What is mathematics?

For all the time schools devote to the teaching of mathematics, very little (if any) is spent trying to convey just what the subject is about. Instead, the focus is on learning and applying various procedures to solve math problems. That's a bit like explaining soccer by saying it is executing a series of maneuvers to get the ball into the goal. Both accurately describe various key features, but they miss the "what?" and the "why?" of the big picture.

Given the demands of the curriculum, I can understand how this happens, but I think it is a mistake. Particularly in today's world, a general understanding of the nature, extent, power, and limitations of mathematics is valuable to any citizen. Over the years, I've met many people who graduated with degrees in such mathematically rich subjects as engineering, physics, computer science, and even mathematics itself, who have told me that they went through their entire school and college-level education without ever gaining a good overview of what constitutes modern mathematics. Only later in life do they sometimes catch a glimpse of the true nature of the subject and come to appreciate the extent of its pervasive role in modern life.

1 More than arithmetic

Most of the mathematics used in present-day science and engineering is no more than three- or four-hundred years old, much of it less than a century old. Yet the typical high school curriculum comprises mathematics at least three-hundred years old—some of it over two-thousand years old!

Now, there is nothing wrong with teaching something so old. As the saying goes, if it ain't broke, don't fix it. The algebra that the Arabic speaking traders developed in the eighth and ninth centuries (the word comes from the Arabic term *al-jabr*) to increase efficiency in their business transactions remains as useful and important today as it did then, even though today we may now implement it in a spreadsheet macro rather than by medieval finger calculation.

But time moves on and society advances. In the process, the need for new mathematics arises and, in due course, is met. Education needs to keep pace.

Mathematics is believed to have begun with the invention of numbers and arithmetic around ten thousand years ago, in order to give the world money. (Yes, it seems it began with money!)

Over the ensuing centuries, the ancient Egyptians and Babylonians expanded the subject to include geometry and trigonometry.[1] In those civilizations, mathematics was largely utilitarian, and very much of a "cookbook" variety. ("Do such and such to a number or a geometric figure and you will get the answer.")

The period from around 500BCE to 300CE was the era of Greek mathematics. The mathematicians of ancient Greece had a particularly high regard for geometry. Indeed, they regarded numbers in a geometric fashion, as measurements of length, and when they discovered that there were lengths to which their numbers did not correspond (the discovery of irrational lengths), their study of number largely came to a halt.[2]

In fact, it was the Greeks who made mathematics into an area of study, not merely a collection of techniques for measuring, counting, and accounting. Around 500BCE, Thales of Miletus (now part of

[1] Other civilizations also developed mathematics; for example the Chinese and the Japanese. But the mathematics of those cultures does not appear to have had a direct influence on the development of modern western mathematics, so in this summary I will ignore them.

[2] There is an oft repeated story that the young Greek mathematician who made this discovery was taken out to sea and drowned, lest the awful news of what he had stumbled upon should leak out. As far as I know, there is no evidence whatsoever to support this fanciful tale. Pity, since it's a great story.

Turkey) introduced the idea that the precisely stated assertions of mathematics could be logically proved by formal arguments. This innovation marked the birth of the theorem, now the bedrock of mathematics. For the Greeks, this approach culminated in the publication of Euclid's *Elements*, reputedly the most widely circulated book of all time after the Bible.[3]

The development of modern place-value arithmetic in India the first half of the First Millennium and its extension (including algebra) by the traders and scholars of the Muslim world in the second half of the Millennium advanced the subject further, and the acquisition of those ideas in southern Europe in medieval times took it still further.[4]

Although mathematics has continued to develop ever since, and shows no sign of stopping, by and large, school mathematics comprises the developments I listed above, together with *just two* further advances, both from the seventeenth century: calculus and probability theory. Virtually nothing from the last three hundred years has found its way into the classroom. Yet most of the mathematics used in today's world was developed in the last two hundred years!

As a result, anyone whose view of mathematics is confined to what is typically taught in schools is unlikely to appreciate that research in mathematics is a thriving, worldwide activity, or to accept that mathematics permeates, often to a considerable extent, most walks of present-day life and society. For example, they are unlikely to know which organization in the United States employs the greatest number of Ph.D.s in mathematics. (The answer is almost certainly the National Security Agency, though the exact number is an official secret. Most of those mathematicians work on code breaking, to enable the agency to read encrypted messages that are intercepted by monitoring systems—at least, that is what is generally assumed, though again the Agency won't say.)

The explosion of mathematical activity that has taken place over the past hundred years or so in particular has been dramatic. At the start of the twentieth century, mathematics could reasonably be regarded as consisting of about twelve distinct subjects: arithmetic, geometry, calculus, and several more. Today, between sixty and seventy distinct categories would be a reasonable figure. Some subjects, like algebra or topology, have split into various subfields; others, such as complexity theory or dynamical systems theory, are completely new areas of study.

The dramatic growth in mathematics led in the 1980s to the emergence of a new definition of mathematics as the *science of patterns*. According to this description, the mathematician identifies and analyzes abstract patterns—numerical patterns, patterns of shape, patterns of motion, patterns of behavior, voting patterns in a population, patterns of repeating chance events, and so on. Those patterns can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative, utilitarian or recreational. They can arise from the world around us, from the pursuit of science, or from the inner workings of the human mind. Different kinds of patterns give rise to different branches of mathematics. For example:

- Arithmetic and number theory study the patterns of number and counting.
- Geometry studies the patterns of shape.
- Calculus allows us to handle patterns of motion.
- Logic studies patterns of reasoning.
- Probability theory deals with patterns of chance.
- Topology studies patterns of closeness and position.
- Fractal geometry studies the self-similarity found in the natural world.

[3]Given today's mass market paperbacks, the definition of "widely circulated" presumably has to incorporate the number of years the book has been in circulation.

[4]See the iTunes University video lectures that accompany this MOOC for further details.

2 Mathematical notation

One aspect of modern mathematics that is obvious to even the casual observer is the use of abstract notations: algebraic expressions, complicated-looking formulas, and geometric diagrams. The mathematicians' reliance on abstract notation is a reflection of the abstract nature of the patterns they study.

Different aspects of reality require different forms of description. For example, the most appropriate way to study the lay of the land or to describe to someone how to find their way around a strange town is to draw a map. Text is far less appropriate. Analogously, annotated line drawings (blueprints) are the appropriate way to specify the construction of a building. And musical notation is the most appropriate way to represent music on paper.

In the case of various kinds of abstract, formal patterns and abstract structures, the most appropriate means of description and analysis is mathematics, using mathematical notations, concepts, and procedures. For instance, the symbolic notation of algebra is the most appropriate means of describing and analyzing general behavioral properties of addition and multiplication.

For example, the commutative law for addition could be written in English as:

When two numbers are added, their order is not important.

However, it is usually written in the symbolic form

$$m + n = n + m$$

Such is the complexity and the degree of abstraction of the majority of mathematical patterns, that to use anything other than symbolic notation would be prohibitively cumbersome. And so the development of mathematics has involved a steady increase in the use of abstract notations.

Though the introduction of symbolic mathematics in its modern form is generally credited to the French mathematician François Viète in the sixteenth century, the earliest appearance of algebraic notation seems to have been in the work of Diophantus, who lived in Alexandria some time around 250CE. His thirteen volume treatise *Arithmetica* (only six volumes have survived) is generally regarded as the first algebra textbook. In particular, Diophantus used special symbols to denote the unknown in an equation and to denote powers of the unknown, and he had symbols for subtraction and for equality.

These days, mathematics books tend to be awash with symbols, but mathematical notation no more is mathematics than musical notation is music. A page of sheet music *represents* a piece of music; the music itself is what you get when the notes on the page are sung or performed on a musical instrument. It is in its performance that the music comes alive and becomes part of our experience; the music exists not on the printed page but in our minds. The same is true for mathematics; the symbols on a page are just a *representation* of the mathematics. When read by a competent performer (in this case, someone trained in mathematics), the symbols on the printed page come alive—the mathematics lives and breathes in the mind of the reader like some abstract symphony.

To repeat, the reason for the abstract notation is the abstract nature of the patterns that mathematics helps us identify and study. For example, mathematics is essential to our understanding the invisible patterns of the universe. In 1623, Galileo wrote,

> The great book of nature can be read only by those who know the language in which it was written. And this language is mathematics.[5]

In fact, physics can be accurately described as the universe seen through the lens of mathematics.

To take just one example, as a result of applying mathematics to formulate and understand the laws of physics, we now have air travel. When a jet aircraft flies overhead, you can't see anything holding it up. Only with mathematics can we "see" the invisible forces that keep it aloft. In this case, those forces were identified by Isaac Newton in the seventeenth century, who also developed the mathematics required to study them, though several centuries were to pass before technology had developed to a point where we could actually use Newton's mathematics (enhanced by a lot of additional mathematics developed in the interim) to build airplanes. This is just one of many illustrations of one of my favorite memes for describing what mathematics does: *mathematics makes the invisible visible.*

[5] *The Assayer.* This is an oft repeated paraphrase of his actual words.

3 Modern college-level mathematics

With that brief overview of the historical development of mathematics under our belts, I can start to explain how modern college math came to differ fundamentally from the math taught in school.

Up to about 150 years ago, although mathematicians had long ago expanded the realm of objects they studied beyond numbers (and algebraic symbols for numbers), they still regarded mathematics as primarily about *calculation.* That is, proficiency at mathematics essentially meant being able to carry out calculations or manipulate symbolic expressions to solve problems. By and large, high school mathematics is still very much based on that earlier tradition.

But during the nineteenth century, as mathematicians tackled problems of ever greater complexity, they began to discover that their intuitions were sometimes inadequate to guide their work. Counter-intuitive (and occasionally paradoxical) results made them realize that some of the methods they had developed to solve important, real-world problems had consequences they could not explain. For example, one such, the Banach–Tarski Paradox, says you can, in principle, take a sphere and cut it up in such a way that you can reassemble it to form two identical spheres each the same size as the original one.

It became clear, then, that mathematics can lead to realms where the only understanding is through the mathematics itself. (Because the mathematics is correct, the Banach–Tarski result had to be accepted as a fact, even though it defies our imagination.) In order to be confident that we can rely on discoveries made by way of mathematics—but not verifiable by other means—mathematicians turned the methods of mathematics inwards, and used them to examine the subject itself.

This introspection led, in the middle of the nineteenth century, to the adoption of a new and different conception of the mathematics, where the primary focus was no longer on performing a calculation or computing an answer, but formulating and understanding abstract concepts and relationships. This was a shift in emphasis from *doing* to *understanding.* Mathematical objects were no longer thought of as given primarily by formulas, but rather as carriers of conceptual properties. Proving something was no longer a matter of transforming terms in accordance with rules, but a process of logical deduction from concepts.

This revolution—for that is what it amounted to—completely changed the way mathematicians thought of their subject. Yet, for the rest of the world, the shift may as well have not occurred. The first anyone other than professional mathematicians knew that something had changed was when the new emphasis found its way into the undergraduate curriculum. If you, as a college math student, find yourself reeling after your first encounter with this "new math," you can lay the blame at the feet of the mathematicians Lejeune Dirichlet, Richard Dedekind, Bernhard Riemann, and all the others who ushered in the new approach.

As a foretaste of what is to come, I'll give one example of the shift. Prior to the nineteenth century, mathematicians were used to the fact that a formula such as $y = x^2 + 3x - 5$ specifies a *function* that produces a new number y from any given number x. Then the revolutionary Dirichlet came along and said, forget the formula and concentrate on what the function *does* in terms of input–output behavior. A *function*, according to Dirichlet, is any rule that produces new numbers from old. The rule does not have to be specified by an algebraic formula. In fact, there's no reason to restrict your attention to numbers. A function can be any rule that takes objects of one kind and produces new objects from them.

This definition legitimizes functions such as the one defined on real numbers by the rule:

$$\text{If } x \text{ is rational, set } f(x) = 0; \text{ if } x \text{ is irrational, set } f(x) = 1.$$

Try graphing that monster!

Mathematicians began to study the properties of such *abstract* functions, specified not by some formula but by their behavior. For example, does the function have the property that when you present it with different starting values it always produces different answers? (This property is called *injectivity.*)

This abstract, conceptual approach was particularly fruitful in the development of the new subject called real analysis, where mathematicians studied the properties of continuity and differentiability of functions as abstract concepts in their own right. French and German mathematicians developed the "epsilon-delta definitions" of continuity and differentiability, that to this day cost each new generation of post-calculus mathematics students so much effort to master.

Again, in the 1850s, Riemann defined a complex function *by its property of differentiability*, rather than a formula, which he regarded as secondary.

The *residue classes* defined by the famous German mathematician Karl Friedrich Gauss (1777–1855), which you are likely to meet in an algebra course, were a forerunner of the approach—now standard—whereby a mathematical structure is defined as a set endowed with certain operations, whose behaviors are specified by axioms.

Taking his lead from Gauss, Dedekind examined the new concepts of *ring*, *field*, and *ideal*—each of which was defined as a collection of objects endowed with certain operations. (Again, these are concepts you are likely to encounter soon in your post-calculus mathematics education.)

And there were many more changes.

Like most revolutions, the nineteenth century change had its origins long before the main protagonists came on the scene. The Greeks had certainly shown an interest in mathematics as a conceptual endeavor, not just calculation, and in the seventeenth century, calculus co-inventor Gottfried Leibniz thought deeply about both approaches. But for the most part, until the nineteenth century, mathematics was viewed primarily as a collection of procedures for solving problems. To today's mathematicians, however, brought up entirely with the post-revolutionary conception of mathematics, what in the nineteenth century was a revolution is simply taken to be what mathematics is. The revolution may have been quiet, and to a large extent forgotten, but it was complete and far reaching. And it sets the scene for this book, the main aim of which is to provide you with the basic mental tools you will need to enter this new world of modern mathematics (or at least to learn to think mathematically).

Although the post-nineteenth century conception of mathematics now dominates the field at the post-calculus, college level, it has not had much influence on high school mathematics—which is why you need a book like this to help you make the transition. There was one attempt to introduce the new approach into school classrooms, but it went terribly wrong and soon had to be abandoned. This was the so-called "New Math" movement of the 1960s. What went wrong was that by the time the revolutionaries' message had made its way from the mathematics departments of the leading universities into the schools, it was badly garbled.

To mathematicians before and after the mid 1800s, both calculation and understanding had always been important. The nineteenth century revolution merely shifted the *emphasis* regarding which of the two the subject was really about and which played the derivative or supporting role. Unfortunately, the message that reached the nation's school teachers in the 1960s was often, "Forget calculation skill, just concentrate on concepts." This ludicrous and ultimately disastrous strategy led the satirist (and mathematician) Tom Lehrer to quip, in his song *New Math*, "It's the method that's important, never mind if you don't get the right answer." After a few sorry years, "New Math" (which was already over a hundred years old, note) was largely dropped from the school syllabus.

Such is the nature of educational policy making in free societies, it is unlikely such a change could ever be made in the foreseeable future, even if it were done properly the second time around. It's also not clear (at least to me) that such a change would be altogether desirable. There are educational arguments (which in the absence of hard evidence either way are hotly debated) that say the human mind has to achieve a certain level of mastery of computation with abstract mathematical entities before it is able to reason about their properties.

4 Why are you having to learn this stuff?

It should be clear by now that the nineteenth century shift from a computational view of mathematics to a conceptual one was a change within the professional mathematical community. Their interest, as professionals, was in the very nature of mathematics. For most scientists, engineers, and others who make use of mathematical methods in their daily work, things continued much as before, and that remains the same today. Computation (and getting the right answer) remains just as important as ever, and even more widely used than at any time in history.

As a result, to anyone outside the mathematical community, the shift looks more like an *expansion* of mathematical activity than a change of focus. Instead of just learning procedures to solve problems, college-level math students today *also* (i.e., *in addition*) are expected to master the underlying concepts and be able to justify the methods they use.

Is it reasonable to require this? Granted that the professional mathematicians—whose job it is to develop new mathematics and certify its correctness—need such conceptual understanding, why make it a requirement for those whose goal is to pursue a career in which mathematics is merely a tool? (Engineering for example.)

There are two answers, both of which have a high degree of validity. (SPOILER: It only appears that there are two answers. On deeper analysis, they turn out to be the same.)

First, education is not solely about the acquisition of specific tools to use in a subsequent career. As one of the greatest creations of human civilization, mathematics should be taught alongside science, literature, history, and art in order to pass along the jewels of our culture from one generation to the next. We humans are far more than the jobs we do and the careers we pursue. Education is a preparation for life, and only part of that is the mastery of specific work skills.

That first answer should surely require no further justification. The second answer addresses the tools-for-work issue.

There is no question that many jobs require mathematical skills. Indeed, in most industries, at almost any level, the mathematical requirements turn out to be higher than is popularly supposed, as many people discover when they look for a job and find their math background lacking.

Over many years, we have grown accustomed to the fact that advancement in an industrial society requires a workforce that has mathematical skills. But if you look more closely, those skills fall into two categories. The first category comprises people who, given a mathematical problem (i.e., a problem already formulated in mathematical terms), can find its mathematical solution. The second category comprises people who can take a new problem, say in manufacturing, identify and describe key features of the problem mathematically, and use that mathematical description to analyze the problem in a precise fashion.

In the past, there was a huge demand for employees with type 1 skills, and a small need for type 2 talent. Our mathematics education process largely met both needs. It has always focused primarily on producing people of the first variety, but some of them inevitably turned out to be good at the second kind of activities as well. So all was well. But in today's world, where companies must constantly innovate to stay in business, the demand is shifting toward type 2 mathematical thinkers—to people who can think outside the mathematical box, not inside it. Now, suddenly, all is not well.

There will always be a need for people with mastery of a range of mathematical techniques, who are able to work alone for long periods, deeply focused on a specific mathematical problem, and our education system should support their development. But in the twenty-first century, the greater demand will be for type 2 ability. Since we don't have a name for such individuals ("mathematically able" or even "mathematician" popularly imply type 1 mastery), I propose to give them one: *innovative mathematical thinkers.*

This new breed of individuals (well, it's not new, I just don't think anyone has shone a spotlight on them before) will need to have, above all else, a good conceptual (in an operational sense) understanding of mathematics, its power, its scope, when and how it can be applied, and its limitations. They will also have to have a solid mastery of some basic mathematical skills, but that skills mastery does not have to be stellar. A far more important requirement is that they can work well in teams, often cross-disciplinary teams, they can see things in new ways, they can quickly learn and come up to speed on a new technique that seems to be required, and they are very good at adapting old methods to new situations.

How do we educate such individuals? We concentrate on the conceptual thinking that lies behind all the specific techniques of mathematics. Remember that old adage, "If you give a man a fish you can keep him alive for a day, but if you teach him how to fish he can keep himself alive indefinitely"? It's the same for mathematics education for twenty-first century life. There are so many different mathematical techniques, with new ones being developed all the time, that it is impossible to cover them all in K-16 education. By the time a college frosh graduates and enters the workforce, many of the specific techniques learned in those four college-years are likely to be no longer as important, while new ones are all the rage. The educational focus has to be on learning how to learn.

The increasing complexity in mathematics led mathematicians in the nineteenth century to shift (broaden, if you prefer) the focus from computational skills to the underlying, foundational, conceptual thinking ability. Now, 150 years later, the changes in society that were facilitated in part by that more complex mathematics, have made that focal shift important not just for professional mathematicians but for everyone who learns math with a view to using it in the world.

So now you know not only why mathematicians in the nineteenth century shifted the focus of mathematical research, but also why, from the 1950s onwards, college mathematics students were expected to master conceptual mathematical thinking as well. In other words, you now know why your college or university wants you to take that transition course, and perhaps work your way through this book. Hopefully, you also now realize why it can be important to YOU in living your life, beyond the immediate need of surviving your college math courses.

NOTE: This course reading is abridged from the course textbook, *Introduction to Mathematical Thinking*, by me (Keith Devlin), available from Amazon as a low-cost, print-on-demand book. You don't need to purchase the book to complete the course, but I know many students like to have a complete textbook. In developing this course, I first wrote the textbook, and then used it to construct all the course materials.