Exam 1

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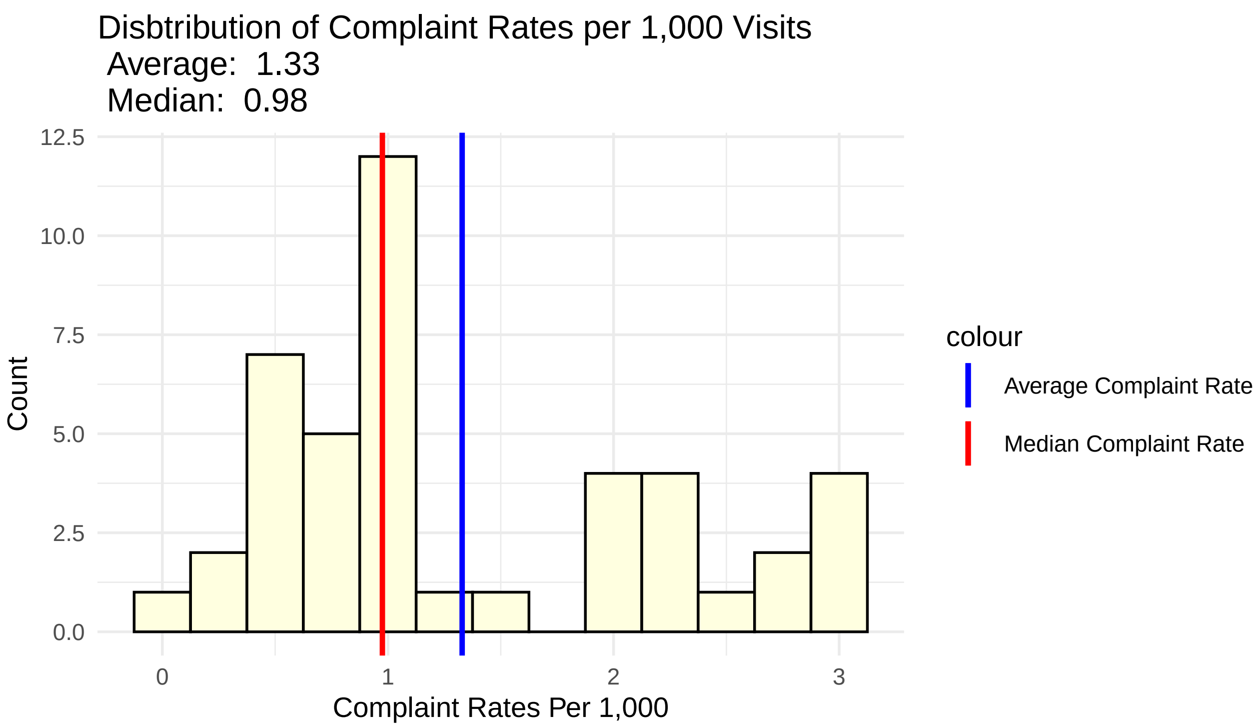
Problem 1

1- A

Before fitting the model I like to explore the distribution shape of the response variable and collect some fundamental summary statistics. Knowing the shape and the spread of the response variable will help us manage the expectation regarding model fit and variance of residuals.



The distribution of complaints per 1,000 visits somewhat balanced without extreme outliers. The mean is pretty close to the median, suggesting again that more extreme values on the upper end of complaints per 1,000 do not knew the mean very much.

The two tables below describe the distribution of numeric variables in the data set, as well as correlation between the three of them.

We can see that the scales of predictors and complaints per 1,000 vary greatly, so we should expect that the coefficients are going to be very small, probably in the 0.001 to 0.0001 range.

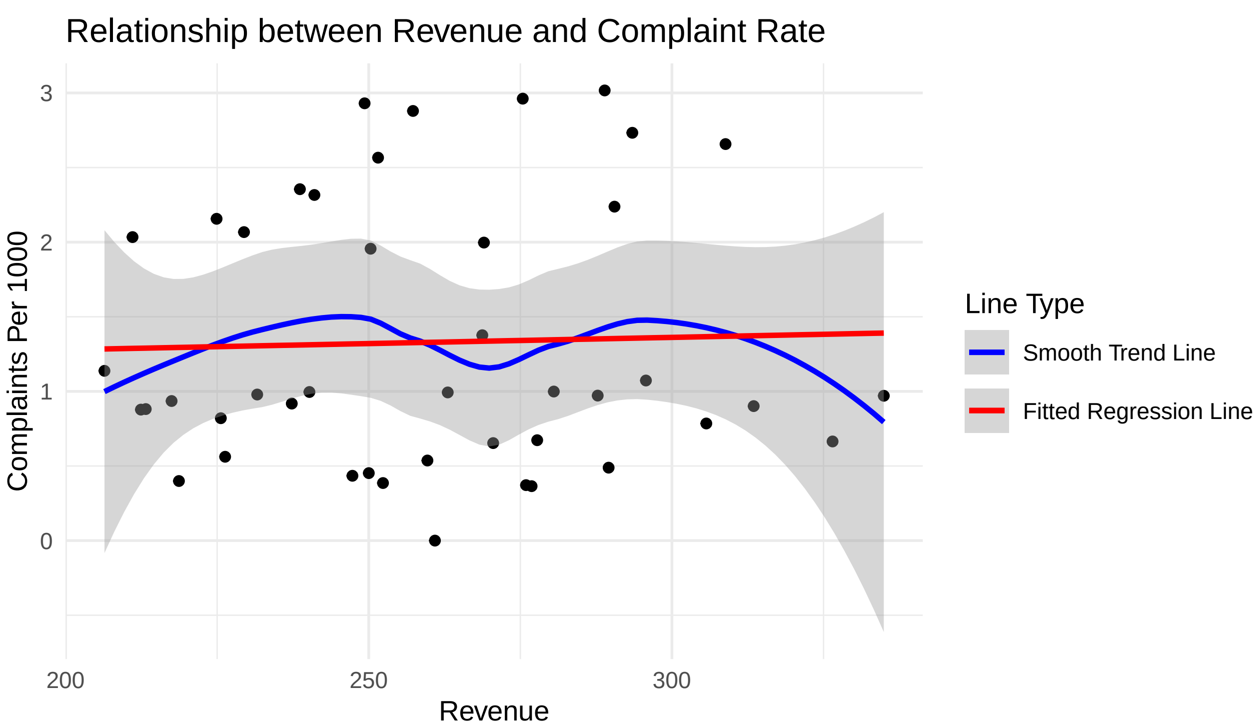
Table 2: Correaltion of Numeric Covariates

	Complaint Rate per 1,000	Revenue	Hours Worked
Complaint Rate per 1,000	1.0000000	0.0305876	0.2788799
Revenue	0.0305876	1.0000000	-0.0405506
Hours Worked	0.2788799	-0.0405506	1.0000000

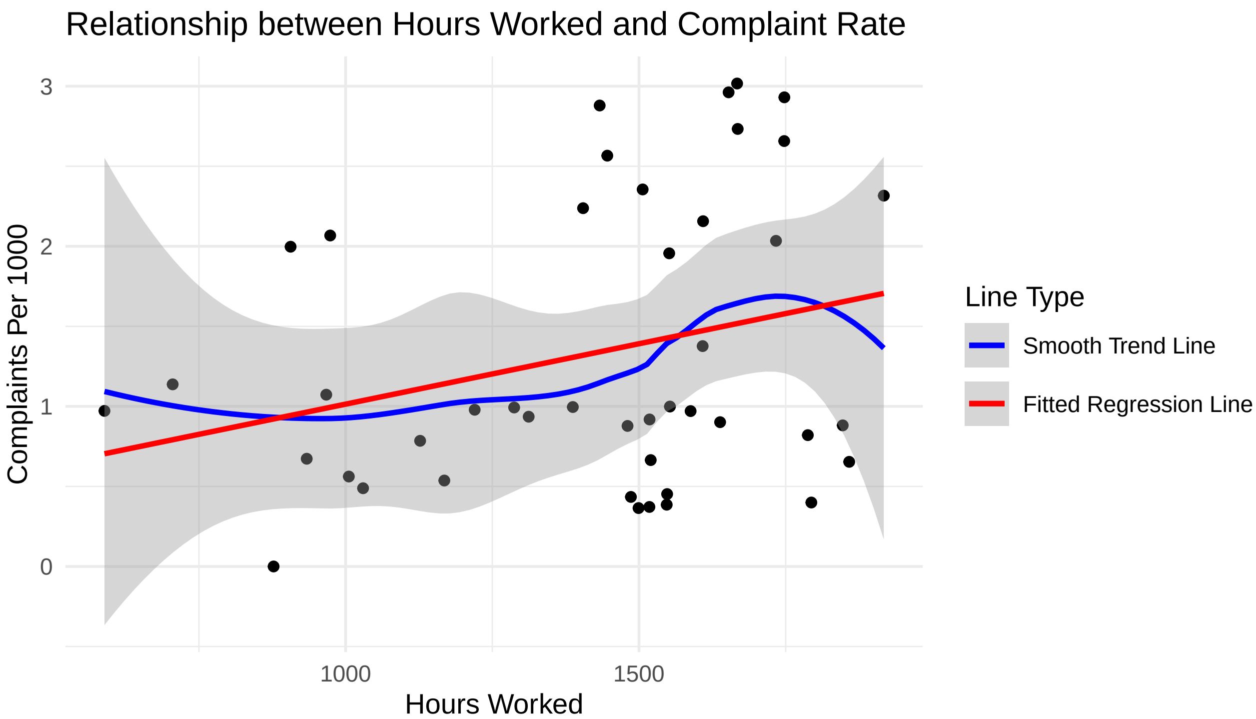
Table 1: Summary of Numeric Variables

Variables	Min	Max	Mean	S.D
complaint_rate_1000	0.00	3.02	1.33	0.88
revenue	206.42	334.94	260.14	32.64
hours	589.00	1917.25	1417.40	326.98

We continue to perform explanatory data analysis in this section by looking at the scatter plots of predictors versus complaint rates. It does not appear that revenue is related to complaint rate at all.



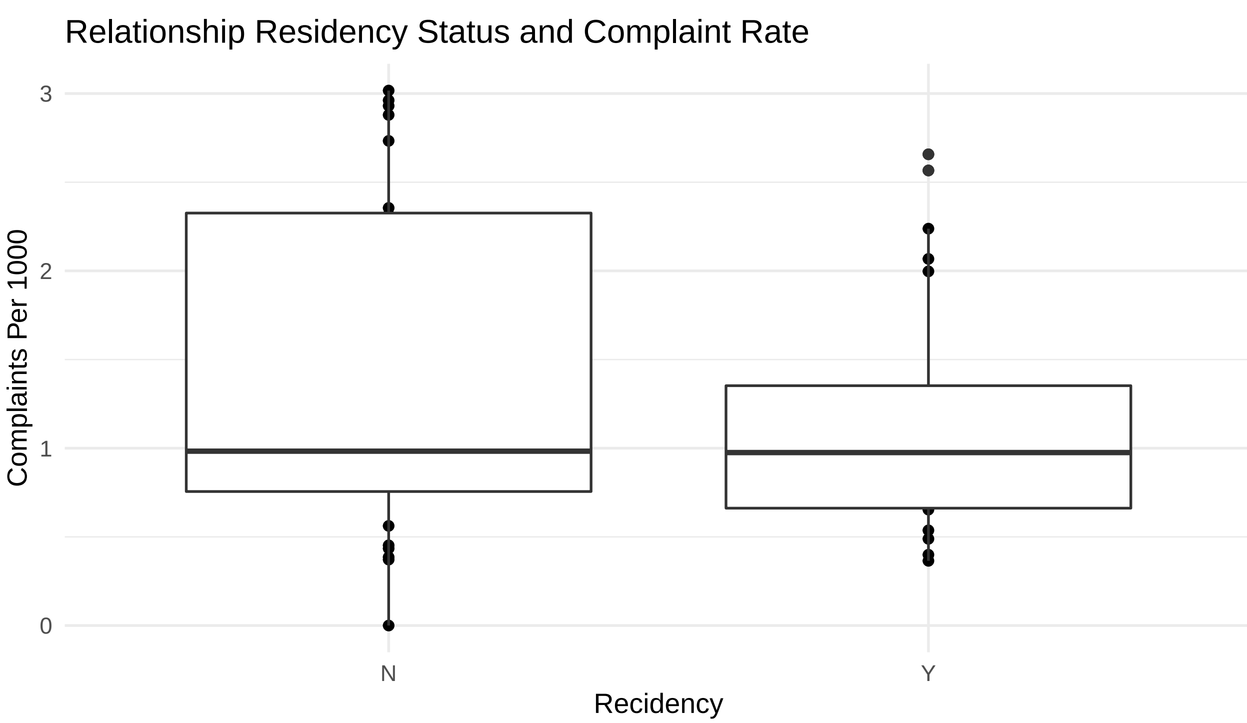
We can see that the number of hours worked is somewhat linearly related to the complaint rates, suggesting the practitioners who work more hours tend to accumulate higher complaint rate, however, the the variance of values is very large around the suggested regression line, so we might not be able to detect a statistically significant relationship when fitting the model.



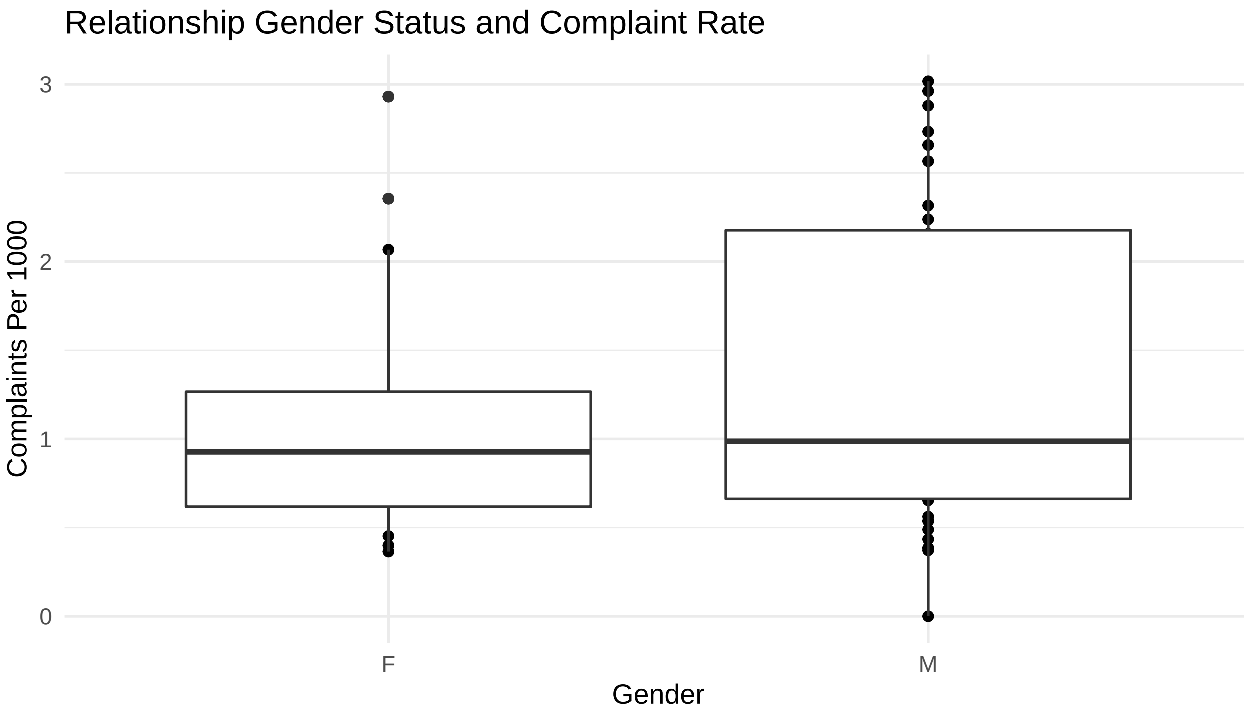
Overall, both plots suggest that linear fit is appropriate for both of these variables. Smooth Loess function does not show any consistent curvature in the data, but rather randomly fluctuates around the fitted regression line.

We have categorical predictors also:

Residency has two levels: Y, N with 54.55%, 45.45% class presence respectively. It does not appear that the median and mean values are different across the two residency levels.



Gender has two levels: F, M with 27.27%, 72.73% class presence respectively. It does not appear that the median and mean values are different across the two gender levels.



Now we are ready to fit and examine the Normal Error Regression Model. When fitting any kind of a model, we need to be careful with the assumptions we take on. **Model Assumptions** are listed below

1. Residuals, Error Terms, are normally distributed with mean $\mu = 0$ and constant variance σ^2.
2. Since fitted values depend on model parameters $\hat{\beta}_i$ and errors e_i , we assume each outcome Y_i comes from a normal distribution with mean $\mu = E[Y_i]$ and variance σ^2.
3. We assume that variance of residuals is constant.
4. Errors are independent and each unit of interest, a data point, is also independent of other observations in the sample.
5. The model is linear because $\hat{Y}_i$ can be expressed as a linear combination of weights, coefficients, $\hat{\beta}_i$ and constant observed data points X_i.
6. Predictors are not correlated or weakly correlated.

After listing model assumptions, we can state the mode:

$$E[Complaint\ \ Rate] = \hat{\beta}_0 + \hat{\beta}_1 * X_1 + \hat{\beta}_2 * X_2 + \hat{\beta}_3 * X_3 + \hat{\beta}_4 * X_4 =$$

$$E[Complaint\ \ Rate] = \hat{\beta}_0 + \hat{\beta}_1 * Revenue + \hat{\beta}_2 * Hours\ \ Worked + \hat{\beta}_3 * Gender + \hat{\beta}_4 * Residency$$

Overall ANOVA

Before investigating individual coefficients and t-test for predictors, we want to look at the overall ANOVA table, and overall F-test. We want to see if the set of all predictors is helpful at explaining the variance of complaint rates per 1,000, and therefore we will know if some of all coefficients are statistically different from 0.

ANOVA table for the F-test is given below:

Source	SSR	DF	MS	F Statistic	P(F* > F)
Regression	3.254294	4	0.8135735	1.04	0.3969
Error	30.386120	39	0.7791313	NA	NA
Total	33.640414	43	NA	NA	NA

- Null Hypothesis: $H_0 : \beta_1 = \beta_2 = ... = \beta_{p-1}$
- Alternative Hypothesis: H_a : Not all coefficients β_i are zero
- $F-$statistic: 1.04
- Cutoff F^*-statistic: 2.6123
- So, $F < F^*$, therefore we do not have enough evidence to reject the null hypothesis to conclude that some or all coefficients β_i are consistently different from zero.
- Moreover, $P(F^* > F) = 0.3969$
- Conclusion: There is not enough statistical evidence that every predictor has a coefficient different from 0. Therefore, we can't reject the null hypothesis. When we look at the individual t tests for coefficients, we might see some suggestive relationships, supported by the somewhat big values of the t-statistic and small p-values, but none of them should be statistically significant.

Table below shows **Regression Coefficients** and model summary. Like we expected, these coefficients are small because the scale and range of predictors and response variable are not the same.

Predictor	Estiamte	Standard Error	T Value	P value
(Intercept)	-0.064405	1.250366	-0.051509	0.959183
revenue	0.001351	0.004610	0.293122	0.770983
hours	0.000676	0.000461	1.467079	0.150373
genderM	0.197338	0.314907	0.626654	0.534537
residencyY	-0.132728	0.329286	-0.403077	0.689093

- R square and 0.0967
- Adjusted R Square 0.0041
- Coefficients explanation:
  - `revenue` is revenue in dollars per hour is a continuous predictor. When revenue increases by 1 dollar per hour, we expect the number of complaints to increase by 0.001351, after adjusting for other predictors.
  - `hours` is the number of hours worded, and is a continuous predictor. With each additional hour of work, we expect the number of complaints to increase by 0.001351, after adjusting for other predictors.
  - `genderM` is a coefficient for the group of men practitioners, when compared with women practitioners, which is a reference level here. Physicians who are men on average are expected to have 0.197338 more complaints per 1,000 when compared with the women physicians after adjusting for other predictors.
  - `residencyY` is a coefficient for the group of practitioners who have a residency fellowship, compared with practitioners who do not participate in such program, which is a reference level here. Physicians with fellowship are expected to have 0.132728 less complaints per 1,000 when compared with the women physicians after adjusting for other predictors.

- As expected, none of these predictors show any evidence of statistical significance, but the results are not contradictory. Doctors who are trained to work in emergency medicine should be able to do their work better, and therefore should have less complaints. Additional hours may result in extra complaints, if the doctor is overworked and their ability to perform reduces.

1- B

A special interest is to investigate how the extra hours of work impact the average complaint rate for each practitioner, given their characteristics we adjust for.

We begin the inference of this variable with a formal t-test.

- Null Hypothesis: $H_0 : \hat{\beta}_4 = 0$
- Alternative Hypothesis: $H_a : \hat{\beta}_4 \neq 0$$
- Test statistic $T : 1.467079$
- $P(t^* > t) = 0.150373$
- Conclusion: p-value is above 0.05 so there is not enough statistical evidence to reject the null hypothesis to conclude that the additional hour of work consistently results in the average increase of complaint rates. However, the p-value is not greatly far for the accepted significance level, so, this relationship is suggestive. Perhaps, with mode data, or a better statistical model we will be able to verify that this relationship is in fact consistent. My recommendation to the managers and decision makers would be to pay close attention to this factor, because even though the test shows no significance, the relationship is perhaps still real, and can't be detected from this sample.

One additional Hour worked results in 0.000676 additional complaints on average.

However, it makes more sense to, say, look at 20 hours. So, an average increase in complaint rates per 1,000 is 0.000676 * 20 = 0.01352.

It is reasonable to expect that a practitioner who is overworked will have an extra 20 hours of work on top of regular hours in one week, especially in a busy or underfunded facility.

C.I.

Using formula $C.I.\ bounds = Estimate \pm 1.96 * Standard\ Error$

C.I. for the estimate 0.000676 with a 0.000461 standard error is (-0.000256, 0.001609)

Similar to the coefficient, we can perform a linear transformation of the lower and upper bounds, and obtain a confidence interval for the effect of extra 20 hours of work. So, an average increase in complaint rates per 1,000 for the extra 20 hours of work is 0.000676 * 20 = 0.01352, with a confidence interval (-0.00512, 0.03218).

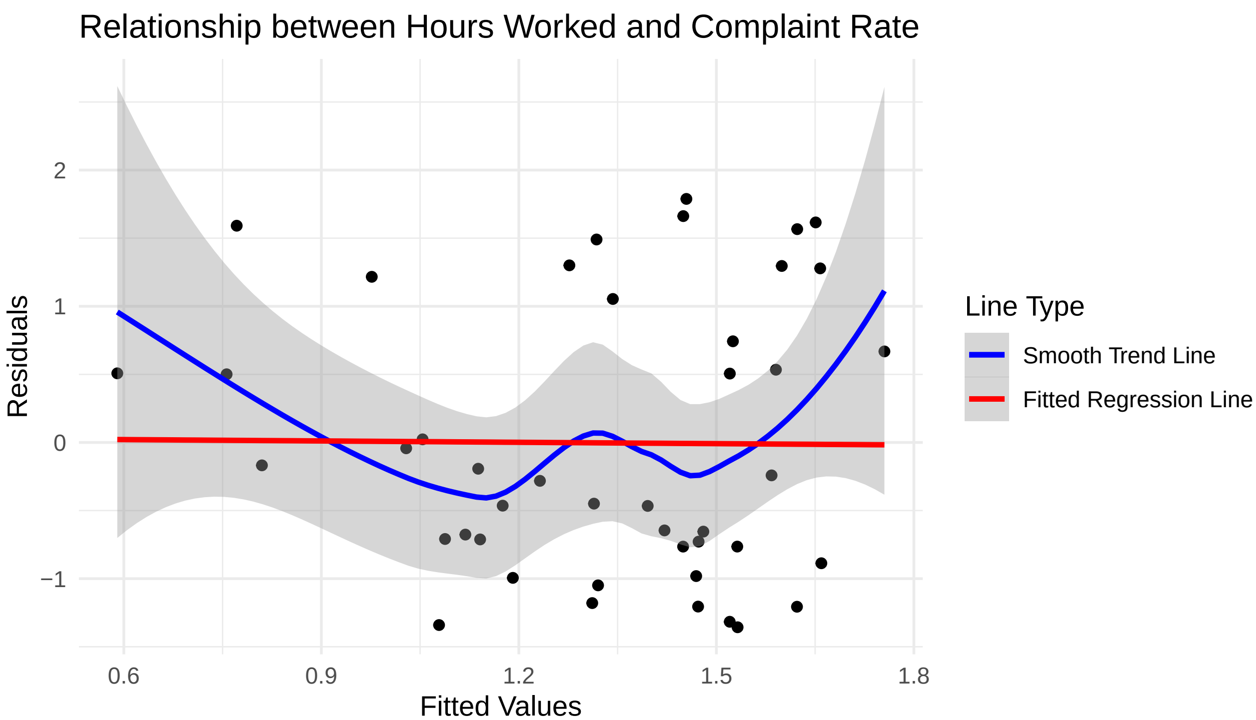
It appears that most of the confidence interval is above 0, in fact, 86.27% of values in the confidence interval are above 0. So, even though this evidence is pretty weak, we would still pay attention to this variables as a source of Y variance explanation.

1- C

The plot below shows the relationship between fitted values and studentized residuals from the regression model we built and evaluated in the previous two sections. There is no linear trend, as evidenced by the flat fitted regression line.

Smooth trend line suggests that either there is some violation of assumptions at the lower and upper ends of the fitted values, or there is simply a small number of values there.

In any case, variance appears to be somewhat constant, we do not see a megaphone or a violin shape. However, residuals above 0 tend to have an upper bound of around 2, whereas residuals below 0 tend to have a lower bound of around 1.5. Overall, this is not a huge cause for concern, but something we should keep an eye.

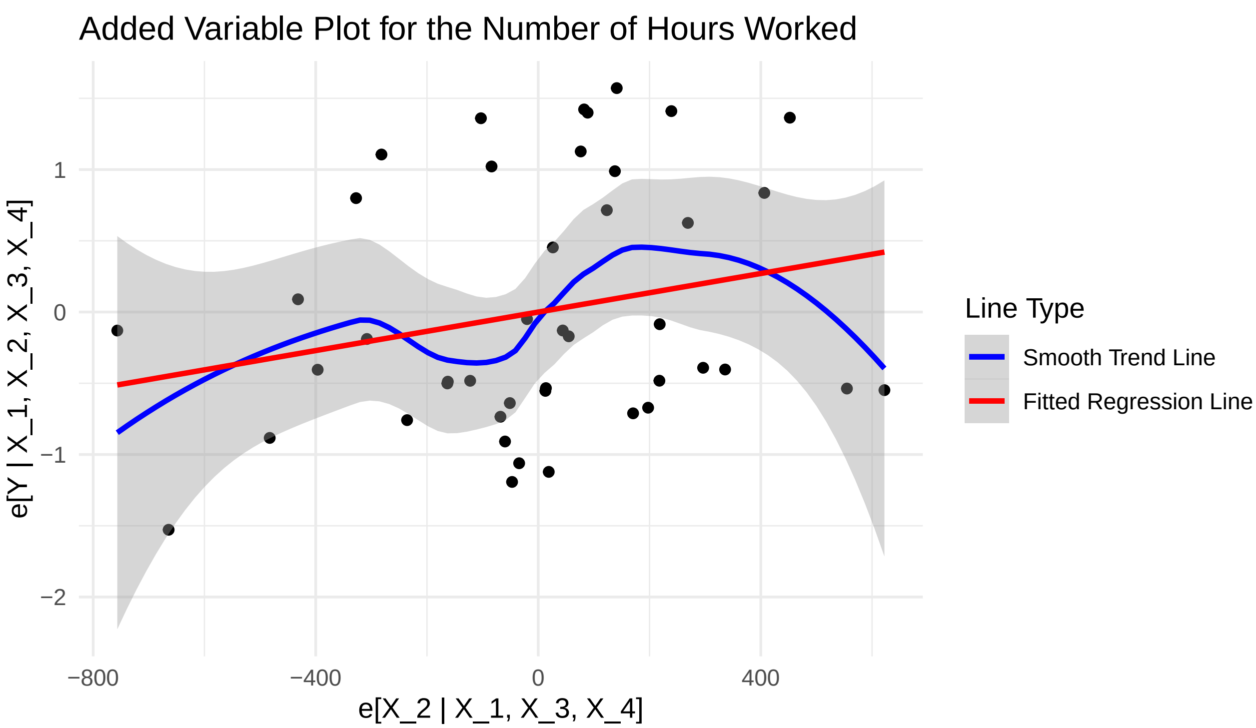


1- D

In order to evaluate the nature of the relationship between complaint rates per 1,000 and the number of hours worked after adjusting for the other 3 predictors we use an added variable plot. In order to do that we will need to obtain two sets of residuals from the two models:

- Model 1: obtains residuals for Y = Complaints per 1,000 visits. We denote these residuals as $\epsilon_Y = e(Y|X_1, ..., X_4)$:
  - $Y = \hat{\beta}_0 + \hat{\beta}_1 * Revenue + \hat{\beta}_2 * Hours\ Worked + \hat{\beta}_3 * Gender + \hat{\beta}_4 * Residency + \epsilon_Y$
- Model 2: obtains residuals for $X_2 = $ Number of Hours Worked. We denote these residuals as $\epsilon_X = e(X_2|X_1, X_3, X_4)$:
  - $X_2 = \hat{\beta}_0 + \hat{\beta}_1 * Revenue + \hat{\beta}_3 * Gender + \hat{\beta}_4 * Residency + \epsilon_x$

Plot below shows the relationship between the two sets of residuals:



This plots gives us two pieces fo evidence that we sue to describe the marginal relationship of X_2 and Y, after adjusting for three other predictors:

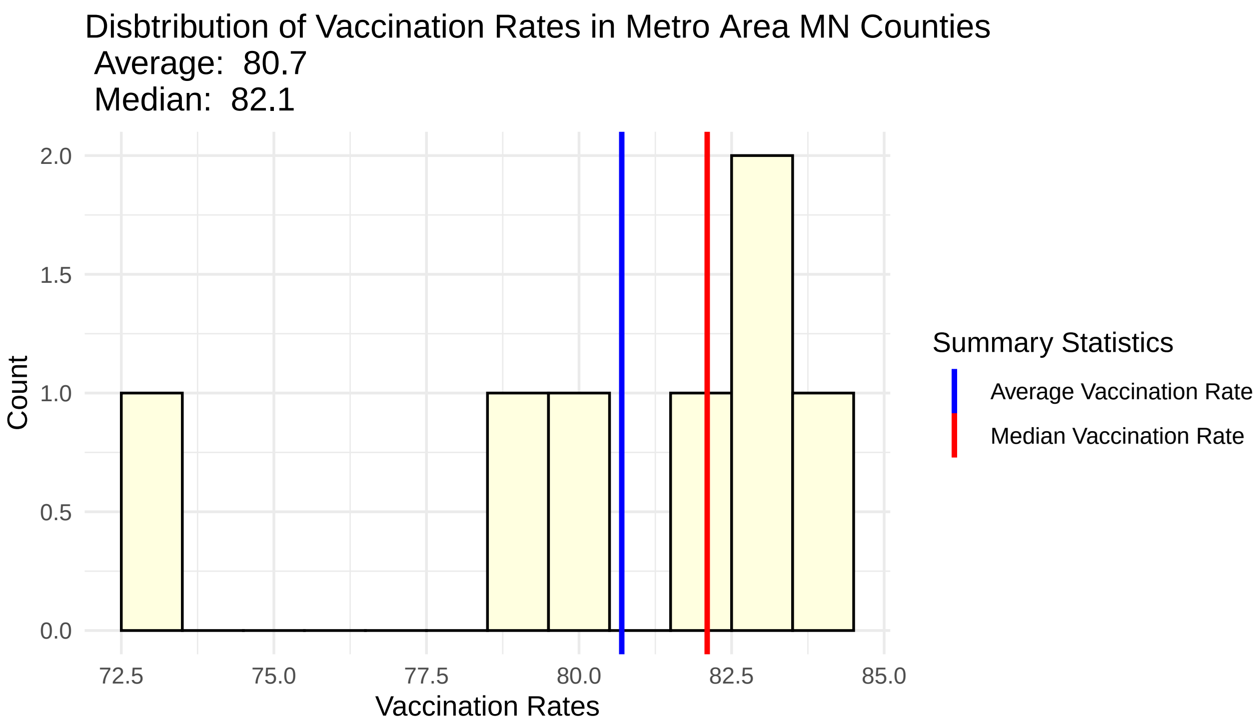
1. The relationship between X_2 and Y, after accounting for other predictors, is linear in its nature. We can see that the smooth trend line fluctuates randomly around the fitted regression line, suggesting that there is no consistent curved, or other non-linear relationship between the number of hours worked and the complaint rate.
2. The fitted regression line that confirm linear relationship has a positive, upward facing, slope, suggesting that the number of hours worked can be used a potentially useful predictor that help increase the percentage of variation of in the complaint rate. This supports our previous conclusion that X_2 may be employed as a useful predictor of Y, but this model does not gives us enough sufficient evidence to make such a claim.

Problem 2

2 - A

In order to pick between the two-sample T-test and Wilcoxon test we need to understand the shape of the distribution, in particular the spread of values, and the effect that extreme values and outliers can have on the t-test. While t-test is robust and produces that we can rely on, it is known that in heavily skewed distributions non-parametric methods that rely on rank of observations will be more effective. On the other hand, if we do not see a heavily skewed distribution, but instead see a distribution that is approximately normal, we want to use a t-test, because for such data Wilcoxon has only 95% of statistical power of the the T-test.

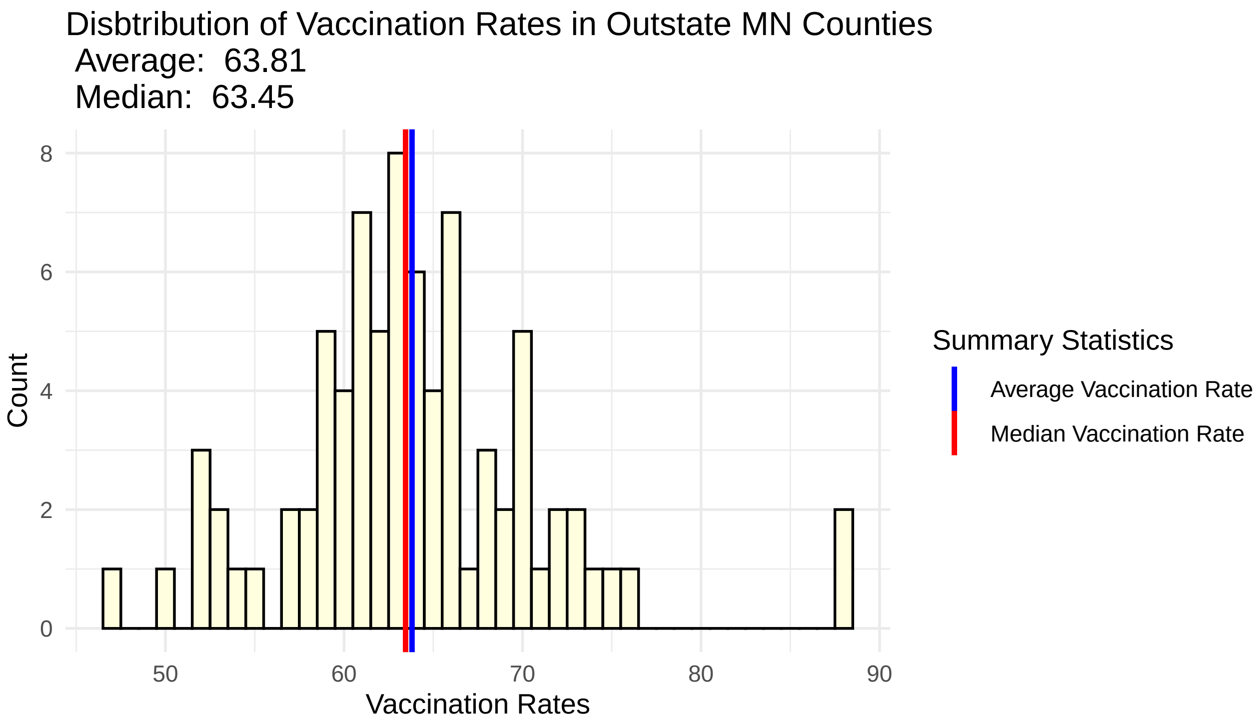
The plot below shows vaccination rates for the metro area counties. It is pretty card to make any conclusions from this plot, and this group of observations, since there are only 7 counties that make up Metro area.



On the other hand, there are 80 Outstate counties. These counties produce a balanced bell-shaped distribution that looks normal. However, there are a few outliers. Outstate counties with unusually high vaccination rates are: Olmsted and Cook with 88.5 and 87.9 vaccination rates, respectively.

Olmsted county includes Rochester, a pretty big city by the outstate standards. Moreover, Mayo clinic is located there, so we can speculate that more people should have more trusting relationship with medicine and public health there.

Cook county is located by the Canadian Border, I am not sure what conclusion we can draw from this fact.



The table below summarized two distributions in terms of most common summary statistics. This table gives me the impression that we will be able to conclude that the two sample means are in fact different because the we have small standard deviations, while the two means are quite different.

Table 3: Vaccination Rates Summary by County Type

Type	N	Mean	Median	S.D.
Outstate	80	63.81	63.45	7.08
Metro	7	80.70	82.10	3.63

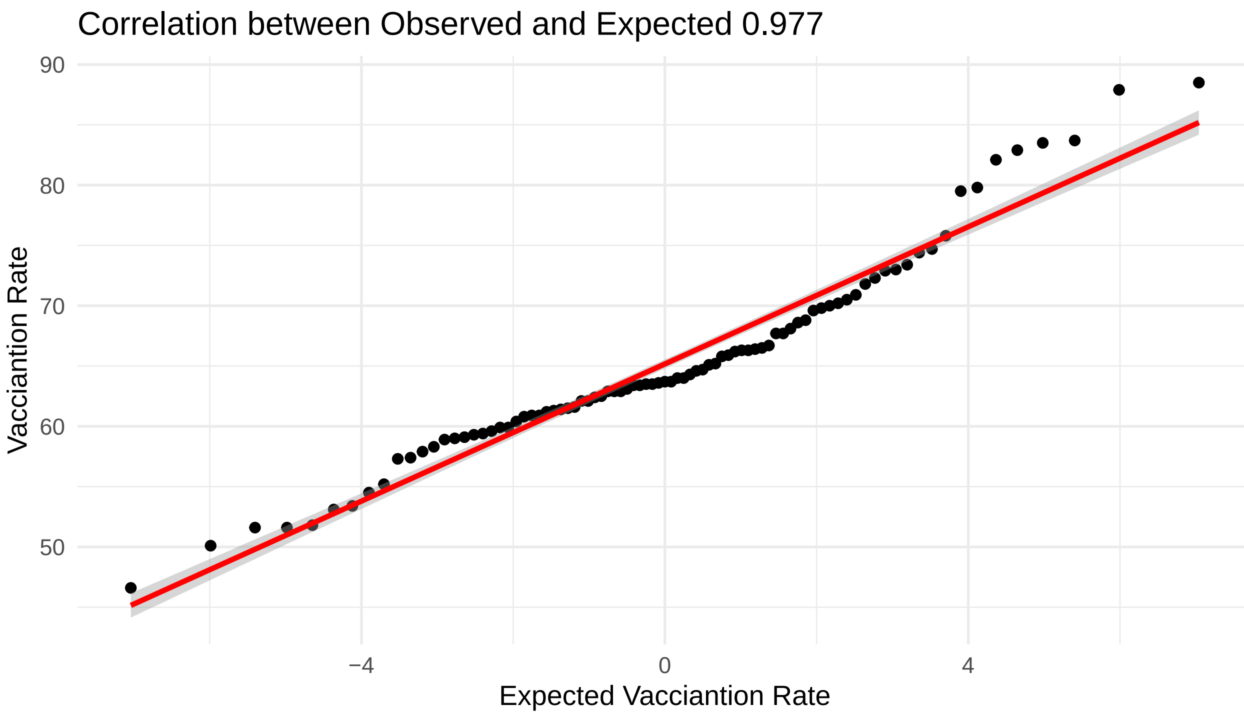
Before conducting the test, we also wish to see if the overall distribution of the two samples combined is normal. Recall, there are only 7 counties in the metro area, so we should combine the two samples for this verification.

We can test the Normality of Vaccination Rates distribution against the expected quantiles of standard normal distribution.

We can calculate these expected values using the formula:

$$\sqrt{Variance} \times z(\frac{Value - .375}{N + .25})$$

It appears that the sample of data we have is approximately normally distributed.



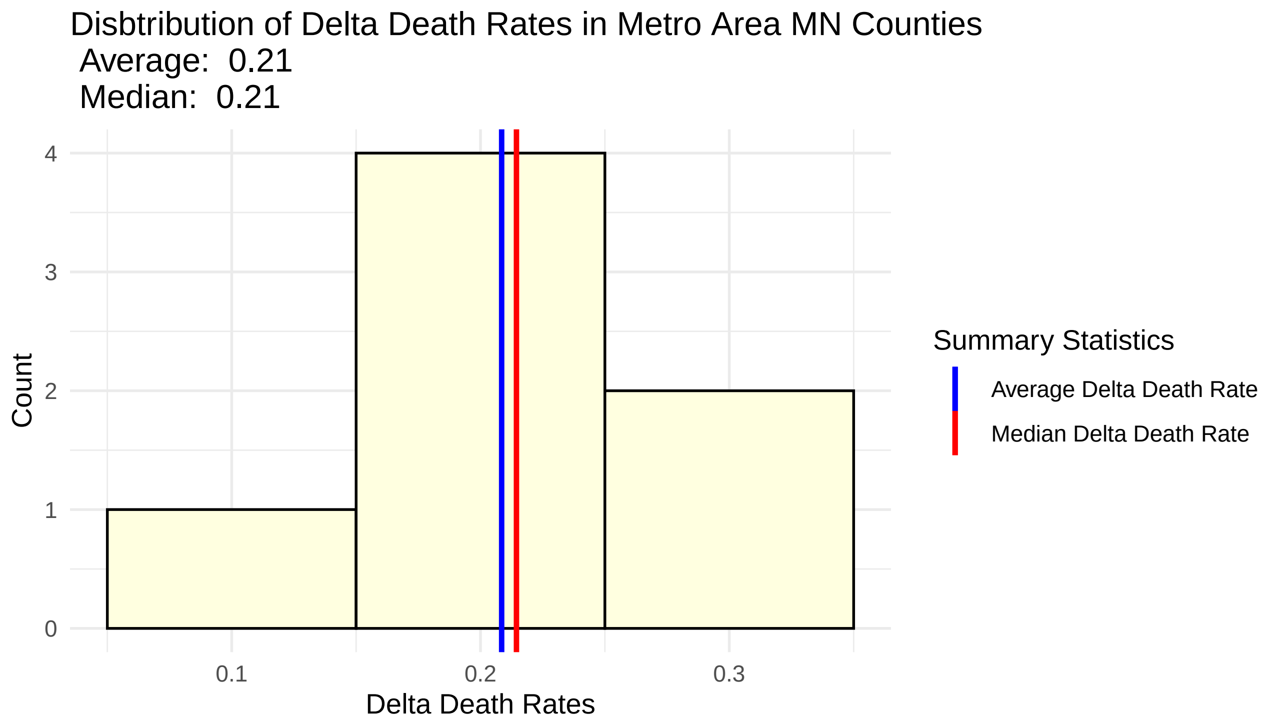
Therefore, we will use T test here.

Test results summary and interpretation are given below:

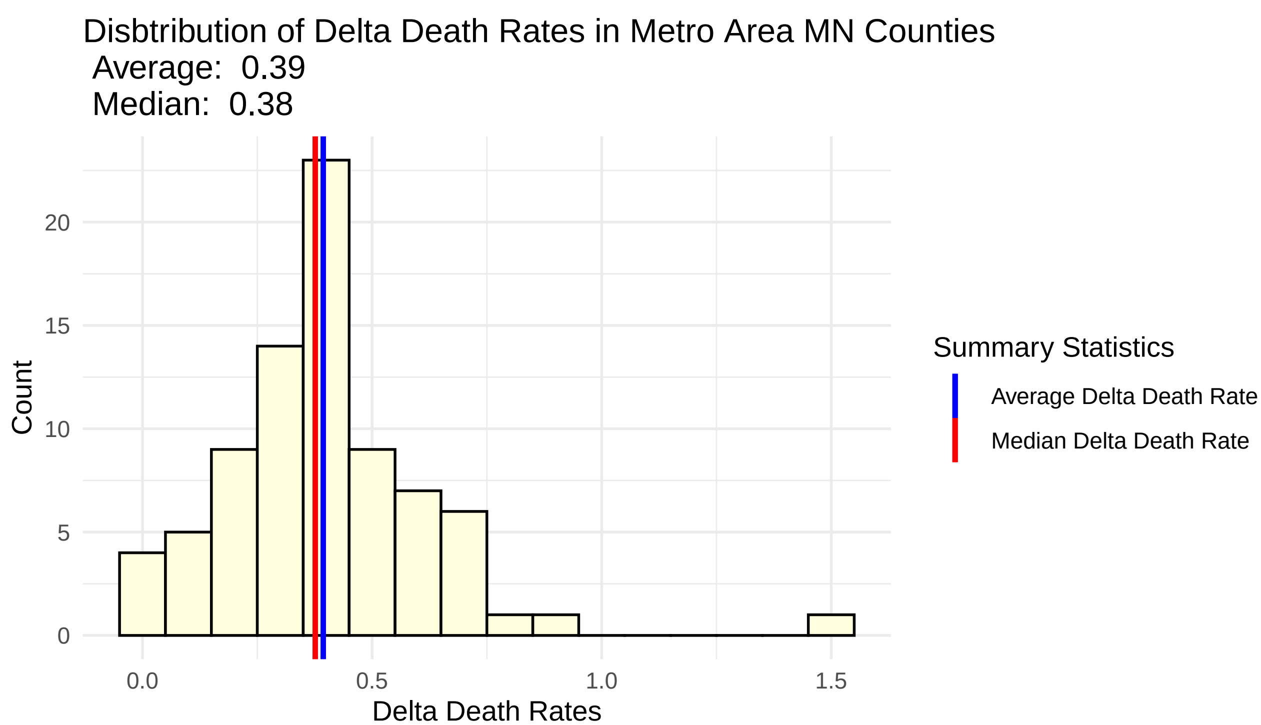
- Null Hypothesis: $H_0 : \mu_{metro\ area} = \mu_{outstate}$
- Test statistic: $H_a : \mu_{metro\ area} \neq \mu_{outstate}$
- Metro area mean vaccination rate is 80.7, while outstate median vaccination mean is 63.81
- Estimated difference is -16.89, bounded by (-20.3959 , -13.3841)
- Test statistic T: -10.6576166
- $P(T^* > T) = 0.000001$
- Conclusion: P-value is small, so we can reject the null hypothesis and conclude that the average difference in vaccination rates on the county level is statistically significant between metro and rural areas. On average, we can expect metro area counties to have 16.89 more vaccines administered per 1,000 county residents.

2 - B

Once again, we begin the problem with the distribution visualization. Due to a small number of observations in the sample of metro counties we have a histogram that does not really provide much information.



Delta variant COVID related death rates for outstate counties looks approximately normally distributed with some heavy and obvious outliers.



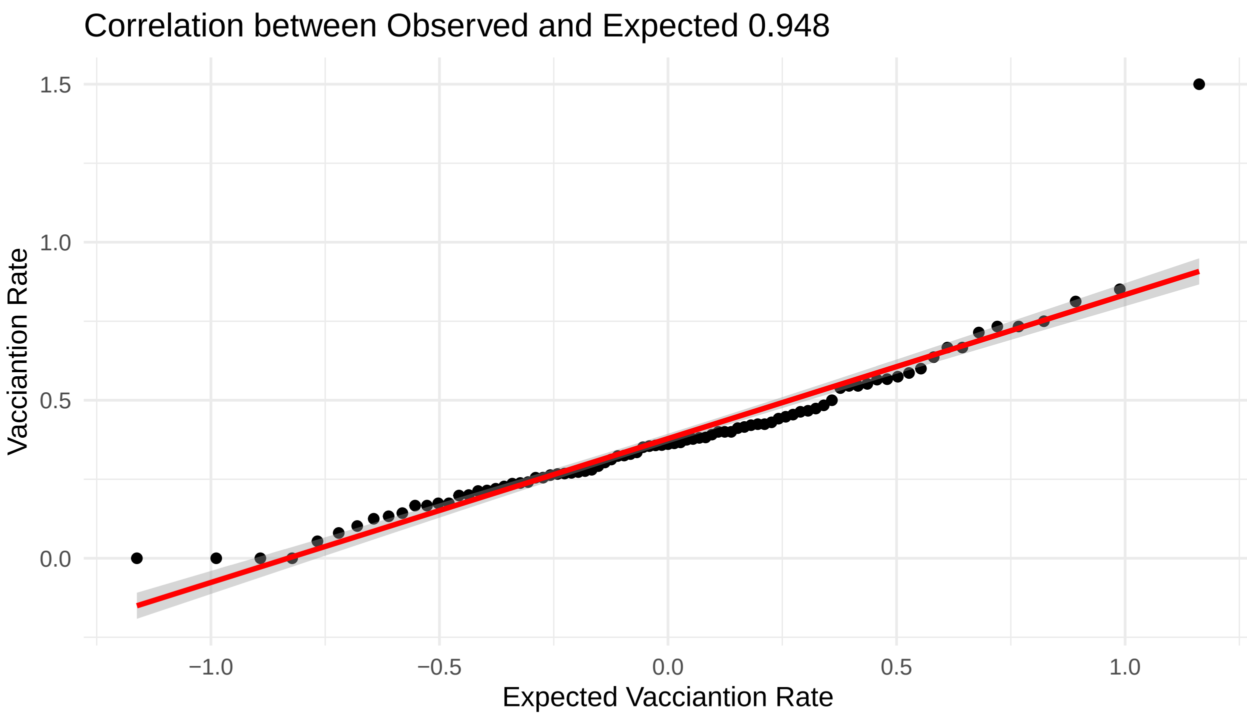
This outlier is a Faribault

Death Rates for this county is 1.5, and the corresponding vaccination rate is 60.9, which corresponds to the 31.25th quantile of vaccination rates for the outstate counties. Perhaps, a lower vaccination causes more COVID related death to happen, which we will verify later in the problem.

Table 4: Death Rates Summary by County Type

Type	N	Mean	Median	S.D.
Outstate	80	0.3936199	0.3761792	0.23
Metro	7	0.2085040	0.2144522	0.04

We visualize the values of death rate to those expected under the normal distribution.



It appears that there are no violations of normality, other than having a huge outlier on the upper end.

Test results summary and interpretation for the average death rates between metro and rural ares is given below:

- Null Hypothesis: $H_0 : \mu_{metro\ area} = \mu_{outstate}$
- Test statistic: $H_a : \mu_{metro\ area} \neq \mu_{outstate}$
- Metro area mean vaccination rate is 0.208504, while outstate median vaccination mean is 0.3936199
- Estimated difference is 0.185116, bounded by (0.1252 , 0.2451)
- Test statistic T: 6.1920794
- $P(T^* > T) = 0$
- Conclusion: P-value is small, so we can reject the null hypothesis and conclude that the average difference in death rates on the county level is statistically significant between metro and rural areas. On average, we can expect metro area counties to have 0.185116 COVID related death per 1,000 county residents.

2 - C

Model Specification

$$E[Death\ \ Rate] = \hat{\beta}_0 + \hat{\beta}_1 * X_1 + \hat{\beta}_2 * X_2 =$$

$$E[Death\ \ Rate] = \hat{\beta}_0 + \hat{\beta}_1 * Vaccination\ \ Rate + \hat{\beta}_2 * Metro\ \ Area\ \ County\ \ Indicator$$

Overall ANOVA test

Source	SSR	DF	MS	F Statistic	P(F* > F)
Regression	0.5740292	2	0.2870146	6.37	0.0027
Error	3.7854492	84	0.0450649	NA	NA
Total	4.3594784	86	NA	NA	NA

- Null Hypothesis: $H_0 : \beta_1 = \beta_2 = ... = \beta_{p-1}$
- Alternative Hypothesis: H_a : Not all coefficients β_i are zero
- $F-$statistic: 6.37
- Cutoff F^*-statistic: 3.1052
- So, $F < F^*$, therefore we do not have enough evidence to reject the null hypothesis to conclude that some or all coefficients β_i are consistently different from zero.
- Moreover, $P(F^* > F) = 0.0027$
- Conclusion: There is enough statistical evidence to reject the null hypothesis and conclude that at least one coefficient is different from 0. Therefore, we will evaluate model output and find what explains variation in death rates on the county level.

Table below shows **Regression Coefficients** and model summary. Like we expected, these coefficients are small because the scale and range of predictors and response variable are not the same.

Predictor	Estiamte	Standard Error	T Value	P value
(Intercept)	0.990663	0.214503	4.618412	0.000014
v_rate	-0.009357	0.003341	-2.800576	0.006329
region	-0.027083	0.100922	-0.268358	0.789081

- R square 0.1317
- Adjusted R Square 0.111

Note that X_2 is the metro/rural area flag indicator variable, so we conduct a formal t-test for this variable below:

- Null Hypothesis: $H_0 : \hat{\beta}_2 = 0$
- Alternative Hypothesis: $H_a : \hat{\beta}_2 \neq 0$
- Test statistic T : -0.268358
- $P(t^* > t) = 0.789081$

- Conclusion: the p-value is quite large, so we can't reject the null hypothesis. The only useful predictor is the vaccination rate, and it is a statistically significant predictor. This is a bit contradictory with the results that we saw in the individual t-tests. So far, we know that vaccination rates are higher for metro counties, while death rates are lower for the metro areas. But the regression model we used does not support the statement that death rates meaningfully differ between the two types of counties. Therefore, we can conclude that vaccination rates is either:
  - a much stronger predictor than a county status
  - a more likely conclusion is that the two are correlated, and the effect of vaccination rate already includes most of county effect into its coefficient

Interpretation of coefficient

Metro Area is expected to have 0.0271 less COVID related deaths per 1,000 residents

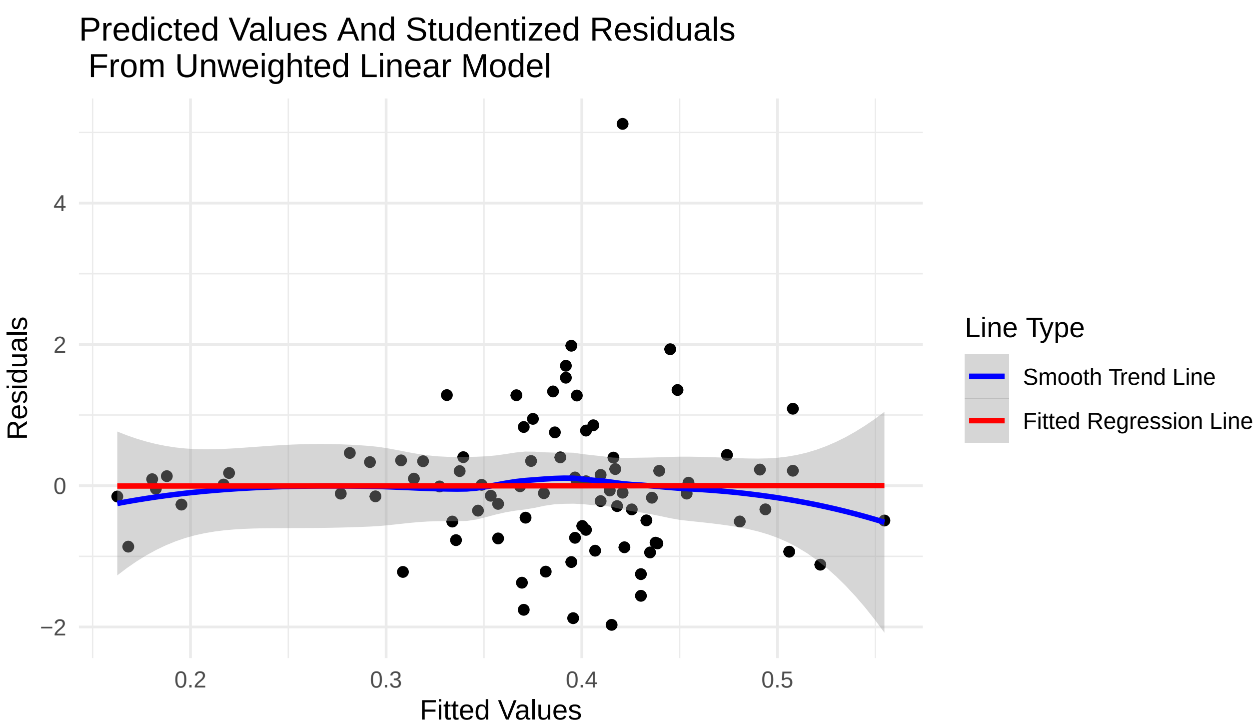
C.I.

C.I. for the estimate -0.027083 with a 0.100922 standard error is (-0.227779, 0.173612). Note now the C.I. is pretty equally balanced around the estimate that is very close to 0, which gives us an even higher degree of certainty that the difference between the metro and rural counties is more likely to be 0, after adjusting for vaccination rate.

2 - D

If we are suggested to use a weighted least squares linear model, then we must have a problem with constant variance assumption.

Let's look at the plot of studentized residuals versus predicted values.



Indeed, we can see that variance of residuals is not constant for different levels of predicted values. We have a megaphone type pattern, and we have a variety of ways how we can construct weights for the updated regression model.

Moreover, there is a very obvious outlier here: a Faribault county.

1. A remedial measure from chapter 11 of the textbook suggests that the can regress absolute values of residuals on the fitted values and obtain weights through the following variance function:

- So we have a regression model $|\epsilon_i| = \hat{\beta}_0 + \hat{\beta}_1 * \hat{Y}_i + error_i$
- Weights for each observation is given by $w_i = \frac{1}{E[|\epsilon_i|]}$

2. Another remedial measure is to fit the variance function using a different way:

- Regression model for the variance function is $\epsilon_i^2 = \hat{\beta}_0 + \hat{\beta}_1 * Vaccination\ Rate + \hat{\beta}_2 * Metro\ Area\ Flag + error_i$
- Weights for each observation is given by $w_i = \frac{1}{E[\epsilon_i^2]}$

Method 1

We re-fit a linear model with weights using method 1 and provide a model summary below:

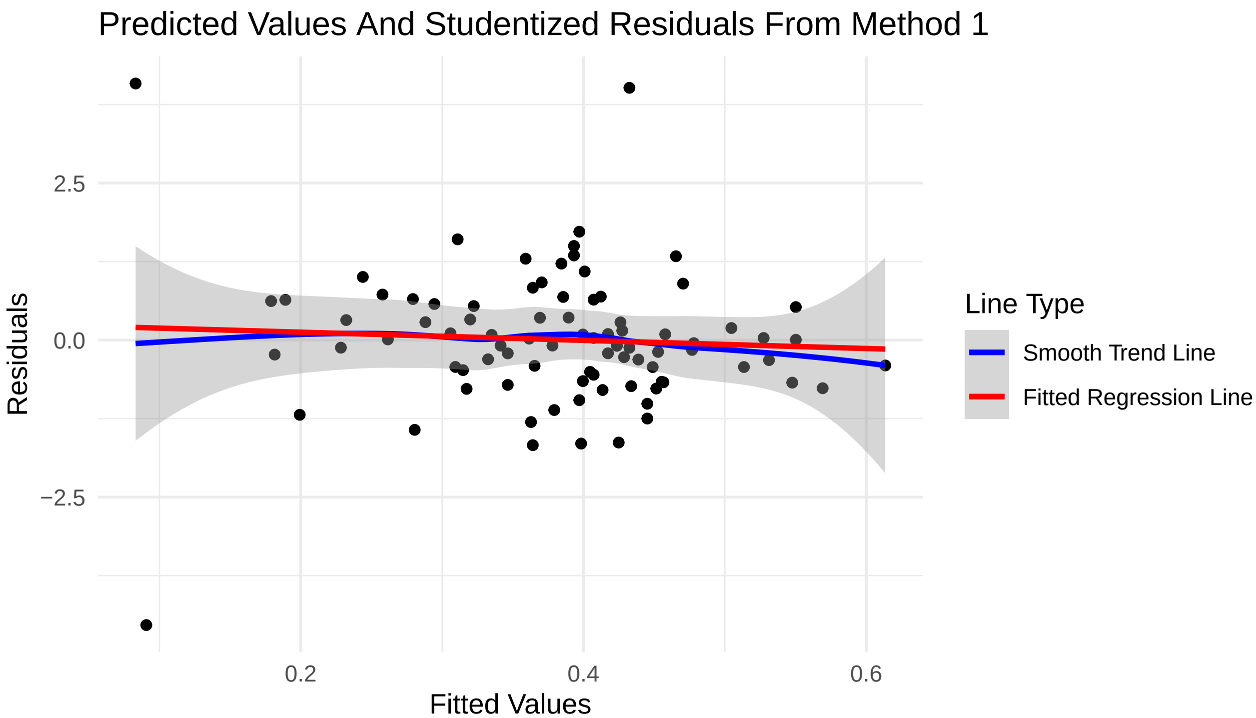
- `v_rate` is a variable for vaccination rates
- `region` is an indicator variable comparing metro areas with the outstate counties, which is a reference level

Predictor	Estiamte	Standard Error	T Value	P value
(Intercept)	1.203182	0.106162	11.333434	0.000000
v_rate	-0.012656	0.001278	-9.901332	0.000000
region	0.035093	0.023223	1.511105	0.134515

After re-weighting the model using method 1, model estimates changed compared with the original model. Key differences are given below:

- R square changed from 0.1317 to 0.5439
- Adjusted R Square changed from 0.111 to 0.533
- Vaccination rates is still a statistically significant predictor, and the coefficient did not change meaningfully.
- County status' p-value dropped meaningfully, bringing closer to the level that will make us believe that the death rates are potentially higher in the metro area, on average, after adjusting for vaccination rates. This is quite a contradictory statement, however, we know the confidence interval is centered at around 0 for this predictor, therefore interpretation of this coefficient can produce misleading results

We also evaluate the plot of studentized residuals with the fitted values from the regression model with weighting described in method 1.



We can see that, overall, the variance of the residuals is in more balance. Yes, there is still some increase in residual values as fitted values increase, but it stops at around fitted values of 0.3 for positive values of residuals. Negative residuals seem to be at around -1 to -1.2 bound. One noticeable problem is the introduction of two more new outliers, where the model predicts a very low fitted value, but the actual observed value differs meaningfully from the prediction.

Method 2

I encountered one issue when implementing method 2. After fitting a regression model with squared residuals as the response variable, and vaccination rates and county status as predictors, I obtained fitted values from the model, and some of them were negative, as evidenced by the summary output below:

```
weights6 <- 1 / lm( (death_lm$residuals)^2 ~  v_rate + region, data = prob_2)$fitted.values
summary(weights6)
```

```
##     Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
## -941.491   18.875   20.923    8.605   23.767  500.826
```

It should not be the case as variance is always greater than 0. There are 4 counties where the output of the variance function is below 0. Those values are: -341.9912795, -470.3466357, -313.475918, -941.4907238, so, for this exercise, I will take an absolute value of these values to make sure all inverse variance weights are above 0.

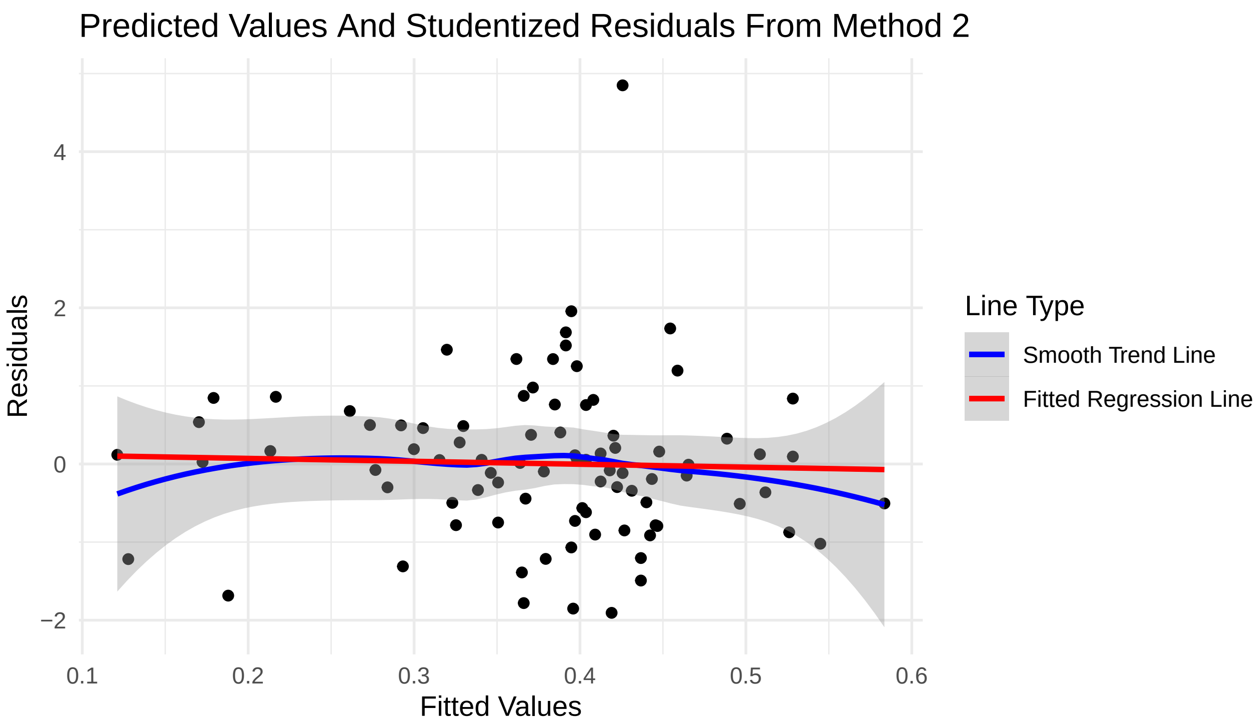
We re-fit a linear model with weights using method 1 and provide a model summary below:

- `v_rate` is a variable for vaccination rates
- `region` is an indicator variable comparing metro areas with the outstate counties, which is a reference level

Predictor	Estiamte	Standard Error	T Value	P value
(Intercept)	1.097770	0.173621	6.322807	0.000000
v_rate	-0.011036	0.002608	-4.231080	0.000059
region	-0.003764	0.050078	-0.075159	0.940267

After re-weighting the model using method 1, model estimates changed compared with the original model. Key differences are given below:

- R square changed from 0.1317 to 0.391
- Adjusted R Square changed from 0.111 to 0.3765
- Vaccination rates is still a statistically significant predictor, and the coefficient did not change meaningfully.



We can see that, overall, the variance of the residuals is in more balance. There is more increase in residual values as fitted values increase for positive values of residuals, compared with method 1. We still could not deal with one outlier that is more than 4 studentized residuals away. Overall, lower bound for residuals does not exhibit any suggestions that there is a violation fo variance assumption, but positive residuals still have non-constant variance, it appears.

Overall, the two methods of weighted linear regression produce three sets of estimates that agree with each other. All three models showed that higher vaccination rates caused lower death rates after adjusting for metro/rural area status.

Metro/Outstate indicator was not statistically significant in any of the models. The point estimate for coefficient fluctuated above and below zero in three models, but none of the test statistics and p-values showed enough evidence that this variable is a useful predictor.