

PUBH 7440: Intro to Bayesian Analysis

Midterm (Take-Home Portion) — Due March 14

Incidence of low weight births in PA: [Insert text saying why looking at the incidence of low birth weight is important]. Here, we let y_{ir} denote the number of low weight births from mothers of race r ($r = 1$ white, $r = 2$ black) in county i out of a total of n_{ir} births. To model these data, we will assume:

$$y_{ir} \sim \text{Bin}(n_{ir}, \pi_{ir}), \text{ where } \text{logit}(\pi_{ir}) = \theta_{ir} \sim \text{Norm}(\beta_{0r}, \sigma_r^2),$$

and where π_{ir} represents the incidence rate. Assuming standard priors for $\beta_{0r} \sim \text{Norm}(0, \tau^2)$ and $\sigma_r^2 \sim \text{IG}(0.001, 0.001)$, with $\tau^2 = 10,000$, answer the following questions:

1. Write the full hierarchical model.
2. Derive the full-conditional distributions for β_{0r} , π_{ir} , and σ_r^2 . Which parameters have full-conditional distributions we can sample from directly, and which parameters require Metropolis steps to sample?
3. Write code to fit the model, and use $\beta_{0r} = 0$ and $\sigma_r^2 = 1$ as initial values.
 - Make history plots of β_{0r} and σ_r^2 for both races and assess model convergence. Is burn-in required? If so, how much?
4. Suppose we're interested in investigating racial disparities in the incidence of low weight births. Using the β_{0r} terms, make a histogram of the posterior distribution of the log odds ratio. Does this indicate evidence of a “significant” racial disparity? (Hint: The log odds ratio is represented by γ_1 in the conventional regression model parameterization, $E[\theta_{ir} | \gamma, \sigma_r^2] = \gamma_0 + \gamma_1 \times (r - 1)$ where $r = 1, 2$, so you'll need to first write γ_1 as a function of the β_{0r} parameters.)
5. Now suppose we're interested in *geographic* trends in the incidence of low weight births by race and in their racial disparities. Using the mapping code from HW3/HW4, make the following maps:
 - The incidence of low weight births for white mothers.
 - The incidence of low weight births for black mothers.
 - The black/white ratio of the incidence of low weight births.
6. Finally, make histograms of posterior distribution of the black/white ratio of the incidence of low weight births in Philadelphia County ($i = 51$) and Sullivan County ($i = 57$) and compare these to their respective crude estimates (i.e., the ratio of the crude incidence rates, y_{ir}/n_{ir} , for black and white mothers in both counties) and the statewide averages (i.e., the ratio of $\sum_i y_{ir} / \sum_i n_{ir}$ for black and white mothers). Are the posterior distributions consistent with either/both of these estimates based on the data? From a statistical perspective, would you have any reservations about presenting these results?

General Details:

- 1) we have $i = 1, 2, \dots, 67$ indexing counties
- 2) within each county we have two racial groups

$$y_{ir} \sim \text{Bin}(n_{ir}, \pi_{ir})$$

$$\text{logit}(\pi_{ir}) = \log\left(\frac{\pi_{ir}}{1-\pi_{ir}}\right) = \theta_{ir} \sim \text{Normal}(\beta_{0r}, \theta_r^2)$$

$$\pi_{ir} = \frac{\exp(\theta_{ir})}{1 + \exp(\theta_{ir})}$$

- 3) we have the same mean and variance for all θ_{ir} within a fixed r .

- 4) So, we need to estimate:

2 total β_{0r}

2 total θ_r^2

67x2 total θ_{ir}

Problem 1:

field model:

$$P(\theta_{ir}, \beta_{or}, \sigma_r^2 | y_{ir, n_{ir}}) \propto$$

$$\prod_{i=1}^{N=67} \prod_{j=1}^{r=2} \text{Bin}(n_{ir}, \pi_{ir}) \times$$

$$\text{Norm}(\beta_{or}, \sigma_r^2) \times$$

$$\text{Norm}(\sigma_r^2) \times$$

$$IG(0.001, 0.001)$$

$$\chi^2 = 10,000$$

Problem 2

1) Full conditional β_{0r} .

We have a total of 2 β_{0r} for

$$r=1 \quad \text{and} \quad r=2$$

- β_{0r} appears in a distribution of θ_{ir} .
- for each β_{0r} we have 67 θ_{ir} 's within each level of r.

$$P(\beta_{0r} | \cdot) \propto \left[\prod_{i=1}^{67} \text{Norm}(\theta_{ir} | \beta_{0r}, \sigma_r^2) \right] \times \text{Norm}(\beta_{0r} | 0, \bar{\sigma}^2) =$$

$$(2\pi\sigma_r^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_r^2} \sum (\theta_{ir} - \beta_{0r})^2\right) \times \\ \times (-2\bar{\sigma}^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\bar{\sigma}^2} \beta_{0r}^2\right) \propto$$

$$\exp \left(-\frac{1}{2\theta^2} \sum (\theta_{ir} - \beta_{0r})^2 - \frac{1}{2\tau^2} \beta_{0r}^2 \right) =$$

$$\exp \left(-\frac{1}{2\theta^2} \cancel{\frac{1}{2} \sum \theta_{ir}^2} + \frac{1}{2\theta^2} \cdot 2\beta_{0r} \sum \theta_{ir} - \frac{1}{2\theta^2} \sum \beta_{0r}^2 - \frac{1}{2\tau^2} \beta_{0r}^2 \right) \propto$$

$$\exp \left(\frac{\beta_{0r}}{\theta^2} \sum \theta_{ir} - \frac{1}{2} \beta_{0r}^2 \left(\frac{n}{\theta^2} + \frac{1}{\tau^2} \right) \right)$$

After considering Metropolis,

I worked out that this update can be done w/ normal distribution.

Details Below ↓

Alternatively for β or consider a general normal distribution:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \propto$$

$$\exp\left(-\frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)\right) \propto$$

$$\exp\left(-\frac{1}{2\sigma^2}x^2 + \frac{1}{\sigma^2}x \cdot \mu\right)$$

This is exactly what we have above:

Change notation of original distribution:

$\theta \rightarrow S$, Then, by matching the terms we get

$$\begin{cases} \frac{1}{2\sigma^2} = \frac{1}{2} \left(\frac{n}{S^2} + \frac{1}{T^2} \right), \\ \frac{\mu}{\sigma^2} = \frac{\sum \theta_{ir}}{S^2}. \end{cases}$$

$$\frac{1}{\sigma^2} = \left(\frac{n}{S^2} + \frac{1}{T^2} \right) \Rightarrow \sigma^2 = \frac{1}{\left(\frac{n}{S^2} + \frac{1}{T^2} \right)}$$

Plug this into the second equation in the system:

$$\frac{M}{\sigma^2} = M \times \left(\frac{n}{S^2} + \frac{1}{\tau^2} \right) = \frac{\sum \theta_{ir}}{S^2},$$

$$M = \frac{\sum \theta_{ir}}{S^2 \left(\frac{n}{S^2} + \frac{1}{\tau^2} \right)} = \frac{\sum \theta_{ir}}{n + \frac{S^2}{\tau^2}}$$

Switch notation back to $S \rightarrow \theta$

Potentially we can sample

$$\text{For from } N \left(\frac{\sum \theta_{ir}}{n + \frac{\theta^2}{\tau^2}}, \frac{1}{\left(\frac{n}{\theta^2} + \frac{1}{\tau^2} \right)} \right)$$

2) Full conditional for π_{ir}

$$Y_{ir} \sim \binom{n_{ir}}{y_{ir}} \pi_{ir}^{y_{ir}} (1-\pi_{ir})^{n_{ir}-y_{ir}}$$

$$\log\left(\frac{\pi_{ir}}{1-\pi_{ir}}\right) = \theta_{ir} \sim \text{Norm}(\beta_0, \theta_i^2)$$

$$\pi_{ir} = \frac{\exp(\theta_{ir})}{1+\exp(\theta_{ir})} \Rightarrow$$

$$Y_{ir} \sim \binom{n_{ir}}{y_{ir}} \left[\frac{\exp(\theta_{ir})}{1+\exp(\theta_{ir})} \right]^{y_{ir}} \left[\frac{1}{1+\exp(\theta_{ir})} \right]^{n_{ir}-y_{ir}}$$

π_{ir} has a deterministic relationship with

θ_{ir} , so we obtain a full conditional for θ_{ir} and obtain π_{ir} using $\exp(\cdot)$ function

$$P(\theta_{ir} | \cdot) \propto$$

$$\left(\frac{u_{ir}}{y_{ir}} \right) \left[\frac{\exp(\theta_{ir})}{1 + \exp(\theta_{ir})} \right]^{y_{ir}} \left[\frac{1}{1 + \exp(\theta_{ir})} \right]^{u_{ir} - y_{ir}} \times$$

$$\exp\left(-\frac{1}{2\sigma^2}(\theta_{ir} - \beta_{or})^2\right) =$$

$$\exp(\theta_{ir})^{y_{ir}} \times (1 - \exp(\theta_{ir}))^{-y_{ir}} \times$$

$$\left(\frac{1}{1 + \exp(\theta_{ir})} \right)^{-u_{ir} + y_{ir}} \times \exp\left(-\frac{1}{2\sigma^2}(\theta_{ir} - \beta_{or})^2\right)$$

$$= \exp(\theta_{ir})^{y_{ir}} \times (1 + \exp(\theta_{ir}))^{-u_{ir}} \times \\ \exp\left(-\frac{1}{2\sigma^2}(\theta_{ir} - \beta_{or})^2\right)$$

Will use metropolis updates
for this one

3) full conditional for θ^2

As known before, this should be

a conjugate prior, so full conditional must be IG as well.

$$p(\theta^2 | \cdot) \propto \prod_{i=1}^{n=67} \text{Norm}(\theta_{ir} / \beta_{0r}, \theta^2) \times \\ \times \text{IG}(\theta^2 / 0.001, 0.001)$$

$$\propto \frac{(2\pi\theta^2)^{-\frac{n}{2}}}{\theta^2} \exp\left(-\frac{1}{2\theta^2} \sum_i (\theta_{ir} - \beta_{0r})^2\right) \times \\ \times \exp(-0.001/\theta^2)$$

Let $\sum_i (\theta_{ir} - \beta_{0r})^2 = A$

$$\propto \frac{1}{\theta^2}^{-\frac{n}{2} - 0.001 - 1} \times$$

$$\exp\left(-\frac{1}{2} A \left[\frac{1}{\theta^2} - 0.001\right]\right)$$

$$= (\theta^2)^{-\left(\frac{u}{2} + 0.001\right) - 1} \times$$

$$\exp\left(-2\left[\frac{1}{2}A + 0.001\right] \cdot \frac{1}{\theta^2}\right)$$

$$\Rightarrow P(\theta^2 | \epsilon) \sim IG\left(\frac{u}{2} + 0.001, \frac{1}{2}A + 0.001\right)$$

Problem 3

Notes for code: we need to get ratios for Metropolis updates.

i) Ratios for β_{or}

$$r(\beta_{or}) =$$

$$\frac{\exp\left(-\frac{\beta_{or}^*}{\sigma^2} \sum \theta_{ir} - \frac{1}{2} \beta_{or}^{*2} \left(\frac{n}{\sigma^2} + \frac{1}{z^2}\right)\right)}{\exp\left(-\frac{\beta_{or}^*}{\sigma^2} \sum \theta_{ir} - \frac{1}{2} \beta_{or}^{*2} \left(\frac{n}{\sigma^2} + \frac{1}{z^2}\right)\right)} =$$

$$\frac{\exp\left(-\frac{\beta_{or}^*}{\sigma^2} \sum \theta_{ir} - \frac{1}{2} \beta_{or}^{*2} \left(\frac{n}{\sigma^2} + \frac{1}{z^2}\right)\right)}{\exp\left(-\frac{\beta_{or}^*}{\sigma^2} \sum \theta_{ir} - \frac{1}{2} \beta_{or}^{*2} \left(\frac{n}{\sigma^2} + \frac{1}{z^2}\right)\right)} =$$

$$\exp\left(\frac{\sum \theta_{ir}}{\sigma^2} (\beta_{or}^* - \beta_{or}^*)\right) \times$$

$$\exp\left(-\frac{1}{2} \left(\frac{n}{\sigma^2} + \frac{1}{z^2}\right) (\beta_{or}^{*2} - \beta_{or}^2)\right)$$

On the log Scale

$$\frac{\sum \theta_{ir} (\beta_{or}^* - \beta_{or})}{\sigma^2} - \frac{1}{2} \left(\frac{n}{\sigma^2} + \frac{1}{T^2} \right) (\beta_{or}^{*2} - \beta_{or}^2)$$

2) Ratios for θ_{ir}^*

$$\exp(\theta_{ir}^*)^{y_{ir}} \times (1 + \exp(\theta_{ir}^*))^{-u_{ir}} \times \\ \exp\left(-\frac{1}{2\sigma^2}(\theta_{ir}^* - \beta_{0r})^2\right)$$

$$r(\theta_{ir}^*) = \frac{\exp(\theta_{ir}^*)^{y_{ir}} \times (1 + \exp(\theta_{ir}^*))^{-u_{ir}} \times \\ \exp\left(-\frac{1}{2\sigma^2}(\theta_{ir}^* - \beta_{0r})^2\right)}{\exp(\theta_{ir}^*)^{y_{ir}} \times (1 + \exp(\theta_{ir}^*))^{-u_{ir}} \times \\ \exp\left(-\frac{1}{2\sigma^2}(\theta_{ir}^* - \beta_{0r})^2\right)} =$$

$$= \frac{\exp(\theta_{ir}^*)}{\exp(\theta_{ir}^*)}^{y_{ir}} \times \left[\frac{1 + \exp(\theta_{ir}^*)}{1 + \exp(\theta_{ir}^*)} \right]^{-u_{ir}} \times \\ \times \exp\left(\frac{1}{2\sigma^2}(\theta_{ir}^* - \beta_{0r})^2 - (\theta_{ir}^* - \beta_{0r})^2\right)$$

On the log Scale:

$$y_{ir}(\theta_{ir}^* - \hat{\theta}_{ir}) = n_{ir} \left[\log(1 + \exp(\theta_{ir}^*)) - \log(1 + \exp(\hat{\theta}_{ir})) \right] -$$
$$- \frac{1}{2} ((\theta_{ir}^* - \beta_{0r})^2 - (\hat{\theta}_{ir} - \beta_{0r})^2)$$

Problem 5

$$\text{logit}(\pi_{ir}) = \log\left(\frac{\pi_{ir}}{1-\pi_{ir}}\right) = \theta_{ir}.$$

$$E[\theta_{ir} | \beta_{0r}, \sigma^2_{\theta r}] = \beta_{0r}$$

$$E[\log(\pi_{ir}) - \log(1-\pi_{ir})] =$$

$$E[\log(\pi_{ir})] - E[\log(1-\pi_{ir})] = E[\theta_{ir}] = \beta_{0r}$$

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$$E\left[\log\left(\frac{\pi_{ir}}{1-\pi_{ir}}\right)\right] = \beta_{0r}$$

Let  $\rho_{0r}$  be the rate of events within group  $r$ .

Then, log odds ratio is given by

$$\log\left[\frac{\frac{\rho_{01}}{1-\rho_{01}}}{\frac{\rho_{02}}{1-\rho_{02}}}\right] = \log\left[\frac{\rho_{01}}{1-\rho_{01}}\right] - \log\left[\frac{\rho_{02}}{1-\rho_{02}}\right]$$

So this basically boils down to

$$\underline{\beta_{01} - \beta_{02}}.$$