PUBH 7440: Intro to Bayesian Analysis
Homework from Week 1 — Due January 30

1. **Exercise #2 from CL3, Chapter 1.6**. Repeat the journal publication thought problem from [Week 1, Lecture 1 slides] for the situation where:

   (a) you have won the lottery on your first try.

   (b) you have correctly predicted the winner of the first game of the World Series (professional baseball).

2. **Exercise #6 from CL3, Chapter 1.6**. In the Normal/Normal example of Subsection 1.5.1 (which I've painstakingly typed below), let $\sigma^2 = 2$, $\mu = 0$, and $\tau^2 = 2$.

   (a) Suppose we observe $y = 4$. What are the mean and variance of the resulting posterior distribution? Sketch the prior, likelihood, and posterior on a single set of coordinate axes.

   (b) Repeat part (a) assuming $\tau^2 = 18$. Explain any resulting differences. Which of these two priors would likely have more appeal for a frequentist statistician?

   Hint: For those of you new to R, here's how you can plot multiple density plots in a single figure

```
rm(list=ls())
curve(dnorm(x,mean=0,sd=0.5),from=-3,to=3,col='black')
curve(dnorm(x,mean=0.5,sd=1),from=-3,to=3,col='red',add=TRUE)
#the "x" in the dnorm will be replaced with numbers between "from" and "to"
#"add=TRUE" is the command to add the new curve to the existing plot
#If you run into issues where a subsequent curve has a higher max density,
# you can set "ylim=c(0,something)" in the initial "curve()" statement,
# where "something" is some number bigger than the height of your curves
```

3. **Balls & Buckets, revisited**: Suppose you have two (relatively large) buckets of red and blue balls.

   - Bucket 1: 35 red balls, 17 blue.
   - Bucket 2: 23 red balls, 37 blue.

   Before picking a ball from a bucket, first you roll a standard six-sided die — if a number 1–4 is on top, you pick a ball from Bucket 1, otherwise you pick from Bucket 2. After rolling the die, you pick a ball from the designated bucket and it's blue.

   (a) What was the probability of drawing a ball from Bucket 2?

   (b) What was the probability of drawing a blue ball?

(c) What is the probability that you drew the ball from Bucket 2 given that you drew a blue ball?

4. **Urban/Rural Disparities in Heart Disease Mortality**: Using the data and code from Thursday's class, answer the following questions:

   (a) Compare the frequency of having higher than average rates in urban counties versus rural counties. Is there a "statistically significant" difference? Note that "average" here means the state-wide average and let's assume that "urban counties" are counties whose combined white and black male population aged 35–54 is larger than 20,000.

   (b) Compare the heart disease mortality rate in urban counties to the mortality rate in rural counties. Is there a "statistically significant" difference? Again, assume that "urban counties" are counties whose combined white and black men population aged 35–54 is larger than 20,000 and assume conjugate Gam (a, b) priors; feel free to use the same sort of an empirical Bayesian approach from class to specify a and b.

   (c) Do the two analyses yield similar answers? If not, which do you think is the more appropriate analysis?

Normal/Normal example from Section 1.5.1: We now consider the case where both the prior and the likelihood are Gaussian (normal) distributions, namely,

$$\theta \sim \text{Norm}\left(\mu, \tau^2\right) \tag{1}$$
$$y \,|\, \theta \sim \text{Norm}\left(\theta, \sigma^2\right). \tag{2}$$

The marginal distribution of y,

$$p(y) = \int p(y \,|\, \theta)\, p(\theta)\, d\theta,$$

turns out to be $\text{Norm}(\mu, \sigma^2 + \tau^2)$, and the posterior distribution,

$$p(\theta \,|\, y) = \frac{p(y \,|\, \theta)\, p(\theta)}{p(y)},$$

is also Gaussian with mean and variance

$$E[\theta \,|\, y] = B\mu + (1 - B)\, y \tag{3}$$
$$V[\theta \,|\, y] = (1 - B)\, \sigma^2, \tag{4}$$

where $B = \sigma^2 / (\sigma^2 + \tau^2)$. Since $0 \leq B \leq 1$, the posterior mean is a weighted average of the prior mean μ and the direct estimate y; the Bayes estimate is pulled back (or *shrunk*) toward the prior mean. Moreover, the weight on the prior mean B depends on the relative variability of the prior distribution and the likelihood. If σ^2 is large relative to τ^2 (i.e., our prior knowledge is more precise than the data information), the B is close to 1, producing substantial shrinkage. If σ^2 is small (i.e., our prior knowledge is imprecise relative to the data information), B is close to 0 and the estimate is moved very little toward the prior mean. As we show in Chapter 5, the shrinkage provides an effective tradeoff between variance and bias, with beneficial effects on the resulting mean squared error.