Problem 6

Given: $\text{logit}(\pi) = \alpha + \beta x$

$x = 0$ or $x = 1$

$x = 0 \Rightarrow n_0$ observations and $p = \frac{y_0}{n_0}$

$x = 1 \Rightarrow n_1$ observations and $p = \frac{y_1}{n_1}$

1) We have more than one observation at each x_0 and x_1, so we can use a formula for likelihood:

Joint mass function: $\prod_{i=0}^{1} \pi(x_i)^{y_i} \left(1 - \pi(x_i)\right)^{n_i - y_i}$

2) We have $i = 1$ or 0, so

$$L = \pi(x_0)^{y_0}\left(1 - \pi(x_0)\right)^{n_0 - y_0} \cdot \pi(x_1)^{y_1} \cdot \left(1 - \pi(x_1)\right)^{n_1 - y_1}$$

3) Now take the log, obtain log-likelihood:

$$\ell = y_0 \cdot \log(\pi(x_0)) + (n_0 - y_0)\log(1 - \pi(x_0)) + \\ + y_1 \log(\pi(x_1)) + (n_1 - y_1)\log(1 - \pi(x_1))$$

4) logit function implies that $\pi(x_i) = \dfrac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}$

So, we can plug it into our ℓ.

when $x = 0$, $\pi(x_0) = \dfrac{e^{\alpha}}{1 + e^{\alpha}}$

$x = 1$, $\pi(x_1) = \dfrac{e^{\alpha + \beta}}{1 + e^{\alpha + \beta}}$

So, $\ell = y_0 \log\left(\frac{e^{\alpha}}{1+e^{\alpha}}\right) + (n_0 - y_0)\log\left(\frac{1}{1+e^{\alpha}}\right) +$

$$+ y_1 \log\left(\frac{e^{\alpha+\beta}}{1+e^{\alpha+\beta}}\right) + (n_1 - y_1)\log\left(\frac{1}{1+e^{\alpha+\beta}}\right) =$$

$$= y_0 \log(e^{\alpha}) - y_0 \log(1+e^{\alpha}) - (n_0 - y_0)\log(1+e^{\alpha}) +$$

$$+ y_1 \log(e^{\alpha+\beta}) - y_1 \log(1+e^{\alpha+\beta}) - (n_1 - y_1)\log(1+e^{\alpha+\beta})$$

$$\boxed{= y_0 \cdot \alpha + y_1(\alpha+\beta) - n_0 \log(1+e^{\alpha}) - n_1 \log(1+e^{\alpha+\beta}) = \ell}$$

we will use this log-likelihood function to find α and β.

5) By definition, we need to solve a system of two equations to find two unknowns:

$$\begin{cases} \dfrac{\partial \ell}{\partial \alpha} = 0 \\ \dfrac{\partial \ell}{\partial \beta} = 0 \end{cases}$$ yields α and β that are MLE estimates.

6) we take a partial derivative w.r.t. β first because it gives us an equation w/ fewer terms:

$$\frac{\partial \ell}{\partial \beta} = y_1 - n_1\left(\frac{e^{\alpha+\beta}}{1+e^{\alpha+\beta}}\right) = 0,$$

$$y_1(1+e^{\alpha+\beta}) = n_1(e^{\alpha+\beta})$$

$$y_1 = (n_1 - y_1)(e^{\alpha+\beta})$$

$\log\left(\frac{y_1}{n_1 - y_1}\right) = \alpha + \beta$ we stop here and move to $\frac{\partial \ell}{\partial \alpha}$

7) $\frac{\partial \ell}{\partial \alpha} = y_0 + y_1 - \frac{n_0 e^{\alpha}}{1+e^{\alpha}} - \frac{n_1 (e^{\alpha+\beta})}{1+e^{\alpha+\beta}} = 0$

we can use the fact that $\alpha+\beta = \log\left(\frac{y_1}{n_1-y_1}\right)$ and plug it in here.

$$y_0 + y_1 - \frac{n_0 e^{\alpha}}{1+e^{\alpha}} - n_1\left(\frac{e^{\log\left(\frac{y_1}{n_1-y_1}\right)}}{1+e^{\log(y_1/(n_1-y_1))}}\right) = 0$$

↳ we now have an equation in terms of α only!

$$y_0 + y_1 - \frac{n_0 e^{\alpha}}{1+e^{\alpha}} - \frac{\left(\frac{n_1 y_1}{n_1-y_1}\right)}{\left(1+\frac{y_1}{n_1-y_1}\right)} = 0,$$

$$y_0 + y_1 - \frac{n_0 e^{\alpha}}{1+e^{\alpha}} - \left(\frac{\cancel{n_1} y_1}{\cancel{n_1-y_1}}\right)\left(\frac{\cancel{n_1-y_1}}{\cancel{n_1}}\right) = 0,$$

$$y_0 + y_1 - \frac{n_0 e^{\alpha}}{1+e^{\alpha}} - y_1 = 0,$$

$$y_0(1+e^{\alpha}) = n_0 e^{\alpha},$$

$$y_0 = e^{\alpha}(n_0 - y_0),$$

$$\left(\frac{y_0}{n_0-y_0}\right) = e^{\alpha} \Rightarrow \alpha = \log\left(\frac{y_0}{n_0-y_0}\right)$$

8) So, we now have

$$\alpha = \log\left(\frac{y_0}{n_0 - y_0}\right) \quad \text{and} \quad \alpha + \beta = \log\left(\frac{y_1}{n_1 - y_1}\right)$$

Therefore, $\beta = \log\left(\frac{y_1}{n_1 - y_1}\right) - \log\left(\frac{y_0}{n_0 - y_0}\right) =$

$$= \log\left[\frac{\left(\frac{y_1}{n_1 - y_1}\right)}{\left(\frac{y_0}{n_0 - y_0}\right)}\right]$$

9) if x_0 has $p = \frac{y_0}{n_0}$, then $q = \frac{n_0 - y_0}{n_0}$, and

odds for x_0 are $\frac{y_0}{n_0 - y_0}$

Similarly, x_1 odds are $\frac{y_1}{n_1 - y_1}$

Then, the log-odds ratio for this sample is:

$$\log\left[\frac{y_1}{n_1 - y_1}\right] - \log\left[\frac{y_0}{n_0 - y_0}\right] = \beta.$$

$\blacksquare$