x, z).x) \in \text{Nat} [\Gamma, x' \in \text{Nat}]} \text{Nat-e)} \\
\frac{\frac{El_{\text{Nat}}(x', 0, (x, z).x) \in \text{Nat} [\Gamma, x' \in \text{Nat}]}{prec(x') \in \text{Nat} [\Gamma, x' \in \text{Nat}]} \text{I-Id)}}{id(prec(x')) \in Id(\text{Nat}, prec(x'), prec(x')) [\Gamma, x' \in \text{Nat}]} \text{I-Id)} \\
\text{(C)}
\end{array}$$

$$\begin{array}{c}
Id(\text{Nat}, prec(\hat{x}), prec(\hat{y})) \text{ type } [\Gamma, \hat{x} \in \text{Nat}, \hat{y} \in \text{Nat}, z \in Id(\text{Nat}, \hat{x}, \hat{y})] \\
\text{(B)}
\end{array}$$

$$\begin{array}{c}
\frac{\begin{array}{c} \text{(B)} \\ \dots \quad succ(x) \in \text{Nat} [\Gamma] \quad succ(y) \in \text{Nat} [\Gamma] \quad b \in Id(\text{Nat}, succ(x), succ(y)) \quad \dots \end{array} \quad \begin{array}{c} \text{(C)} \\ \dots \end{array}}{El_{Id}(b, (x').id(prec(x')) \in Id(\text{Nat}, prec(succ(x)), prec(succ(y))) [\Gamma, b \in Id(\text{Nat}, succ(x), succ(y))]} \text{E-Id)} \\
\frac{El_{Id}(b, (x').id(prec(x')) \in Id(\text{Nat}, x, y) [\Gamma, b \in Id(\text{Nat}, succ(x), succ(y))]}{\lambda b. El_{Id}(b, (x').id(prec(x')) \in Id(\text{Nat}, succ(x), succ(y)) \rightarrow Id(\text{Nat}, x, y) [\Gamma]} \text{I-}\rightarrow) \\
\text{(A)}
\end{array}$$

$$\begin{array}{c}
\text{(Equality preservation among programs 1)} \\
\text{E-Id)} \frac{El_{Id}(a, (z).id(succ(z))) \in Id(\text{Nat}, succ(x), succ(y)) [\Gamma, a \in Id(\text{Nat}, x, y)]}{\lambda a. El_{Id}(a, (z).id(succ(z))) \in Id(\text{Nat}, x, y) \rightarrow Id(\text{Nat}, succ(x), succ(y)) [\Gamma]} \text{(A)} \\
\text{I-}\rightarrow) \frac{\lambda a. El_{Id}(a, (z).id(succ(z))) \in Id(\text{Nat}, x, y) \rightarrow Id(\text{Nat}, succ(x), succ(y)) [\Gamma]}{\langle \lambda a. \dots, \lambda b. \dots \rangle \in Id(\text{Nat}, x, y) \leftrightarrow Id(\text{Nat}, succ(x), succ(y)) [x \in \text{Nat}, y \in \text{Nat}]} \text{I-}\times) \\
\frac{\langle \lambda a. \dots, \lambda b. \dots \rangle \in Id(\text{Nat}, x, y) \leftrightarrow Id(\text{Nat}, succ(x), succ(y)) [x \in \text{Nat}, y \in \text{Nat}]}{\lambda y. \langle \lambda a. \dots, \lambda b. \dots \rangle \in \Pi_{y \in \text{Nat}} Id(\text{Nat}, x, y) \leftrightarrow Id(\text{Nat}, succ(x), succ(y)) [x \in \text{Nat}]} \text{I-II)} \\
\frac{\lambda x. \lambda y. \langle \lambda a. \dots, \lambda b. \dots \rangle \in \Pi_{x \in \text{Nat}} \Pi_{y \in \text{Nat}} Id(\text{Nat}, x, y) \leftrightarrow Id(\text{Nat}, succ(x), succ(y)) []}{\lambda x. \lambda y. \langle \lambda a. \dots, \lambda b. \dots \rangle \in \Pi_{x \in \text{Nat}} \Pi_{y \in \text{Nat}} Id(\text{Nat}, x, y) \leftrightarrow Id(\text{Nat}, succ(x), succ(y)) []} \text{I-II)} \\
\text{(Ax7)}
\end{array}$$

$$\begin{array}{c}
(2) \quad prec \equiv El_{\text{Nat}}(x, 0, (x, z).x) \in \text{Nat} [x \in \text{Nat}] \text{ Dove :} \\
El_{\text{Nat}}(succ(x), 0, (x, z).x) = x \in \text{Nat} [x \in \text{Nat}] \\
El_{\text{Nat}}(0, 0, (x, z).x) = 0 \in \text{Nat} [x \in \text{Nat}]
\end{array}$$

$$\begin{array}{c}
\text{(Nat terms inequality 8)} \\
\frac{\lambda w. Ap(\pi_1(k(\hat{N}_1, \hat{N}_2, p_f(w))), \star) \in Id(\text{Nat}, 0, succ(x)) \rightarrow N_0 [x \in \text{Nat}]}{\lambda x. \lambda w. Ap(\pi_1(k(\hat{N}_1, \hat{N}_2, p_f(w))), \star) \in \Pi_{x \in \text{Nat}} Id(\text{Nat}, 0, succ(x)) \rightarrow N_0 []} \text{I-II)} \\
\text{(Ax8)}
\end{array}$$

$$\begin{array}{c}
\frac{\pi_1 \quad \frac{[] \text{ cont}}{\text{Nat type } []} \text{Nat-F)} \quad \frac{\pi_2 \quad \pi_3}{0 \in D [\Gamma] \quad succ(x') \in D [x \in \text{Nat}, x' \in \text{Nat}, z \in \text{Nat}]} \text{Nat-e)}}{D \text{ type } [\Gamma] \quad \frac{x \in \text{Nat} [x \in \text{Nat}]}{El_{\text{Nat}}(x, 0, (x', z).succ(x')) \in D [x \in \text{Nat}]} \text{var)}} \\
\text{(Ax9)}
\end{array}$$

10 Propositional Equality among sum operators [Nat]

$$\mathbf{pf} \in Id(Nat, x' +_1 y', x' +_2 y') \ [x' \in Nat, y' \in Nat]$$

Date le seguenti definizioni:

$$\begin{aligned} s &\equiv succ \\ x' +_1 y' &\equiv El_{Nat}(y', x', (x, z).s(z)) \\ x' +_2 y' &\equiv El_{Nat}(x', y', (x, z).s(z)) \\ Id_N(a, b) &\equiv Id(Nat, a, b) \\ c &\equiv id(s(x)) \\ e &\equiv El_{Id}(z, c) \\ w &\equiv El_{Nat}(x', id(s(y)), e) \\ e' &\equiv Ap(\lambda w'_1. Ap(\lambda w'_2. Ap(El_{Id}(w'_2, \lambda w'. w'), w'_1), w), e) \end{aligned}$$

$$\frac{\text{(Transitivity 4)}}{\frac{Ap(El_{Id}(w'_2, \lambda w'. w'), w'_1) \in Id_N(s(x' +_1 y), x' +_2 s(y)) \ [\Gamma, w_2 \in Id_N(s(x' +_2 y), x' +_2 s(y))]}{\lambda w'_2. Ap(El_{Id}(w'_2, \lambda w'. w'), w'_1) \in Id_N(s(x' +_2 y), x' +_2 s(y)) \rightarrow Id_N(s(x' +_1 y), x' +_2 s(y)) \ [\Gamma]} \text{I-}\rightarrow)} \text{(D)}$$

$$\frac{\text{(Equality preservation 1)}}{\frac{El_{Id}(z, c) \in Id_N(s(s(x +_2 y)), s(x +_2 s(y))) \ [\Gamma]}{El_{Id}(z, c) \in Id_N(s(s(x) +_2 y), s(x) +_2 s(y)) \ [\Gamma, x \in Nat, z' \in Id_N(s(x +_2 y), x +_2 s(y))]} \text{(E)}}$$

$$\frac{\begin{array}{c} id(s(y)) \in Id_N(s(y), s(y)) \ [\Gamma] \quad \text{(E)} \\ \dots \quad id(s(y)) \in Id_N(s(0 +_2 y), 0 +_2 s(y)) \ [\Gamma] \quad \dots \end{array}}{\frac{El_{Nat}(x', id(s(y)), e) \in Id_N(s(x' +_2 y), x' +_2 s(y)) \ [\Gamma]}{w \in Id_N(s(x' +_2 y), x' +_2 s(y)) \ [\Gamma]} \text{Nat-e}} \text{(C)}$$

$$\frac{\begin{array}{c} \text{(C)} \quad \text{(D)} \\ w \in Id_N(s(x' +_2 y), x' +_2 s(y)) \ [\Gamma] \quad \dots \end{array}}{\frac{Ap(\lambda w'_2. Ap(\dots), w) \in Id_N(s(x' +_1 y), x' +_2 s(y)) \ [\Gamma, w_1 \in Id_N(s(x' +_1 y), s(x' +_2 y))]}{\lambda w_1. Ap(\lambda w'_2. Ap(\dots), w) \in Id_N(s(x' +_1 y), s(x' +_2 y)) \rightarrow Id_N(s(x' +_1 y), x' +_2 s(y)) \ [\Gamma]} \text{E-}\rightarrow)} \text{(B)}$$

$$\frac{\begin{array}{c} \text{(Equality preservation 1)} \quad \text{(B)} \\ e \in Id_N(s(x' +_1 y), s(x' +_2 y)) \ [\Gamma] \quad \dots \end{array}}{\frac{Ap(\lambda w'_1. Ap(\dots), e) \in Id_N(s(x' +_1 y), x' +_2 s(y)) \ [\Gamma]}{e' \in Id_N(x' +_1 s(y), x' +_2 s(y)) \ [\Gamma, y \in Nat, z \in Id_N(x' +_1 y, x' +_2 y)]} \text{E-}\rightarrow)} \text{(A)}$$

$$\frac{\begin{array}{c} \text{(Propositional Eq. (Basic) 5)} \\ \text{Nat-e)} \quad \frac{El_{Nat}(x', id(0), e) \in Id_N(x', x' +_2 0) \ [x', y']}{El_{Nat}(x', id(0), e) \in Id_N(x' +_1 0, x' +_2 0) \ [x', y']} \quad \text{(A)} \\ \dots \end{array}}{\frac{El_{Nat}(y', El_{Nat}(x', id(0), e), e') \in Id_N(x' +_1 y', x' +_2 y') \ [x' \in Nat, y' \in Nat]}{\text{Nat-e)}} \text{(A)}$$

11 Product η -conversion

$$\mathbf{pf} \in Id(A \times B, \langle \pi_1(z), \pi_2(z) \rangle, z) [z \in A \times B]$$

Dati i seguenti lemmi:

- (Lemma 1), dato $\pi_1) a = b \in A$ derivo $id(a) \in Id(A, a, b)$ su contesto $[\Gamma, a \in A, b \in B]$

Il primo punto dell'esercizio richiede anche di provare $\langle \pi_1(z), \pi_2(z) \rangle = z \in A \times B [z \in A \times B]$, ma questo non si avvera dato che le loro forme normali non sono equivalenti, ovvero $nf(\langle \pi_1(z), \pi_2(z) \rangle) \equiv \langle \pi_1(z), \pi_2(z) \rangle \neq z \equiv nf(z)$. Questo deriva dal teorema:

$$\begin{array}{c} a = b \in A [] \text{ derivabile} \\ \text{\textit{sse}} \\ p \in Id(A, a, b) [] \text{ derivabile per qualche } p \\ \text{\textit{sse}} \\ nf(a) \equiv nf(b) \end{array}$$

La prova riportata di seguito mostra che la seconda ipotesi del teorema è derivabile quindi rende falsa l'equivalenza definizionale.

$$\begin{array}{c} \frac{\frac{a \in A [\Gamma]}{id(a) \in Id(A, a, a) [\Gamma]} \text{I-Id)} \quad \frac{\frac{a \in A [\Gamma]}{a = a \in A [\Gamma]} \text{ref)} \quad \frac{\pi_1}{a = b \in A [\Gamma]} \text{sub)}}{\frac{Id(A, a, a) = Id(A, a, b) \text{ type } [\Gamma]}{id(a) \in Id(A, a, b) [a \in A, b \in A]} \text{conv)}} \\ \text{(Lemma 1)} \\ \\ \frac{\frac{a \in A [\Gamma] \quad b \in B [\Gamma]}{\langle a, b \rangle \in A \times B [\Gamma]} \text{F-}\times)}{\frac{id(\langle a, b \rangle) \in Id(A \times B, \langle a, b \rangle, \langle a, b \rangle) [\Gamma, b \in B]}{\text{(B)}}} \text{I-Id)} \\ \\ \frac{\frac{\pi_2(\langle a, y \rangle) \in B [\Gamma] \quad y \in B [\Gamma] \quad \frac{\frac{a \in A [\Gamma] \quad y \in B [\Gamma]}{\pi_2(\langle a, y \rangle) = y \in B [\Gamma]} \beta_2\text{-}\times)}{id(y) \in Id(B, \pi_2(\langle a, y \rangle), y) [\Gamma]} \text{L1)} \quad \text{(B)}}{\frac{El_{Id}(id(y), \langle b \rangle, id(\langle a, b \rangle)) \in Id(A \times B, \langle a, \pi_2(\langle a, y \rangle) \rangle, \langle a, y \rangle) [\Gamma, a \in A]}{\text{(A)}}} \text{E-Id)}^{(2)} \\ \\ \frac{\frac{\frac{x \in A [\Gamma] \quad y \in B [\Gamma]}{\pi_1(\langle x, y \rangle) = x \in B [\Gamma]} \beta_1\text{-}\times)}{\frac{\pi_1(\langle x, y \rangle) \in A [\Gamma] \quad x \in A [\Gamma]}{id(x) \in Id(A, \pi_1(\langle x, y \rangle), x) [\Gamma]} \text{L1)} \quad \text{(A)}}{\frac{El_{Id}(id(x), El_{Id}(\dots)) \in Id(A \times B, \langle \pi_1(\langle x, y \rangle), \pi_2(\langle x, y \rangle) \rangle, \langle x, y \rangle) [z, x \in A, y \in B]}{\text{E-Id)}^{(1)}}} \text{E-II)} \\ \frac{z \in A \times B [z \in A \times B]}{El_{\Sigma}(z, El_{Id}(id(x), \langle a \rangle, El_{Id}(id(y), \langle b \rangle, id(\langle a, b \rangle)))) \in Id(A \times B, \langle \pi_1(z), \pi_2(z) \rangle, z) [z \in A \times B]} \end{array}$$

$$\begin{array}{ll} (1) & C(a, b, p) \equiv Id(A \times B, \langle a, \pi_2(\langle b, y \rangle) \rangle, \langle b, y \rangle) \\ \text{equivalentemente} & C(a, b, p) \equiv Id(A \times B, \langle \pi_1(\langle x, y \rangle), \pi_2(\langle x, y \rangle) \rangle, \langle x, y \rangle) \{ \pi_1(\langle x, y \rangle) \mapsto a, x \mapsto b \} \\ \\ (2) & C(a', b', p) \equiv Id(A \times B, \langle a, a' \rangle, \langle a, b' \rangle) \\ \text{equivalentemente} & C(a', b', p) \equiv Id(A \times B, \langle a, \pi_2(\langle a, y \rangle) \rangle, \langle a, y \rangle) \{ \pi_2(\langle a, y \rangle) \mapsto a', y \mapsto b' \} \end{array}$$