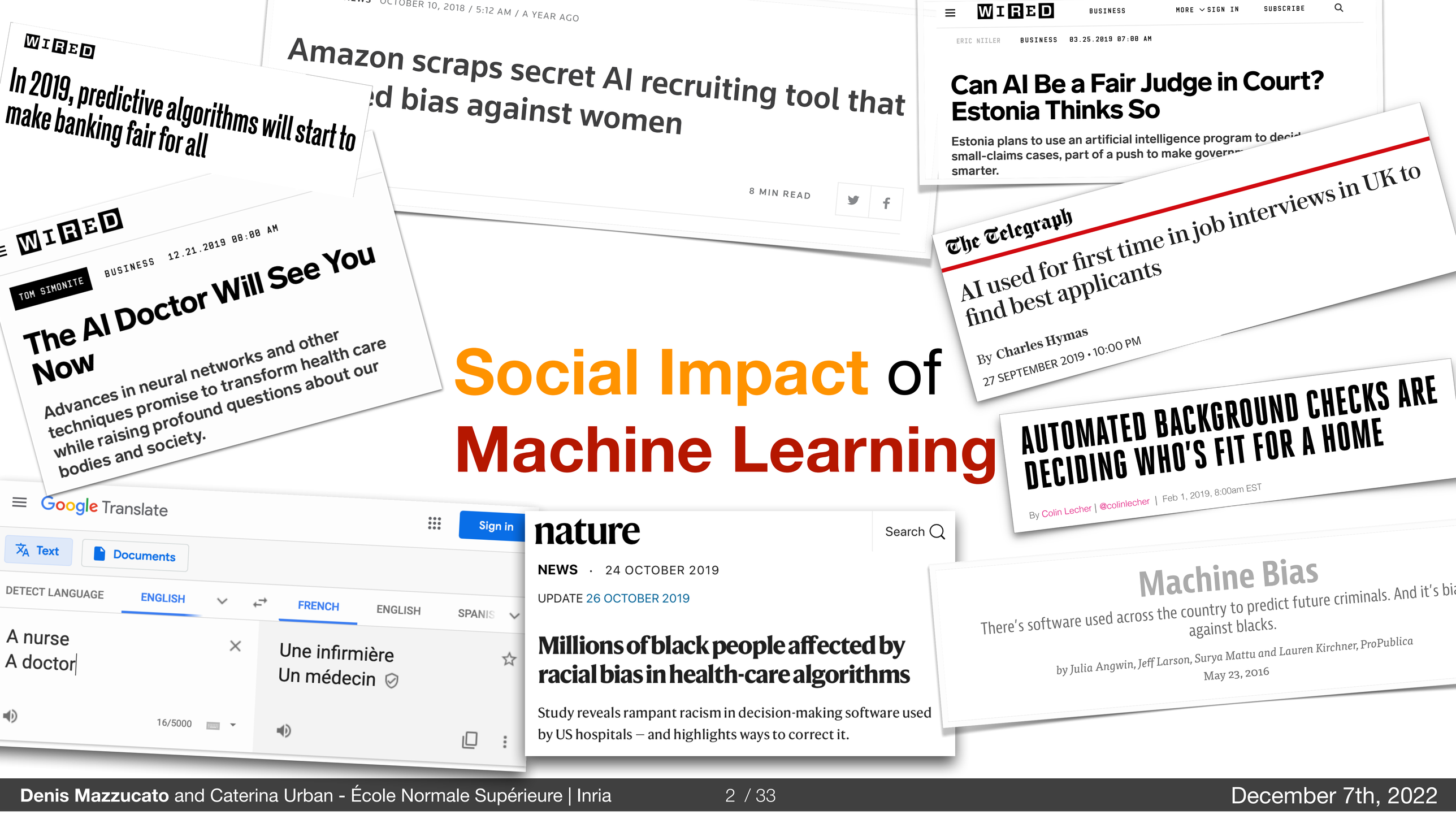
# Reduced Products of Abstract Domains for Fairness Certification of Neural Networks




Denis Mazzucato and Caterina Urban
École Normale Supérieure | Inria

**COVID Time Papers In Person**
**SPLASH 2022 - December 7th, 2022**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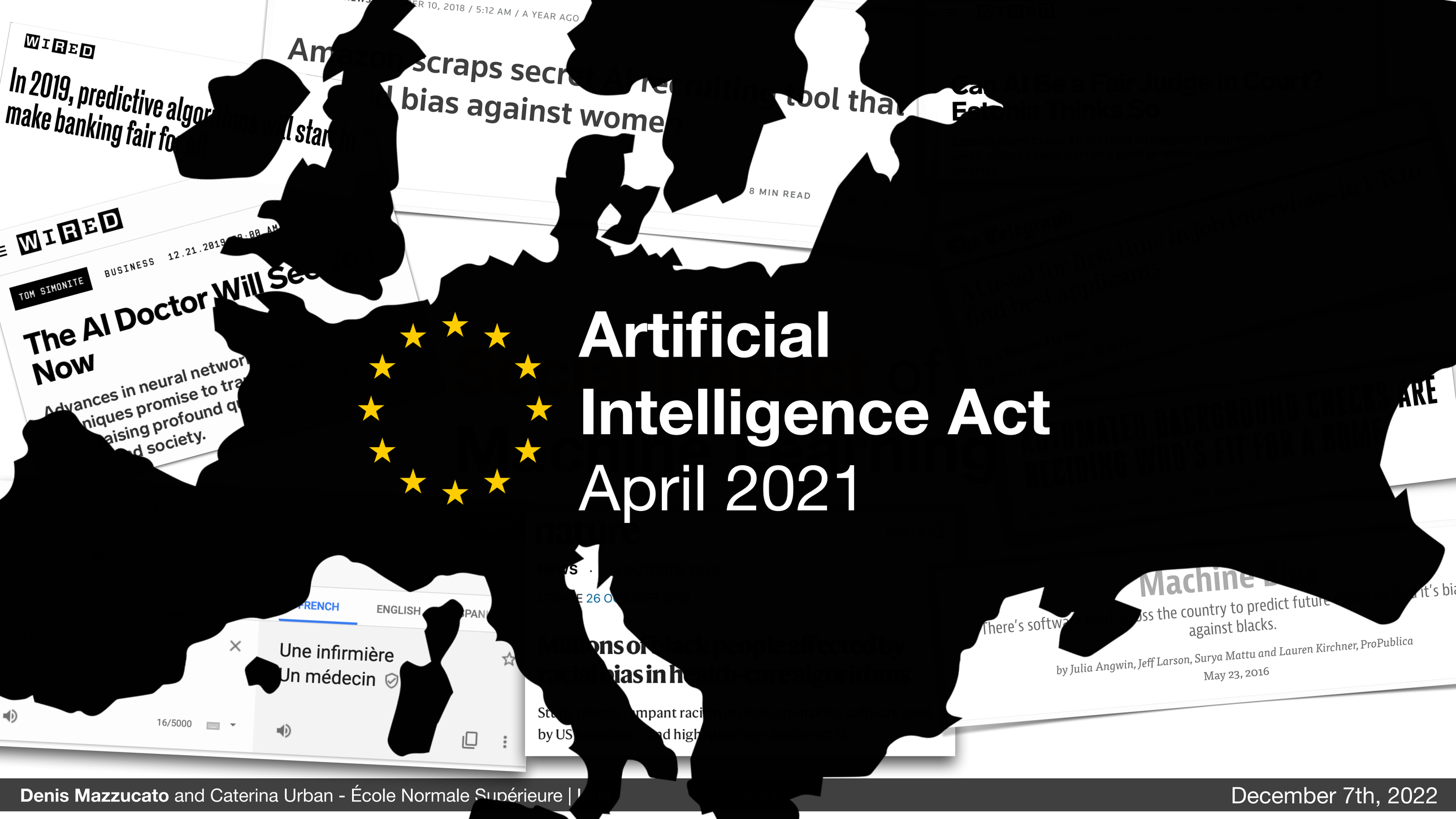
Machine Bias
There's software used across the country to predict future criminals. And it's bia
against blacks.
by Julia Angwin, Jeff Larson, Surya Mattu and Lauren Kirchner, ProPublica
May 23, 2016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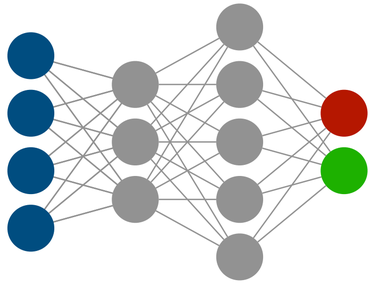

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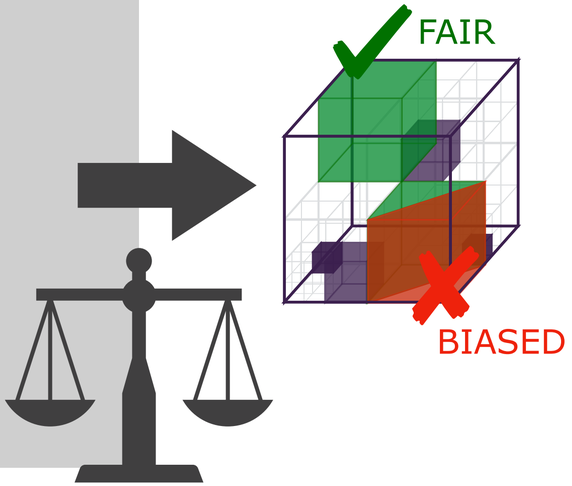
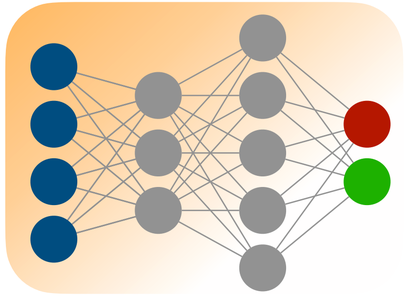
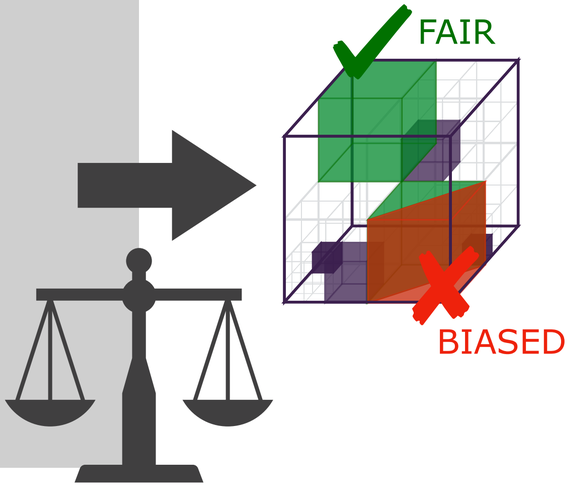


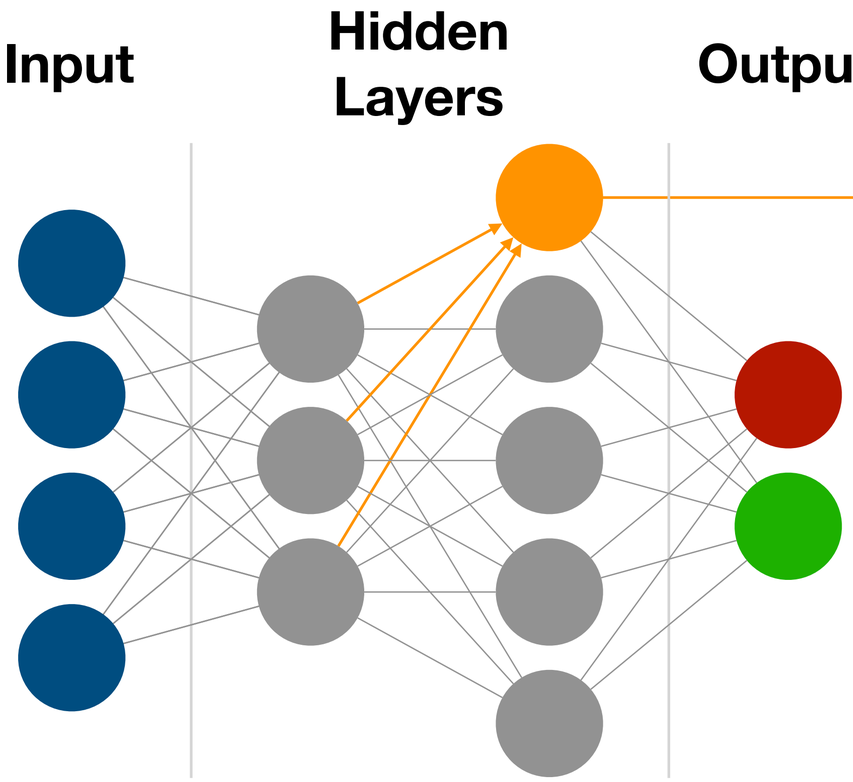


Neural Network

# Libra

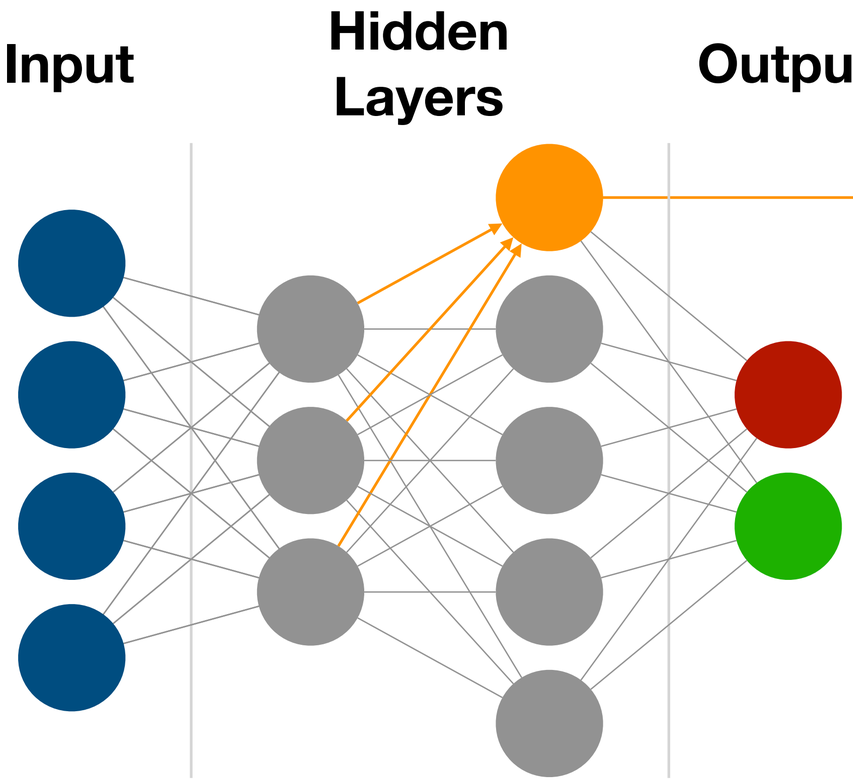


# Feed-Forward **Neural Networks** with **ReLU Activations**



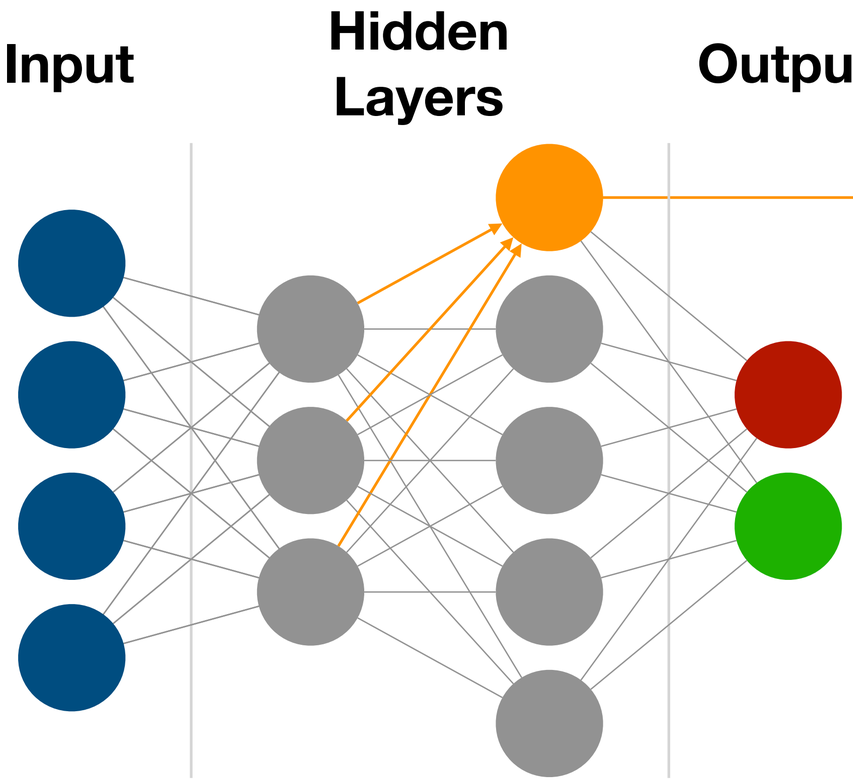

$$x_{i,j} = \text{ReLU}(\sum_k w_k \cdot x_{i-1,k} + b_{i,j})$$

# Feed-Forward **Neural Networks** with **ReLU Activations**




$$x_{i,j} = \text{ReLU}(\sum_k w_k \cdot x_{i-1,k} + b_{i,j})$$
$$= \max(\sum_k w_k \cdot x_{i-1,k} + b_{i,j}, 0)$$

# Feed-Forward **Neural Networks** with **ReLU Activations**

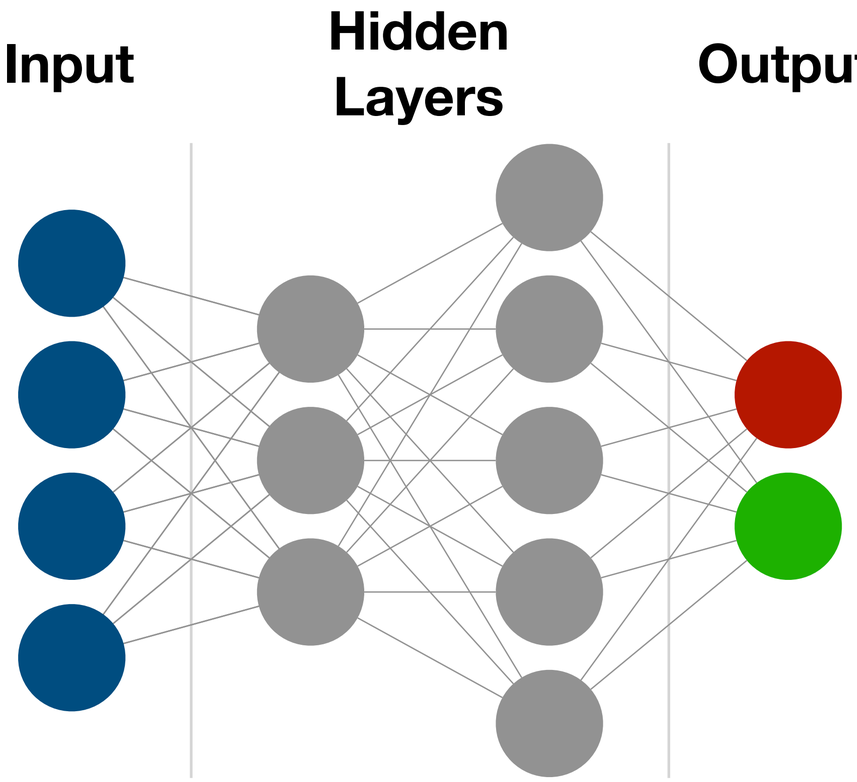



$$x_{i,j} = \text{ReLU}(\sum_k w_k \cdot x_{i-1,k} + b_{i,j})$$

$$= \max(\underbrace{\sum_k w_k \cdot x_{i-1,k} + b_{i,j}}, 0)$$

$$\hat{x}_{i,j} = \sum_k w_k \cdot x_{i-1,k} + b_{i,j}$$

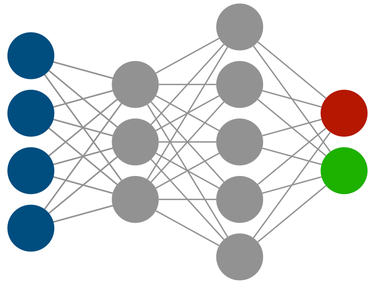
# Feed-Forward **Neural Networks** with **ReLU Activations**




**Classification Outcome**

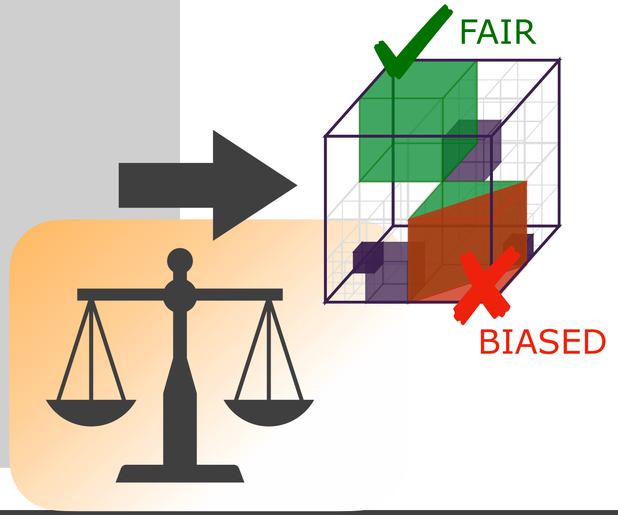
$$\underset{j}{\text{argmax}}\ x_{i,j}$$

for $x_{i,j}$ outputs



Neural Network

# Libra

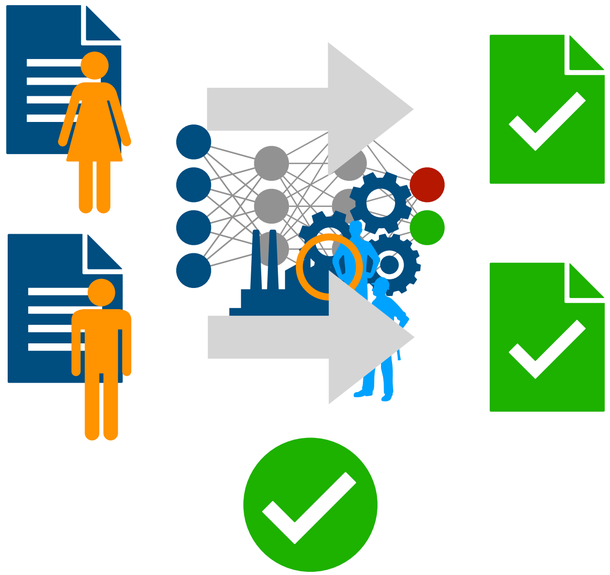


# Dependency **Fairness**

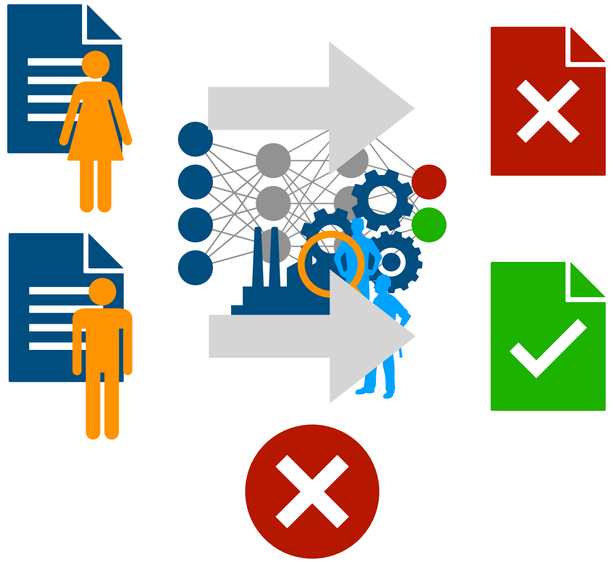
The classification outcome is
**Independent** on the
**Sensitive Features**

# Recruiting Process

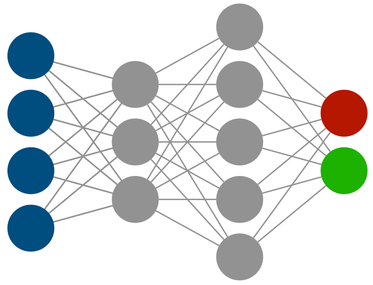
# Recruiting Process



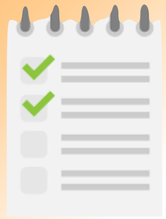
Fair



Unfair

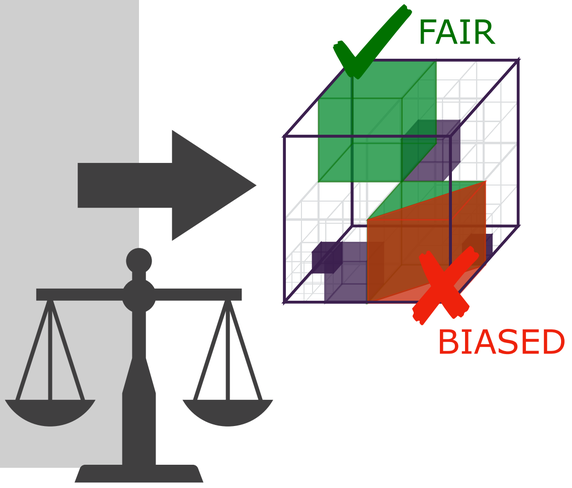


Neural Network



Specification

Libra



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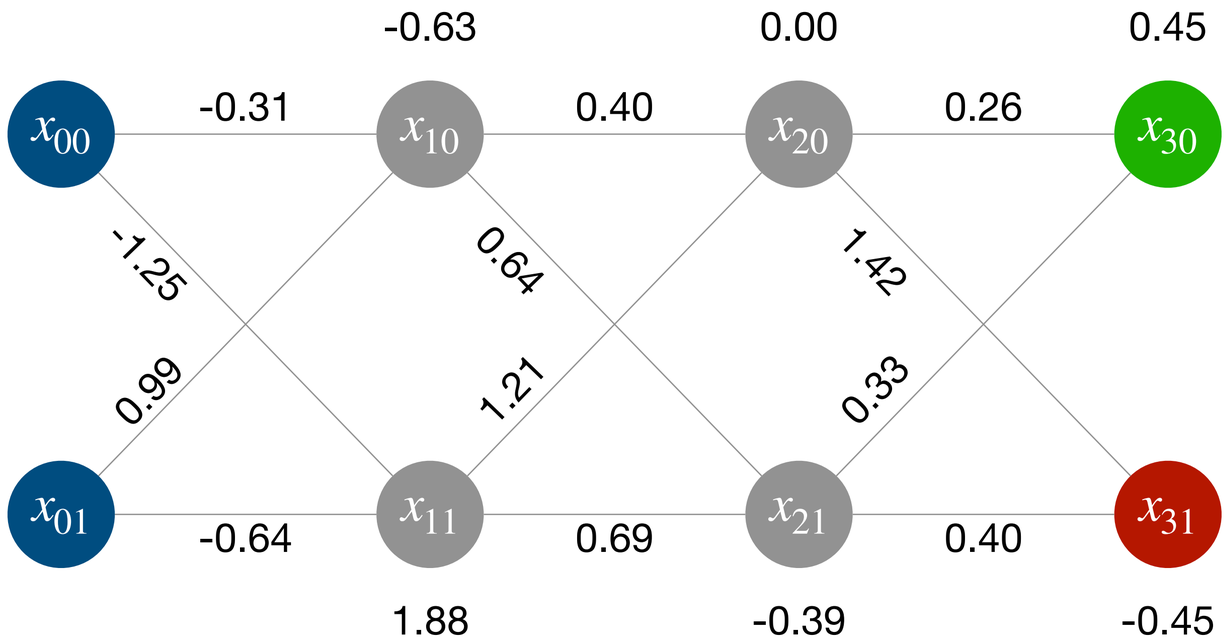

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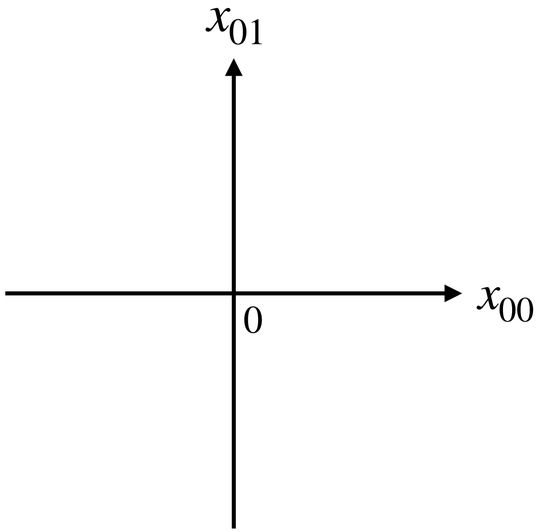


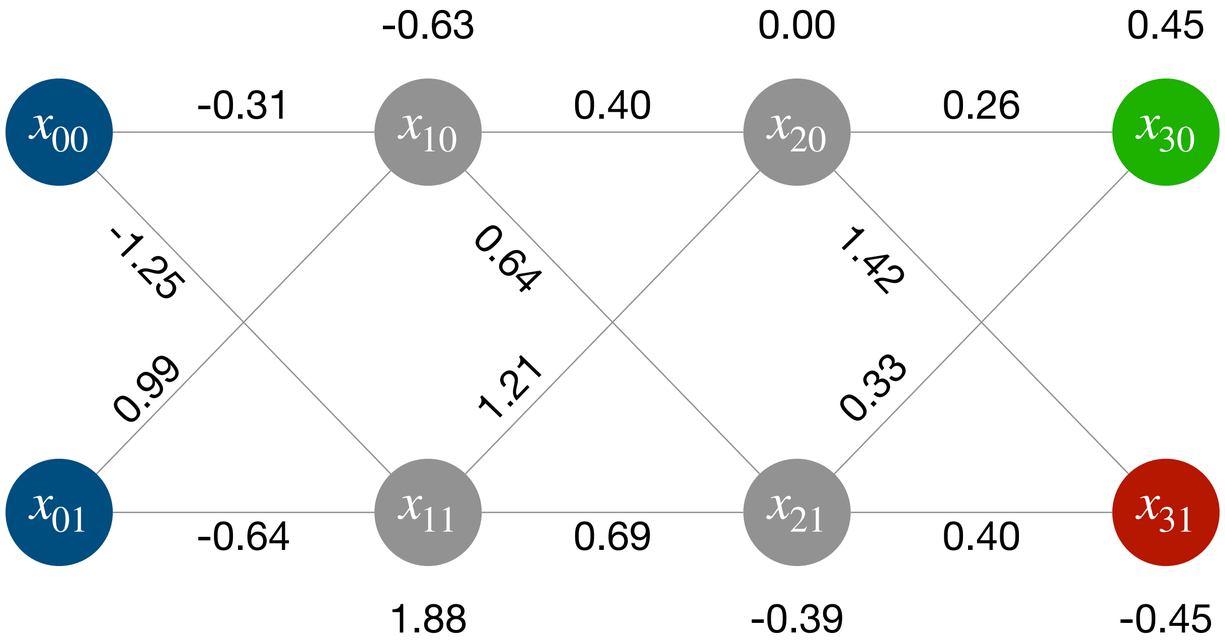

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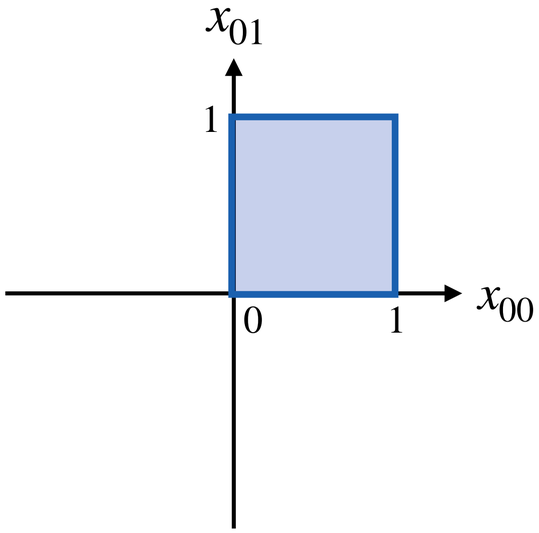


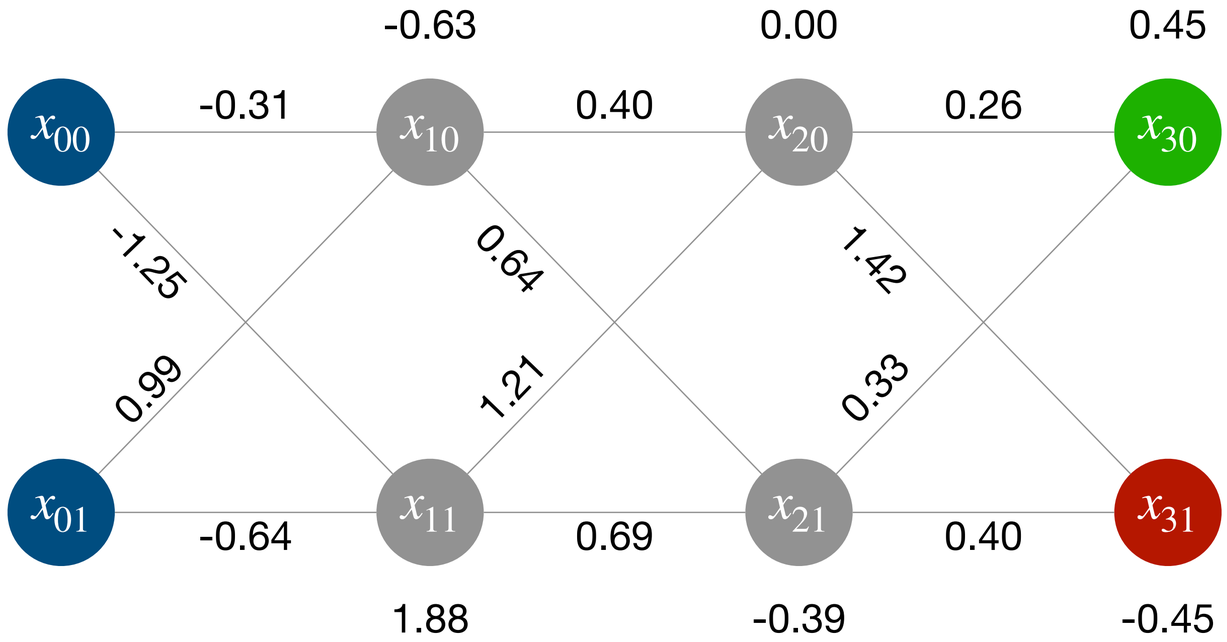

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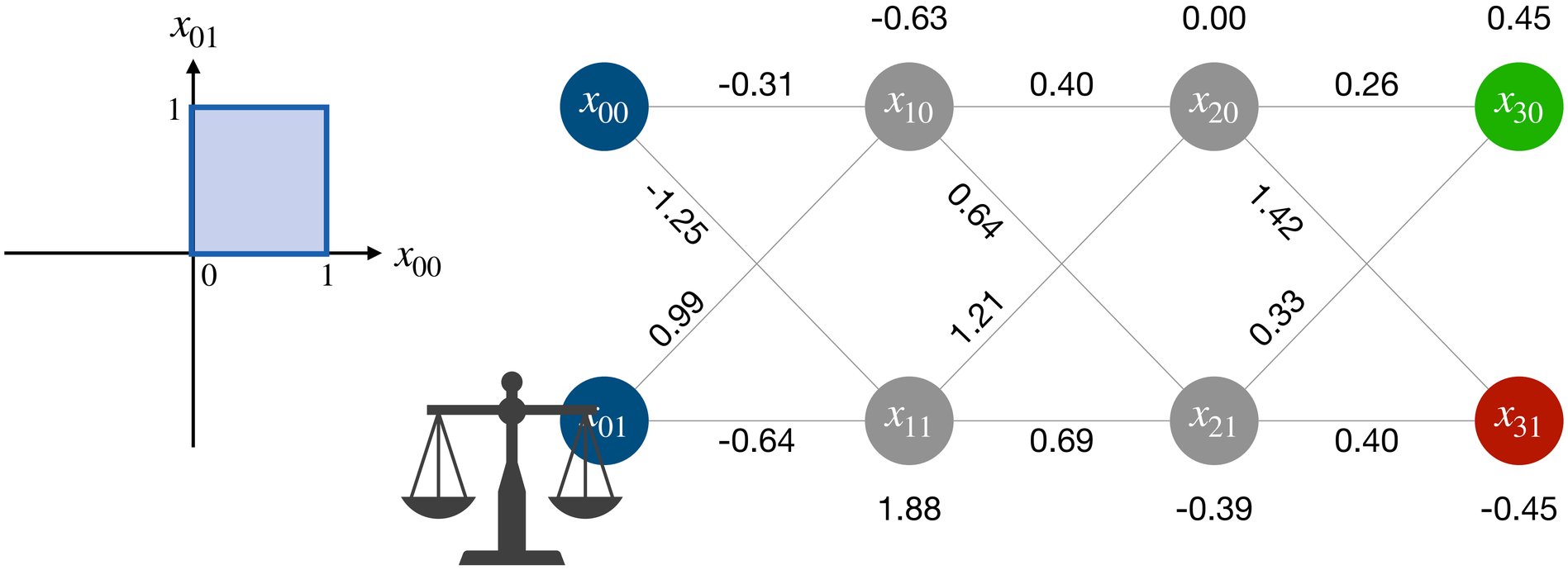
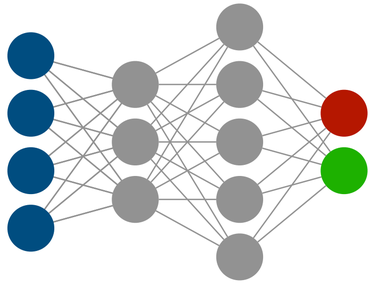
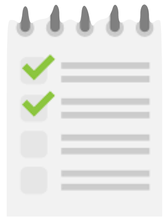
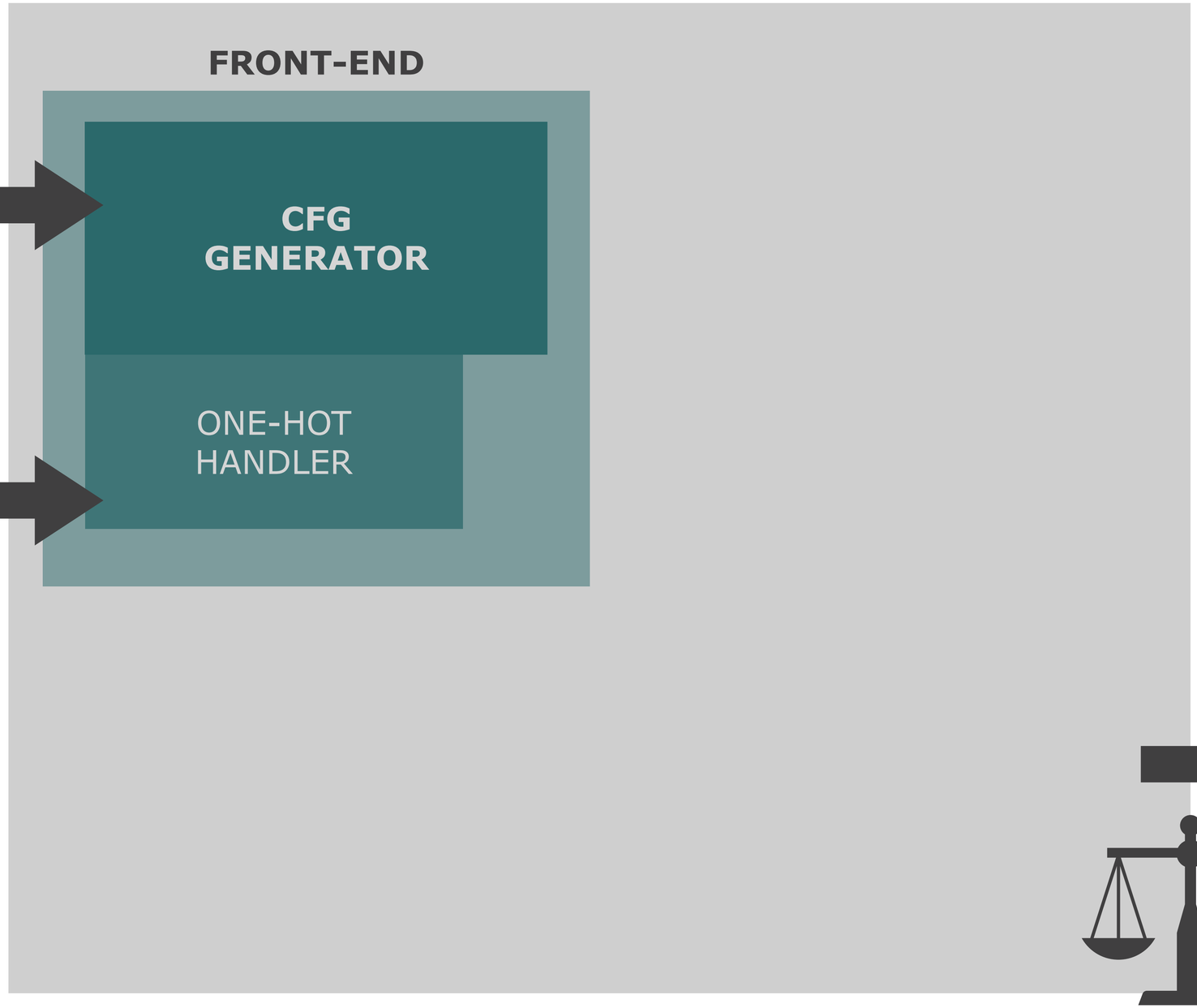


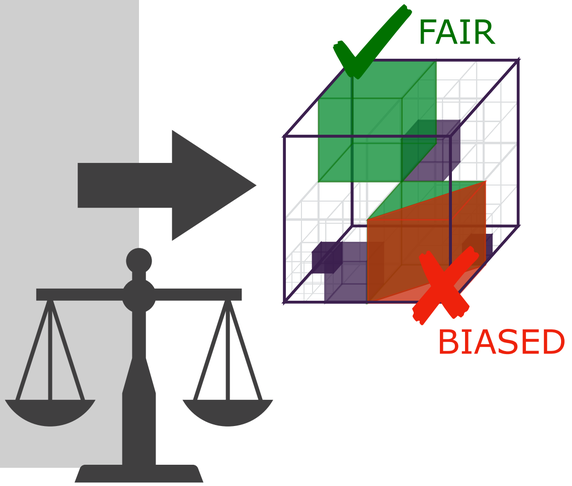


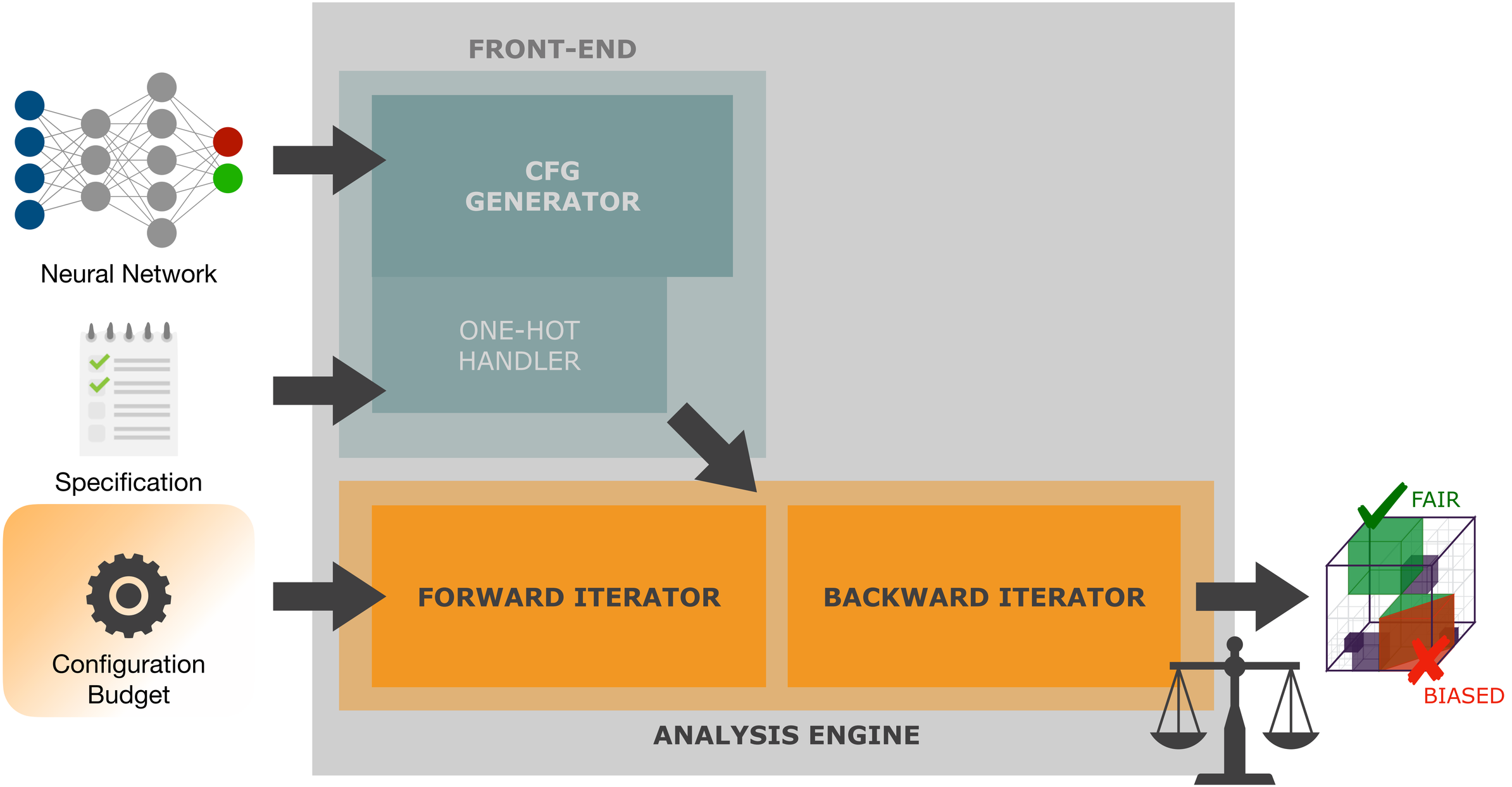
Neural Network



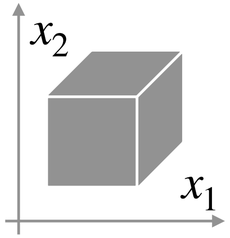
Specification



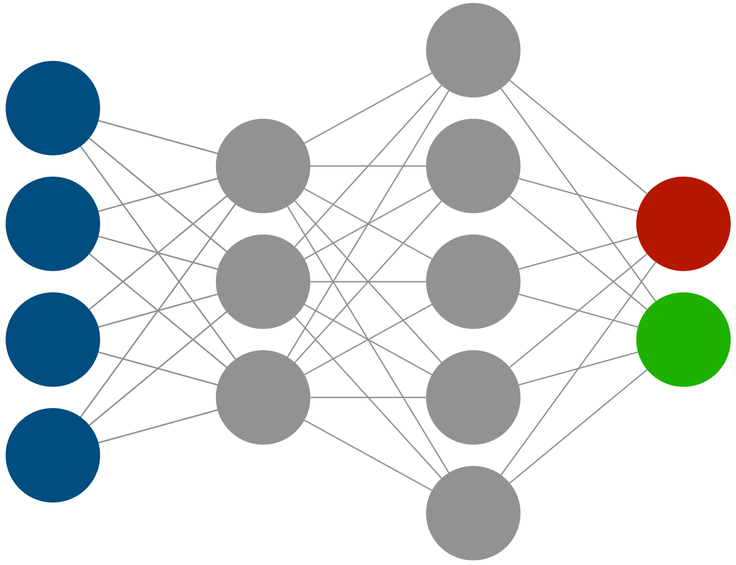



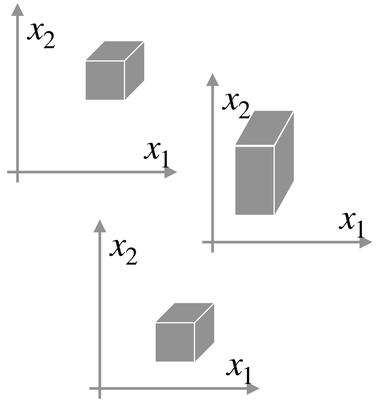

FRONT-END
CFG GENERATOR
ONE-HOT HANDLER
Neural Network
Specification
Configuration Budget
FORWARD ITERATOR
BACKWARD ITERATOR
ANALYSIS ENGINE
FAIR
BIASED

Cheap **Forward Pre-Analysis**

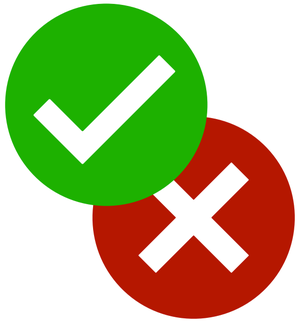



Input Space



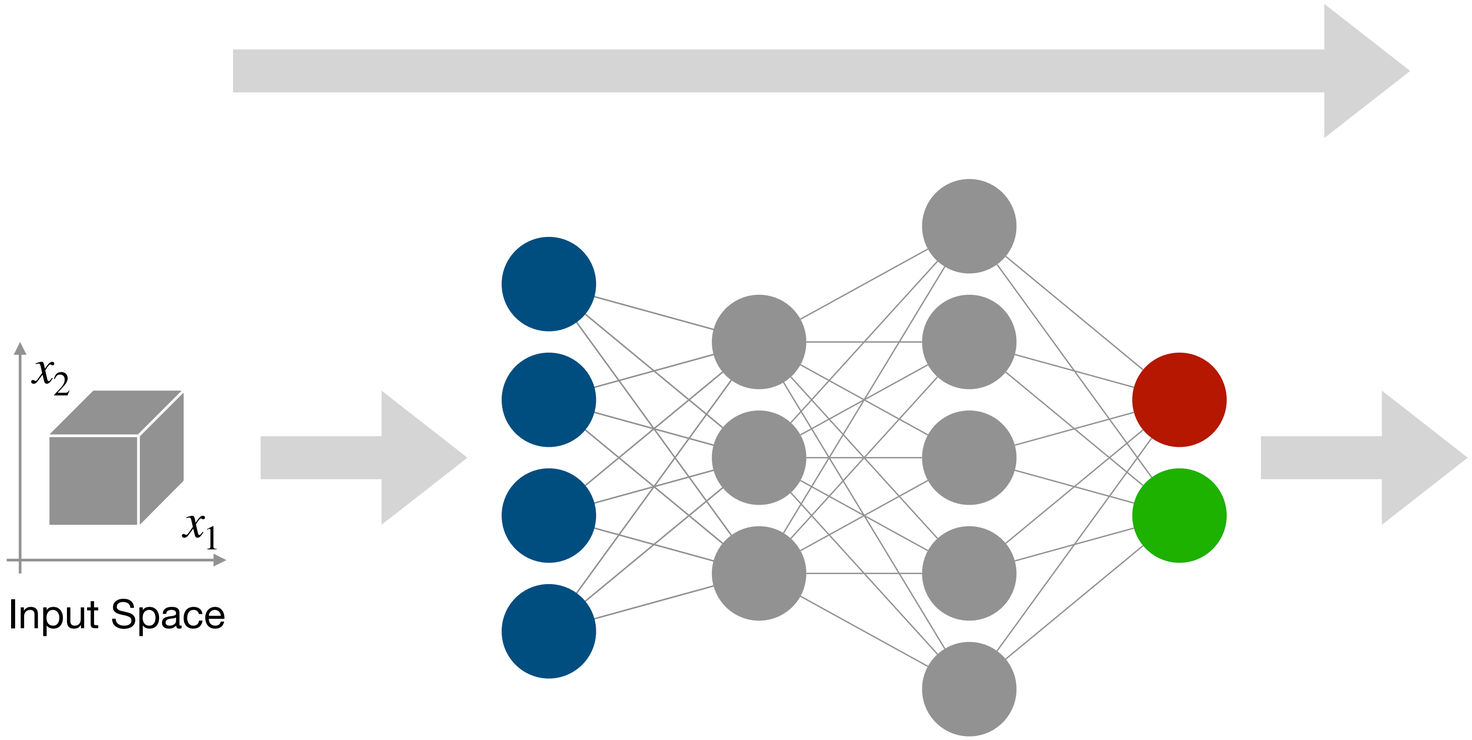



Partitions



Exact **Backward Analysis**
using **Polyhedra**

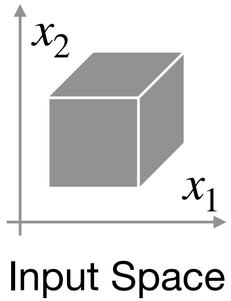
Cheap **Forward Pre-Analysis**



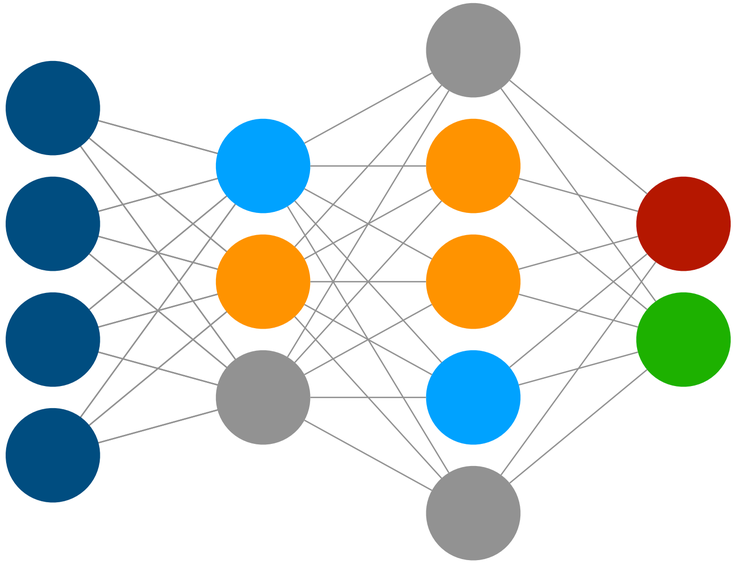

- **Fair**

Propagate the partition through the network via **abstract domains**

# Cheap **Forward Pre-Analysis**




Input Space



- **Fair**
- Or, …

**Activation Patterns**

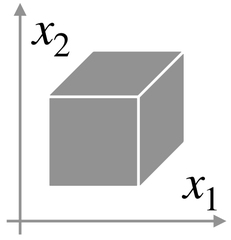
**Always Negative** $\max(\hat{x},0) = 0$

**Always Positive** $\max(\hat{x},0) = \hat{x}$

**Uncertain** $\hat{x} \in [a, b]$ for $a < 0 < b$

**Fixed Activation Status**

# Cheap **Forward Pre-Analysis**

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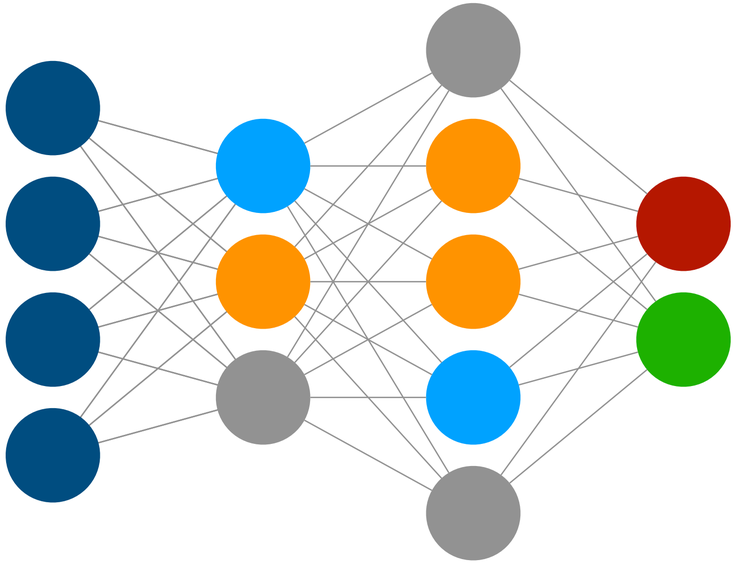
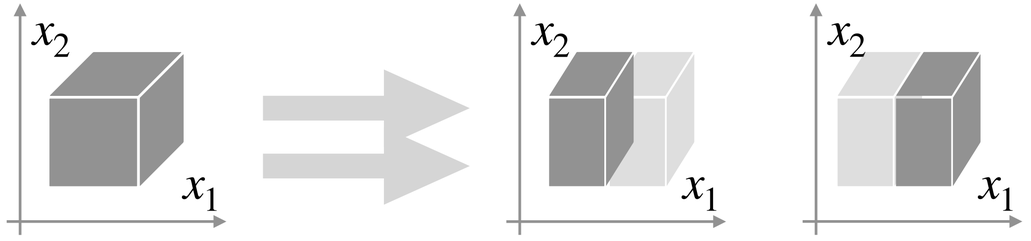


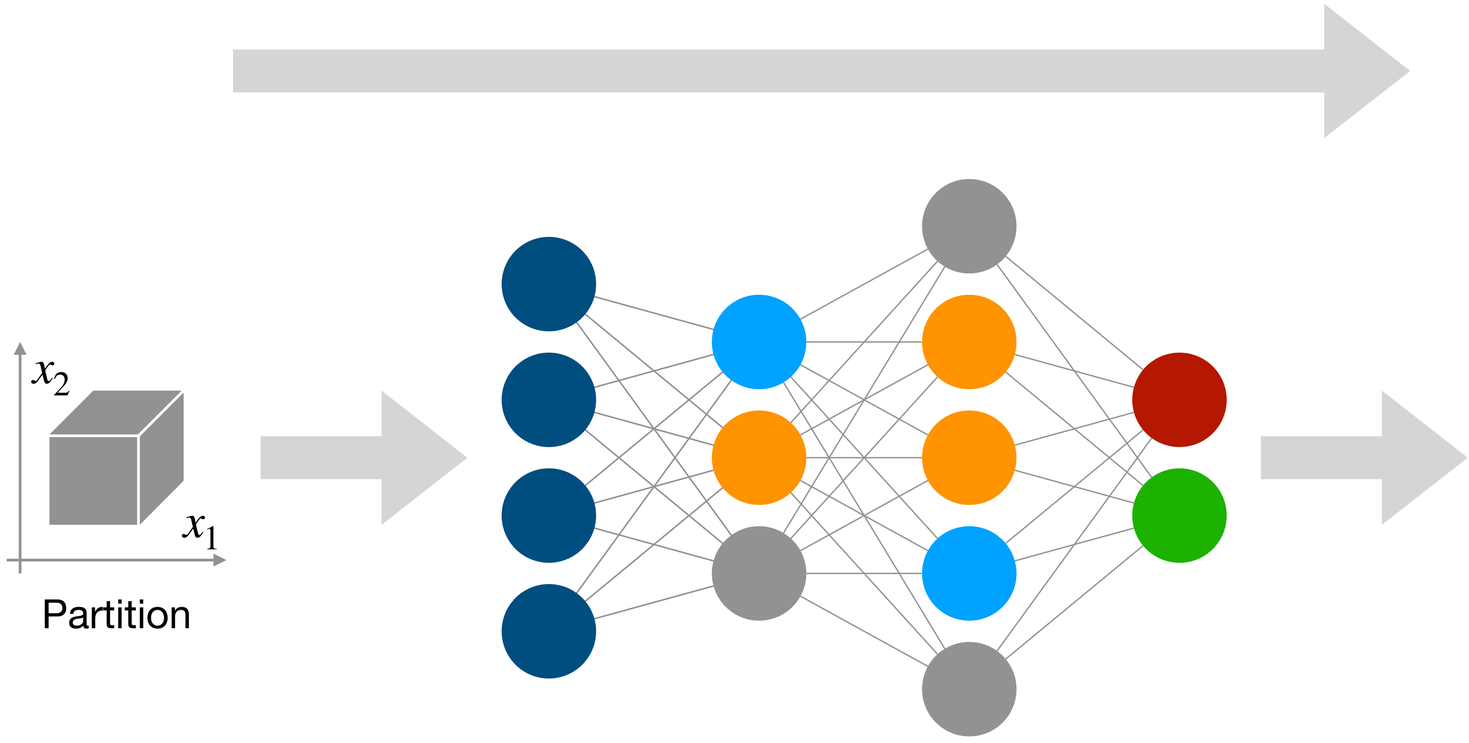
- **Fair**
  - **Partitioned**

$\sum$ ● $\geq U$




Along non-sensitive features only

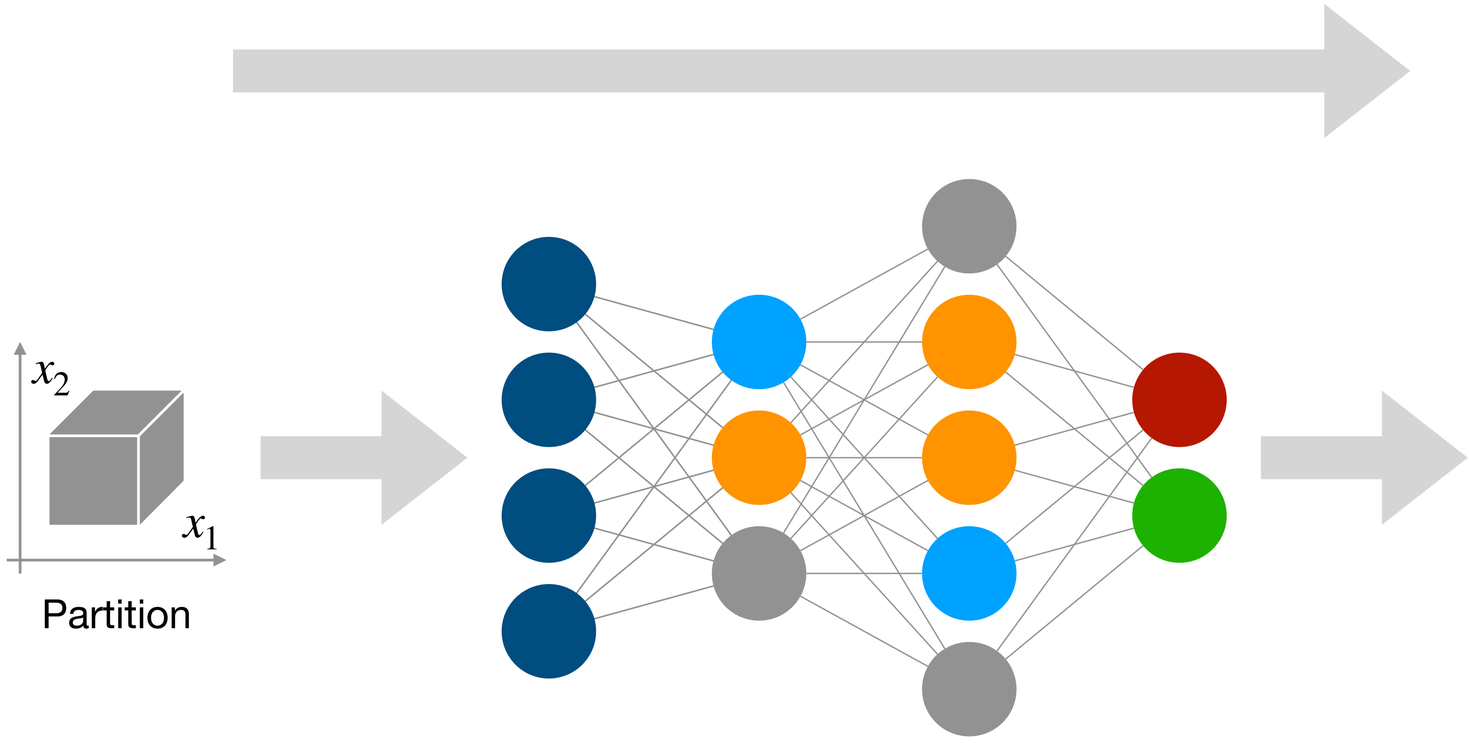
# Cheap **Forward Pre-Analysis**




- **Fair**
- **Partitioned**
  - **Feasible**

$$\sum \bigcirc \leq U$$

# Cheap **Forward Pre-Analysis**

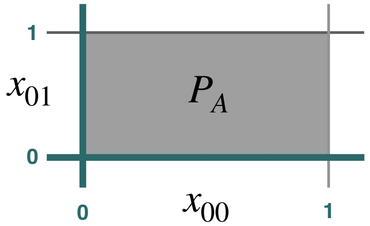


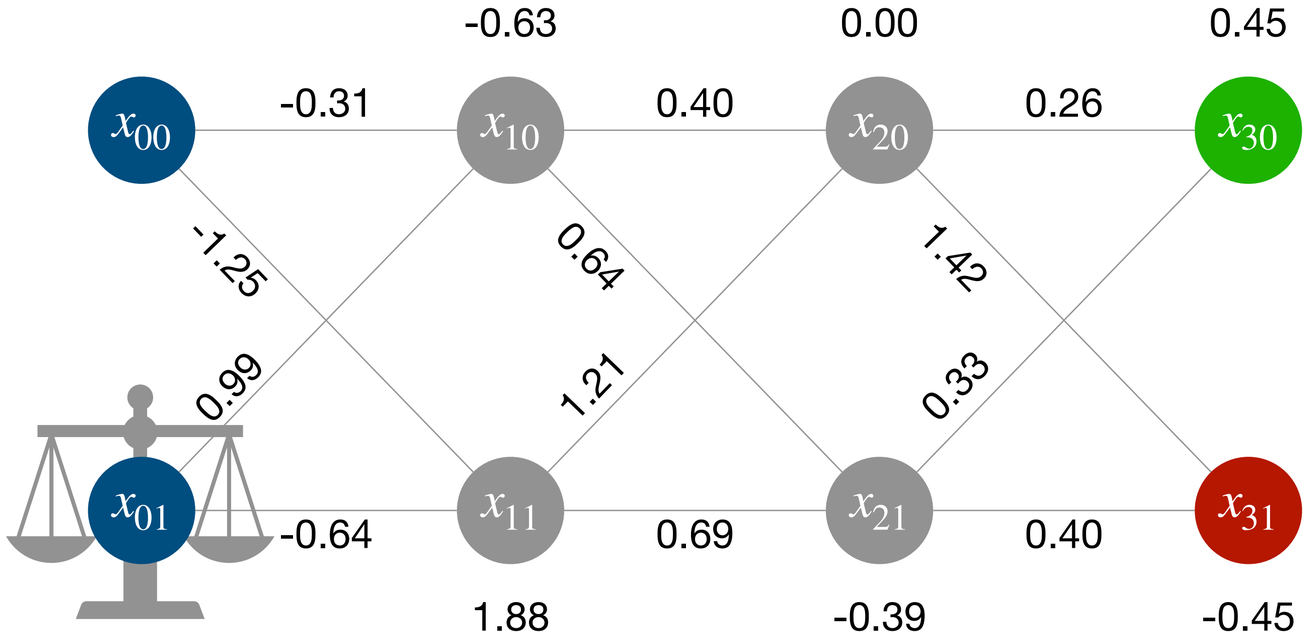

- **Fair**
- **Partitioned**
- **Feasible**
- ▸ **Excluded**

$\sum$ ● $\geq U$, and the partition becomes smaller than $L$

# Forward Analysis

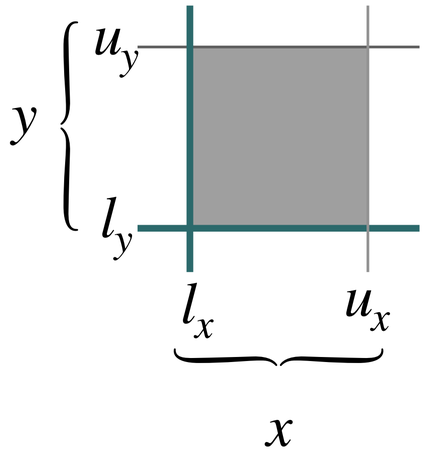
$L = 0.25, U = 2$

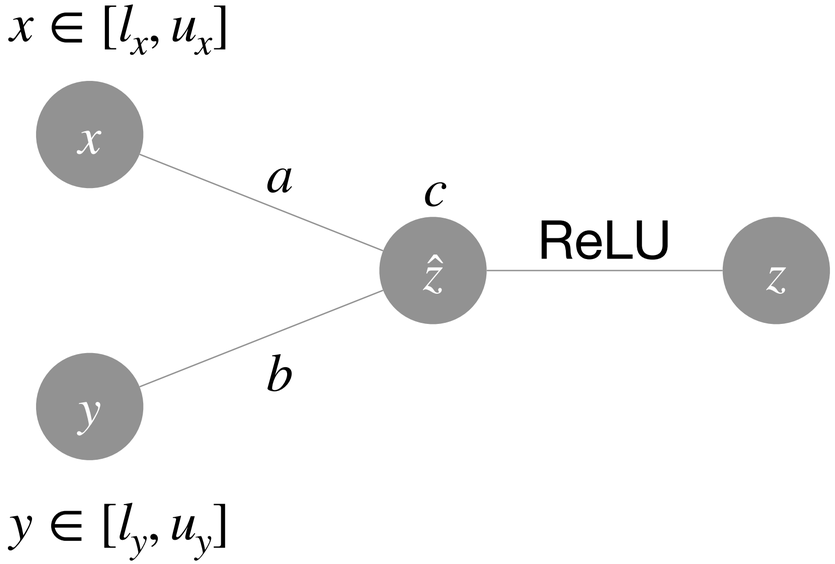


# Forward Analysis

Boxes domain

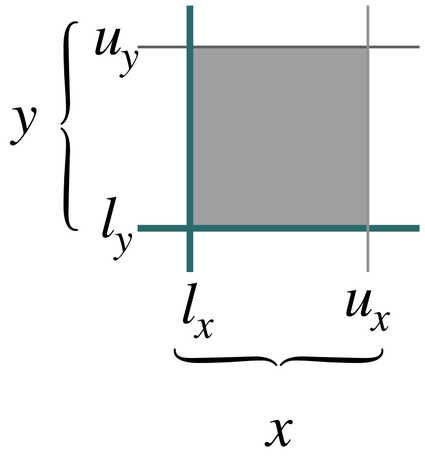


# Forward Analysis

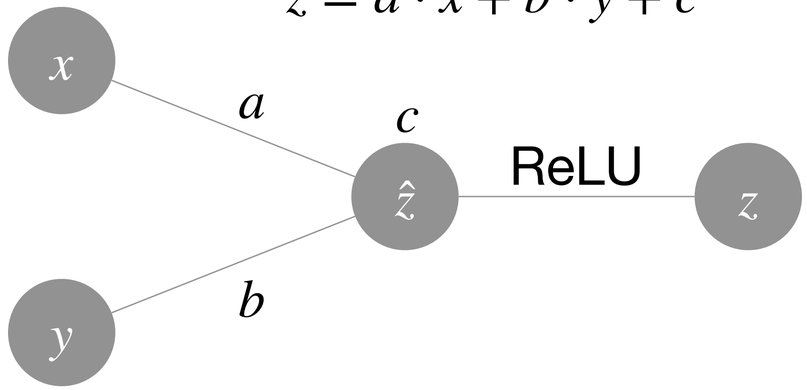


Boxes domain




$x \in [l_x, u_x]$

$\hat{z} = a \cdot x + b \cdot y + c$

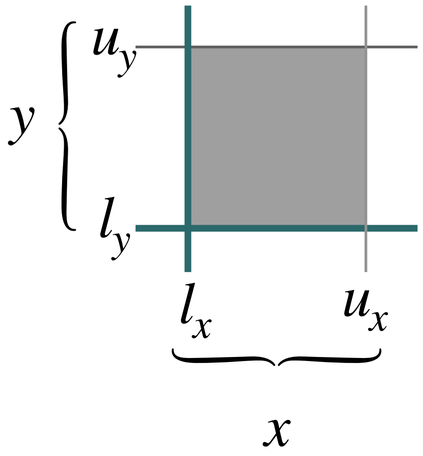


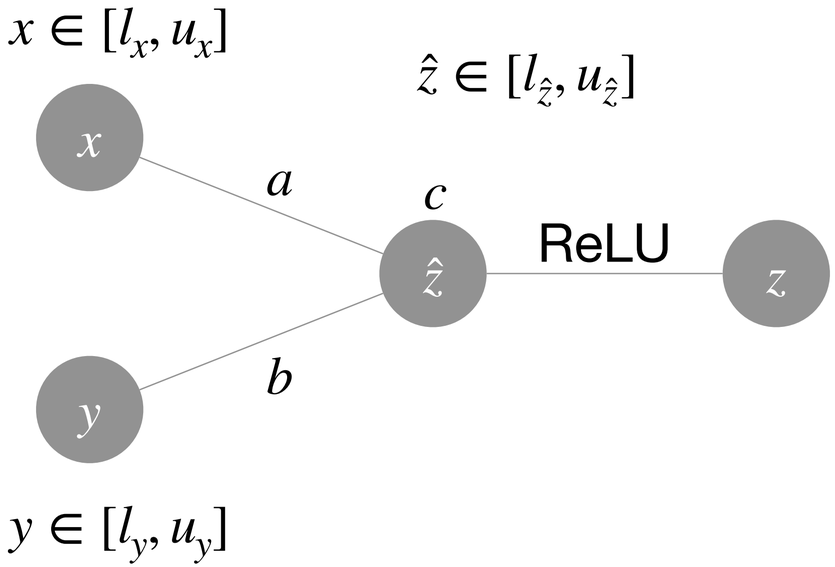

$y \in [l_y, u_y]$

# Forward Analysis



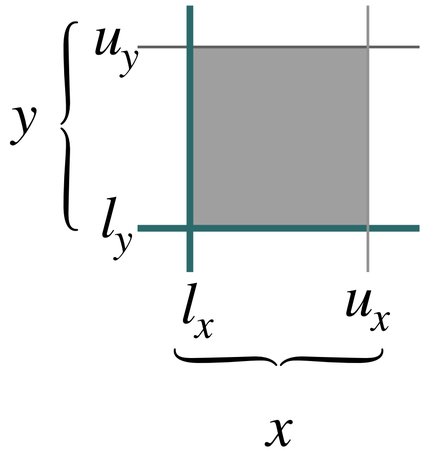
Boxes domain

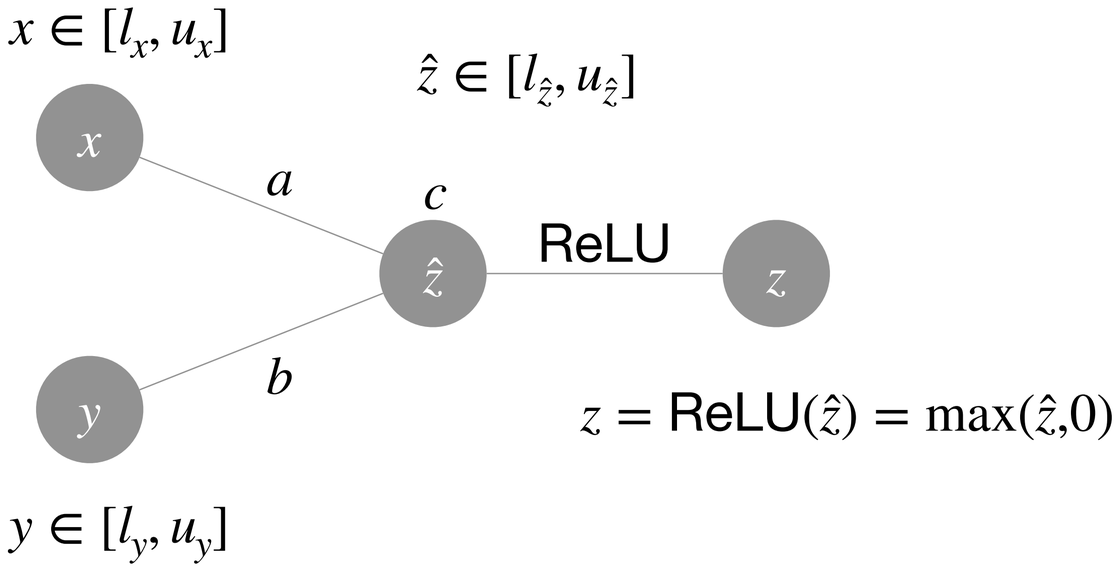


# Forward Analysis

Boxes domain

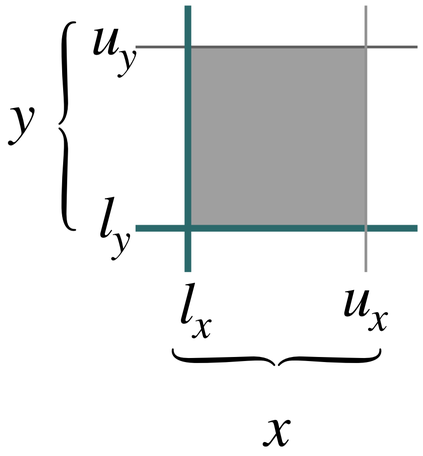


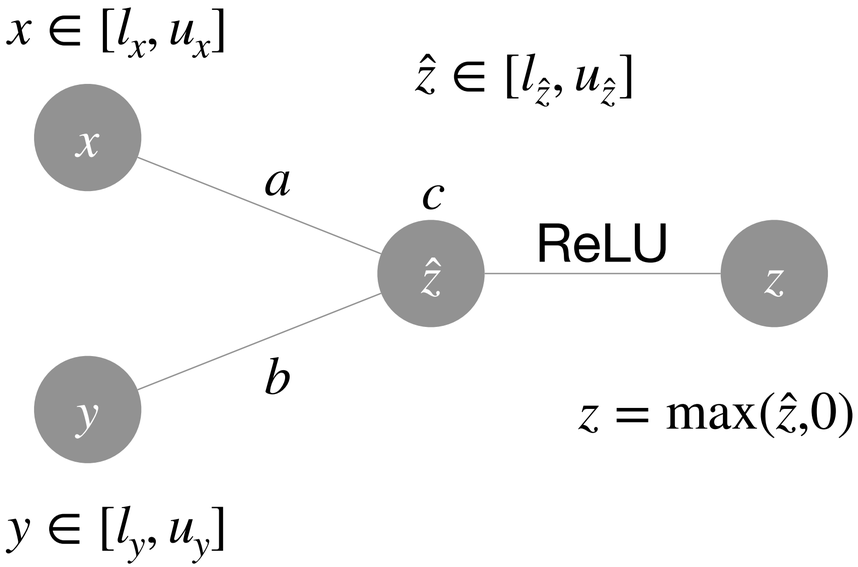



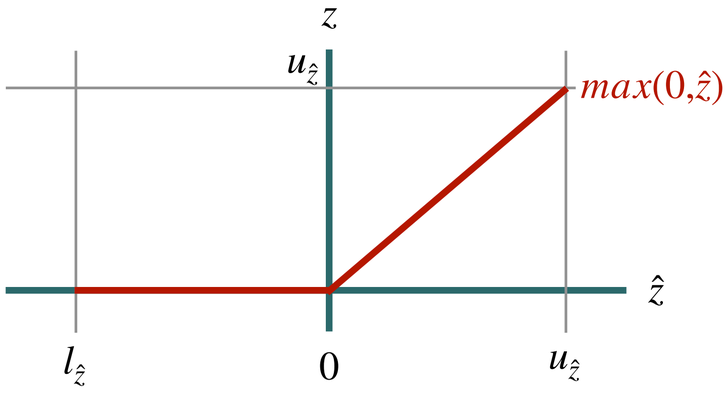
# Forward Analysis



Boxes domain

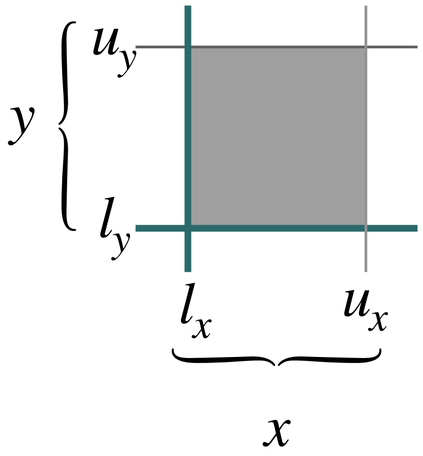


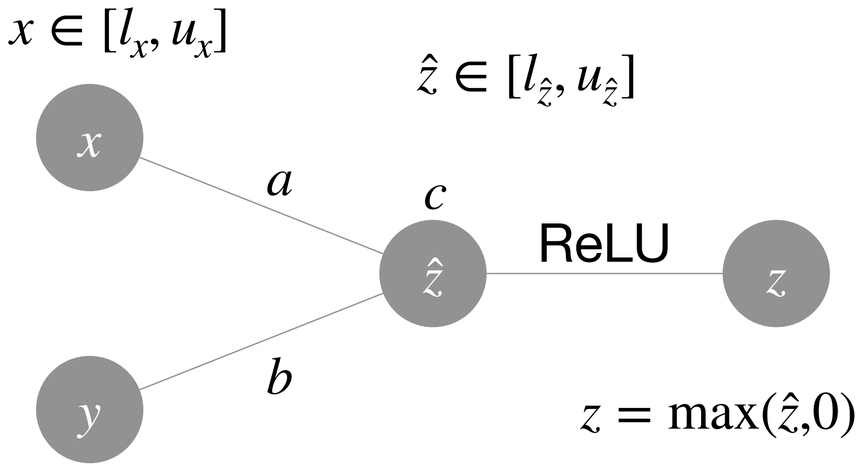



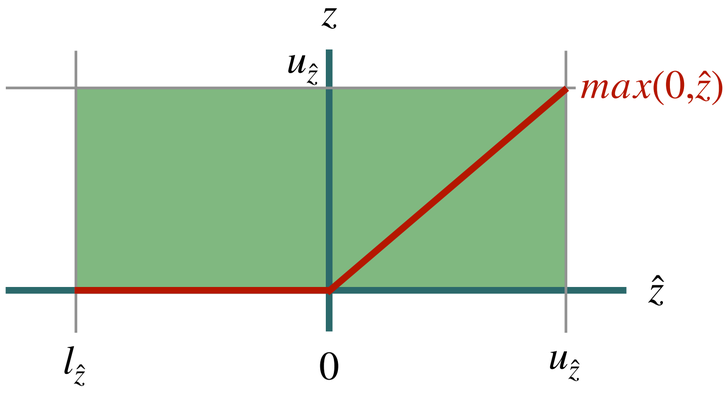



# Forward Analysis

Boxes domain



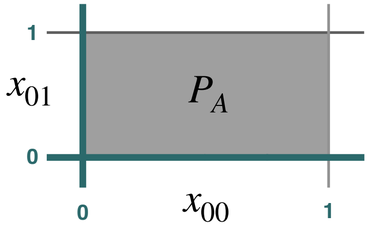



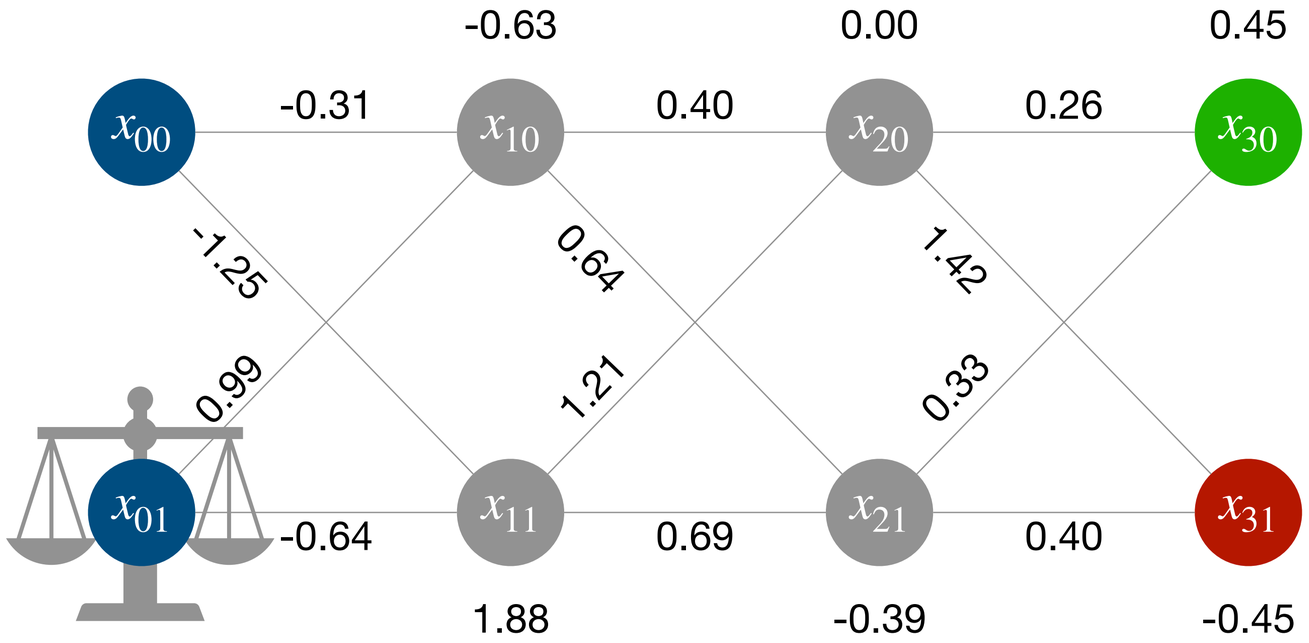

$z \in [0,u_{\hat{z}}]$

$0 \leq z \leq u_{\hat{z}}$

# Forward Analysis

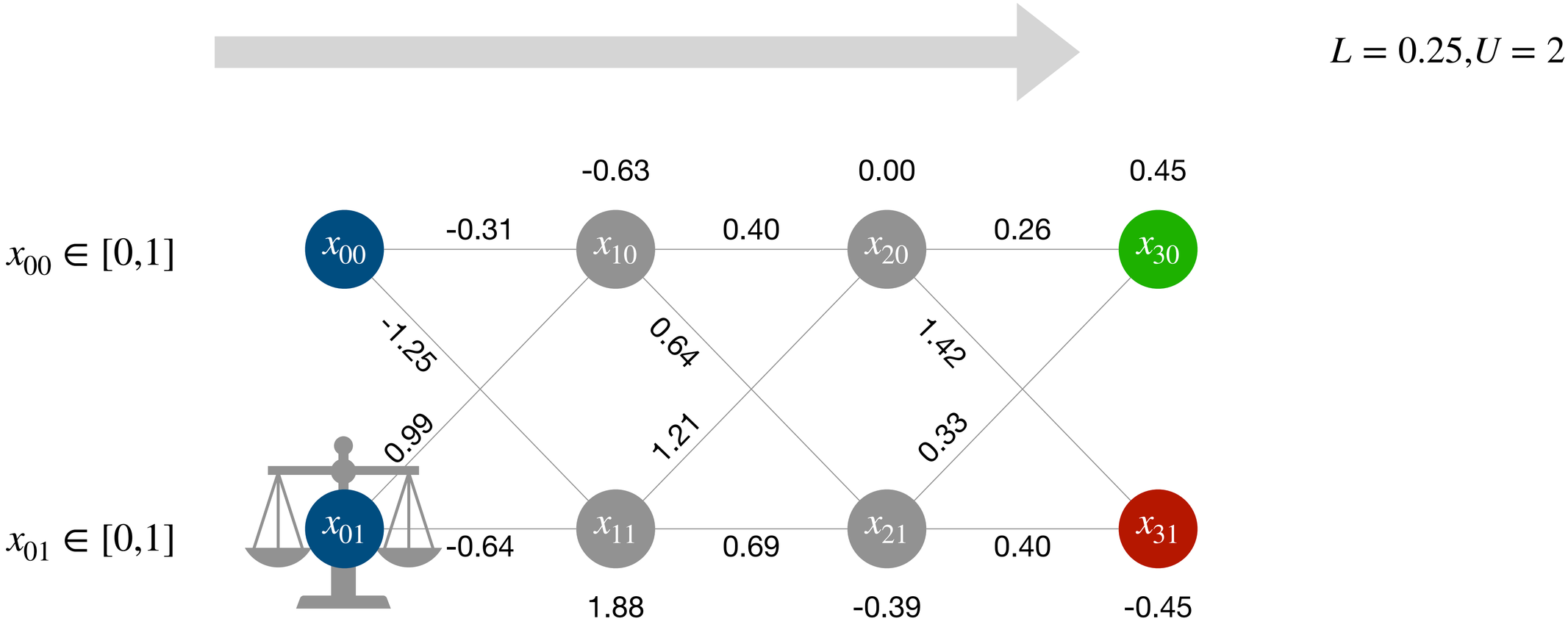
$L = 0.25, U = 2$



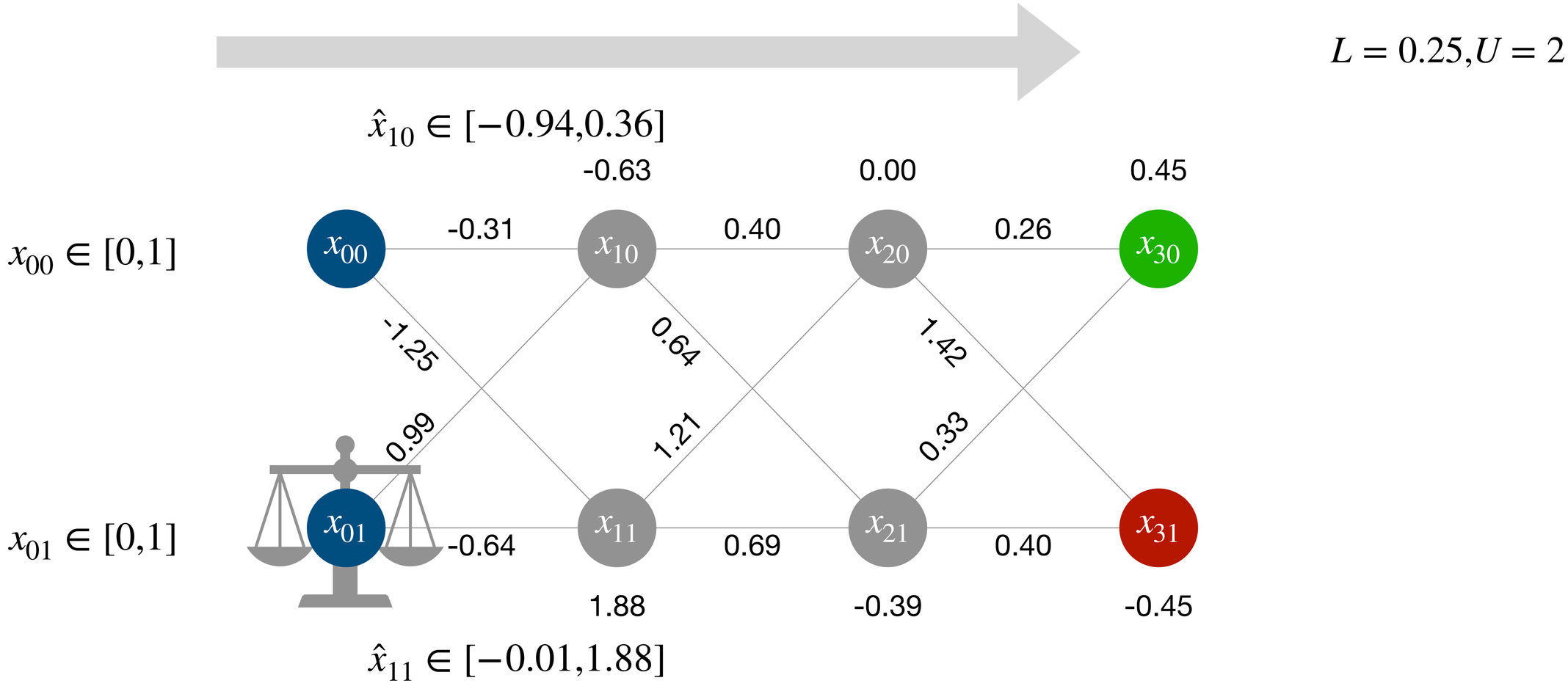



# Forward Analysis

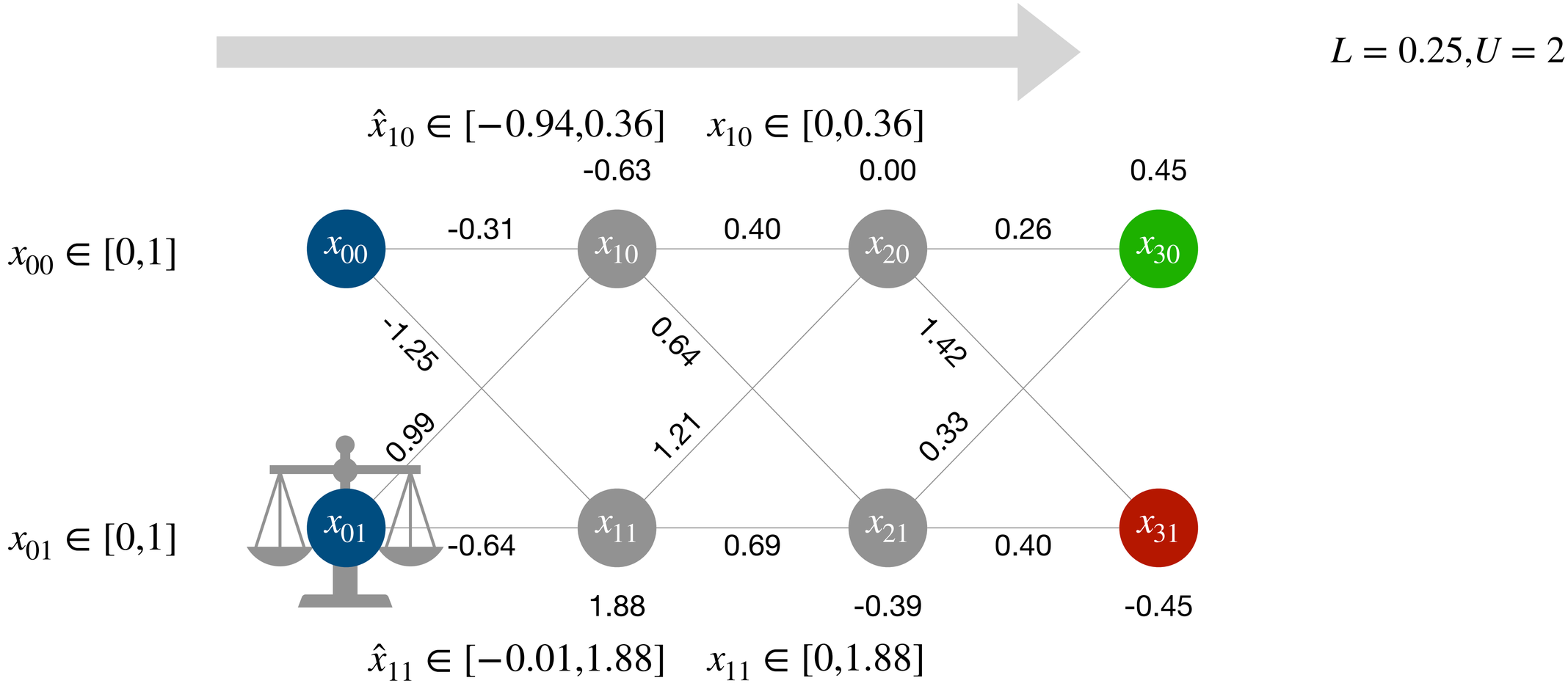
$L = 0.25, U = 2$



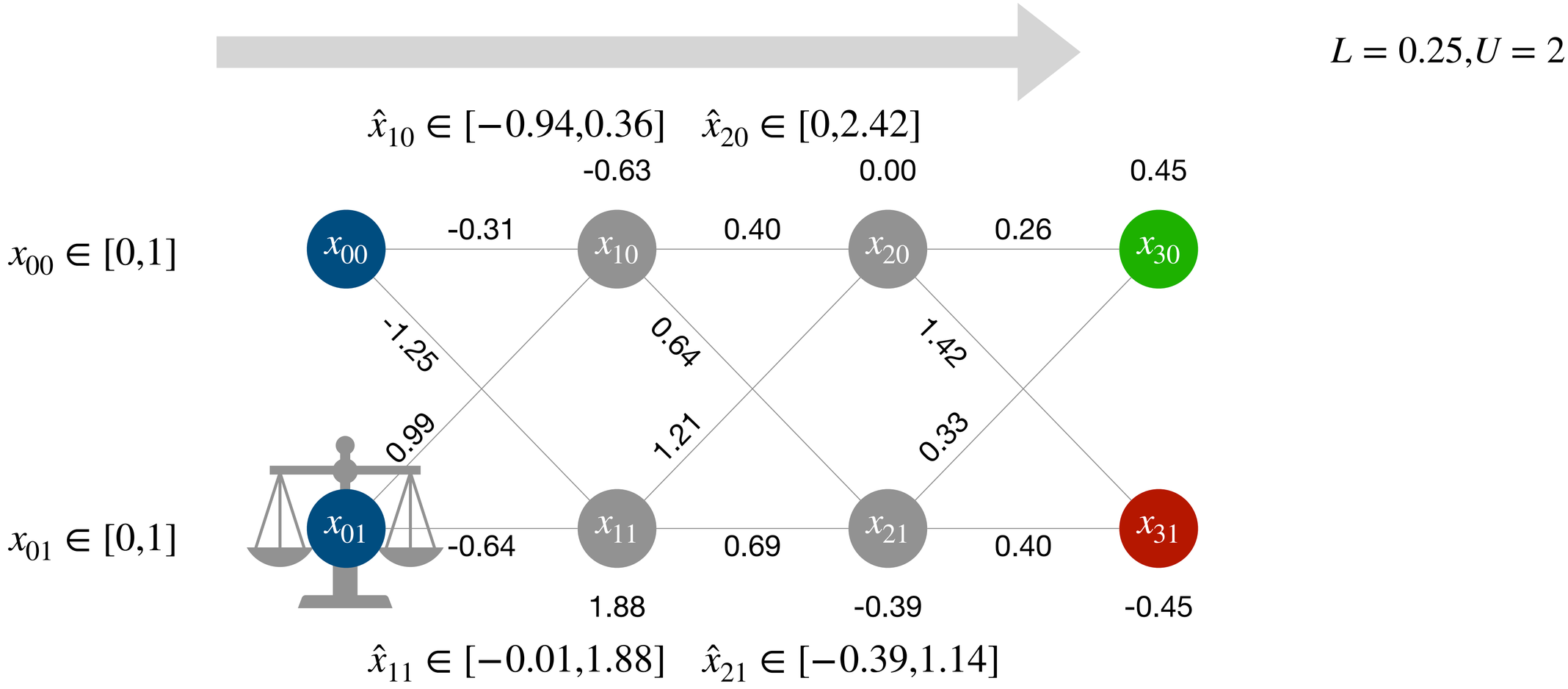
# Forward Analysis



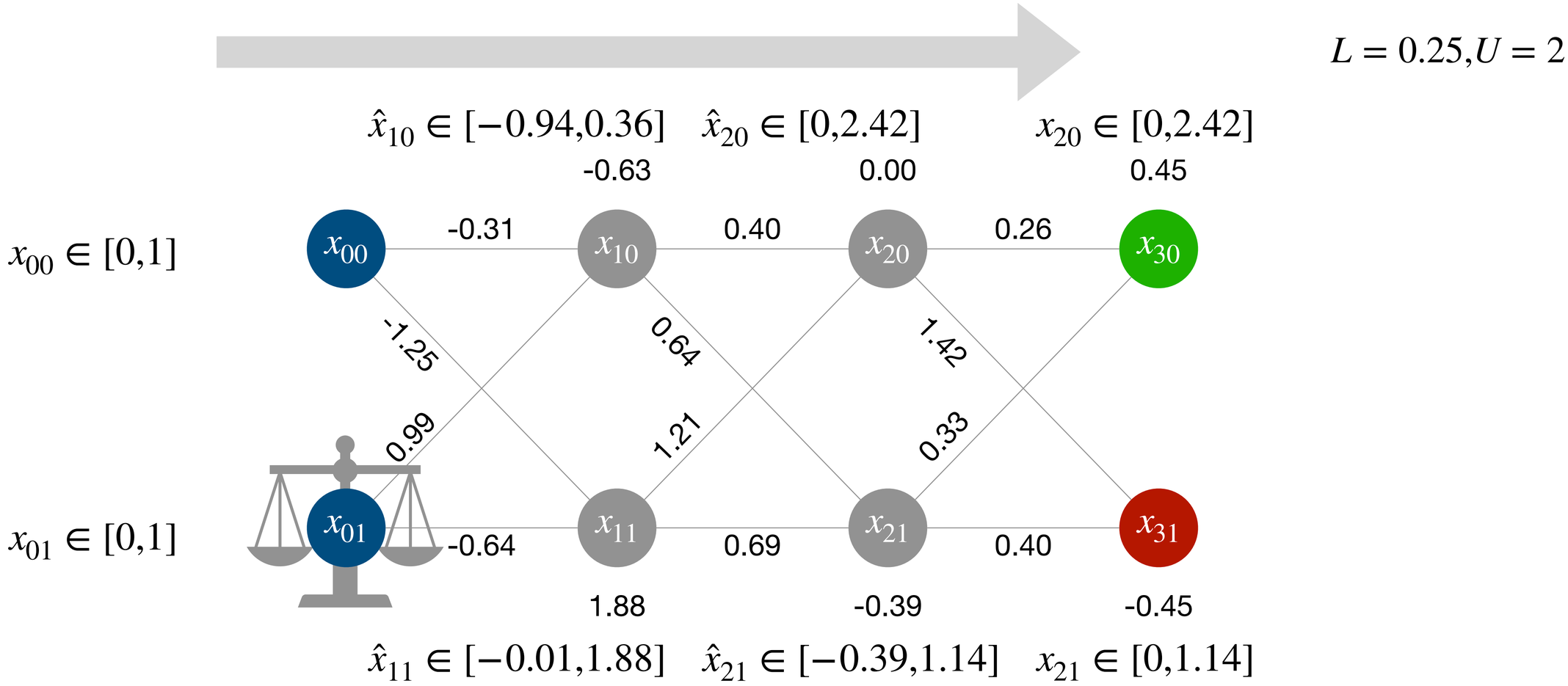
# Forward Analysis



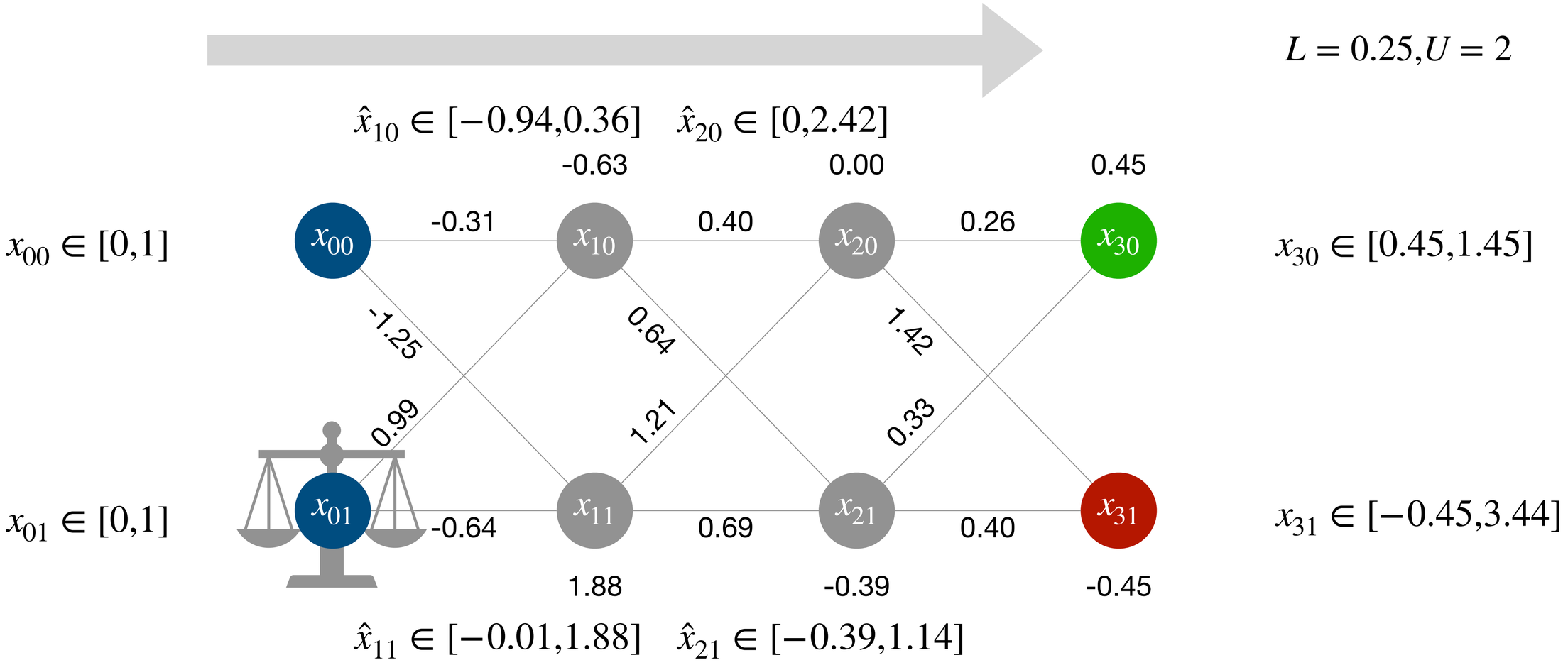
# Forward Analysis



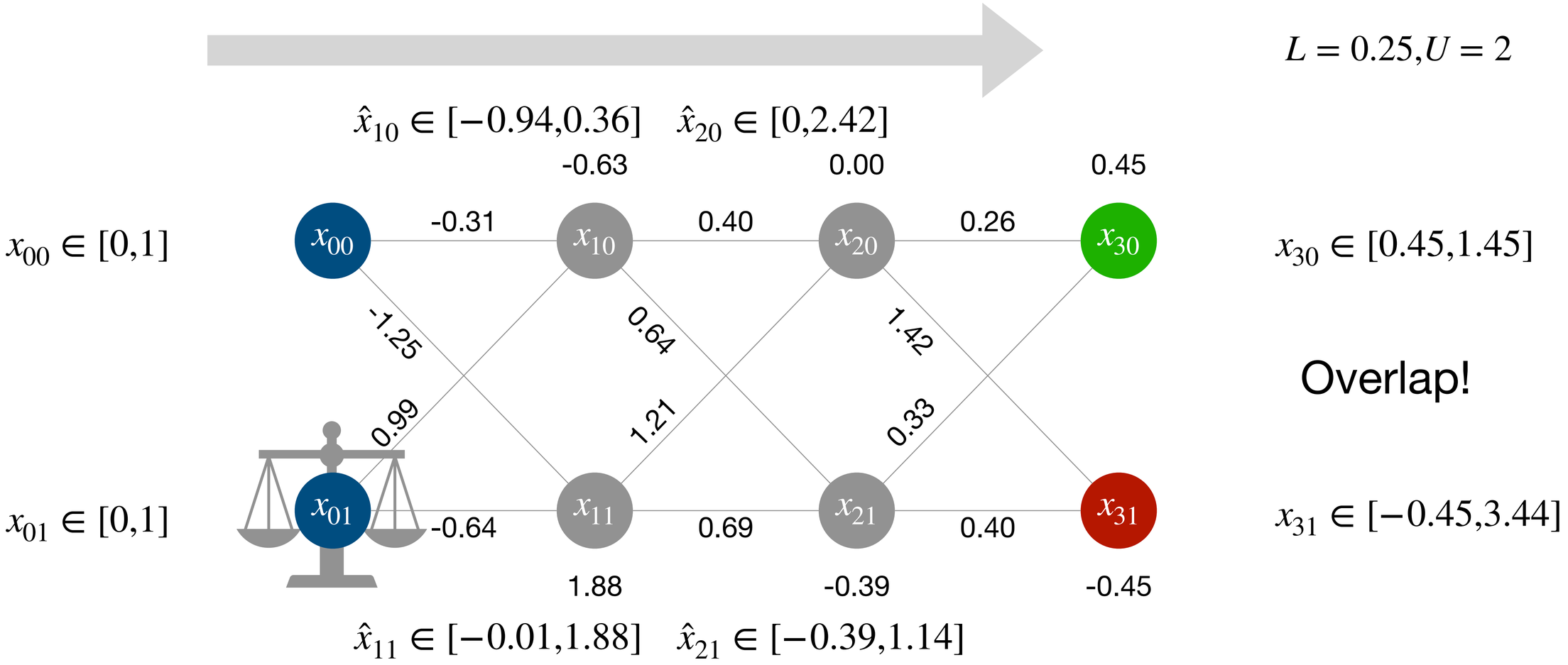
# Forward Analysis



# Forward Analysis

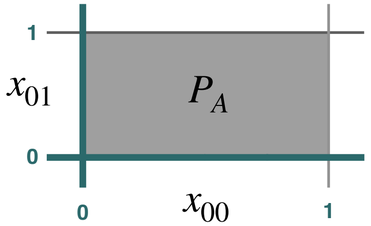


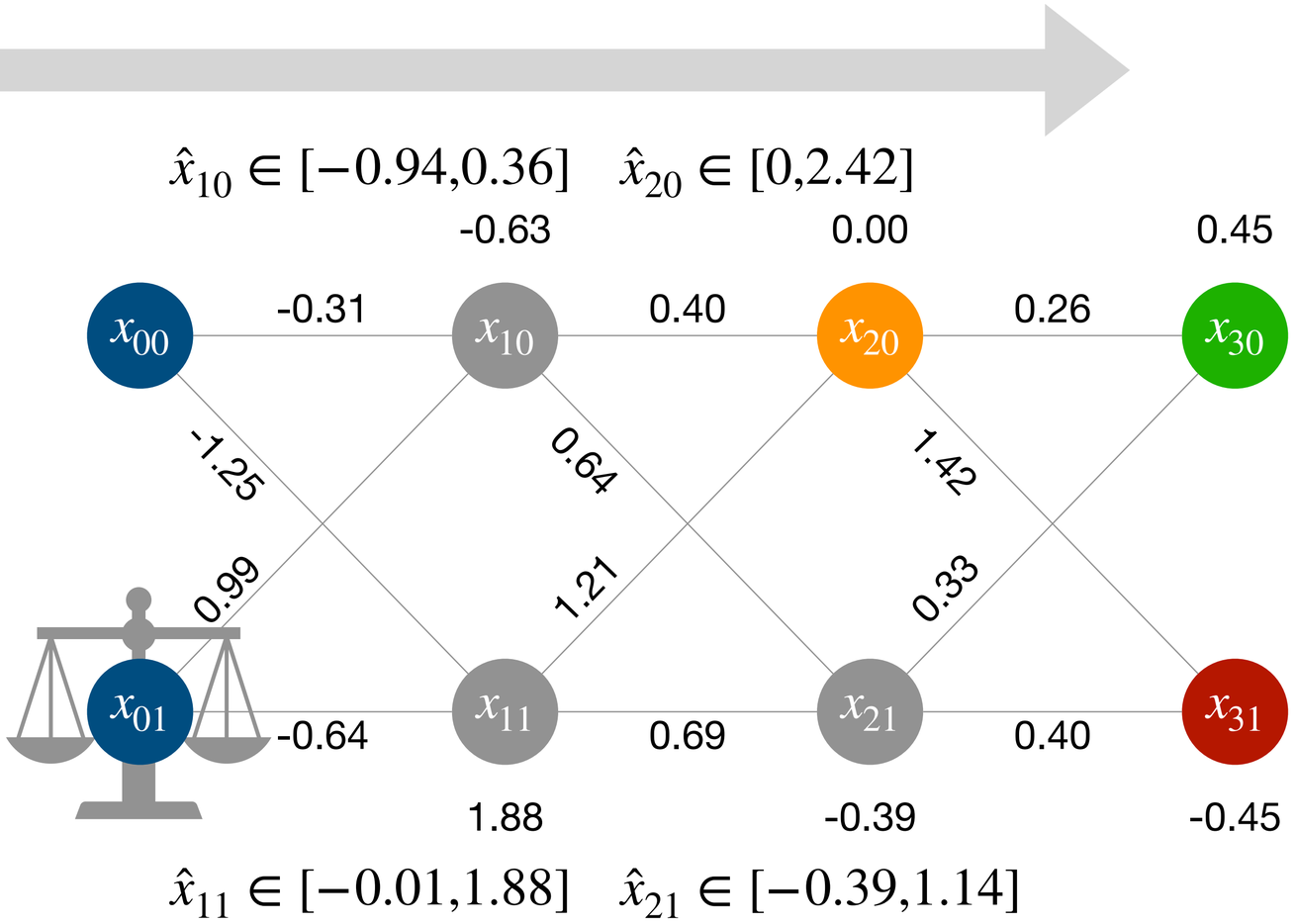
# Forward Analysis



# Forward Analysis

$L = 0.25, U = 2$



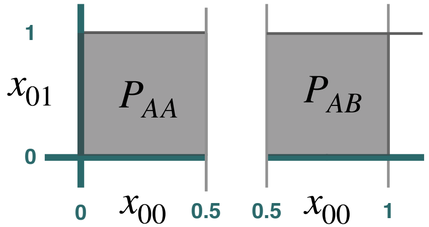



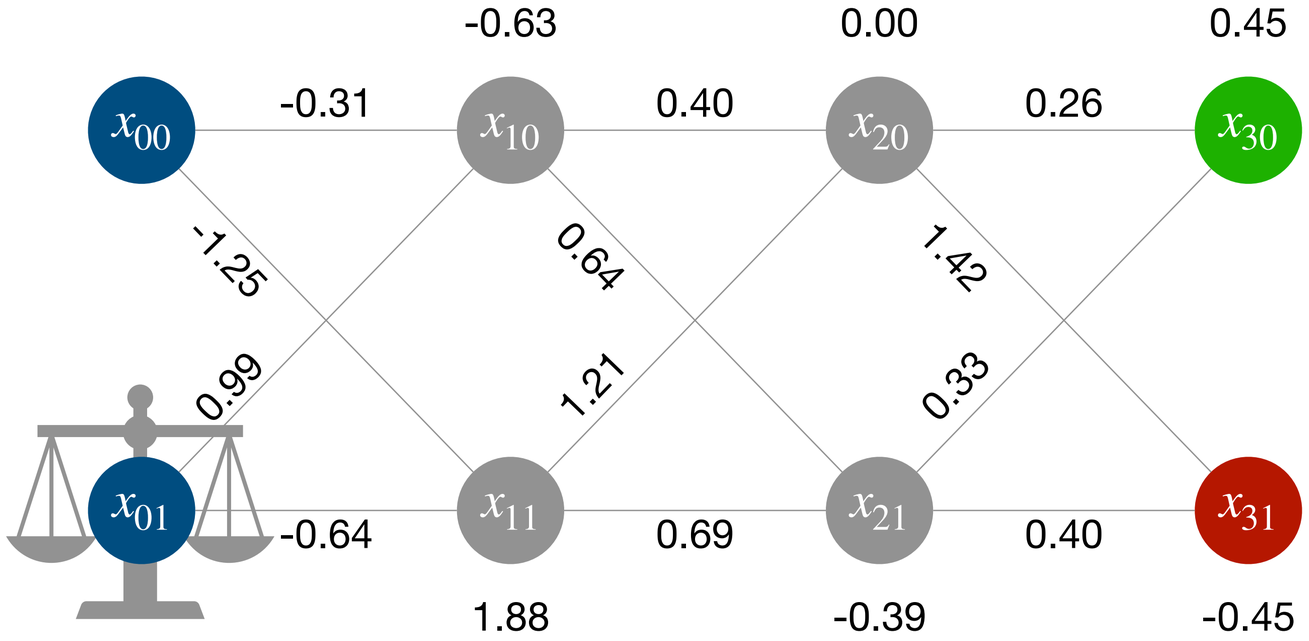


Activation Pattern

# Forward Analysis

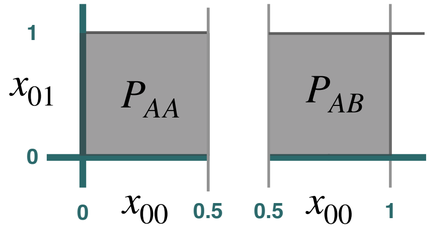
$L = 0.25, U = 2$

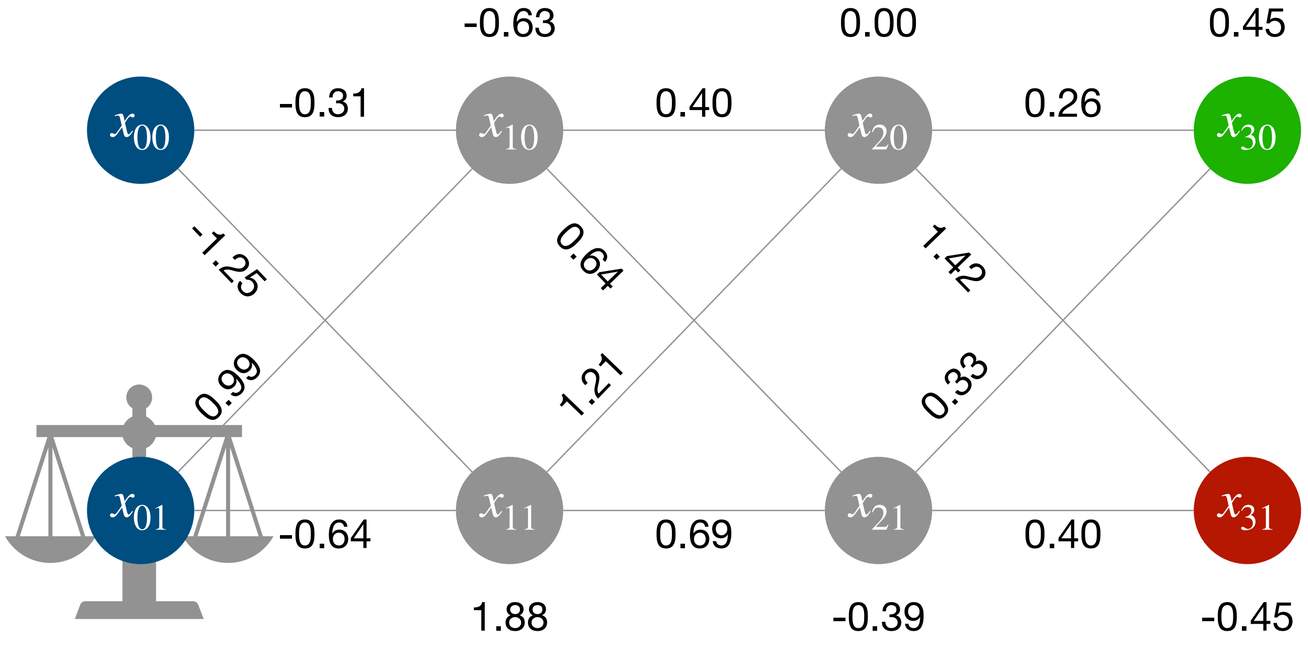


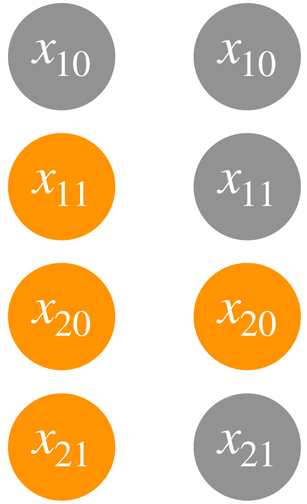



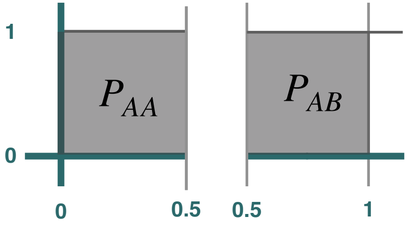
# Forward Analysis

$L = 0.25, U = 2$



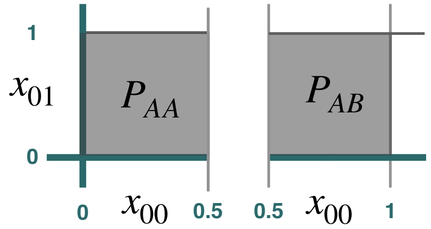



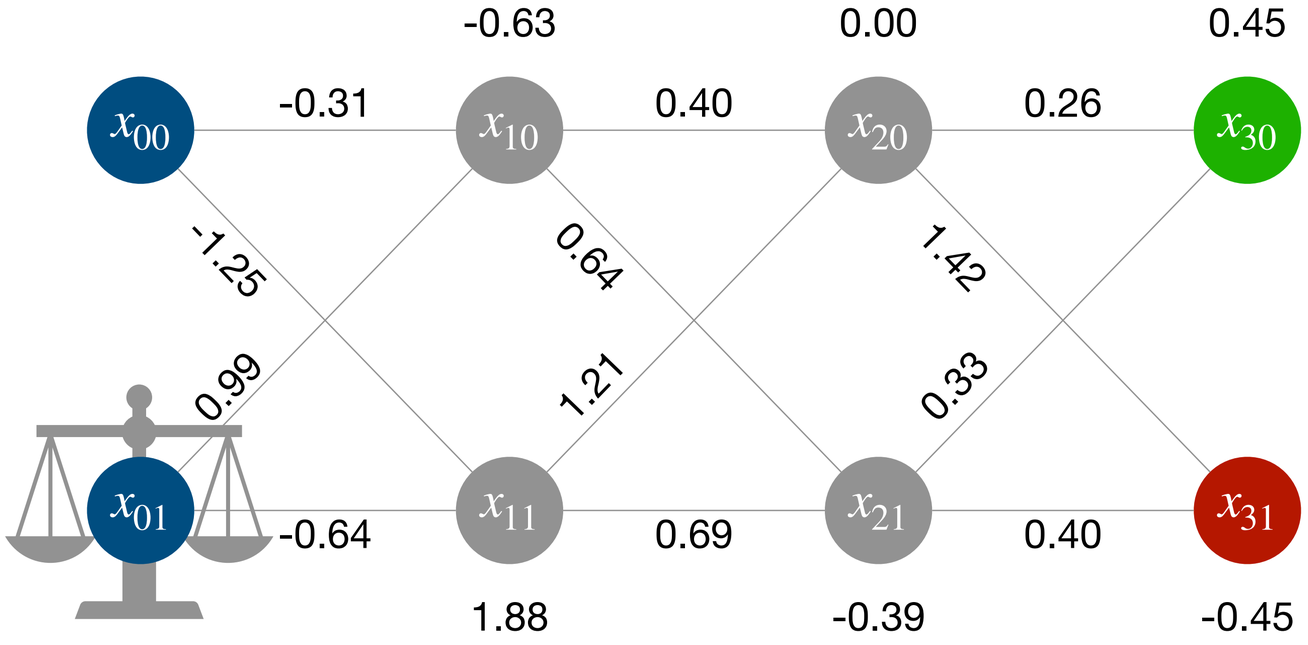



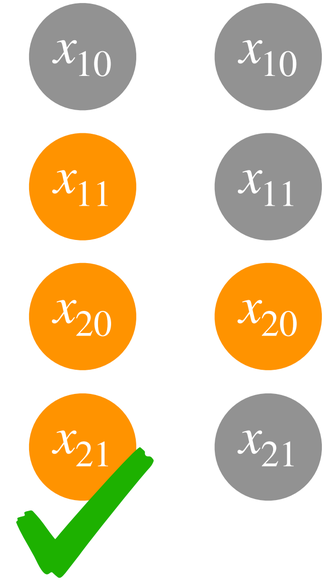



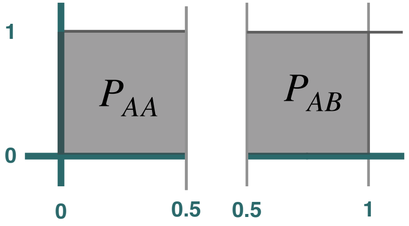
# Forward Analysis

$L = 0.25, U = 2$



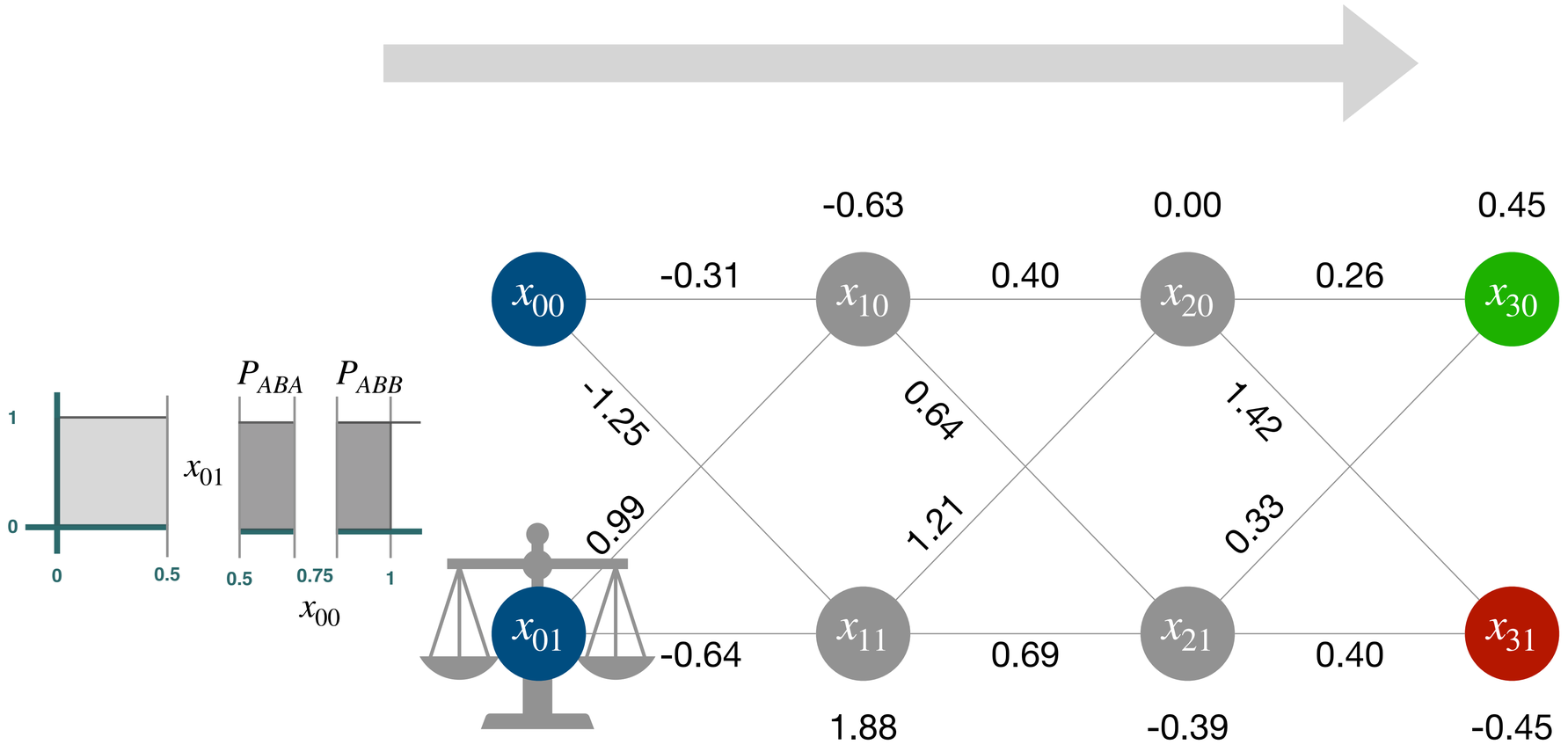



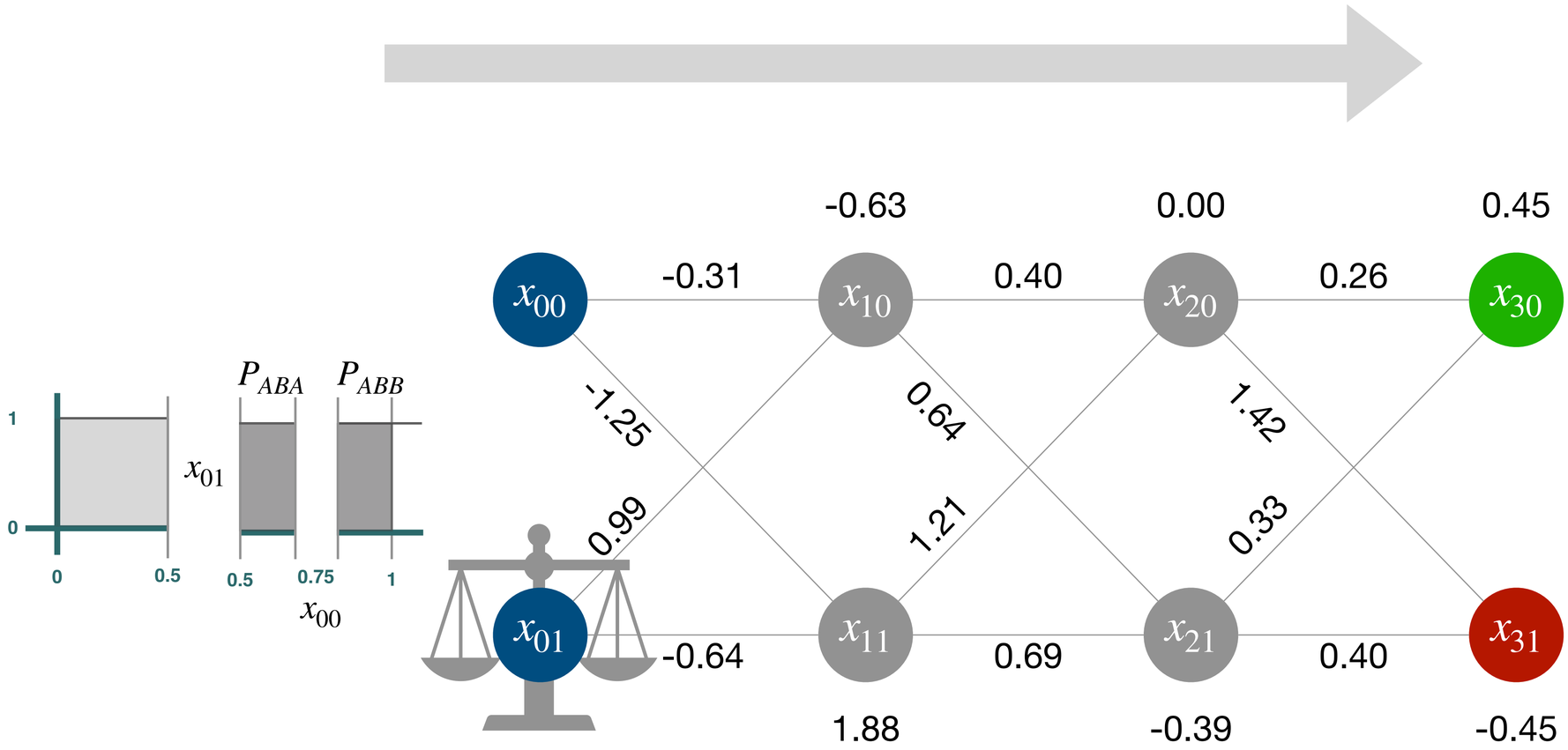



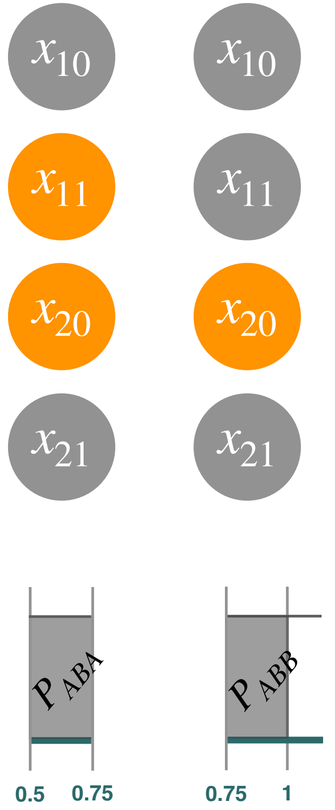
# Forward Analysis

$L = 0.25, U = 2$



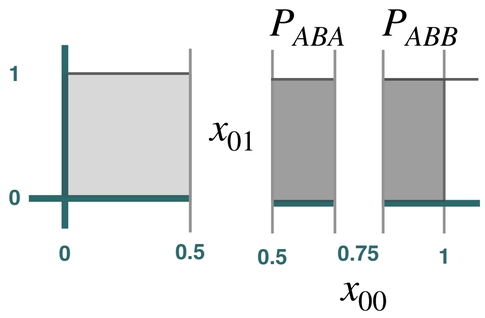
# Forward Analysis

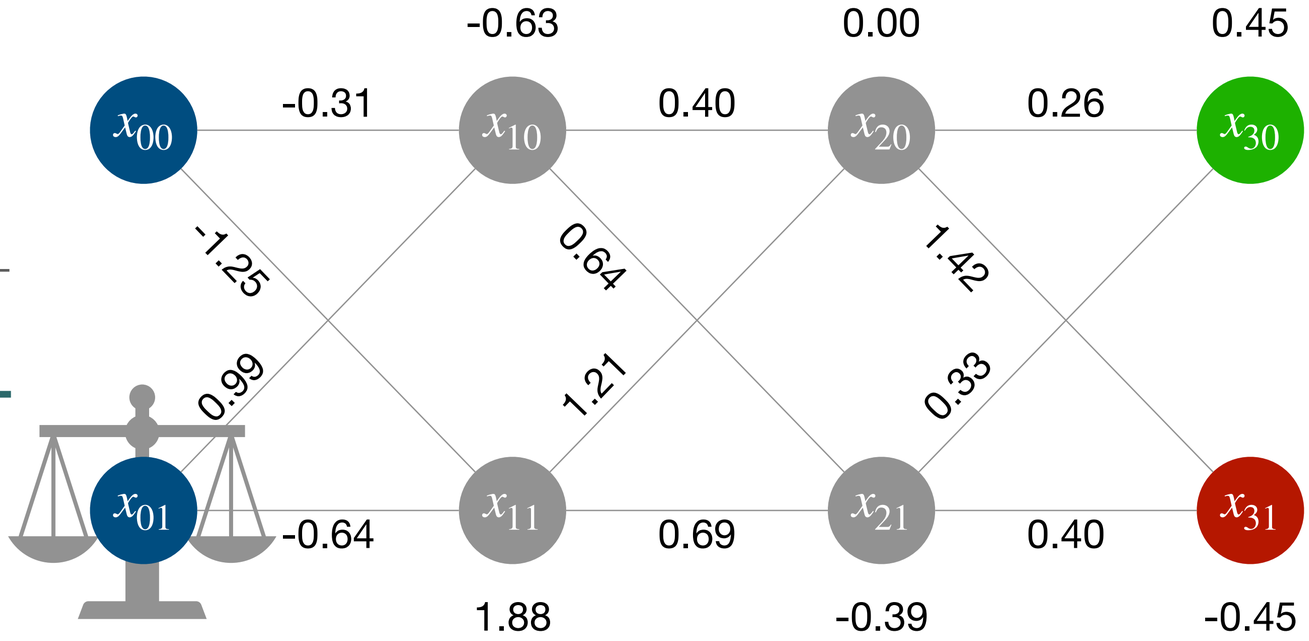


# Forward Analysis

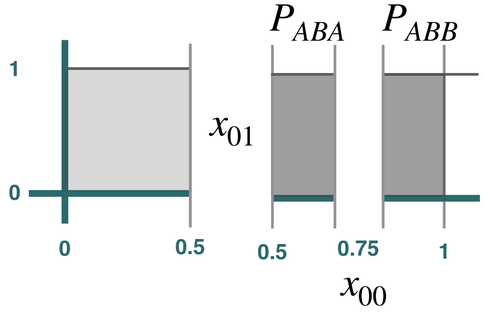
$L = 0.25, U = 2$

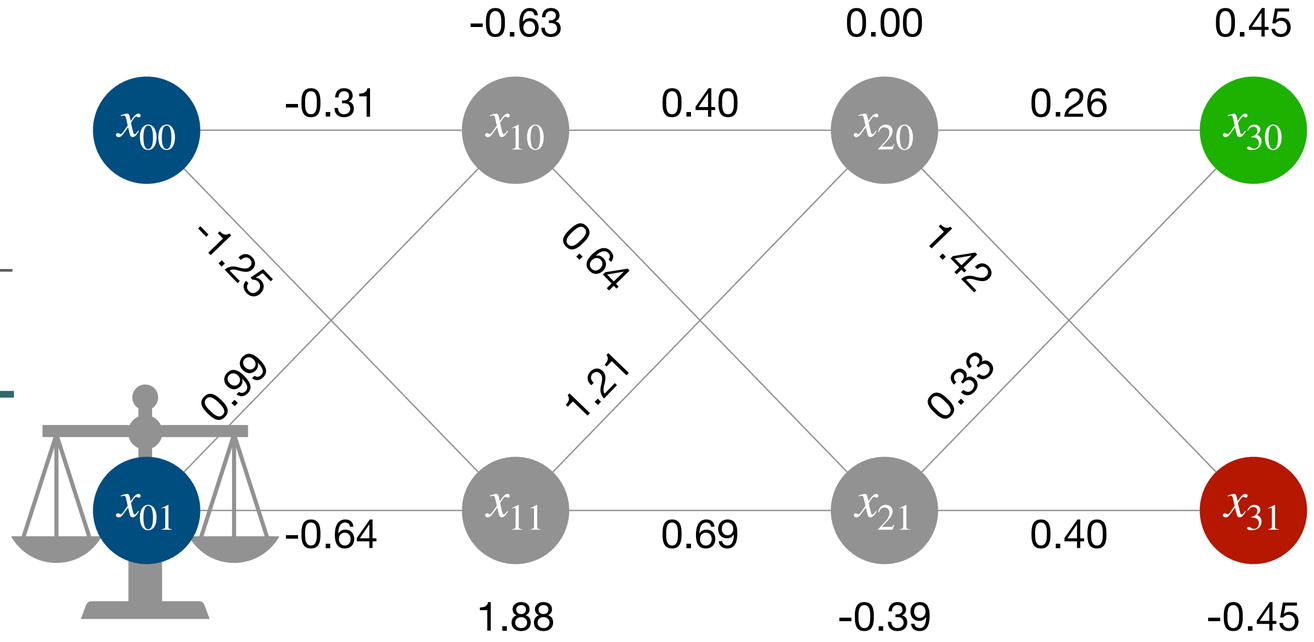


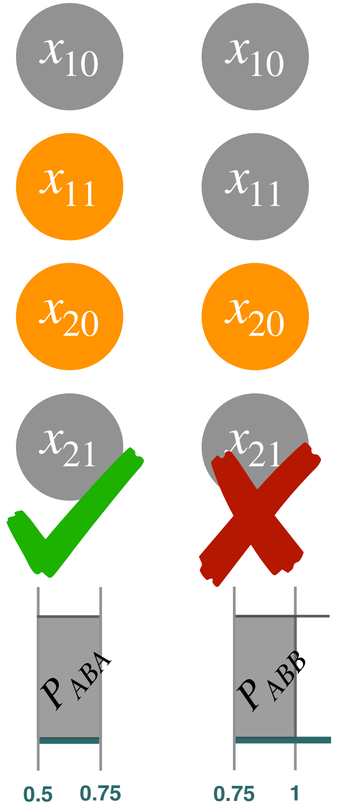




# Forward Analysis

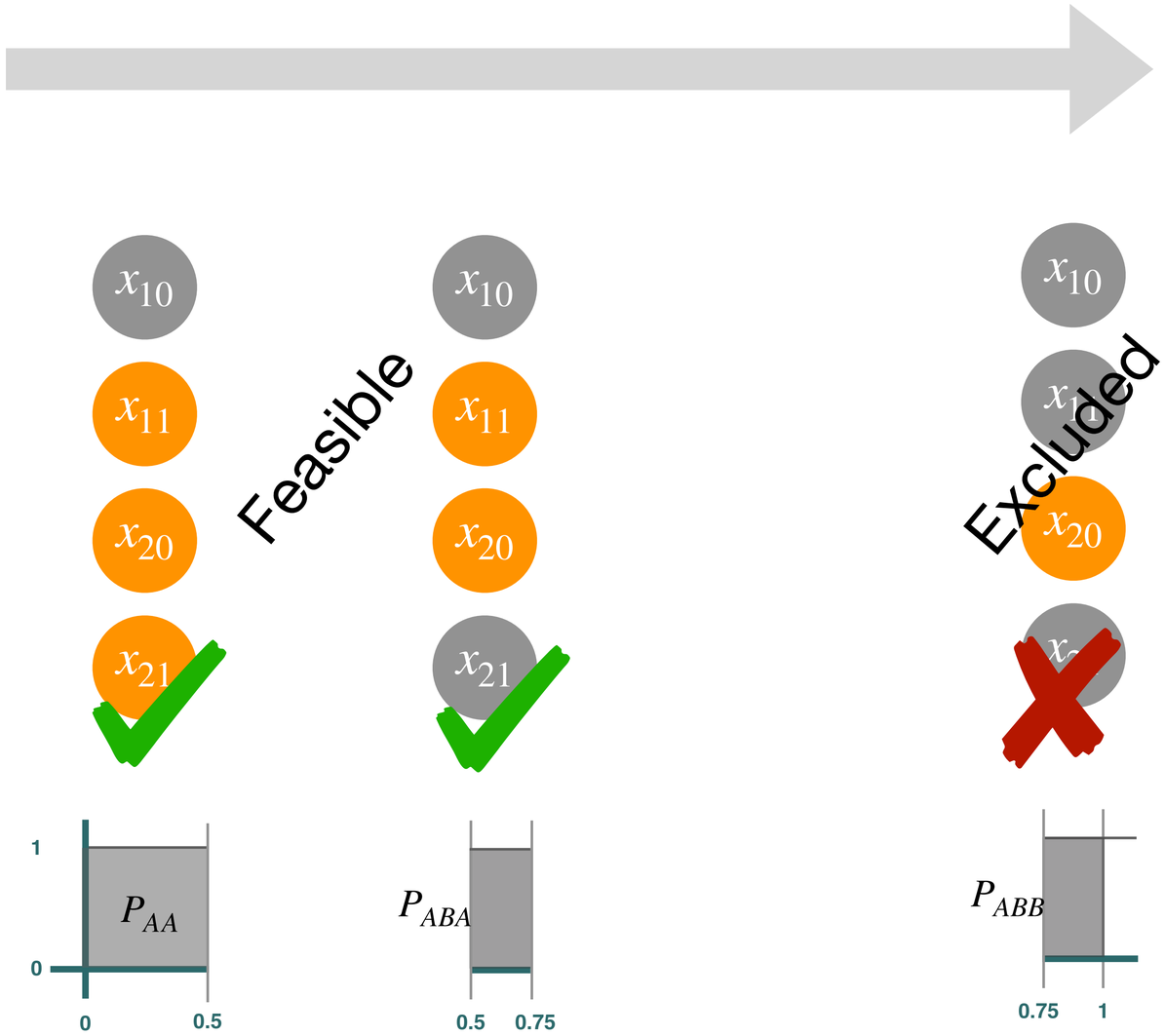
$L = 0.25, U = 2$



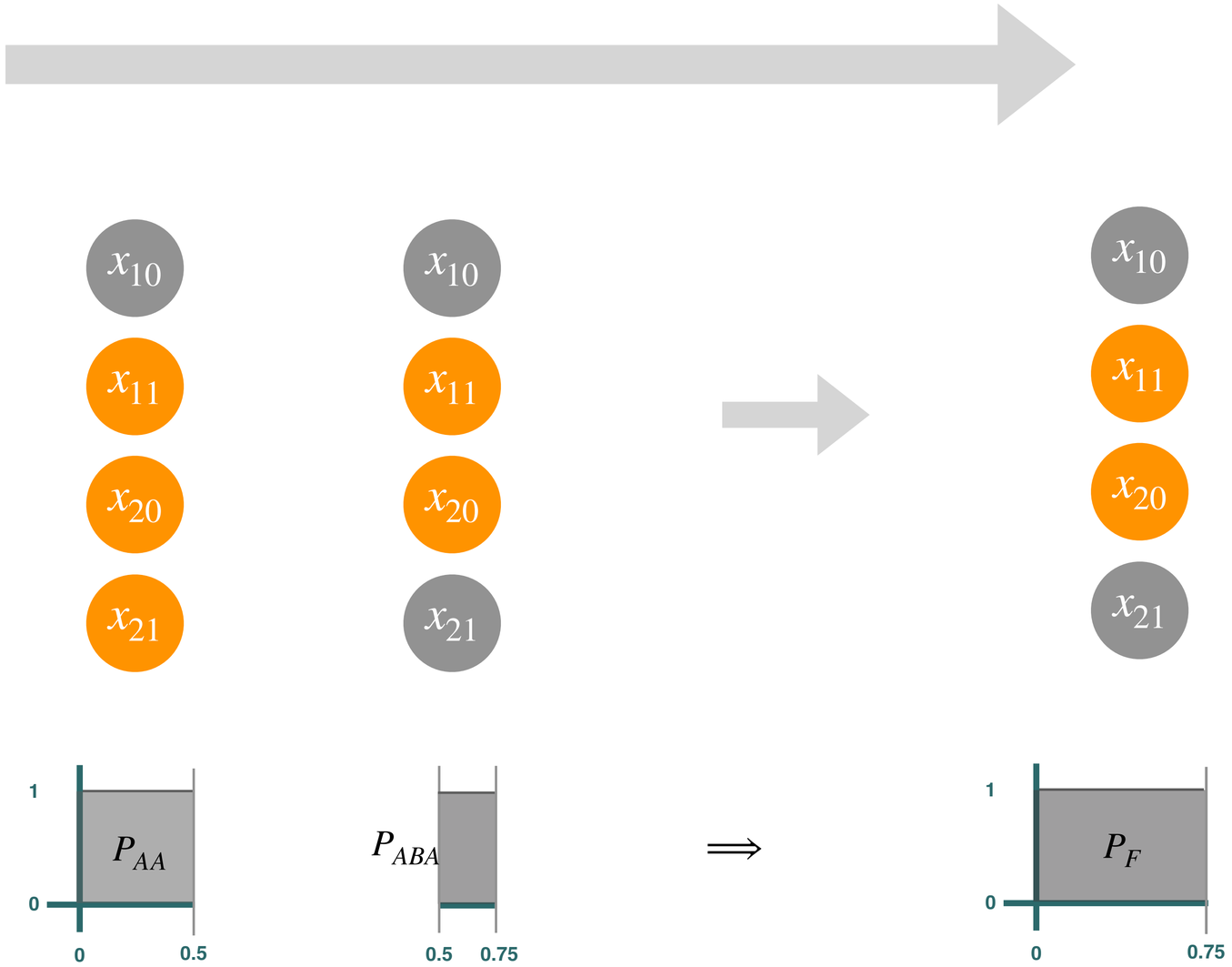



# Forward Analysis



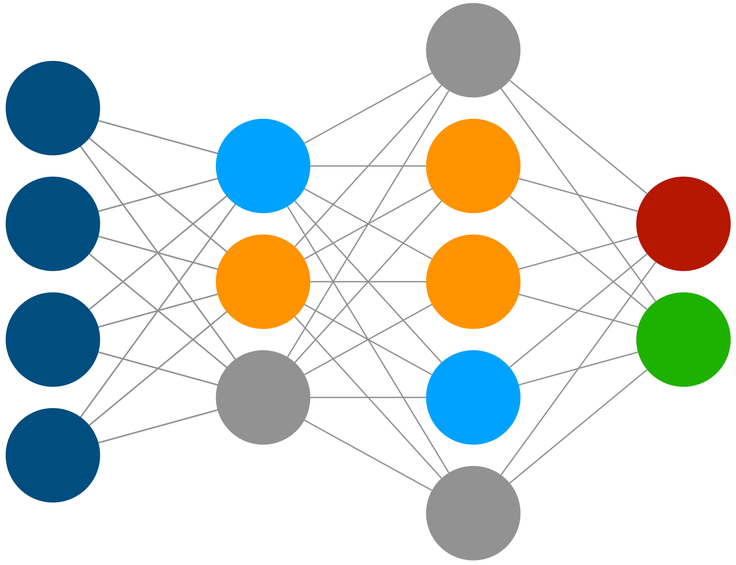
# Forward Analysis

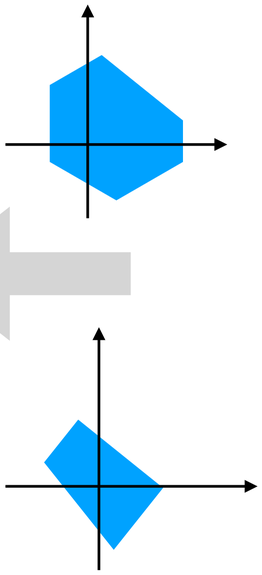


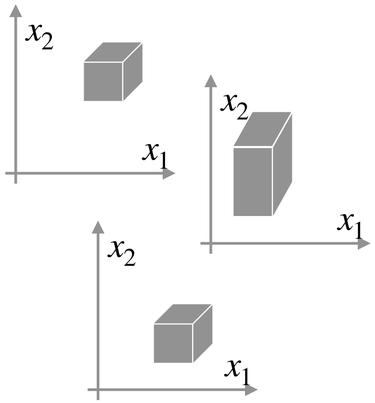
# Proceed **Backwards**
for each **Feasible** partitions
for each possible **Outcome**

Empty Intersection

Otherwise





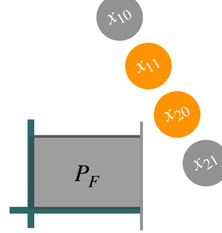



Feasible Partitions

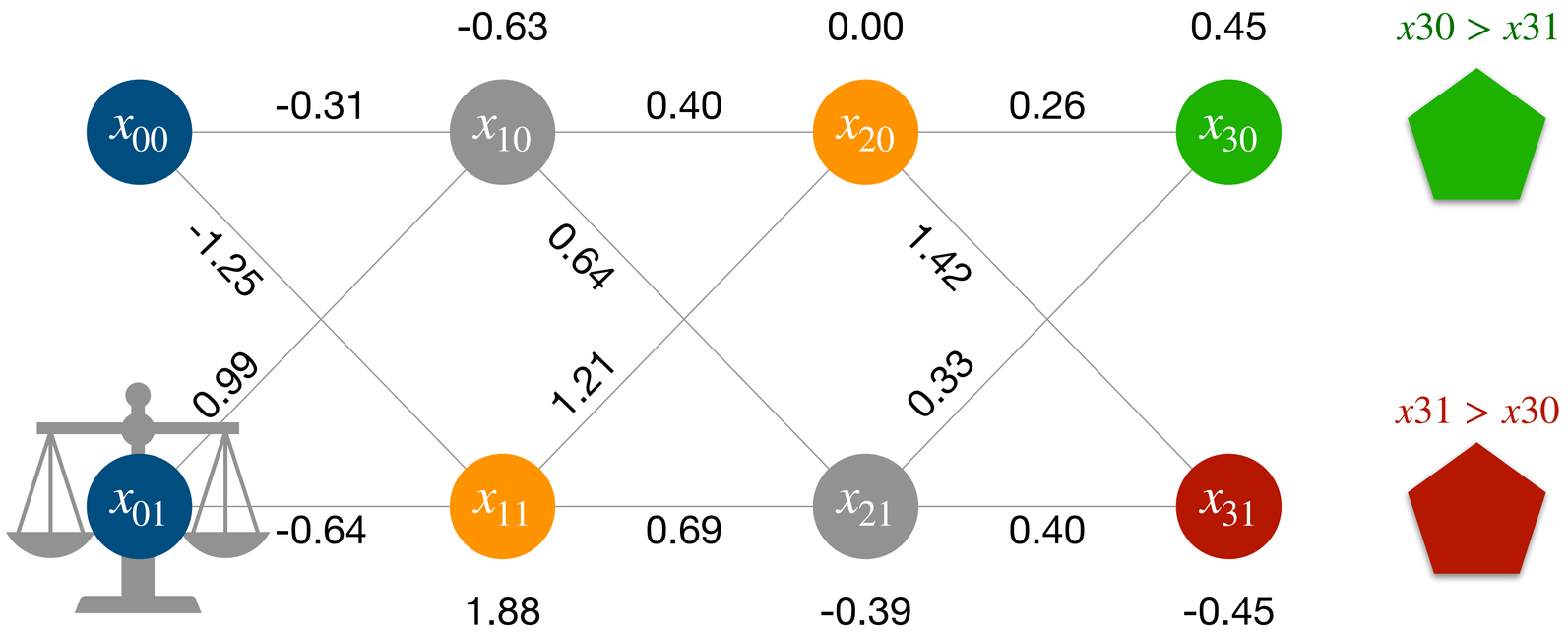
Exact **Backward Analysis**
using **Polyhedra**

# Backward Analysis

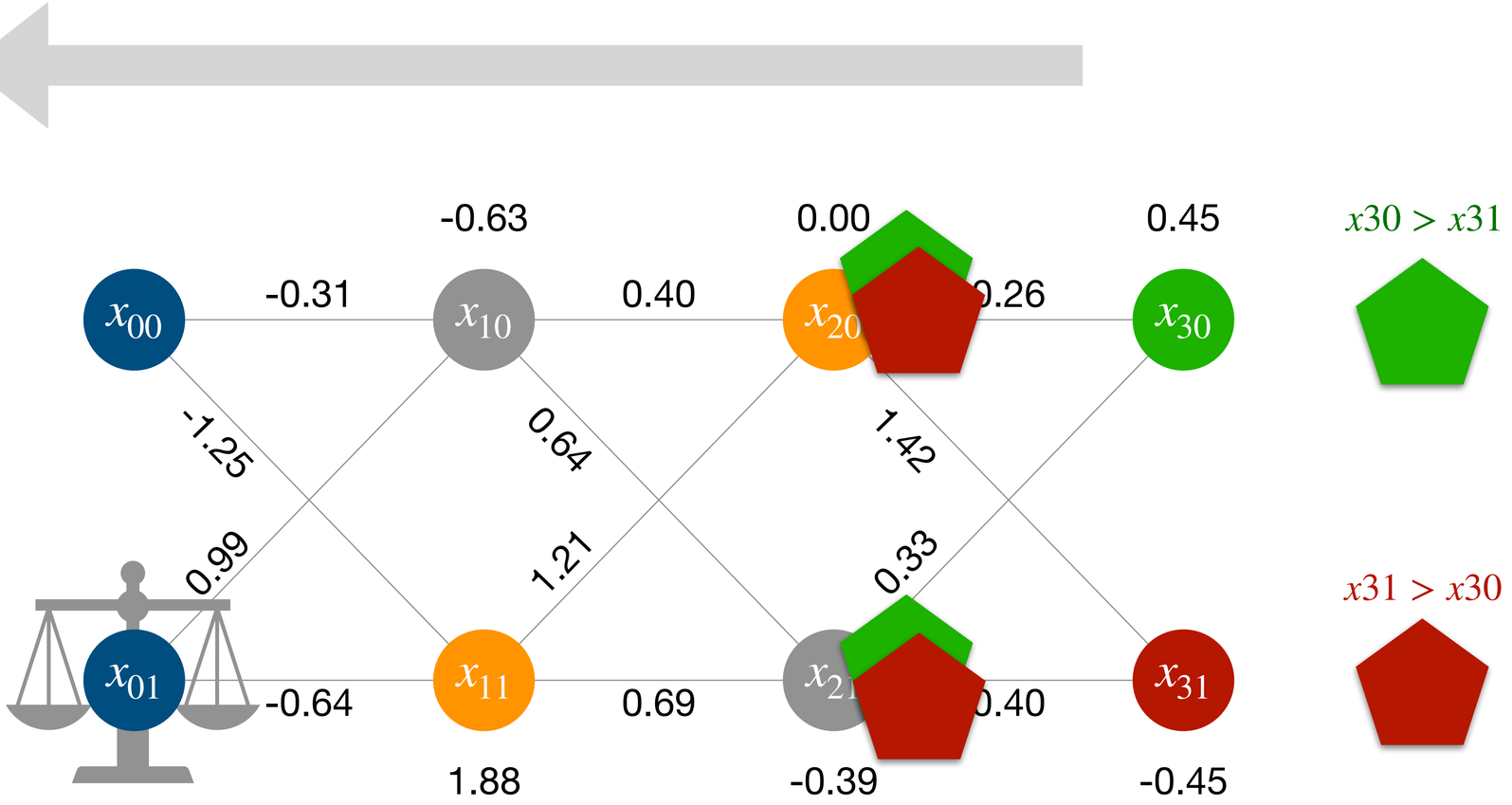
From the forward analysis



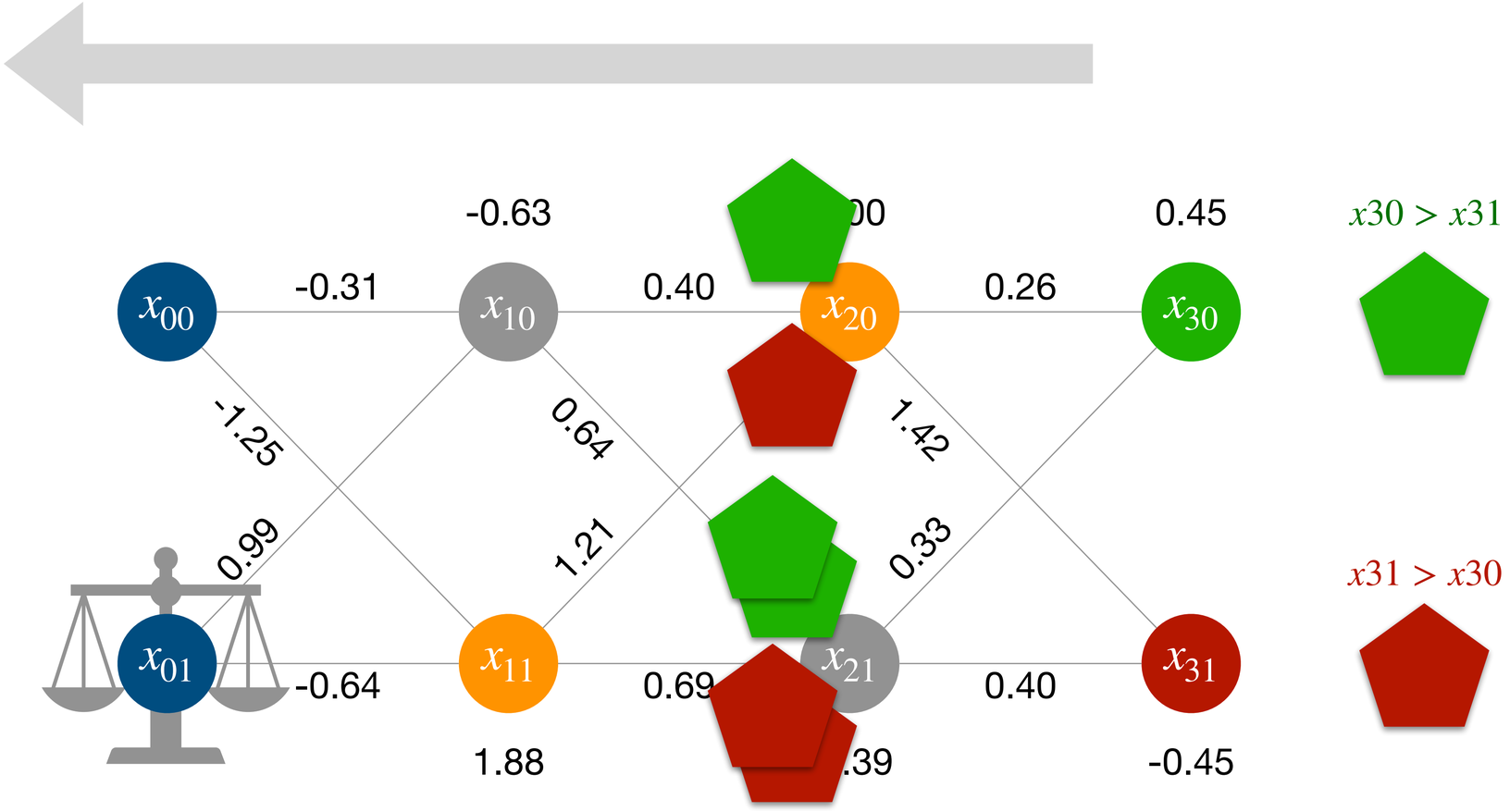


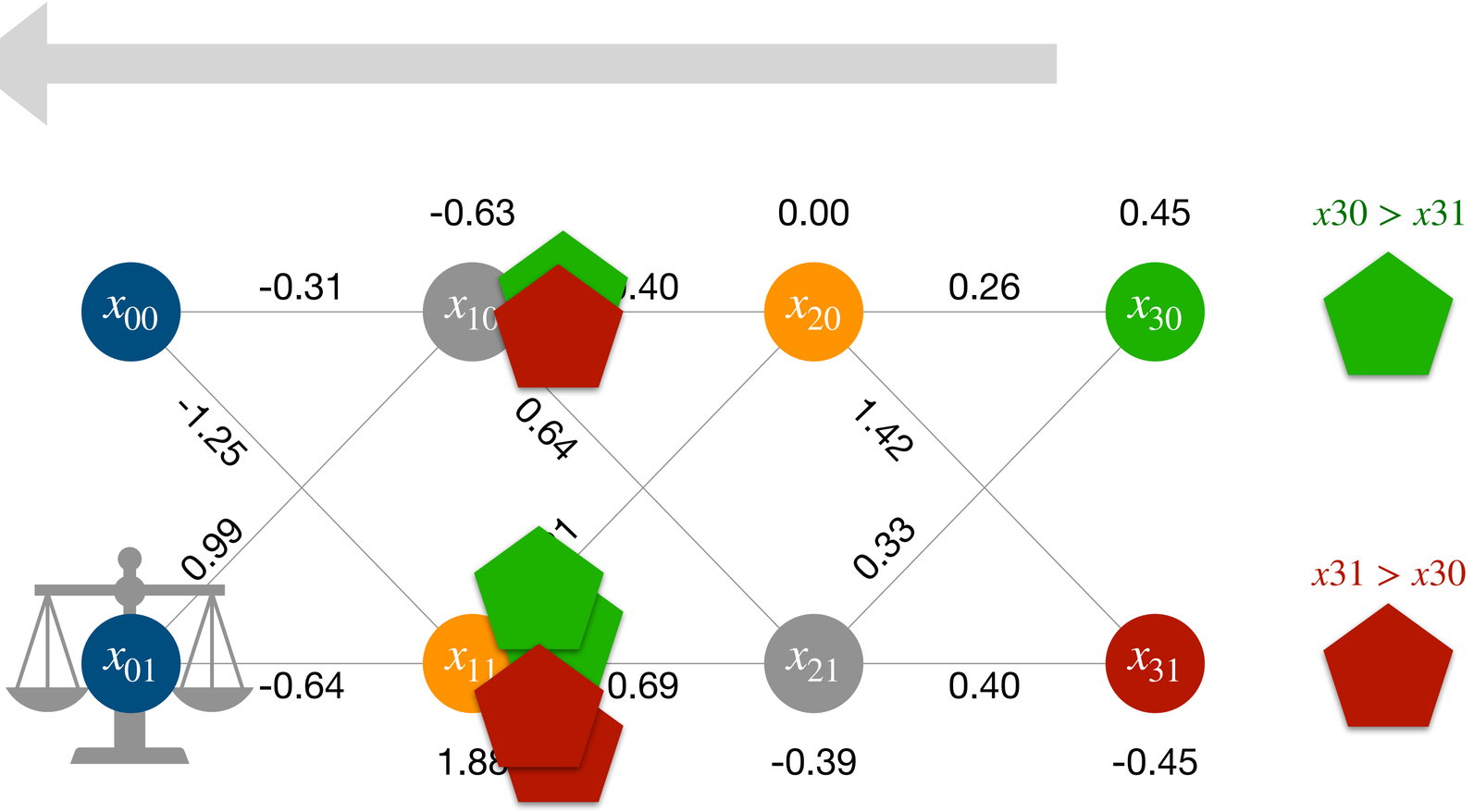
# Backward Analysis



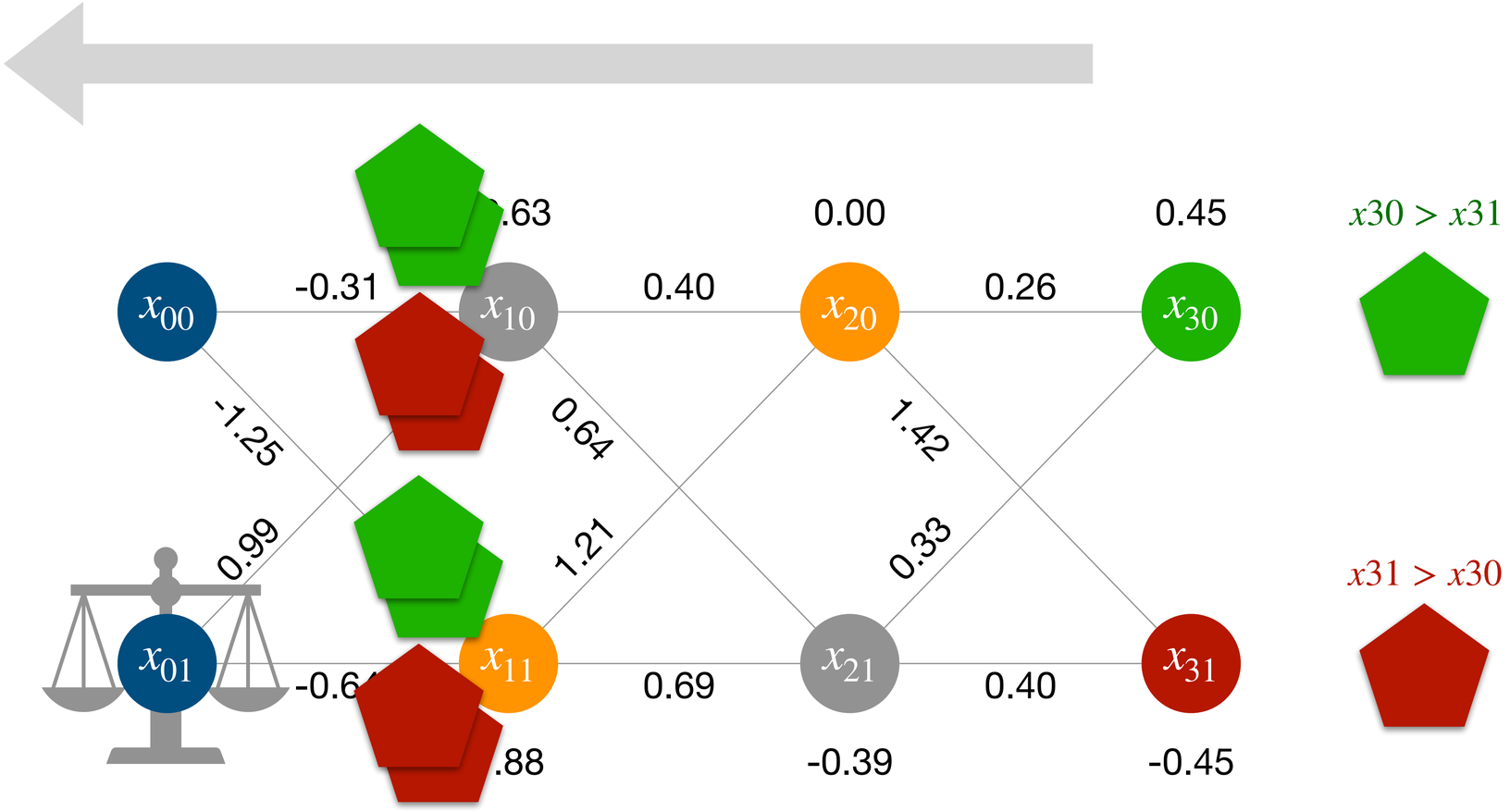
# Backward Analysis



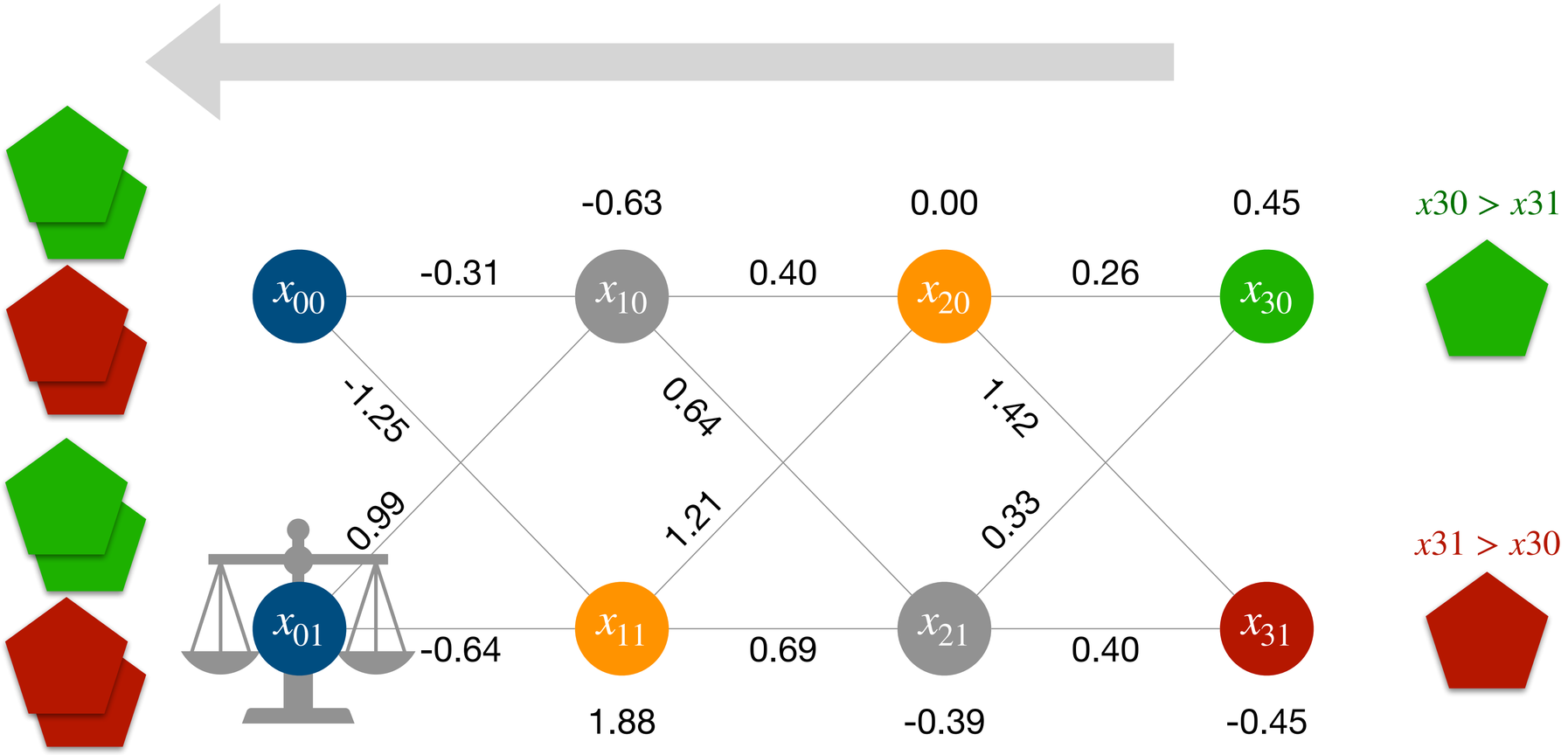
# Backward Analysis



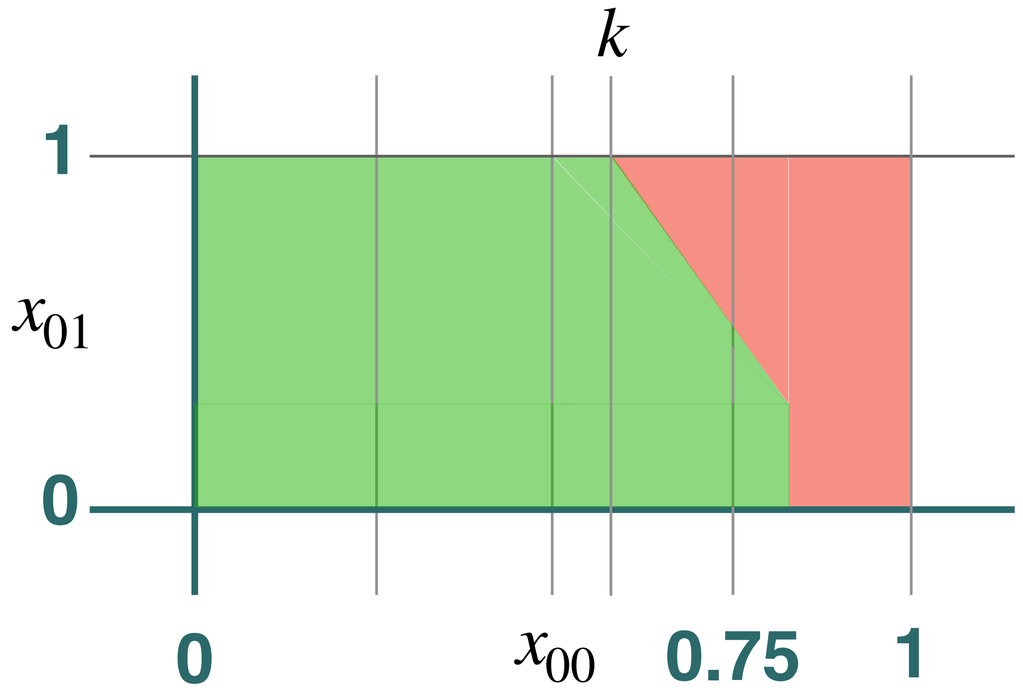
# Backward Analysis



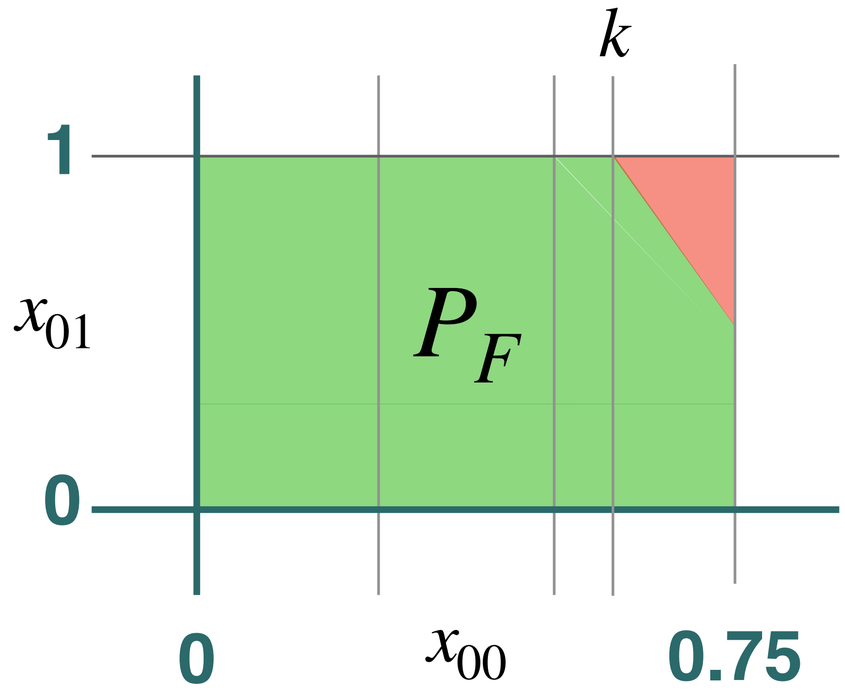
# Backward Analysis



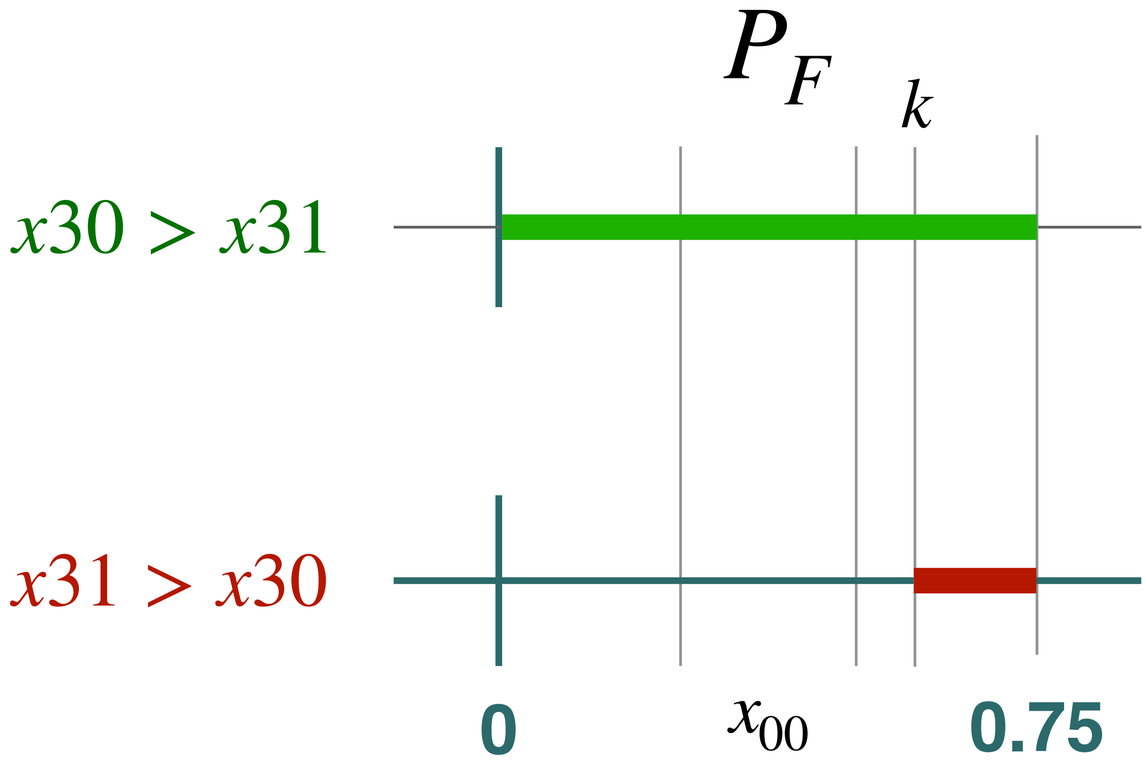
# Analysis Result



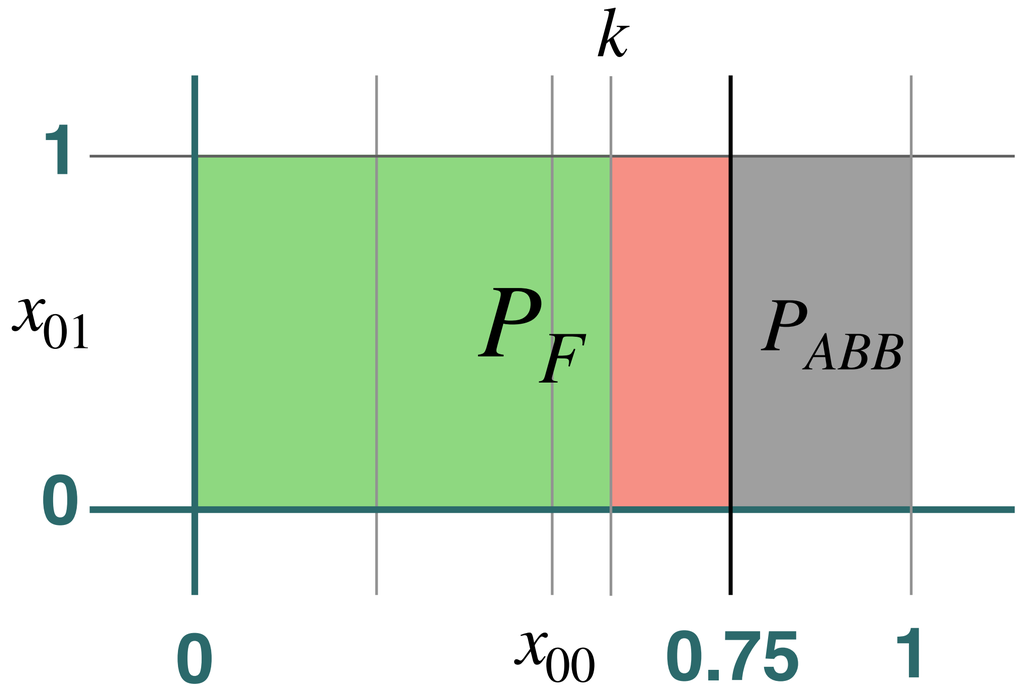
# Analysis Result

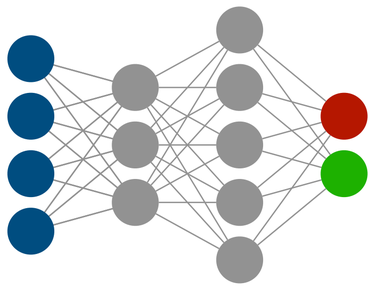


# Analysis Result

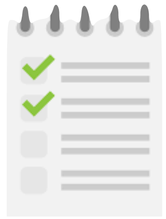


# Analysis Result



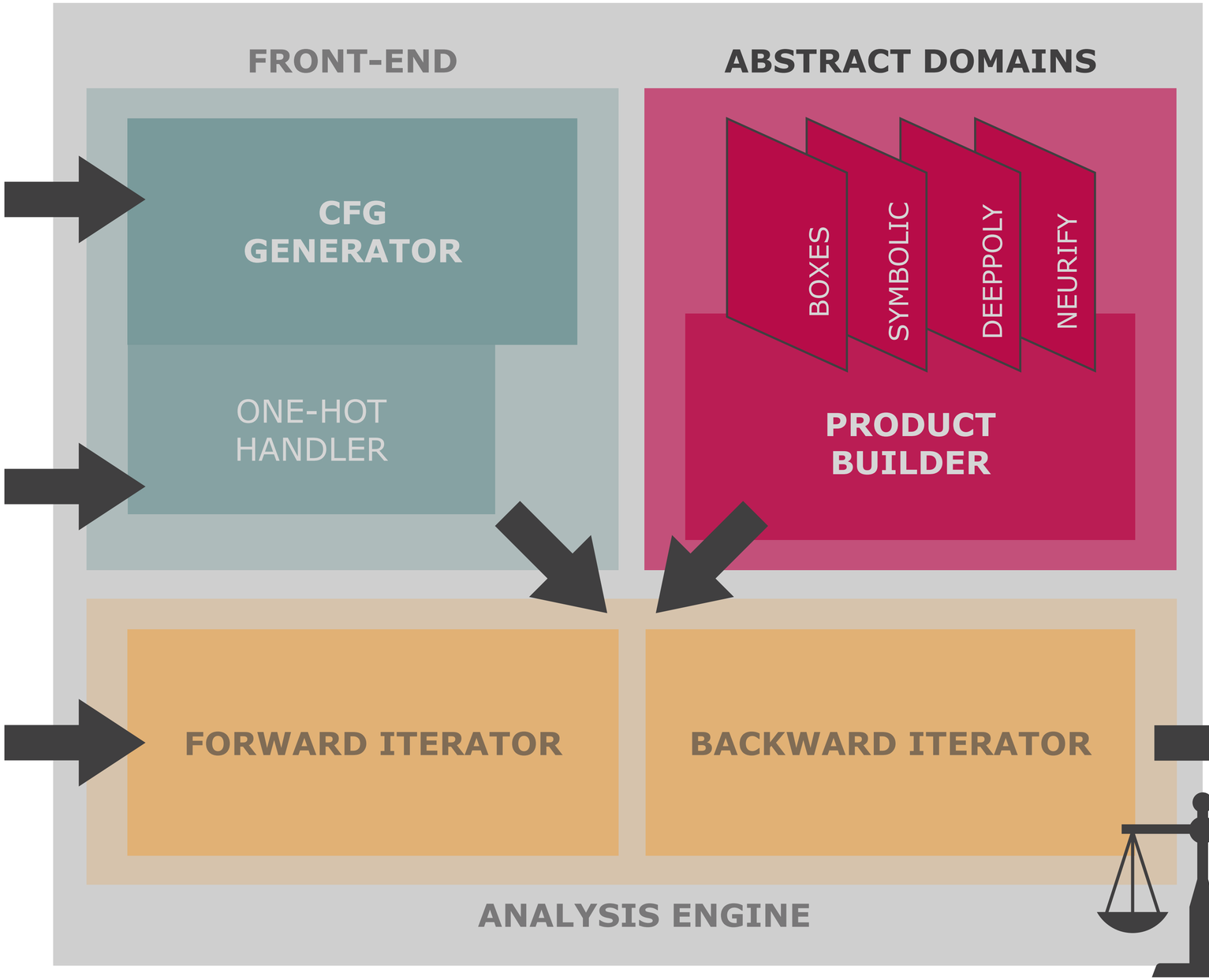


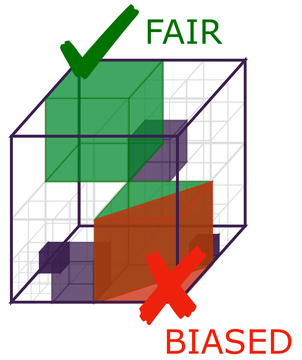
Neural Network

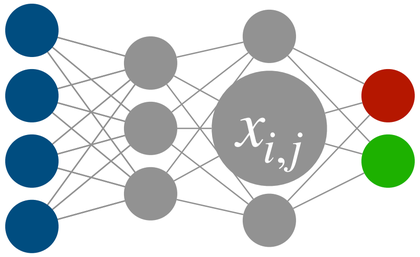


Specification

Configuration
Budget






## Symbolic

Li et al. @ SAS 2019

$$[l, u]$$

$$\sum_k m_k \cdot x_k + q$$

## DeepPoly

Singh et al. @ POPL 2019

$$[l, u]$$

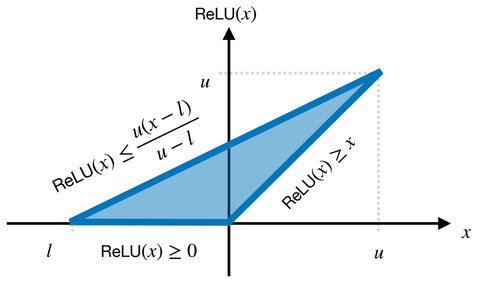
$$[\text{eq}_{\text{low}}, \text{eq}_{\text{up}}]$$

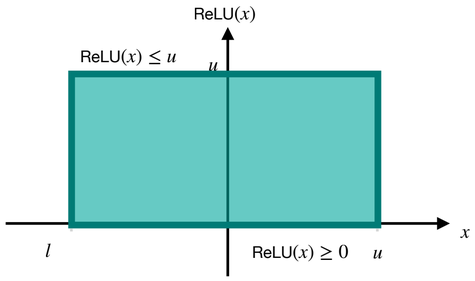
## Neurify

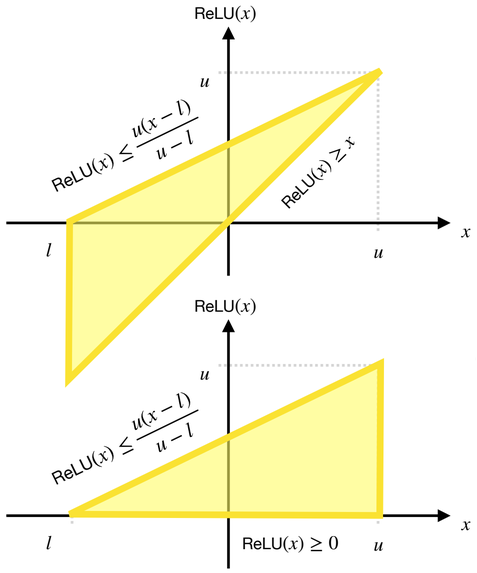
Wang et al. @ NeurIPS 2018

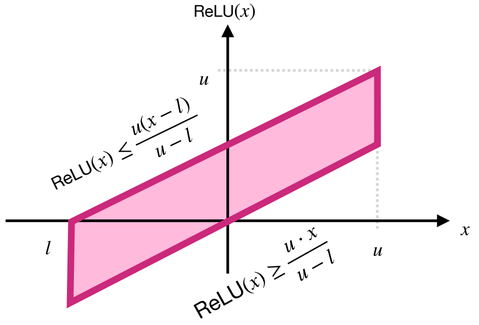
$$[l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}]$$

$$[\text{eq}_{\text{low}}, \text{eq}_{\text{up}}]$$



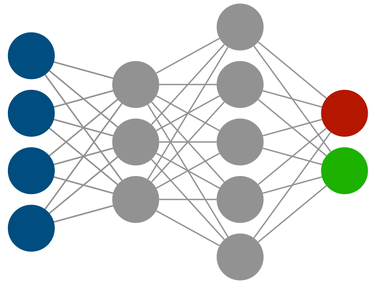




## Reduced Product

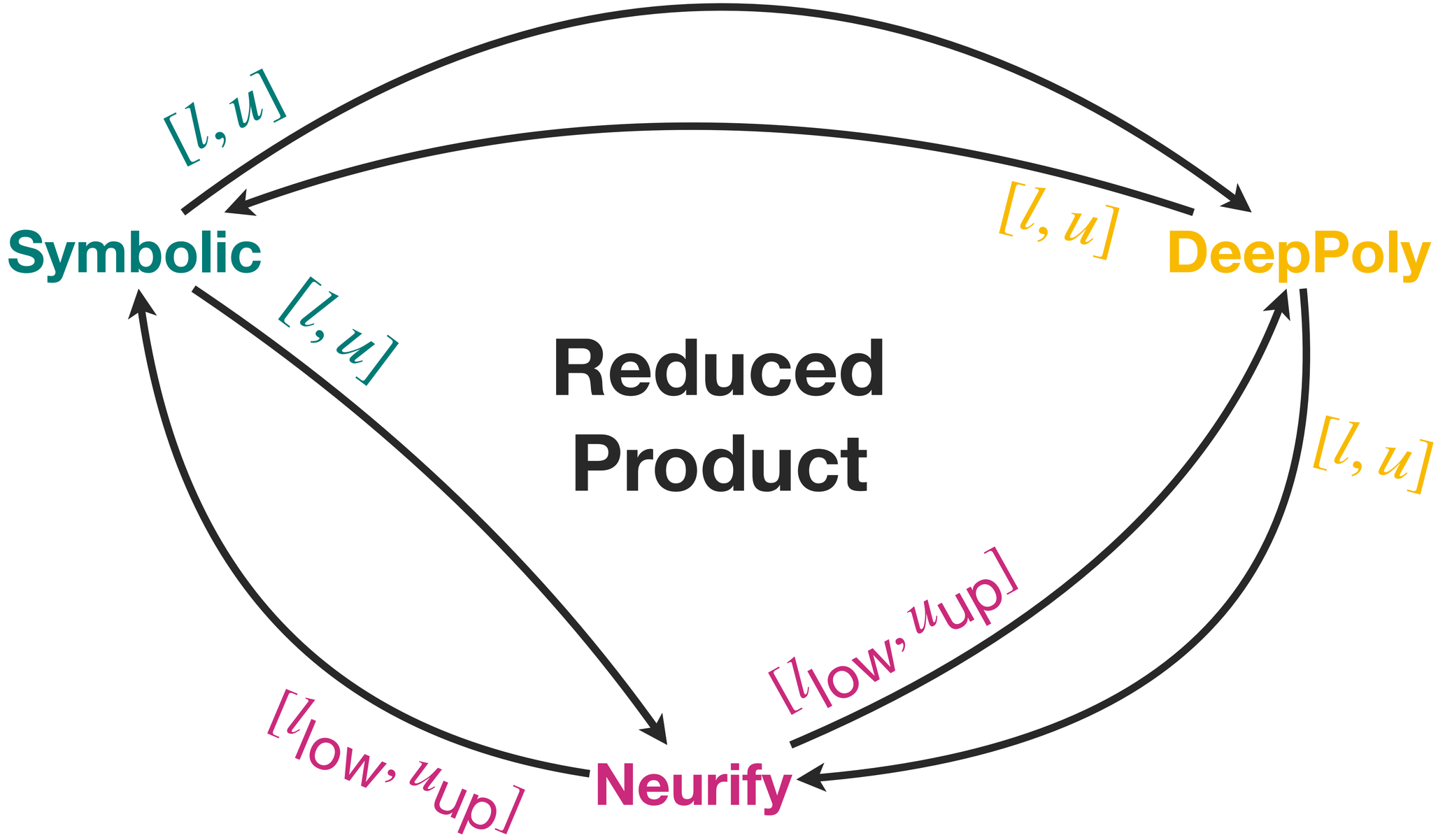
Mazzucato, Urban @ SAS 2021

# Precision-vs-Scalability

| L | U | Symbolic | DeepPoly | Neurify |
|---|---|---|---|---|
| 0.5 | 3 | 48.78% | 49.01% | 46.49% |
| | 5 | 56.11% | 56.15% | 53.06% |
| 0.25 | 3 | 83.63% | 81.82% | 81.40% |
| | 5 | 91.67%% | 91.58% | 92.33% |



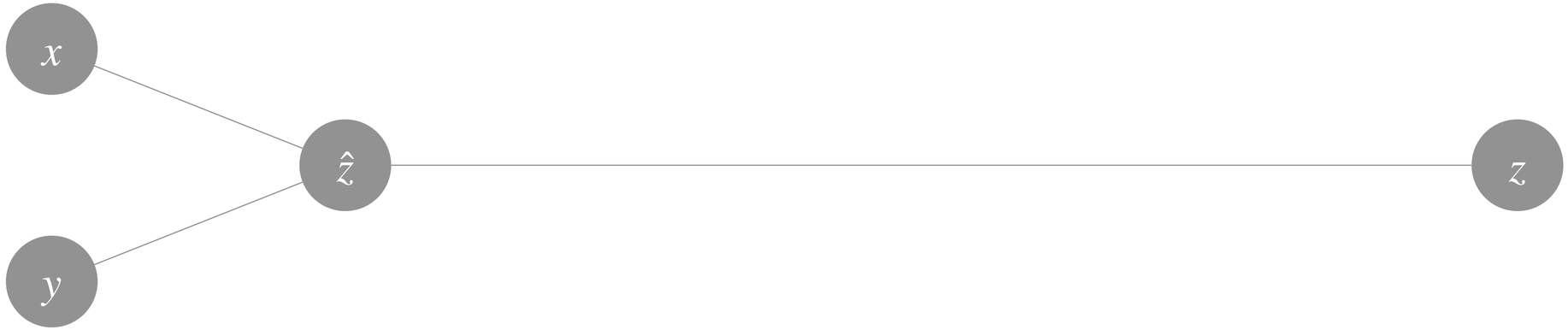
- 4 Hidden Layers
- 5 Neuron per Layer
- 23 inputs $\in [0,1]$
- 2 Output classes


Symbolic
DeepPoly
Neurify
Reduced Product
[l, u]
[l, u]
[l, u]
[l, u]
[l_low, u_up]
[l_low, u_up]

# Reduced Product

Mazzucato, Urban @ SAS 2021

$$x \in \begin{cases} [l_x, u_x] & \textbf{Symbolic} \\ [l_x, u_x] & \textbf{DeepPoly} \\ [l_{x\text{low}}, l_{x\text{up}}, u_{x\text{low}}, u_{x\text{up}}] & \textbf{Neurify} \end{cases}$$



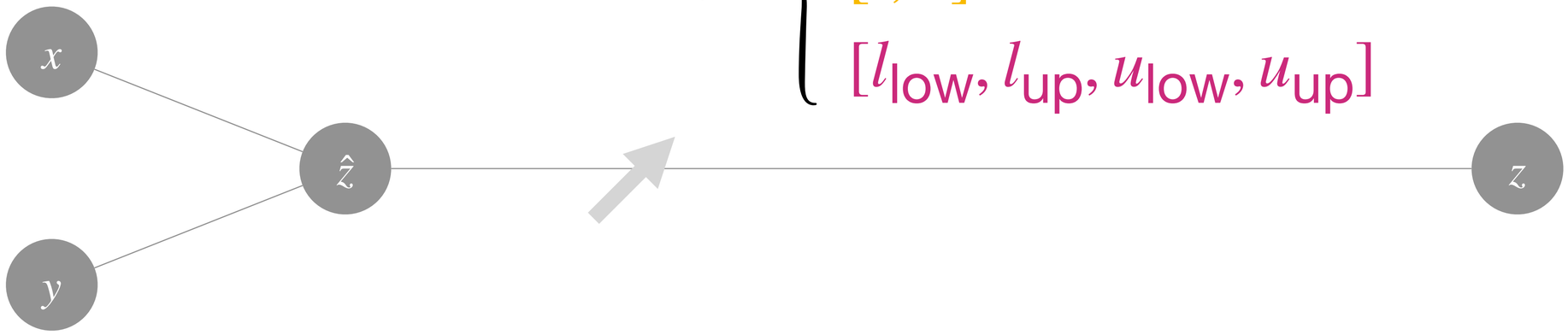

$$y \in \begin{cases} [l_y, u_y] & \textbf{Symbolic} \\ [l_y, u_y] & \textbf{DeepPoly} \\ [l_{y\text{low}}, l_{y\text{up}}, u_{y\text{low}}, u_{y\text{up}}] & \textbf{Neurify} \end{cases}$$

# Reduced Product

Mazzucato, Urban @ SAS 2021

$$x \in \begin{cases} [l_x, u_x] \\ [l_x, u_x] \\ [l_{x\text{low}}, l_{x\text{up}}, u_{x\text{low}}, u_{x\text{up}}] \end{cases}$$

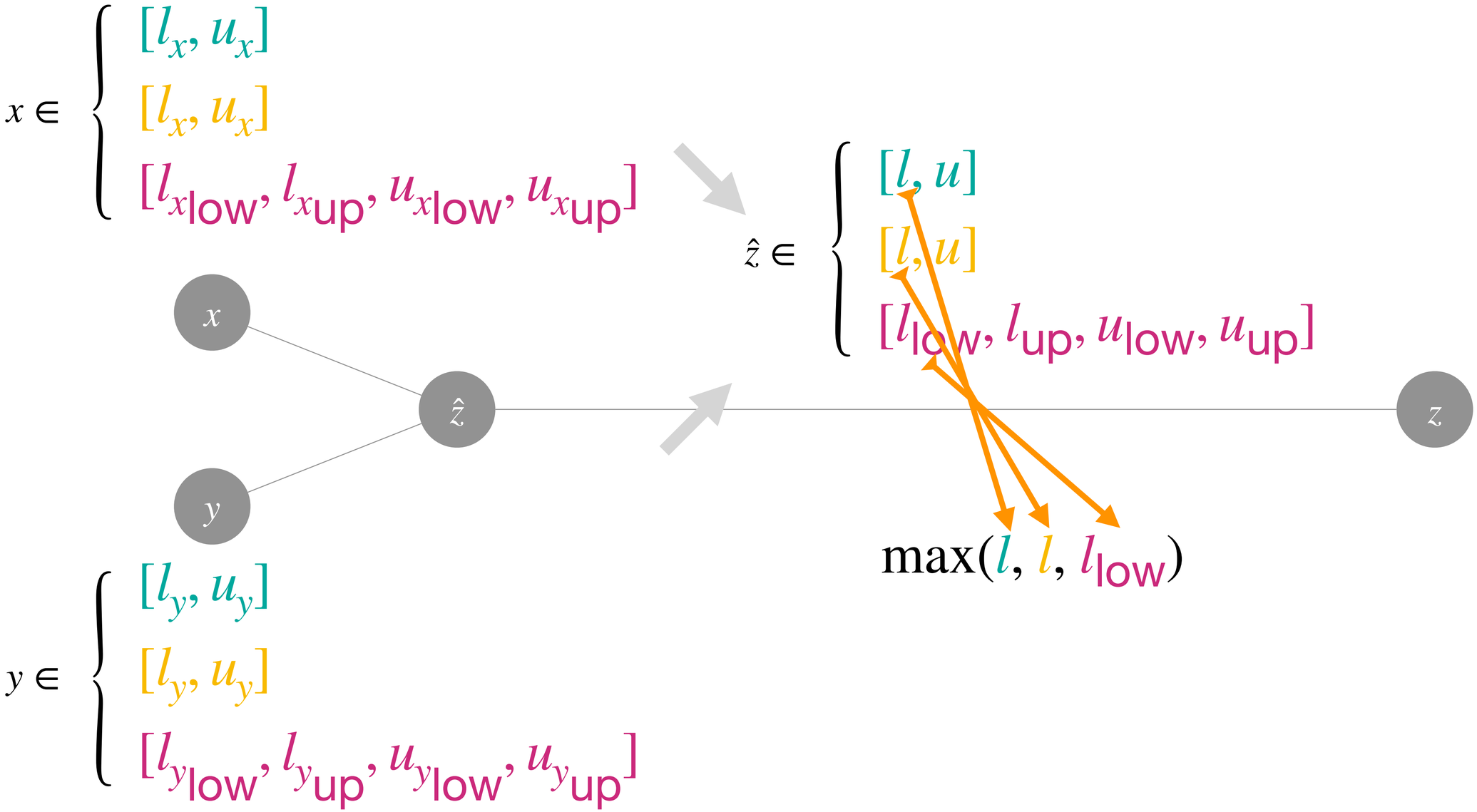
$$\hat{z} \in \begin{cases} [l, u] \\ [l, u] \\ [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}] \end{cases}$$




$$y \in \begin{cases} [l_y, u_y] \\ [l_y, u_y] \\ [l_{y\text{low}}, l_{y\text{up}}, u_{y\text{low}}, u_{y\text{up}}] \end{cases}$$

# Reduced Product

Mazzucato, Urban @ SAS 2021

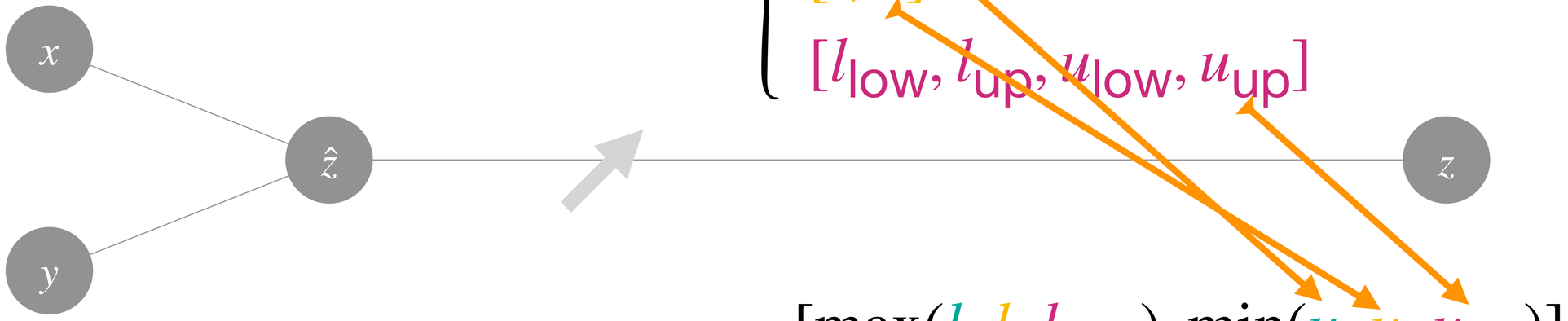


# Reduced Product

Mazzucato, Urban @ SAS 2021

$$x \in \begin{cases} [l_x, u_x] \\ [l_x, u_x] \\ [l_{x\text{low}}, l_{x\text{up}}, u_{x\text{low}}, u_{x\text{up}}] \end{cases}$$

$$\hat{z} \in \begin{cases} [l, u] \\ [l, u] \\ [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}] \end{cases}$$




$$y \in \begin{cases} [l_y, u_y] \\ [l_y, u_y] \\ [l_{y\text{low}}, l_{y\text{up}}, u_{y\text{low}}, u_{y\text{up}}] \end{cases}$$

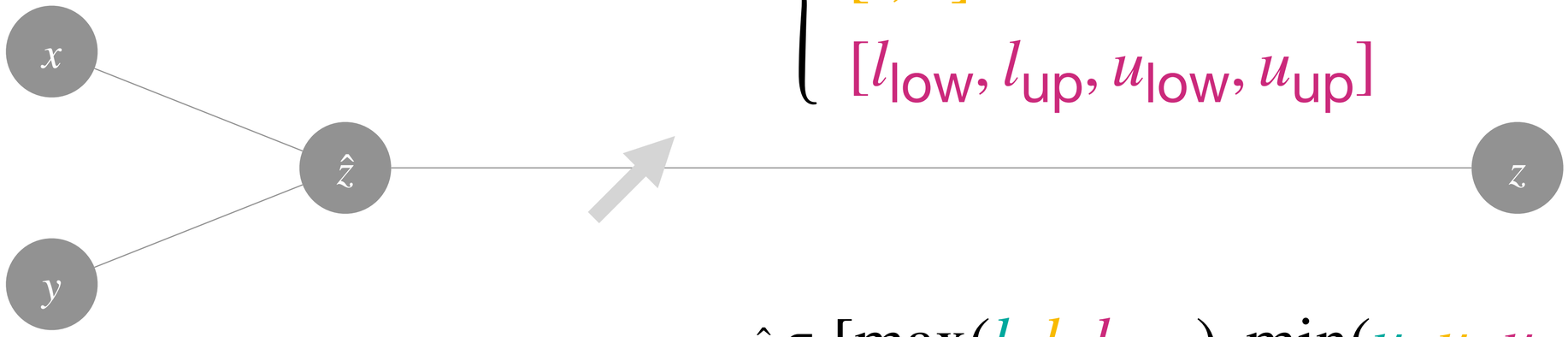
$$[\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})]$$

# Reduced Product

Mazzucato, Urban @ SAS 2021

$$x \in \begin{cases} [l_x, u_x] \\ [l_x, u_x] \\ [l_{x\text{low}}, l_{x\text{up}}, u_{x\text{low}}, u_{x\text{up}}] \end{cases}$$

$$\hat{z} \in \begin{cases} [l, u] \\ [l, u] \\ [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}] \end{cases}$$




$$y \in \begin{cases} [l_y, u_y] \\ [l_y, u_y] \\ [l_{y\text{low}}, l_{y\text{up}}, u_{y\text{low}}, u_{y\text{up}}] \end{cases}$$

$$\hat{z} \in [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})]$$

# Reduced Product

Mazzucato, Urban @ SAS 2021

$$x \in \begin{cases} [l_x, u_x] \\ [l_x, u_x] \\ [l_{x\text{low}}, l_{x\text{up}}, u_{x\text{low}}, u_{x\text{up}}] \end{cases}$$

$$y \in \begin{cases} [l_y, u_y] \\ [l_y, u_y] \\ [l_{y\text{low}}, l_{y\text{up}}, u_{y\text{low}}, u_{y\text{up}}] \end{cases}$$


$$\hat{z} \in \begin{cases} [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})] \end{cases}$$

**Symbolic**

$$\hat{z} \in [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})]$$

# Reduced Product

Mazzucato, Urban @ SAS 2021

$$x \in \begin{cases} [l_x, u_x] \\ [l_x, u_x] \\ [l_{x\text{low}}, l_{x\text{up}}, u_{x\text{low}}, u_{x\text{up}}] \end{cases}$$

$$\hat{z} \in \begin{cases} [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})] \\ [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})] \end{cases}$$

DeepPoly


$$\hat{z} \in [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})]$$

$$y \in \begin{cases} [l_y, u_y] \\ [l_y, u_y] \\ [l_{y\text{low}}, l_{y\text{up}}, u_{y\text{low}}, u_{y\text{up}}] \end{cases}$$

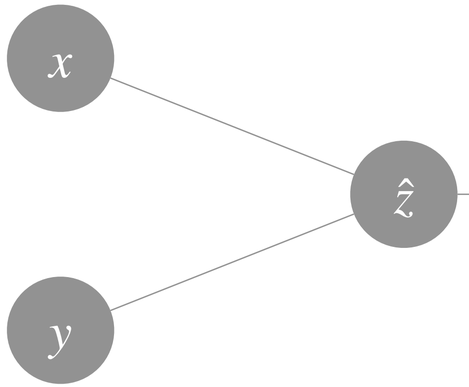
# Reduced Product

Mazzucato, Urban @ SAS 2021

$$x \in \begin{cases} [l_x, u_x] \\ [l_x, u_x] \\ [l_{x\text{low}}, l_{x\text{up}}, u_{x\text{low}}, u_{x\text{up}}] \end{cases}$$

$$\hat{z} \in \begin{cases} [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})] \\ [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})] \\ [\max(l, l, l_{\text{low}}), \ldots, \ldots, \min(u, u, u_{\text{up}})] \end{cases}$$

**Neurify**



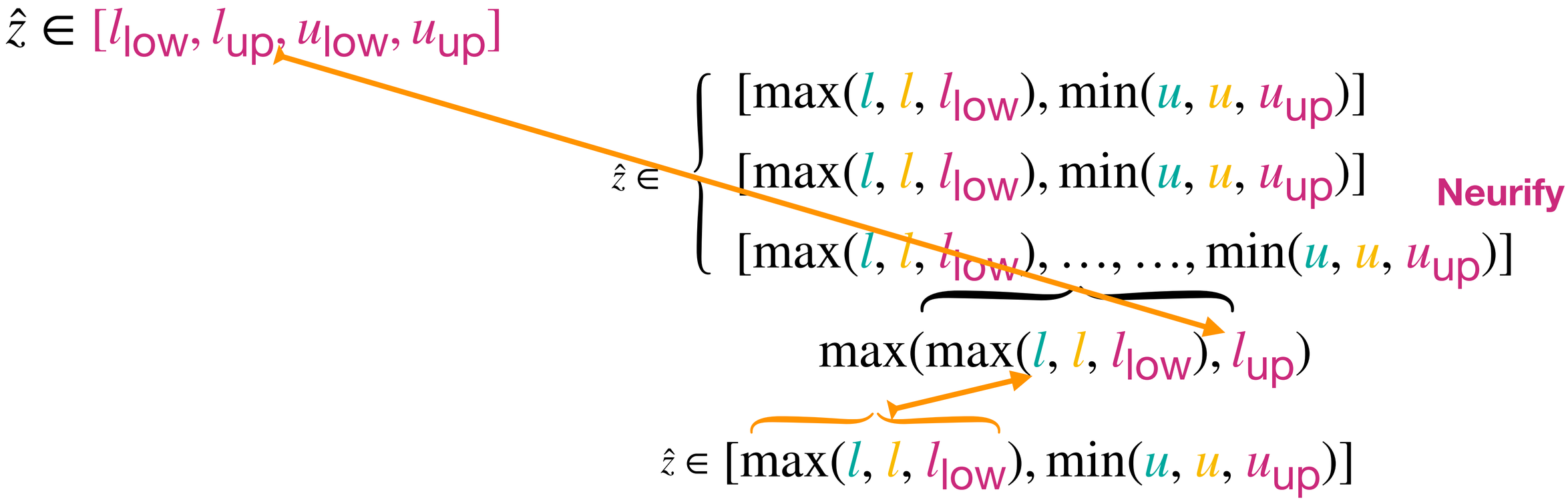


$$\hat{z} \in [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})]$$

$$y \in \begin{cases} [l_y, u_y] \\ [l_y, u_y] \\ [l_{y\text{low}}, l_{y\text{up}}, u_{y\text{low}}, u_{y\text{up}}] \end{cases}$$

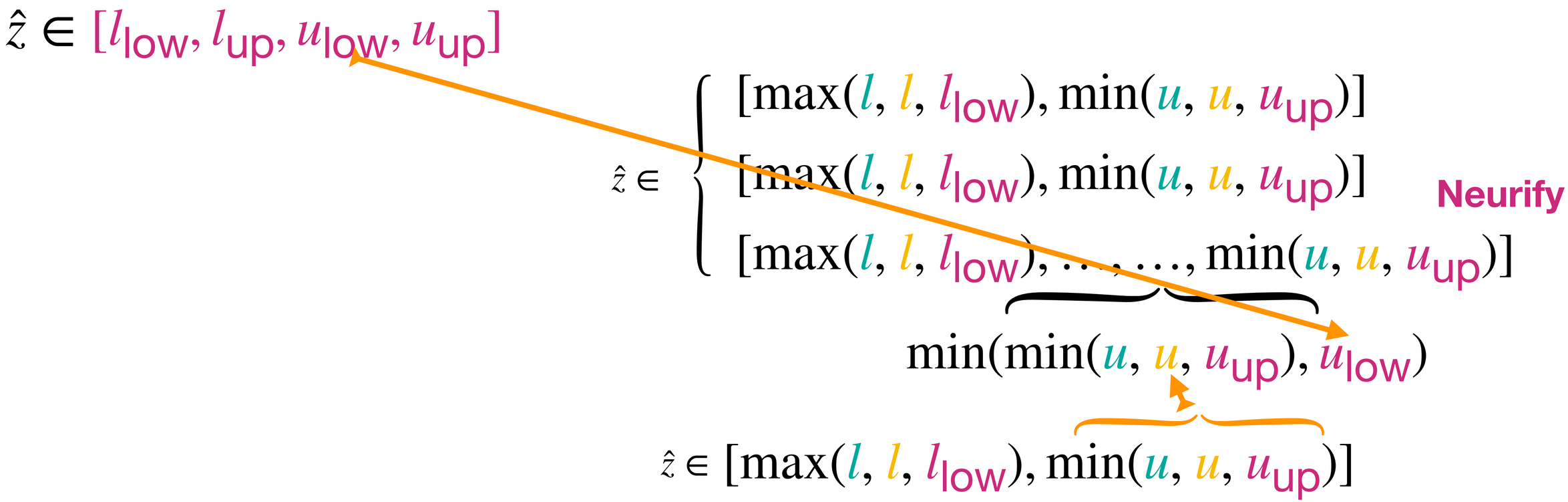
# Reduced Product

Mazzucato, Urban @ SAS 2021



# Reduced Product

Mazzucato, Urban @ SAS 2021



# Reduced Product

Mazzucato, Urban @ SAS 2021

$$\hat{z} \in \begin{cases} [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})] \\ [\max(l, l, l_{\text{low}}), \min(u, u, u_{\text{up}})] \\ [\max(l, l, l_{\text{low}}), \max(\max(l, l, l_{\text{low}}), l_{\text{up}}), \\ \quad \min(\min(u, u, u_{\text{up}}), u_{\text{low}}), \min(u, u, u_{\text{up}})] \end{cases}$$

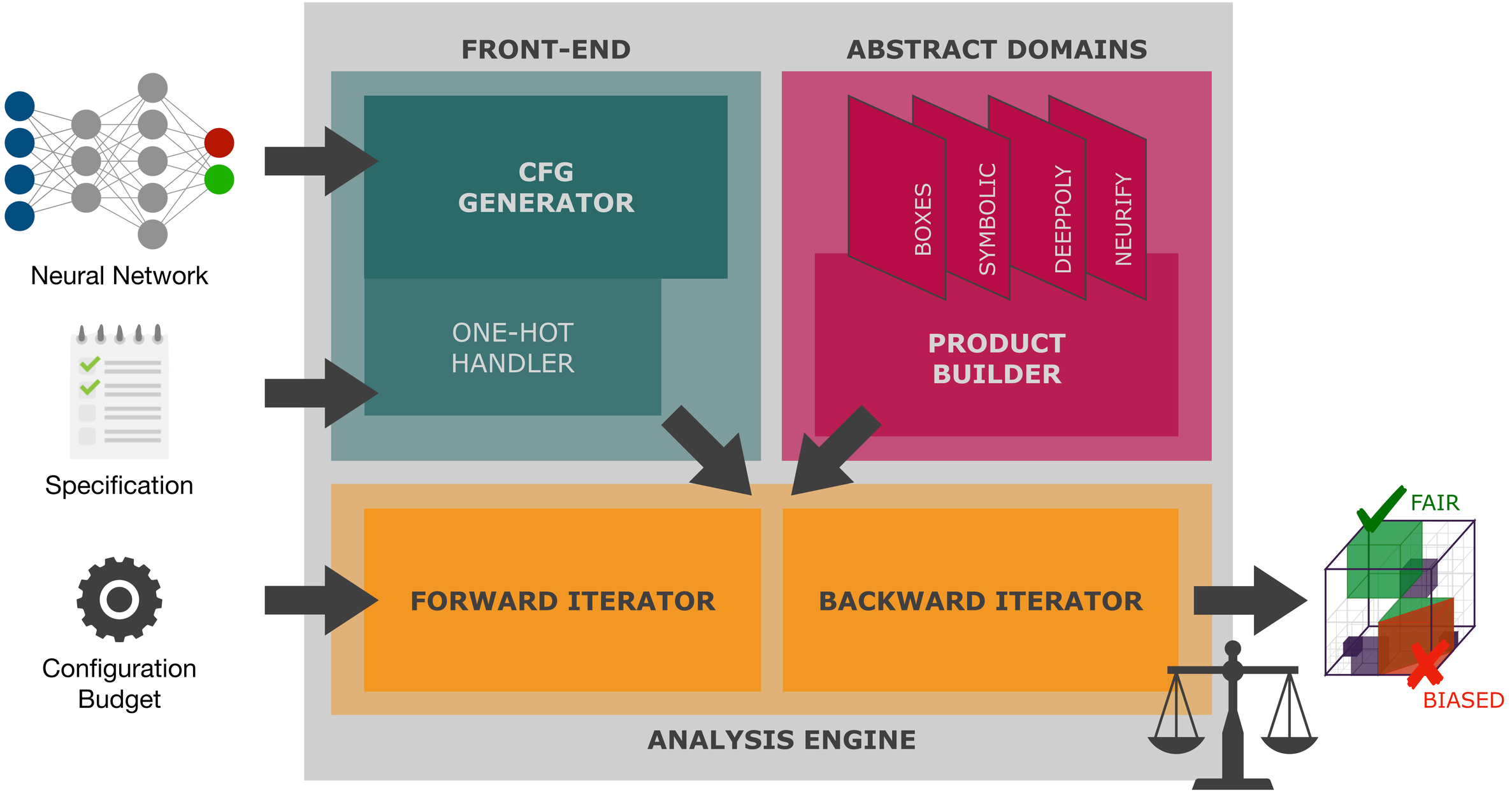
**Symbolic**

**DeepPoly**

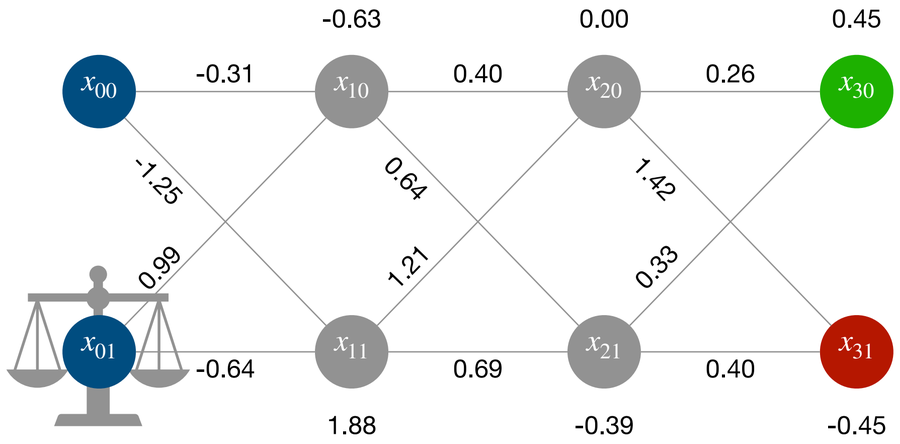
**Neurify**

# Precision-vs-Scalability

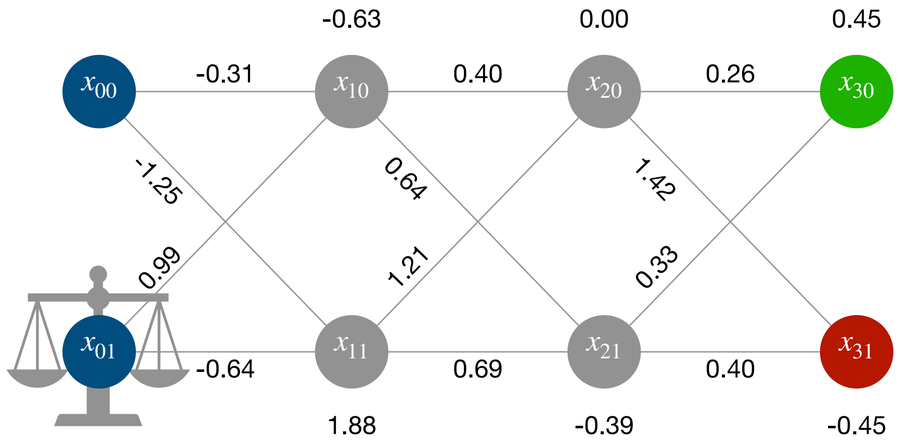
| L | U | Symbolic | DeepPoly | Neurify | Product |  |
|---|---|---|---|---|---|---|
| 0.5 | 3 | 48.78% | 49.01% | 46.49% | 59.20% | +10.3% |
| | 5 | 56.11% | 56.15% | 53.06% | 68.23% | +11.9% |
| 0.25 | 3 | 83.63% | 81.82% | 81.40% | 87.04% | +3.4% |
| | 5 | 91.67%% | 91.58% | 92.33% | 95.48% | +3.2% |


FRONT-END
ABSTRACT DOMAINS
CFG GENERATOR
ONE-HOT HANDLER
BOXES
SYMBOLIC
DEEPPOLY
NEURIFY
PRODUCT BUILDER
Neural Network
Specification
Configuration Budget
FORWARD ITERATOR
BACKWARD ITERATOR
ANALYSIS ENGINE
FAIR
BIASED

# Ongoing work

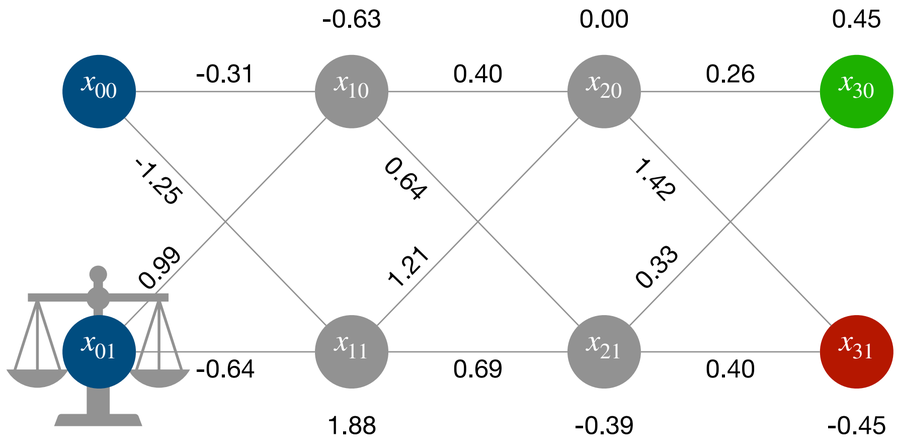


# Ongoing work




```
 1| x00 = input()
 2| x01 = input()
 3|
 4| x10 = ReLU(-0.31*x00 +0.99*x01 -0.63)
 5| x11 = ReLU(-1.25*x00 -0.64*x01 +1.88)
 6|
 7| x20 = ReLU(0.40*x10 +1.21*x11)
 8| x21 = ReLU(0.64*x10 +0.69*x11 -0.39)
 9|
10| x30 = 0.26*x20 +0.33*x21 +0.45
11| x31 = 1.42*x20 +0.40*x21 -0.45
```

# Ongoing work



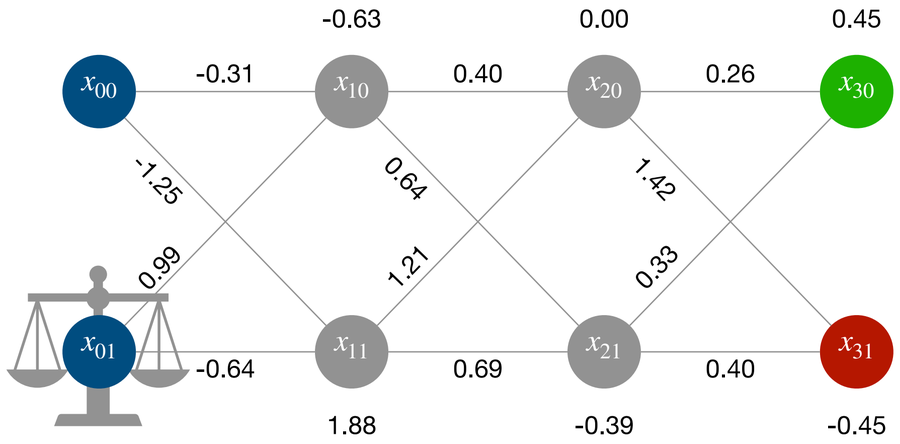

**Fair**

## Input Data Usage Framework

**[Urban18]**

```
 1| x00 = input()
 2| x01 = input() Unused
 3|
 4| x10 = ReLU(-0.31*x00 +0.99*x01 -0.63)
 5| x11 = ReLU(-1.25*x00 -0.64*x01 +1.88)
 6|
 7| x20 = ReLU(0.40*x10 +1.21*x11)
 8| x21 = ReLU(0.64*x10 +0.69*x11 -0.39)
 9|
10| x30 = 0.26*x20 +0.33*x21 +0.45
11| x31 = 1.42*x20 +0.40*x21 -0.45
```

# Ongoing work




**Fair**

## Input Data Usage Framework

[Urban18]

```
 1| x00 = input()
 2| x01 = input()  Unused
 3|
 4| x10 = ReLU(-0.31*x00 +0.99*x01 -0.63)
 5| x11 = ReLU(-1.25*x00 -0.64*x01 +1.88)
 6|
 7| x20 = ReLU(0.40*x10 +1.21*x11)
 8| x21 = ReLU(0.64*x10 +0.69*x11 -0.39)
 9|
10| x30 = 0.26*x20 +0.33*x21 +0.45
11| x31 = 1.42*x20 +0.40*x21 -0.45
12|
13| output(x30 > x31)
```

# Ongoing work

## Input Data Usage Framework

[Urban18]

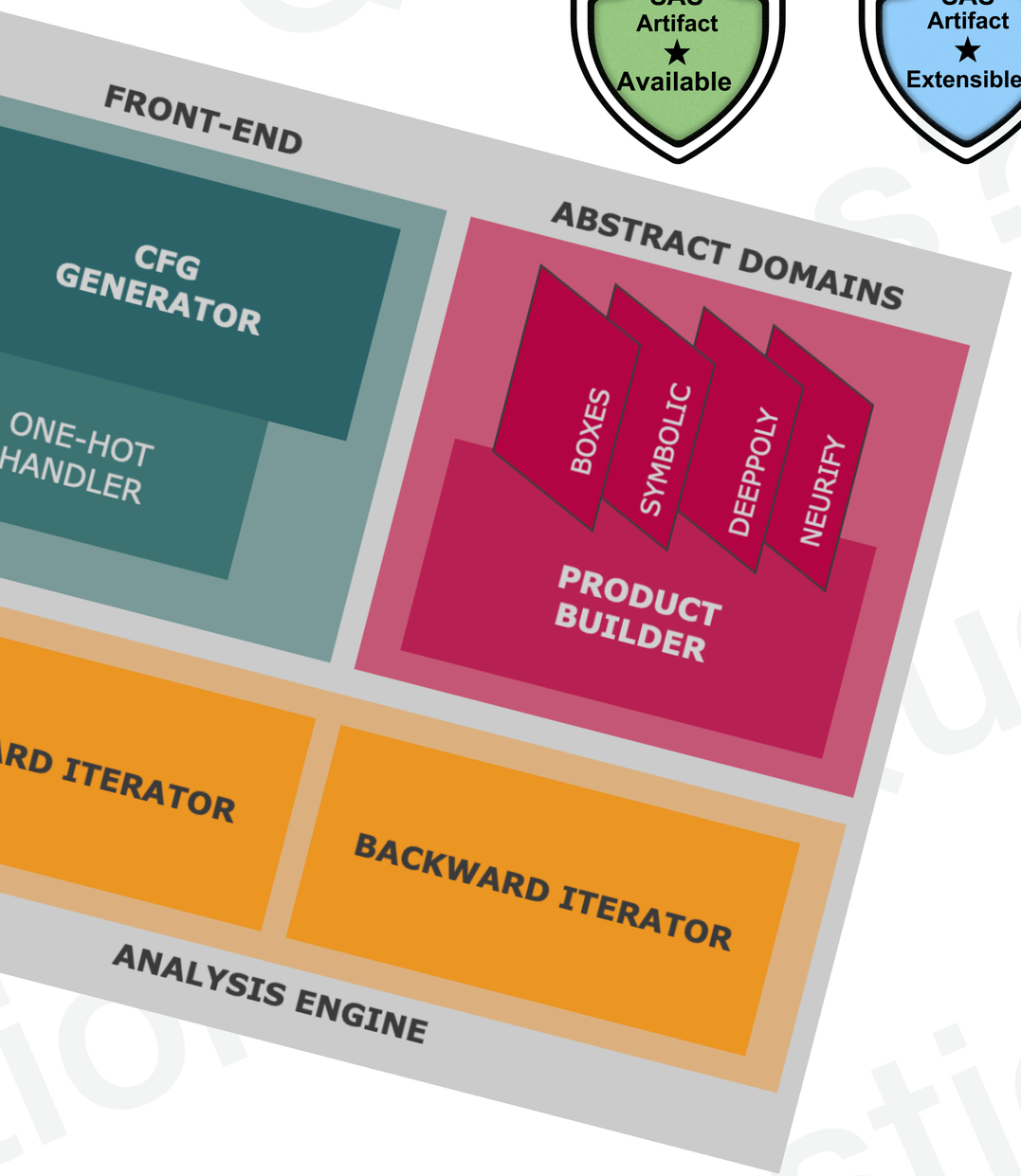
### Quantitative Input Data Usage

Unused $\Longrightarrow$ no impact on the outcome

$\Downarrow$

Used $\Longrightarrow$ how much?

```
 1| x00 = input()
 2| x01 = input()
 3|
 4| x10 = ReLU(-0.31*x00 +0.99*x01 -0.63)
 5| x11 = ReLU(-1.25*x00 -0.64*x01 +1.88)
 6|
 7| x20 = ReLU(0.40*x10 +1.21*x11)
 8| x21 = ReLU(0.64*x10 +0.69*x11 -0.39)
 9|
10| x30 = 0.26*x20 +0.33*x21 +0.45
11| x31 = 1.42*x20 +0.40*x21 -0.45
12|
13| output(x30 > x31)
```











Ready-to-go **Docker image** at
https://doi.org/10.5281/zenodo.4737450

Check it out on **GitHub**!
https://github.com/caterinaurban/libra

# Ongoing work

## Quantitative Input Data Usage

# Backup

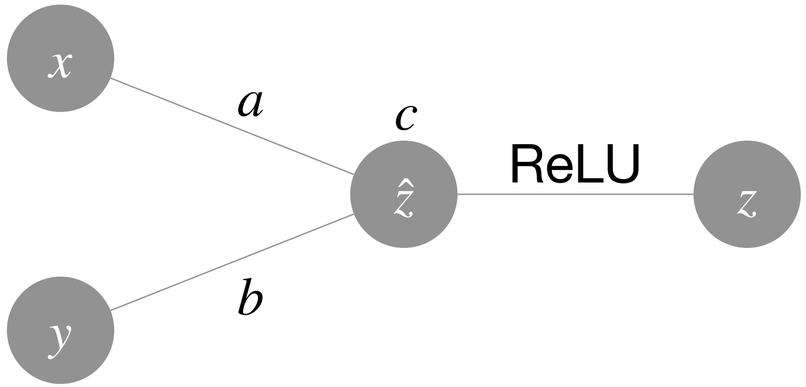
# Symbolic

Li et al. @ SAS 2019

$$\hat{z} = a \cdot x + b \cdot y + c$$

$$x \in [l_x, u_x]$$

$$x \leftarrow \sum_j m_j \cdot x_j + p_j$$




$$y \in [l_y, u_y]$$

$$y \leftarrow \sum_j n_j \cdot x_j + q_j$$

# Symbolic

Li et al. @ SAS 2019

$$x \in [l_x, u_x]$$

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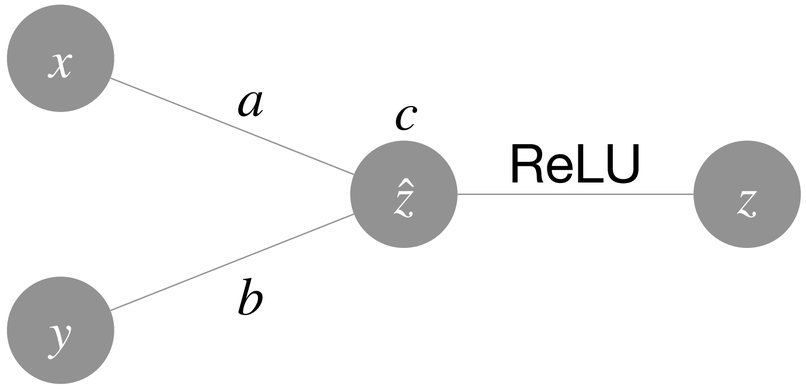


$$\hat{z} = a \cdot x + b \cdot y + c$$

$$= a \cdot (\sum_j m_j \cdot x_j + p_j) + b \cdot (\sum_j n_j \cdot x_j + q_j) + c$$

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Li et al. @ SAS 2019

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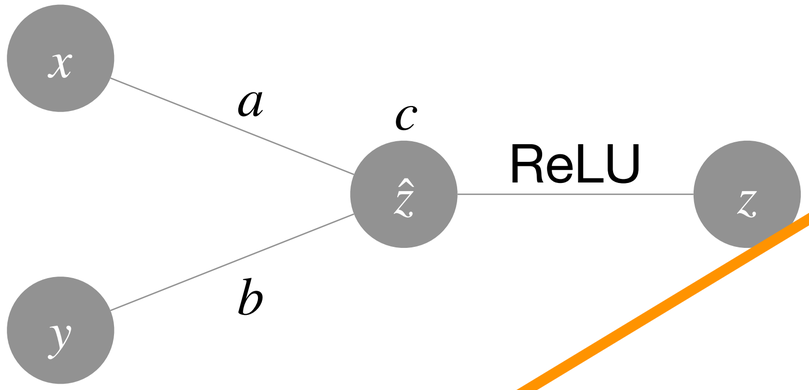

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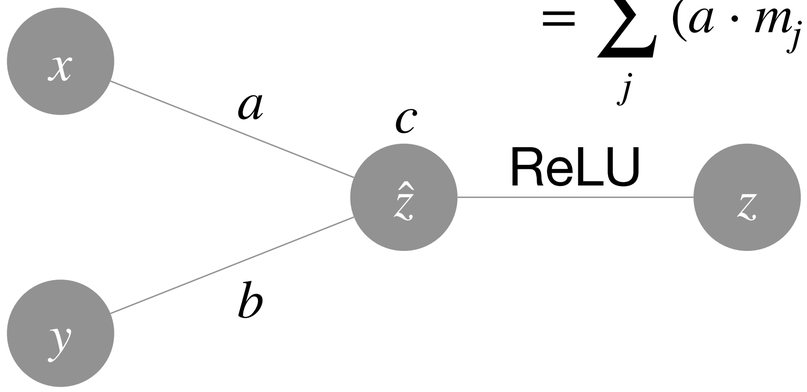


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$$= \sum_j (a \cdot m_j + b \cdot n_j) \cdot x_j + a \cdot p_j + b \cdot q_j + c$$

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Li et al. @ SAS 2019

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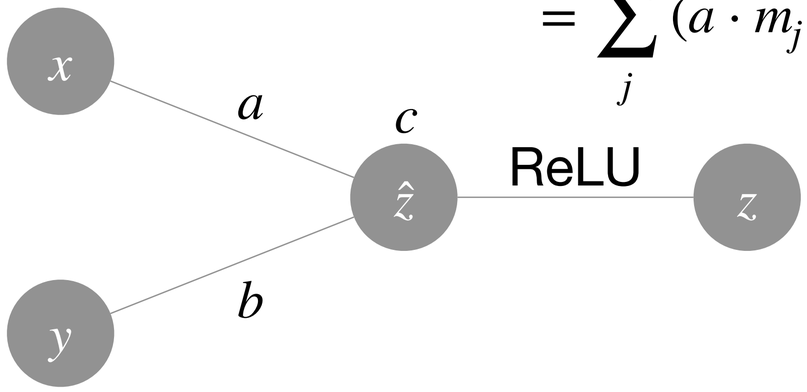


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Li et al. @ SAS 2019

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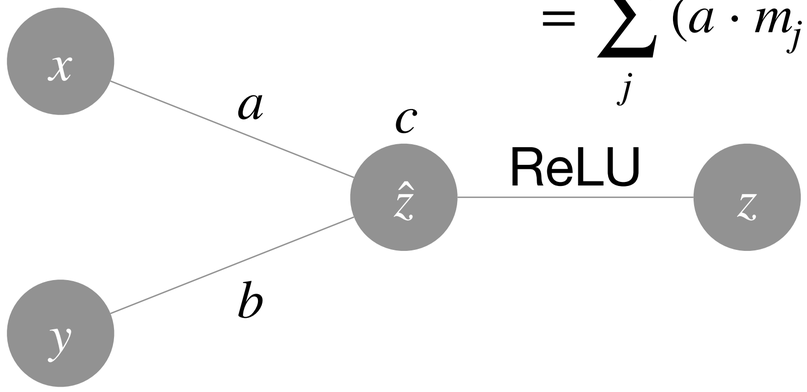


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Dependency Problem

$$x \in [0,2], x \leftarrow m + n$$

$$y \in [0,2], y \leftarrow m + n$$

$$\rightarrow \quad z = x - y$$

# Symbolic

Li et al. @ SAS 2019

$$x \in [l_x, u_x]$$

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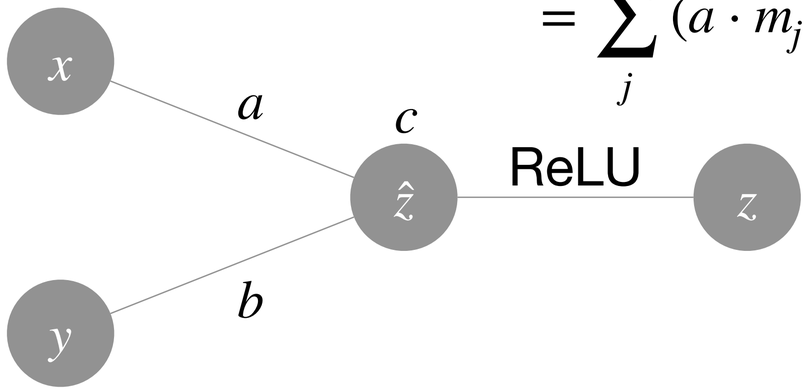


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Dependency Problem

$$x \in [0,2], x \leftarrow m + n$$

$$y \in [0,2], y \leftarrow m + n$$

$$\rightarrow \quad z = x - y$$

**Boxes**

$$\searrow \quad z \in [-2,2]$$

# Symbolic

Li et al. @ SAS 2019

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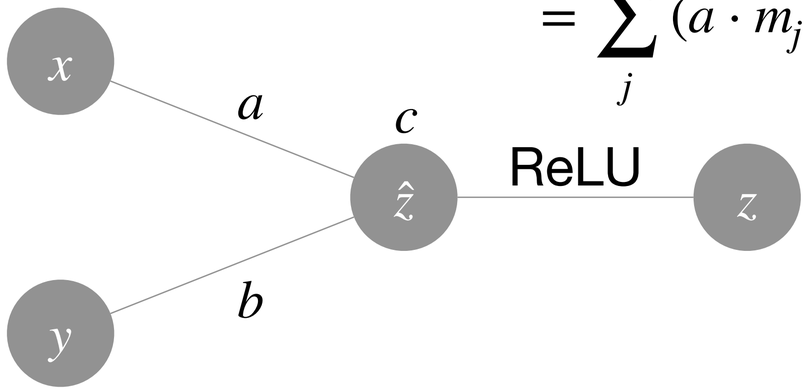


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## Dependency Problem

$$x \in [0,2], x \leftarrow m + n$$

$$y \in [0,2], y \leftarrow m + n$$

$$z = x - y$$

**Symbolic**

$$z = (m + n) - (m + n)$$
$$= (1 - 1) \cdot m + (1 - 1) \cdot n$$
$$= 0 \cdot m + 0 \cdot n = 0 \in [0,0]$$

**Boxes**

$$z \in [-2,2]$$

# Symbolic

Li et al. @ SAS 2019

$$x \in [l_x, u_x]$$

$$x \leftarrow \sum_j m_j \cdot x_j + p_j$$

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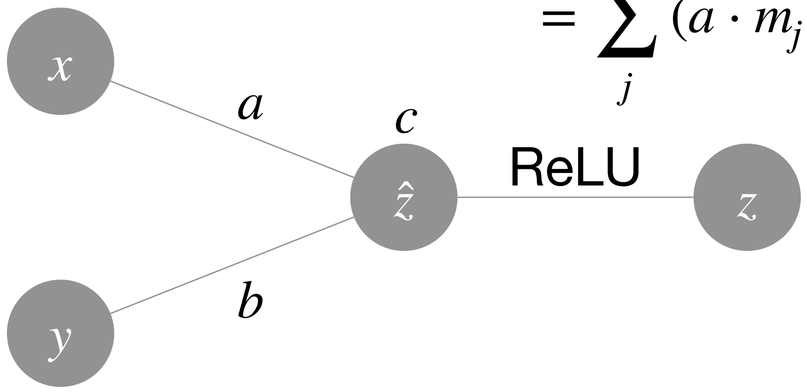


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## Dependency Problem

$$x \in [0,2], x \leftarrow m + n$$

$$y \in [0,2], y \leftarrow m + n$$

$$z = x - y$$

**Symbolic**

$$z = (m + n) - (m + n)$$
$$= (1 - 1) \cdot m + (1 - 1) \cdot n$$
$$= 0 \cdot m + 0 \cdot n = 0 \in [0,0]$$

$$\cap$$

**Boxes**

$$z \in [-2,2]$$

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Li et al. @ SAS 2019

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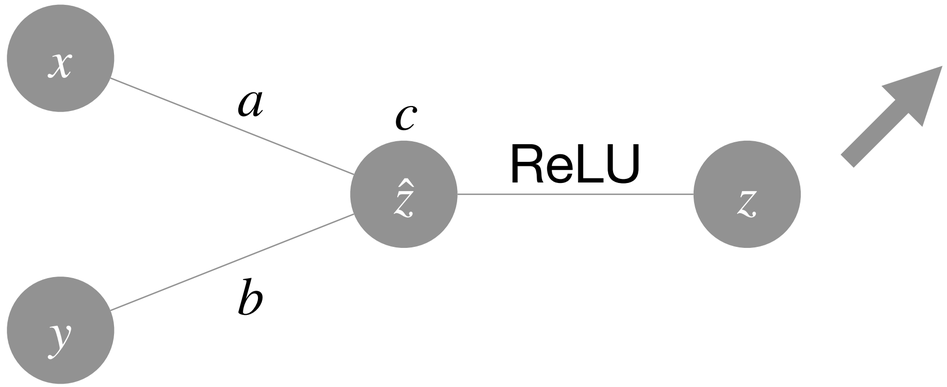


$$u_{\hat{z}} \leq 0 \implies z \leftarrow 0$$

$$y \in [l_y, u_y]$$

$$y \leftarrow \sum_j n_j \cdot x_j + q_j$$

# Symbolic

Li et al. @ SAS 2019

$$x \in [l_x, u_x]$$

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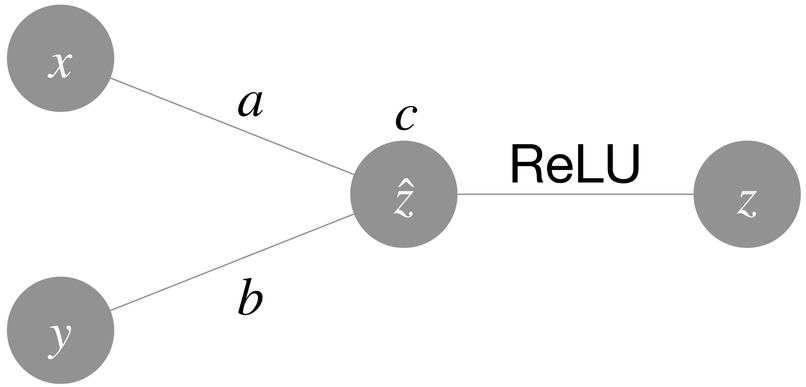


$$u_{\hat{z}} \leq 0 \implies z \leftarrow 0$$

$$l_{\hat{z}} \geq 0 \implies z \leftarrow a \cdot x + b \cdot y + c$$

$$y \in [l_y, u_y]$$

$$y \leftarrow \sum_j n_j \cdot x_j + q_j$$

# Symbolic

Li et al. @ SAS 2019

$$x \in [l_x, u_x]$$

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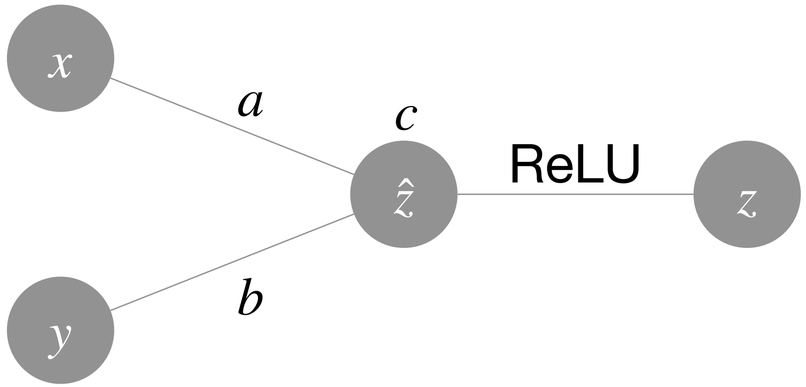


$$\hat{z} = a \cdot x + b \cdot y + c = \ldots \in [l_{\hat{z}}, u_{\hat{z}}]$$

$$u_{\hat{z}} \leq 0 \implies z \leftarrow 0$$

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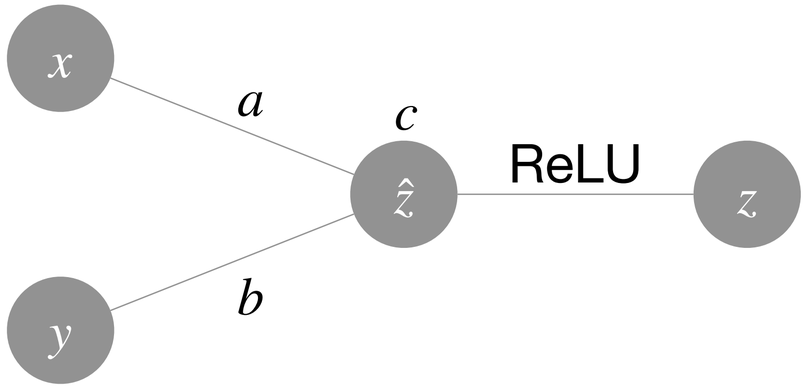
# Symbolic

Li et al. @ SAS 2019

$x \in [l_x, u_x]$

$x \leftarrow \sum_j m_j \cdot x_j + p_j$

$\hat{z} = a \cdot x + b \cdot y + c = \ldots \in [l_{\hat{z}}, u_{\hat{z}}]$




$u_{\hat{z}} \leq 0 \implies z \leftarrow 0$

$l_{\hat{z}} \geq 0 \implies z \leftarrow a \cdot x + b \cdot y + c$

$l_{\hat{z}} \leq 0 \wedge u_{\hat{z}} \geq 0 \implies z \leftarrow s \wedge s \in [0, u_{\hat{z}}]$

$y \in [l_y, u_y]$

$y \leftarrow \sum_j n_j \cdot x_j + q_j$

$s$ is a fresh variable

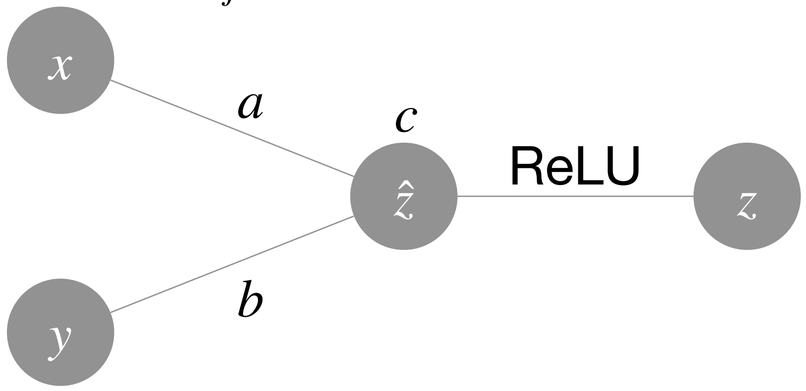
# DeepPoly

Singh et al. @ POPL 2019

$$x \in [l_x, u_x]$$

$$\sum_j \underline{m}_j \cdot x_j + \underline{p}_j \leq x \leq \sum_j \overline{m}_j \cdot x_j + \overline{p}_j$$

$$\hat{z} = a \cdot x + b \cdot y + c$$




$$y \in [l_y, u_y]$$

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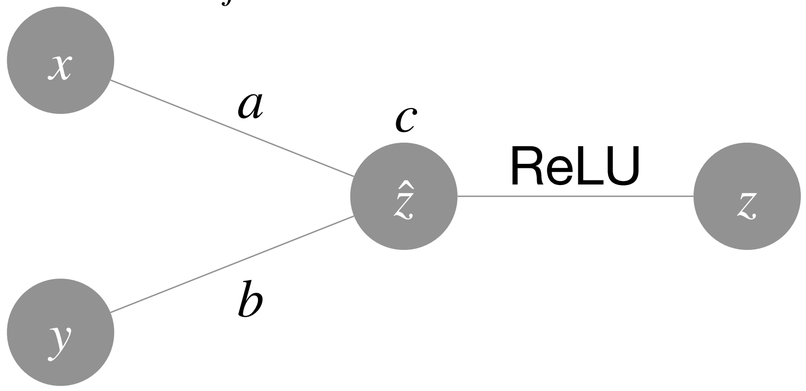


$$y \in [l_y, u_y]$$

$$\sum_j \underline{n}_j \cdot y_j + \underline{q}_j \leq y \leq \sum_j \overline{n}_j \cdot y_j + \overline{q}_j$$

$$a \geq 0 \wedge b \geq 0 \implies a \cdot (\sum_j \underline{m}_j \cdot x_j + \underline{p}_j) + b \cdot (\sum_j \underline{n}_j \cdot x_j + \underline{q}_j) + c \leq$$

$$\hat{z} \leq a \cdot (\sum_j \overline{m}_j \cdot x_j + \overline{p}_j) + b \cdot (\sum_j \overline{n}_j \cdot x_j + \overline{q}_j) + c$$

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Singh et al. @ POPL 2019

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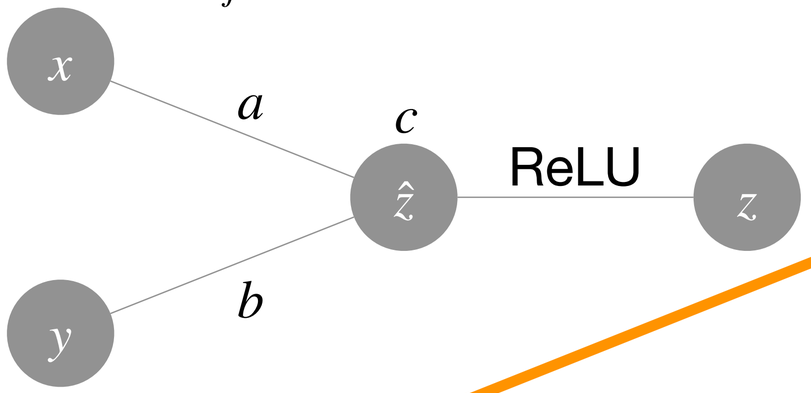


$$a \geq 0 \wedge b \geq 0 \implies a \cdot (\sum_j \underline{m}_j \cdot x_j + \underline{p}_j) + b \cdot (\sum_j \underline{n}_j \cdot x_j + \underline{q}_j) + c \leq$$

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Singh et al. @ POPL 2019

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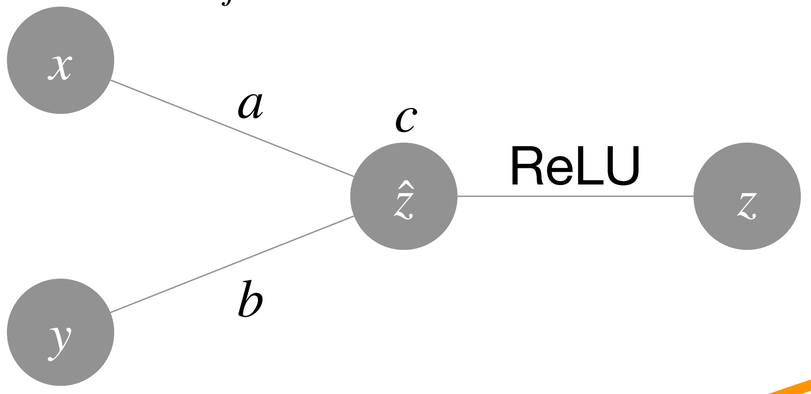


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# DeepPoly

Singh et al. @ POPL 2019

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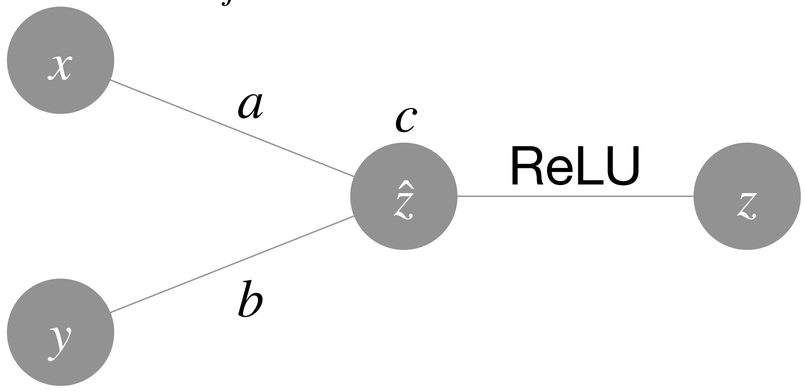


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$$\hat{z} \leq \sum_j (a \cdot \overline{m}_j + b \cdot \overline{n}_j) \cdot x_j + a \cdot \overline{p}_j + b \cdot \overline{q}_j + c$$

# DeepPoly

Singh et al. @ POPL 2019

$$x \in [l_x, u_x]$$

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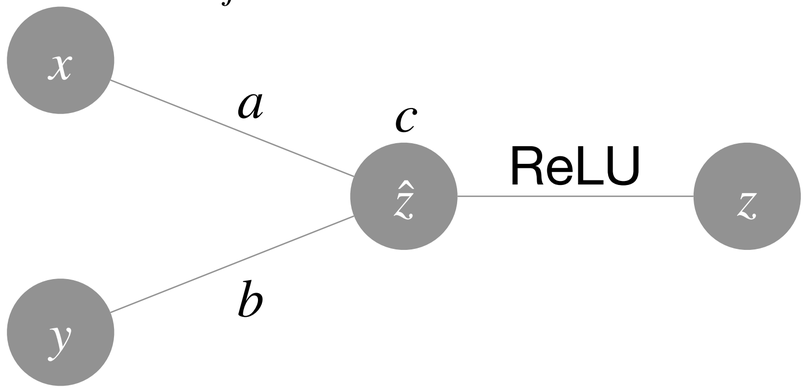


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$$\hat{z} \leq a \cdot (\sum_j \overline{m}_j \cdot x_j + \overline{p}_j) + b \cdot (\sum_j \overline{n}_j \cdot x_j + \overline{q}_j) + c$$

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$$\hat{z} \leq \sum_j (a \cdot \overline{m}_j + b \cdot \overline{n}_j) \cdot x_j + a \cdot \overline{p}_j + b \cdot \overline{q}_j + c$$

$$\iff \ldots \in [l_{\hat{z}}, u_{\hat{z}}]$$

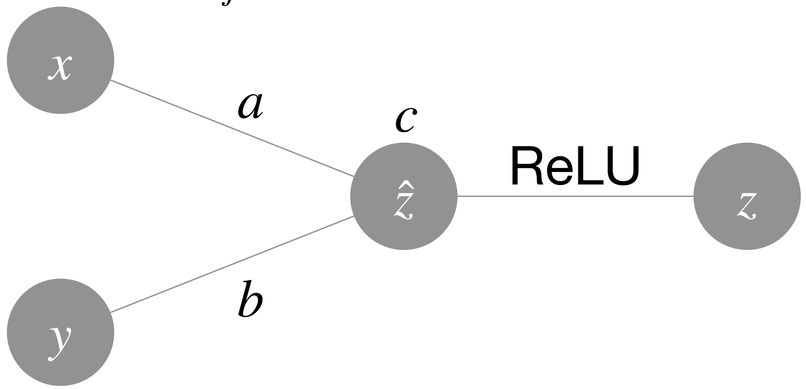
# DeepPoly

Singh et al. @ POPL 2019

$$a \cdot x + b \cdot y + c \leq \hat{z} \leq a \cdot x + b \cdot y + c$$

$$x \in [l_x, u_x]$$

$$\sum_j \underline{m}_j \cdot x_j + \underline{p}_j \leq x \leq \sum_j \overline{m}_j \cdot x_j + \overline{p}_j$$




$$u_{\hat{z}} \leq 0 \implies 0 \leq z \leq 0$$

$$l_{\hat{z}} \geq 0 \implies a \cdot x + b \cdot y + c \leq z \leq a \cdot x + b \cdot y + c$$

$$y \in [l_y, u_y]$$

$$\sum_j \underline{n}_j \cdot y_j + \underline{q}_j \leq y \leq \sum_j \overline{n}_j \cdot y_j + \overline{q}_j$$

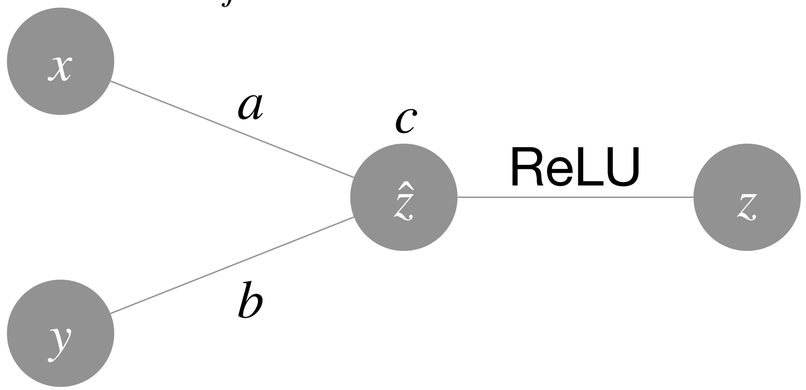
# DeepPoly

Singh et al. @ POPL 2019

$$a \cdot x + b \cdot y + c \leq \hat{z} \leq a \cdot x + b \cdot y + c$$

$$x \in [l_x, u_x]$$

$$\sum_j \underline{m}_j \cdot x_j + \underline{p}_j \leq x \leq \sum_j \overline{m}_j \cdot x_j + \overline{p}_j$$

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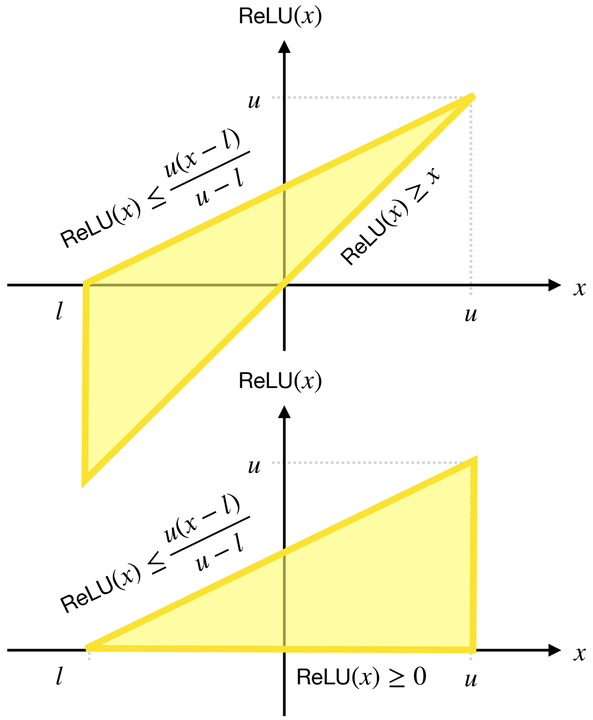
$$l_{\hat{z}} \geq 0 \implies a \cdot x + b \cdot y + c \leq z \leq a \cdot x + b \cdot y + c$$

$$l_{\hat{z}} \leq 0 \wedge u_{\hat{z}} \geq 0 \implies ?$$

# DeepPoly

Singh et al. @ POPL 2019




$$l_{\hat{z}} \leq 0 \wedge u_{\hat{z}} \geq 0 \wedge -l_{\hat{z}} \leq u_{\hat{z}} \implies \begin{cases} z \leq \dfrac{u_{\hat{z}}(\hat{z} - l_{\hat{z}})}{u_{\hat{z}} - l_{\hat{z}}} \\ z \geq \hat{z} \end{cases}$$

$$l_{\hat{z}} \leq 0 \wedge u_{\hat{z}} \geq 0 \wedge -l_{\hat{z}} > u_{\hat{z}} \implies \begin{cases} z \leq \dfrac{u_{\hat{z}}(\hat{z} - l_{\hat{z}})}{u_{\hat{z}} - l_{\hat{z}}} \\ z \geq 0 \end{cases}$$

# DeepPoly

Singh et al. @ POPL 2019

$$a \cdot x + b \cdot y + c \leq \hat{z} \leq a \cdot x + b \cdot y + c$$

$$x \in [l_x, u_x]$$

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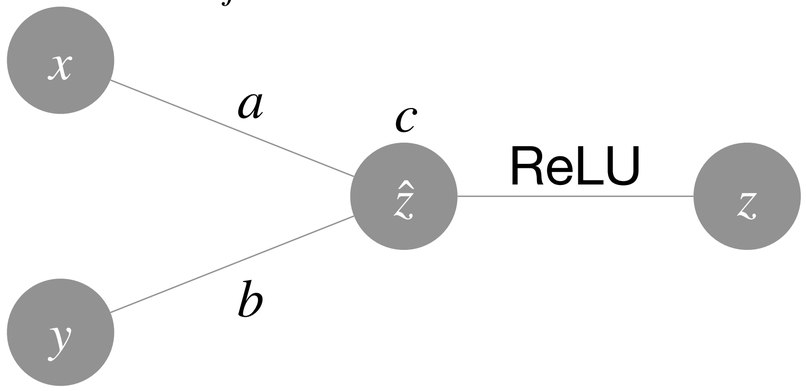


$$y \in [l_y, u_y]$$

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$$l_{\hat{z}} \leq 0 \wedge u_{\hat{z}} \geq 0 \implies \begin{cases} \hat{z} \leq z \leq \frac{u_{\hat{z}}(\hat{z} - l_{\hat{z}})}{u_{\hat{z}} - l_{\hat{z}}} & -l_{\hat{z}} \leq u_{\hat{z}} \\ 0 \leq z \leq \frac{u_{\hat{z}}(\hat{z} - l_{\hat{z}})}{u_{\hat{z}} - l_{\hat{z}}} & -l_{\hat{z}} > u_{\hat{z}} \end{cases}$$

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Singh et al. @ POPL 2019

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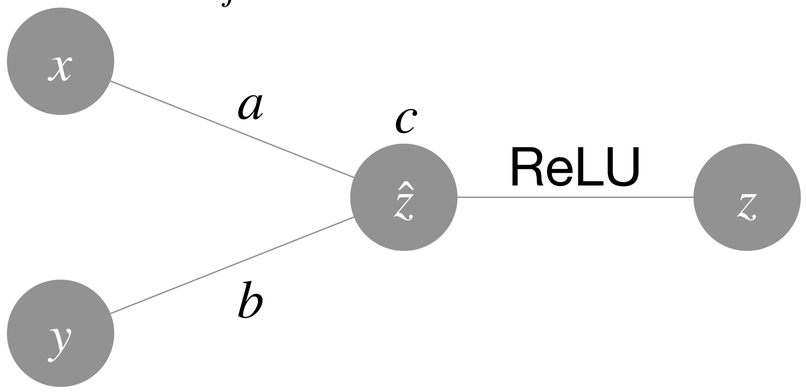


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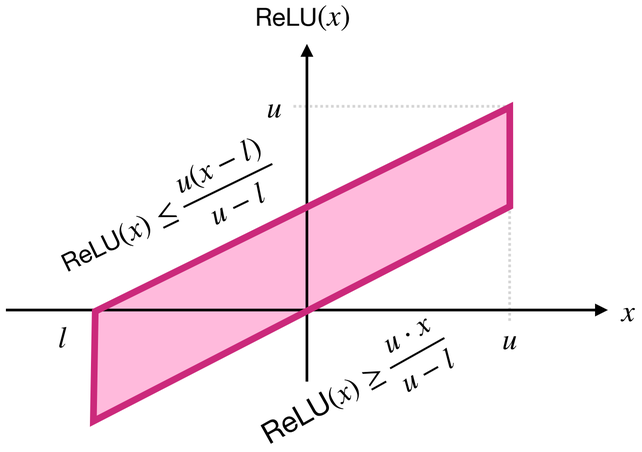
$$\bigcap$$

**Symbolic** $\implies z \leftarrow s$

# Neurify

Wang et al. @ NeurIPS 2018

$$z \leftarrow \text{ReLU}(\hat{z}) \qquad \hat{z} \in [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}]$$



# Neurify

Wang et al. @ NeurIPS 2018

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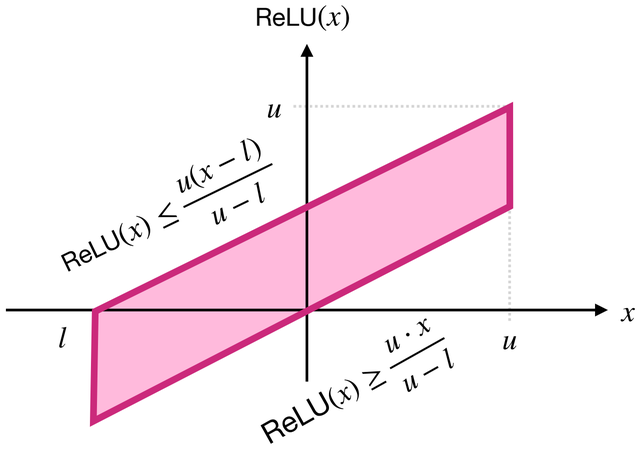
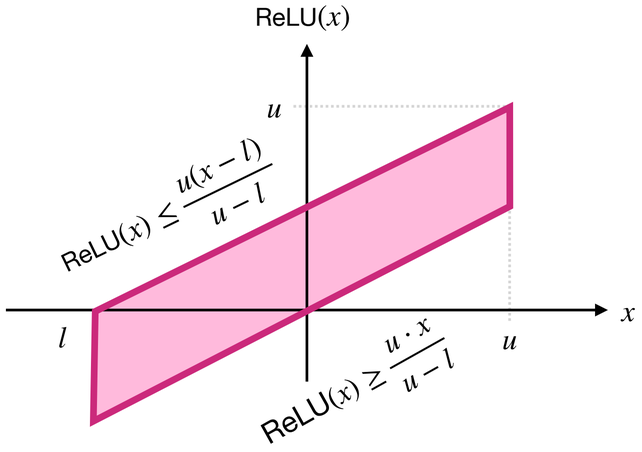



$$\underline{\text{ReLU}}(\hat{z}) \leq z \leq \overline{\text{ReLU}}(\hat{z})$$

# Neurify

Wang et al. @ NeurIPS 2018

$$z \leftarrow \text{ReLU}(\hat{z}) \qquad \hat{z} \in [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}]$$




$$\underline{\text{ReLU}}(\hat{z}) \leq z \leq \overline{\text{ReLU}}(\hat{z})$$

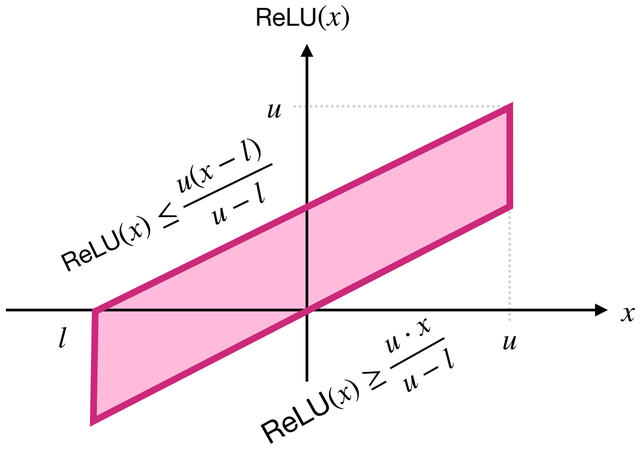
$$l_{\text{low}} \leq 0 \wedge l_{\text{up}} \geq 0 \implies \underline{\text{ReLU}}(\hat{z}) \leq z$$

$$l_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 \wedge$$

# Neurify

Wang et al. @ NeurIPS 2018

$$z \leftarrow \text{ReLU}(\hat{z}) \qquad \hat{z} \in [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}]$$




$$\underline{\text{ReLU}}(\hat{z}) \leq z \leq \overline{\text{ReLU}}(\hat{z})$$

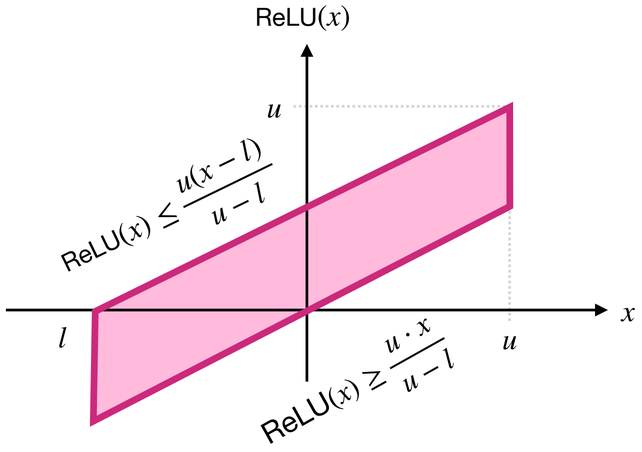
$$l_{\text{low}} \leq 0 \wedge l_{\text{up}} \geq 0 \implies \underline{\text{ReLU}}(\hat{z}) \leq z$$
$$l_{\text{low}} \leq 0 \wedge l_{\text{up}} \leq 0 \implies 0 \leq z$$

$$l_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 \wedge$$

# Neurify

Wang et al. @ NeurIPS 2018

$$z \leftarrow \text{ReLU}(\hat{z}) \qquad \hat{z} \in [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}]$$




$$\underline{\text{ReLU}}(\hat{z}) \leq z \leq \overline{\text{ReLU}}(\hat{z})$$

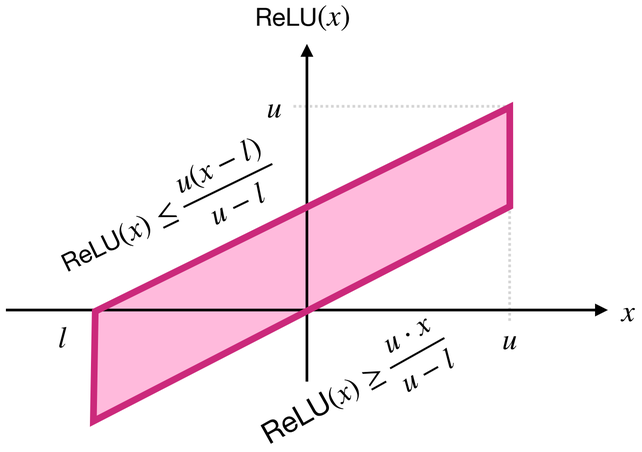
$$l_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 \wedge$$

$$l_{\text{low}} \leq 0 \wedge l_{\text{up}} \geq 0 \implies \underline{\text{ReLU}}(\hat{z}) \leq z$$
$$l_{\text{low}} \leq 0 \wedge l_{\text{up}} \leq 0 \implies 0 \leq z$$
$$u_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 \implies z \leq \overline{\text{ReLU}}(\hat{z})$$

# Neurify

Wang et al. @ NeurIPS 2018

$$z \leftarrow \text{ReLU}(\hat{z}) \qquad \hat{z} \in [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}]$$




$$\underline{\text{ReLU}}(\hat{z}) \leq z \leq \overline{\text{ReLU}}(\hat{z})$$

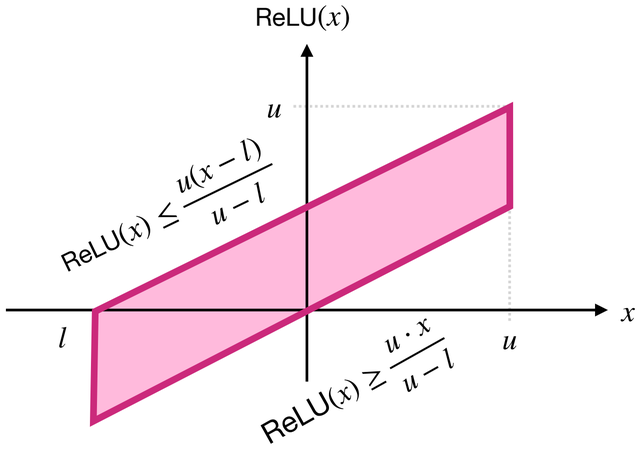
$$l_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 \wedge$$

$$\begin{aligned}
l_{\text{low}} \leq 0 \wedge l_{\text{up}} \geq 0 &\implies \underline{\text{ReLU}}(\hat{z}) \leq z \\
l_{\text{low}} \leq 0 \wedge l_{\text{up}} \leq 0 &\implies 0 \leq z \\
u_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 &\implies z \leq \overline{\text{ReLU}}(\hat{z}) \\
u_{\text{low}} \geq 0 \wedge u_{\text{up}} \geq 0 &\implies z \leq \hat{z}
\end{aligned}$$

# Neurify

Wang et al. @ NeurIPS 2018

$$z \leftarrow \text{ReLU}(\hat{z}) \qquad \hat{z} \in [l_{\text{low}}, l_{\text{up}}, u_{\text{low}}, u_{\text{up}}]$$




$$\underline{\text{ReLU}}(\hat{z}) \leq z \leq \overline{\text{ReLU}}(\hat{z})$$

$l_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 \wedge$

$$l_{\text{low}} \leq 0 \wedge l_{\text{up}} \geq 0 \implies \underline{\text{ReLU}}(\hat{z}) \leq z$$

$$l_{\text{low}} \leq 0 \wedge l_{\text{up}} \leq 0 \implies 0 \leq z$$ ← No relaxation

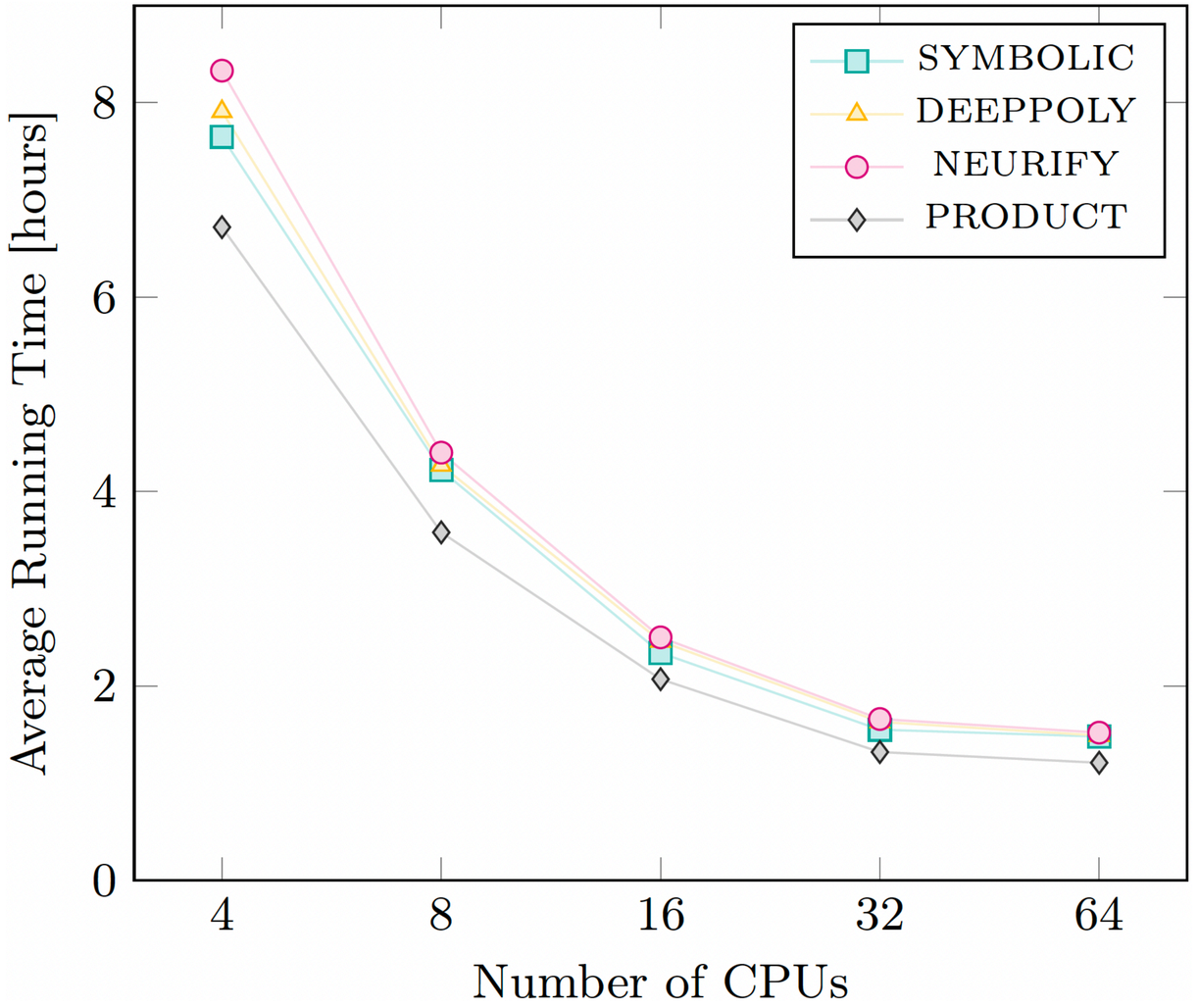
$$u_{\text{low}} \leq 0 \wedge u_{\text{up}} \geq 0 \implies z \leq \overline{\text{ReLU}}(\hat{z})$$

$$u_{\text{low}} \geq 0 \wedge u_{\text{up}} \geq 0 \implies z \leq \hat{z}$$ ← No relaxation

# Effect of Neural Network Structure

| Size | Symbolic | DeepPoly | Neurify | Product |  |
|---|---|---|---|---|---|
| **10** | 98.72% | 98.37% | 98.51% | 99.44% | **+0.7%** |
| **12** | 76.70% | 66.39% | 64.58% | 77.29% | **+0.6%** |
| **20** | 56.11% | 56.10% | 53.06% | 68.23% | **+12.1%** |
| **40** | 34.72% | 38.69% | 41.22% | 51.18% | **+10%** |
| **45** | 43.78% | 51.21% | 50.59% | 55.53% | **+4.3%** |

# Leveraging Multiple CPUs



# Perfectly Parallelisation