PUBH 7440: Intro to Bayesian Analysis
Homework from Week 5 — Due Feb 22nd/27th

Stroke mortality in PA: Using the same data as HW 3, we want to fit the following two models:

- $Y_{ia} \sim \text{Pois}\,(n_{ia}\lambda_{ia})$ where $\log \lambda_{ia} = \beta_{0a} + z_{ia}$ and $z_{ia} \sim \text{Norm}\,(0, \sigma_a^2)$.
- $Y_{ia} \sim \text{Pois}\,(n_{ia}\lambda_{ia})$ where $\log \lambda_{ia} = \theta_{ia} \sim \text{Norm}\,(\beta_{0a}, \sigma_a^2)$.

In both cases, use standard $\beta_{0a} \sim \text{Norm}\,(0, \tau_a^2)$ and $\sigma_a^2 \sim \text{IG}\,(0.001, 0.001)$ priors, with $\tau_a^2 = 10{,}000$, and answer the following questions:

1. Write the full hierarchical model. As with HW 3, account for the suppression of $Y_{ia} < 10$.
2. Derive the full-conditional distributions for $\{\boldsymbol{\beta}_0, \mathbf{z}, \sigma_a^2\}$ and $\{\boldsymbol{\beta}_0, \boldsymbol{\theta}, \sigma_a^2\}$ for the two models, respectively. Which parameters have full-conditional distributions we can sample from directly, and which parameters require Metropolis steps to sample?
3. Write code to fit the two models. How do the results compare? Examples of "results" to compare include:
   - Posterior summaries for β_{0a} from the two models.
   - Posterior summaries for $\lambda_{i\cdot}$, the age-adjusted rates (you can use the same code as in HW 3 to make maps of these rates).

   Note: Use symmetric candidate densities — e.g., $\theta^* \sim \text{Norm}\left(\theta^{(\ell-1)}, q\right)$ — to generate proposed values for all Metropolis steps. Don't worry too much about acceptance rates, though the closer you can get to 44% acceptance rates, the better your convergence will be. If you're having issues with this, let me know.

My plan will be to have the *first model* (the one with z_{ia}) due on Thursday Feb 22nd and the second model due Tuesday Feb 27th. I suppose that means that when you turn in the results for Model 1, you need to report the various summaries for #3, but there won't be a "compare" part of that question until you've done Model 2.

Problem 1

$$Y_{i\alpha} \sim Pois(n_{i\alpha} \cdot \lambda_i), \quad \log \lambda_{i\alpha} = \theta_{i\alpha} \Rightarrow$$

$$\lambda_i = \exp(\theta_{i\alpha})$$

$$\theta_{i\alpha} \sim Norm(\beta_{0\alpha}, \sigma^2_\alpha)$$

$$\beta_{0\alpha} \sim Norm(0, \tau^2_\alpha)$$

$$\sigma^2_\alpha \sim IG(a, b) \qquad a = b = 0.001$$

$$p(Y_{i\alpha} \mid \cdot) = \frac{e^{-\Lambda} \Lambda^{Y_{i\alpha}}}{Y_{i\alpha}!} =$$

$$= \frac{\exp(-n_{i\alpha} \exp(\theta_{i\alpha})) \times (n_{i\alpha} \exp(\theta_{i\alpha}))^{Y_{i\alpha}}}{Y_{i\alpha}!}$$

Full hierarchical model:

$$p(\theta_{i\alpha}, \beta_{0\alpha}, \sigma_\alpha^2 \mid \mathbf{Y}) =$$

$$\prod_{i,\alpha} \text{Pois}(y_{i\alpha} \mid n_{i\alpha} \exp(\theta_{i\alpha}))^{I(y_{i\alpha} < 10)^{1-d_{i\alpha}}} \times \text{Norm}(\beta_{0\alpha}, \sigma_\alpha^2)$$

$$\times \prod_{\alpha} \text{Norm}(0, \tau_\alpha^2) \times IG(a, b)$$

where $\tau_\alpha^2 = 10{,}000$ and a, b <u>and</u>

$$d_{i\alpha} = \begin{cases} 1 & \text{if } y_{i\alpha} \geq 10 \\ 0 & \text{if } y_{i\alpha} < 0 \end{cases}$$

Problem 2

Full conditional for $\theta_{i\alpha}$.

Need one θ for each i, α, so

$\theta_{i\alpha}$ shows up in pmf for $y_{i\alpha}$ and in its own prior

$$p(\theta_{i\alpha} \mid \cdot) \propto$$

$$\frac{\exp(-n_{i\alpha}\exp(\theta_{i\alpha})) \times (\cancel{n_{i\alpha}}\exp(\theta_{i\alpha}))^{y_{i\alpha}}}{\cancel{y_{i\alpha}!}} \times$$

$$\cancel{\frac{1}{\sqrt{2\pi\theta_\alpha^2}}}\, e^{-\frac{1}{2\theta_\alpha^2}(\theta_{i\alpha} - \beta_{0\alpha})^2} \propto$$

$$e^{-n_{i\alpha} e^{\theta_{i\alpha}}} \times e^{y_{i\alpha}\theta_{i\alpha}} \times e^{-\frac{1}{2\theta_\alpha^2}(\theta_{i\alpha} - \beta_{0\alpha})^2}$$

$$= e^{-n_{i\alpha} e^{\theta_{i\alpha}} + y_{i\alpha}\theta_{i\alpha} - \frac{1}{2\theta^2}\theta_{i\alpha}^2 + \frac{1}{\theta_\alpha^2}\theta_{i\alpha}\beta_{0\alpha} - \cancel{\frac{1}{2\theta}}}$$

$$\propto e^{-n_{i\alpha} e^{\theta_{i\alpha}} + \theta_{i\alpha}\left(y_{i\alpha} - \frac{1}{2\theta_\alpha^2}\theta_{i\alpha} + \frac{1}{\theta_\alpha^2}\beta_{0\alpha}\right)}$$

This is a distribution we do not recognize, so we need to use Metropolis updates.

Full conditional for $\beta_{0\alpha}$

- $\beta_{0\alpha}$ shows up in prior of $\theta_{i\alpha}$ as mean, and in its own density.
- Note that we have 67 $\theta_{i\alpha}$ for an age group α. So we need a joint likelihood of these $\theta_{i\alpha}$.
- So, $\ell = \prod_{i=1}^{N=67} \frac{1}{(2\pi\theta_\alpha^2)^{\frac{1}{2}}} e^{-\frac{1}{2\theta_\alpha^2}(\theta_{i\alpha} - \beta_{0\alpha})^2}$

$$= \cancel{\left(2\pi\theta_\alpha^2\right)^{-\frac{n}{2}}} e^{-\frac{1}{2\theta_\alpha^2}\sum_i (\theta_{i\alpha} - \beta_{0\alpha})^2}$$

Don't need

$$p(\beta_{0\alpha} \mid \tau_\alpha^2) = \cancel{\frac{1}{\sqrt{2\pi\tau_\alpha^2}}} e^{-\frac{1}{2\tau_\alpha^2}\beta_{0\alpha}^2}$$

Don't need

- Therefore:

$$p(\beta_{0\alpha} \mid \cdot) \propto e^{-\frac{1}{2\theta_\alpha^2}\sum_i (\theta_{i\alpha} - \beta_{0\alpha})^2 - \frac{1}{2\tau_\alpha^2}\beta_{0\alpha}^2}$$

Working with exponential from ℓ:

$$\sum_i^n (\theta_{i\alpha} - \beta_{0\alpha})^2 = \sum_i^n (\theta_{i\alpha}^2 - 2\theta_{i\alpha}\beta_{0\alpha} + \beta_{0\alpha}^2) =$$

$$= \sum_i \theta_{i\alpha}^2 - 2\beta_{0\alpha}\sum_i \theta_{i\alpha} + n\beta_{0\alpha}^2 .$$

So, the whole contribution from ℓ can be written as

$$e^{-\frac{1}{2\theta_\alpha^2}\left(\sum \theta_{i\alpha}^2 - 2\beta_{0\alpha}\sum\theta_{i\alpha} + n\beta_{0\alpha}^2\right)} =$$

$$= e^{-\frac{\sum\theta_{i\alpha}^2}{2\theta_\alpha^2} + \frac{\beta_{0\alpha}\sum\theta_{i\alpha}}{\theta_\alpha^2} - \frac{n\beta_{0\alpha}^2}{2\theta_\alpha^2}}$$

Return to $p(\beta_{0\alpha}|\cdot)$

$$p(\beta_{0\alpha}|\cdot) \propto e^{\left\{\cancel{-\sum\frac{\theta_{i\alpha}^2}{2\theta_\alpha^2}} + \frac{\beta_{0\alpha}\sum\theta_{i\alpha}}{\theta_\alpha^2} - \frac{n\beta_{0\alpha}^2}{2\theta_\alpha^2} - \frac{1}{2\tau_\alpha^2}\beta_{0\alpha}^2\right\}}$$

can remove b.c. no β_0

$$\propto e^{\frac{\beta_{0\alpha}\sum\theta_{i\alpha}}{\theta_\alpha^2} - \frac{n\beta_{0\alpha}^2}{2\theta_\alpha^2} - \frac{1}{2\tau_\alpha^2}\beta_{0\alpha}^2} =$$

$$= e^{\beta_{0\alpha}\left(\frac{\sum\theta_{i\alpha}}{\theta_\alpha^2}\right) - \beta_{0\alpha}^2\left(\frac{n}{2\theta_\alpha^2} - \frac{1}{2\tau_\alpha^2}\right)}$$

Stop here for now

Full conditional for σ^2_α.

σ^2_α prior IG

σ^2_α shows up in ℓ of $\theta_{i\alpha}$ for fixed α

$$p(\sigma^2_\alpha \mid \cdot) \propto (2\pi\sigma^2_\alpha)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2_\alpha}\sum_i (\theta_{i\alpha} - \beta_{0\alpha})^2} \times (\sigma^2_\alpha)^{-a-1} e^{-b/\sigma^2_\alpha}$$

$$\propto (\sigma^2_\alpha)^{-a-1-\frac{n}{2}} \times e^{-\frac{1}{\sigma^2_\alpha}\left[\frac{1}{2}\sum(\theta_{i\alpha} - \beta_{0\alpha})^2 + b\right]}$$

$$= (\sigma^2_\alpha)^{-\underbrace{\left(\frac{n}{2}+a\right)}_{\text{param. } \alpha^*}-1} \times e^{-\underbrace{\left[\frac{1}{2}\sum(\theta_{i\alpha} - \beta_{0\alpha})^2 + b\right]}_{\text{param } \beta^*}/\sigma^2_\alpha}$$

as expected we obtained an Inverse Gamma