PUBH 7440: Intro to Bayesian Analysis
Homework from Week 5 — Due Feb 22nd/27th

Stroke mortality in PA: Using the same data as HW 3, we want to fit the following two models:

- $Y_{ia} \sim \text{Pois}\left(n_{ia}\lambda_{ia}\right)$ where $\log \lambda_{ia} = \beta_{0a} + z_{ia}$ and $z_{ia} \sim \text{Norm}\left(0, \sigma_a^2\right)$.
- $Y_{ia} \sim \text{Pois}\left(n_{ia}\lambda_{ia}\right)$ where $\log \lambda_{ia} = \theta_{ia} \sim \text{Norm}\left(\beta_{0a}, \sigma_a^2\right)$.

In both cases, use standard $\beta_{0a} \sim \text{Norm}\left(0, \tau_a^2\right)$ and $\sigma_a^2 \sim \text{IG}\left(0.001, 0.001\right)$ priors, with $\tau_a^2 = 10{,}000$, and answer the following questions:

1. Write the full hierarchical model. As with HW 3, account for the suppression of $Y_{ia} < 10$.
2. Derive the full-conditional distributions for $\{\boldsymbol{\beta}_0, \mathbf{z}, \sigma_a^2\}$ and $\{\boldsymbol{\beta}_0, \boldsymbol{\theta}, \sigma_a^2\}$ for the two models, respectively. Which parameters have full-conditional distributions we can sample from directly, and which parameters require Metropolis steps to sample?
3. Write code to fit the two models. How do the results compare? Examples of "results" to compare include:
   - Posterior summaries for β_{0a} from the two models.
   - Posterior summaries for $\lambda_{i\cdot}$, the age-adjusted rates (you can use the same code as in HW 3 to make maps of these rates).

   Note: Use symmetric candidate densities — e.g., $\theta^* \sim \text{Norm}\left(\theta^{(\ell-1)}, q\right)$ — to generate proposed values for all Metropolis steps. Don't worry too much about acceptance rates, though the closer you can get to 44% acceptance rates, the better your convergence will be. If you're having issues with this, let me know.

My plan will be to have the *first model* (the one with z_{ia}) due on Thursday Feb 22nd and the second model due Tuesday Feb 27th. I suppose that means that when you turn in the results for Model 1, you need to report the various summaries for #3, but there won't be a "compare" part of that question until you've done Model 2.

Getting Guesses for β_{0a}, Z_{ia}.

$$\log \lambda_{ia} = \beta_{0a} + Z_{ia}.$$

Problem 1

Full hierarchical model:

- $Y_{ia} \sim Pois(u_{ia} \lambda_{ia})$

  $\log \lambda_{ia} = \beta_{0a} + Z_{ia}$

  $\lambda_{ia} = \exp(\beta_{0a} + Z_{ia}) \implies$

  $Y_{ia} \sim Pois(u_{ia} \exp(\beta_{0a} + Z_{ia}))$

- Full model

$$p(\beta_{0a}, Z_{ia}, \sigma_a^2 \mid Y_{ia}) \propto h(\vec{\theta}) =$$

$$\left[\prod_{i,a} Pois(Y_{ia} \mid \beta_{0a}, Z_{ia}) \right]^{\bar{1}} \times \underline{1}(Y_{ia} < 10)^{1-d_{ia}} \times$$

$$\times Norm(\beta_{0a} \mid 0, \tau_0^2) \times$$

$$\times Norm(Z_{ia} \mid 0, \sigma_a^2) \times$$

$$\times IG(\sigma_a^2 \mid a, b)$$

where $\tau_0^2 = 10,000$

$a = 0.001$

$b = 0.001$

Problem 3

likelihood of data is:

$$L = \prod_{i,a} \Big[\exp(-n_{ia} \exp(\beta_{0a} + z_{ia})) \times (n_{ia} \exp(\beta_{0a} + z_{ia}))^{y_{ia}} \times \frac{1}{y_{ia}!} \Big]$$

$$= \exp\Big(-\sum_{i,a} n_{ia} \exp(\beta_{0a} + z_{ia})\Big) \times \prod_{i,a} \times n_{ia}^{y_{ia}} \times \exp(y_{ia}(\beta_{0a} + z_{ia})) \times \frac{1}{y_{ia}!}$$

$$= \exp\Big(-\sum_{i,a} n_{ia} \exp(\beta_{0a} + z_{ia})\Big) \times \exp\Big(\sum y_{ia}(\beta_{0a} + z_{ia})\Big) \times \prod \frac{n_{ia}^{y_{ia}}}{y_{ia}!}$$

$$\propto \exp\Big(\sum y_{ia}(\beta_{0a} + z_{ia}) - \sum n_{ia} \exp(\beta_{0a} + z_{ia})\Big)$$

- Given a new version of data likelihood, derive a new ratio form

$$h(\beta_{0a}) = \exp\left(\sum y_{ia}(\beta_{0a} + z_{ia}) - \sum u_{ia}\exp(\beta_0 + z_{ia})\right) \times \frac{1}{\sqrt{2\pi\tau_a^2}} e^{-\frac{1}{2}\frac{\beta_{0a}^2}{\tau_a^2}}$$

ratio: $r(\beta^*_{0a}) =$

$$\frac{\exp\left(\sum y_{ia}(\beta^*_{0a} + z_{ia}) - \sum u_{ia}\exp(\beta^*_{0a} + z_{ia})\right) \times e^{-\frac{1}{2}\beta^{*2}_{0a}/\tau_a^2}}{\exp\left(\sum y_{ia}(\beta'_{0a} + z_{ia}) - \sum u_{ia}\exp(\beta'_{0a} + z_{ia})\right) \times e^{-\frac{1}{2}\beta'^2/\tau_a^2}}$$

$$= \exp\left(\sum y_{ia}(\beta^a_{0a} - \beta'_{0a})\right) \times$$

$$\exp\left(\sum u_{ia}\exp(\beta'_{0a} + z_{ia}) - u_{ia}\exp(\beta^*_{ia} + z_{ia})\right) \times$$

$$\exp\left(\frac{1}{2\tau_a^2} \left(\beta_{0a}^{*2} - \beta_{0a}'^{2} \right)\right)$$

$$= \exp\left(\sum y_{ia} \left(\beta^{a}_{0a} - \beta'_{0a}\right) \times \right.$$

$$\exp\Big(\sum u_{ia} \exp(\beta'_{0a}) \exp(z_{ia}) -$$

$$u_{ia} \exp(\beta^{*}_{ia}) \exp(z_{ia}) \Big) \times$$

$$\exp\left(-\frac{1}{2\tau_a^2} \left(\beta_{0a}^{*2} - \beta_{0a}'^{2} \right)\right)$$

$$= \exp\left(\sum y_{ia} \left(\beta^{*}_{0a} - \beta'_{0a}\right)\right) \times$$

$$\exp\left(\left[\sum u_{ia} \exp(z_{ia}) \right] \left(\exp(\beta'_{0a}) - \exp(\beta^{*}_{ia}) \right)\right) \cancel{\times}$$

$$\exp\left(-\frac{1}{2\tau_a^2} \left(\beta^{*2}_{0a} - \beta'^{2}_{0a} \right)\right) = R$$

- Translate to the log scale
- But, the result will need to be exponentiated anyway.

$$\log R =$$

$$\sum Y_{ia} (\beta^{*}_{0a} - \beta^{\prime}_{0a}) +$$

$$\left[\sum u_{ia} \exp(z_{ia}) \cdot \left(\exp(\beta^{\prime}_{0a}) - \exp(\beta^{*}_{0a}) \right) \right] -$$

$$\frac{1}{2\tau^2_a} \left(\beta^{*2}_{0a} - \beta^{\prime 2}_{0a} \right)$$

Update Z_{ia}.

Z_{ia} is specific to each county and age group. Therefore, I don't think we need to use likelihood, just the dens.

- $$Y_{ia} \sim \exp(-n_{ia} \exp(\beta_{0a} + Z_{ia})) \times (n_{ia} \exp(\beta_{0a} + Z_{ia}))^{Y_{ia}} \times \frac{1}{(Y_{ia}!)}$$

$$Z_{ia} \sim \frac{1}{\sqrt{2\pi\theta_a^2}} e^{-\frac{1}{2}\frac{Z_{ia}^2}{\theta_a^2}}$$

- ratio:

$$r(Z_{ia}^*) = \frac{\exp(-n_{ia} \exp(\beta_{0a} + Z_{ia}^*)) \times (\cancel{n_{ia}} \exp((\beta_{0a} + Z_{ia}^*)))^{Y_{ia}} \times \cancel{\frac{1}{Y_{ia}!}} \times \cancel{\frac{1}{\sqrt{2\pi\theta_a^2}}} e^{-Z^{*2}/\sigma}}{\exp(-n_{ia} \exp(\beta_{0a} + Z_{ia}')) \times (\cancel{n_{ia}} \exp((\beta_{0a} + Z_{ia}')))^{Y_{ia}} \times \cancel{\frac{1}{Y_{ia}!}} \times \cancel{\frac{1}{\sqrt{2\pi\theta}}} e^{-Z'^2/\sigma}}$$

$$= \exp\left(-n_{ia}\left(\exp\left(\beta_{0a} + z^*_{ia}\right) + \exp\left(\beta_{0a} + z'_{ia}\right)\right)\right) \times$$

$$\exp\left(y_{ia}\left(z^*_{ia} - z'_{ia}\right)\right) \times$$

$$\exp\left(-\frac{1}{2\sigma_a^2}\left(z^{*2}_{ia} - z'^{2}_{ia}\right)\right) = R$$

$$\log R =$$

$$\left(-n_{ia}\left[\exp\left(\beta_{0a} + z^*_{ia}\right) + \exp\left(\beta_{0a} + z'_{ia}\right)\right]\right) +$$

$$y_{ia}\left(z^*_{ia} - z'_{ia}\right) +$$

$$\left(-\frac{1}{2\sigma_a^2}\left(z^{*2}_{ia} - z'^{2}_{ia}\right)\right)$$

Update σ^2

we now need to use candidate density in there because IG is not a symmetric candidate density

σ^2_α is observed in $p(z_{i\alpha} | \alpha, \sigma^2_\alpha)$

σ^2_α is a term in its own prior IG( )

and because of non-symmetry we use $IG(q, q\sigma_\alpha^{2\,(\ell-1)})$

where $\sigma_\alpha^{2\,(\ell-1)}$ is the most recent value of σ^2_α

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write out densities:

$$z_{i\alpha} \sim \frac{1}{\sqrt{2\pi\sigma^2_\alpha}} e^{-\frac{1}{2}\frac{z_{i\alpha}^2}{\sigma^2_\alpha}}$$

$$\sigma^2_\alpha \sim \frac{b^a}{\Gamma(a)} (\sigma^2_\alpha)^{-a-1} e^{-b/\sigma^2_\alpha}$$

$$\sigma^2_\alpha \sim IG(q, q\sigma^2_\alpha) \leftarrow \text{Candidate density}$$

we have 67 for $i = 1, 2, \ldots, 67$ z_i for each θ^2.

So, I think we need to have z_{ia} represented as a likelihood function.

$$\ell(z_i) = \prod_{i=1}^{67} \frac{1}{\sqrt{2\pi\theta_a^2}} e^{-\frac{1}{2} z_{ia}^2 / \theta_a^2}$$

$$= \left(\frac{1}{2\pi\theta_a^2}\right)^n \cdot e^{-\frac{1}{2\theta_a^2} \sum_i z_{ia}^2}$$

So, a new update function for θ_a^2 is:

{ keep in mind that this needs to be done 3 times for each θ_a^2 $a = 1, 2, 3$ }

$$r(\theta^{2*}) = \frac{\left(\frac{1}{\sqrt{2\pi\theta_a^{2*}}}\right)^n e^{-\frac{1}{2\theta_a^{2*}} \sum z_{ia}^2} \times \frac{b^a}{\Gamma(a)} (\theta_a^{2*})^{-a-1} e^{-b/\theta_a^{2*}}}{\left(\frac{1}{\sqrt{2\pi\theta_a^{2\prime}}}\right)^n e^{-\frac{1}{2\theta_a^{2\prime}} \sum z_{ia}^2} \times \frac{b^a}{\Gamma(a)} (\theta_a^{2\prime})^{-a-1} e^{-b/\theta_a^{2\prime}}} \times$$

$$\frac{\times\,(\cancel{q}\,\theta_a^{2*})^{q}\quad(\theta_a^{2'})^{-q-1}\quad e^{\left(-\frac{q\theta_a^{2*}}{\theta_a^{2'}}\right)}}{(\cancel{q}\,\theta_a^{2'})^{q}\quad(\theta_a^{2*})^{-q-1}\quad e^{\left(-\frac{q\theta_a^{2'}}{\theta_a^{2*}}\right)}} =$$

$$= \left(\frac{\theta_a^{2*}}{\theta_a^{2'}}\right)^{-n/2} e^{-\frac{1}{2}\sum z_{ia}^2\left(\frac{1}{\theta_a^{2*}}-\frac{1}{\theta_a^{2'}}\right)} \times$$

$$\times\left(\frac{\theta_a^{2*}}{\theta_a^{2'}}\right)^{-a-1} \times e^{-b\left(\frac{1}{\theta_a^{2*}}-\frac{1}{\theta_a^{2'}}\right)} \times$$

$$\times\left(\frac{\theta_a^{2*}}{\theta_a^{2'}}\right)^{q} \times \left(\frac{\theta_a^{2*}}{\theta_a^{2'}}\right)^{q+1} \times e^{-q\left(\frac{\theta_a^{2*}}{\theta_a^{2'}}-\frac{\theta_a^{2'}}{\theta_a^{2*}}\right)}$$

$$-\frac{n}{2}-a-1+q+q+1$$

$$= \boxed{\begin{aligned}&\left(\frac{\theta_a^{2*}}{\theta_a^{2'}}\right)^{2q-a-\frac{n}{2}} \times e^{-\frac{1}{2}\sum z_{ia}^2\left(\frac{1}{\theta_a^{2}}-\frac{1}{\theta_a^{2'}}\right)}\\ &\times e^{-b\left(\frac{1}{\theta_a^{2*}}-\frac{1}{\theta_a^{2'}}\right)} \times e^{-q\left(\frac{\theta_a^{2*}}{\theta_a^{2'}}-\frac{\theta_a^{2'}}{\theta_a^{2*}}\right)}\end{aligned}}$$

Update σ^2

$\sigma^2 \sim IG$, and it shows up in z_i with unknown variance. So, we can treat IG as a conjugate prior

Lesson learned: before lengthy calculation always look for the fact if something is a conj. prior

σ_a^2 is observed in $p(z_{ia} | a, \sigma_a^2)$

σ_a^2 is a term in its own prior IG( )

write out densities:

$$z_{ia} \sim \frac{1}{\sqrt{2\pi\sigma_a^2}} e^{-\frac{1}{2}\frac{z_{ia}^2}{\sigma_a^2}} \Rightarrow \ell = \left(\frac{1}{\sqrt{2\pi\sigma_a^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i z_{ia}^2}$$

$$\sigma_a^2 \sim \frac{b^a}{\Gamma(a)} (\sigma_a^2)^{-a-1} e^{-b/\sigma_a^2}$$

rewrite ℓ:

$$(2\pi)^{-\frac{n}{2}} (\sigma_a^2)^{-\frac{n}{2}} e^{-\frac{\sum z_{ia}^2}{2\sigma^2}} \propto (\sigma_a^2)^{-\frac{n}{2}} e^{-\frac{\sum z^2}{2} \cdot \frac{1}{\sigma^2}}$$

So, full conditional

$$p(\sigma^2 \mid \cdot) \propto (\sigma_a^2)^{-\frac{n}{2}} \, e^{-\frac{\sum z_i^2}{2} \cdot \frac{1}{\sigma_a^2}} \times (\sigma_a^2)^{-a-1} \, e^{-b/\sigma_a^2} =$$

$$= (\sigma_a^2)^{-\underbrace{\left(\frac{n}{2}+a\right)}_{a^*}-1} \, e^{-\underbrace{\left(b+\frac{\sum z_i^2}{2}\right)}_{b^*}\frac{1}{\sigma_a^2}}$$