

PUBH 7440: Intro to Bayesian Analysis

Homework from Week 5 — Due Feb 22nd/27th

Stroke mortality in PA: Using the same data as HW 3, we want to fit the following two models:

- $Y_{ia} \sim \text{Pois}(n_{ia}\lambda_{ia})$ where $\log \lambda_{ia} = \beta_{0a} + z_{ia}$ and $z_{ia} \sim \text{Norm}(0, \sigma_a^2)$.
- $Y_{ia} \sim \text{Pois}(n_{ia}\lambda_{ia})$ where $\log \lambda_{ia} = \theta_{ia} \sim \text{Norm}(\beta_{0a}, \sigma_a^2)$.

In both cases, use standard $\beta_{0a} \sim \text{Norm}(0, \tau_a^2)$ and $\sigma_a^2 \sim \text{IG}(0.001, 0.001)$ priors, with $\tau_a^2 = 10,000$, and answer the following questions:

1. Write the full hierarchical model. As with HW 3, account for the suppression of $Y_{ia} < 10$.
2. Derive the full-conditional distributions for $\{\beta_0, \mathbf{z}, \sigma_a^2\}$ and $\{\beta_0, \boldsymbol{\theta}, \sigma_a^2\}$ for the two models, respectively. Which parameters have full-conditional distributions we can sample from directly, and which parameters require Metropolis steps to sample?
3. Write code to fit the two models. How do the results compare? Examples of “results” to compare include:
 - Posterior summaries for β_{0a} from the two models.
 - Posterior summaries for λ_i , the age-adjusted rates (you can use the same code as in HW 3 to make maps of these rates).

Note: Use symmetric candidate densities — e.g., $\theta^* \sim \text{Norm}(\theta^{(\ell-1)}, q)$ — to generate proposed values for all Metropolis steps. Don’t worry too much about acceptance rates, though the closer you can get to 44% acceptance rates, the better your convergence will be. If you’re having issues with this, let me know.

My plan will be to have the *first model* (the one with z_{ia}) due on Thursday Feb 22nd and the second model due Tuesday Feb 27th. I suppose that means that when you turn in the results for Model 1, you need to report the various summaries for #3, but there won’t be a “compare” part of that question until you’ve done Model 2.

Both VI and 12 derivations
are * correct *.

The issue was a bug in my code.

Problem 1

$$Y_{i\alpha} \sim \text{Pois}(n_{i\alpha}, \lambda_i) , \log \lambda_{i\alpha} = \theta_{i\alpha} \Rightarrow \lambda_i = \exp(\theta_{i\alpha})$$

$$\theta_{i\alpha} \sim \text{Norm}(\beta_{0\alpha}, \sigma_{\alpha}^2)$$

$$\beta_{0\alpha} \sim \text{Norm}(0, \sigma_0^2)$$

$$\sigma_{\alpha}^2 \sim \text{IG}(\alpha, b) \quad \alpha = b = 0.001$$

$$p(Y_{i\alpha} | \cdot) = \frac{e^{-\lambda} \lambda^{Y_{i\alpha}}}{Y_{i\alpha}!} =$$

$$= \frac{\exp(-n_{i\alpha} \exp(\theta_{i\alpha})) \times (n_{i\alpha} \exp(\theta_{i\alpha}))^{Y_{i\alpha}}}{Y_{i\alpha}!}$$

Full hierarchical model:

$$P(\theta_{i2}, \beta_{02}, \sigma^2_\alpha | \mathbf{y}) = \underbrace{\text{I}(y_{i2} < 10)}_{\downarrow}$$

$$\prod_{i,2} \text{Pois}(y_{i2} | \mu_{i2} \exp(\theta_{i2})) \times \text{Norm}(\beta_{02}, \sigma^2_\alpha)$$

$$\times \prod_{i,2} \text{Norm}(0, \sigma^2_\alpha) \times \text{IG}(a, b)$$

where $\sigma^2_\alpha = 10,000$ and a, b and

$$\delta_{i2} = \begin{cases} 1 & \text{if } y_{i2} \geq 10 \\ 0 & \text{if } y_{i2} < 0 \end{cases}$$

Problem 2

Full conditional for θ_{i2} .

Need one θ for each i, t , so

θ_{i2} shows up in pdf for y_{it} and in its own prior

$$p(\theta_{i2} | \cdot) \propto$$

$$\frac{\exp(-n_{i2} \exp(\theta_{i2})) \times (\text{un}/\exp(\theta_{i2}))^{y_{i2}}}{\cancel{y_{i2}}} \times$$

$$\cancel{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\theta_{i2} - \beta_{02})^2}} \propto$$

$$e^{-n_{i2} e^{\theta_{i2}}} \times e^{y_{i2} \theta_{i2}} \times e^{-\frac{1}{2\sigma^2}(\theta_{i2} - \beta_{02})^2}$$

$$= e^{-n_{i2} e^{\theta_{i2}} + y_{i2} \theta_{i2} - \frac{1}{2\sigma^2} \theta_{i2}^2 + \frac{1}{\sigma^2} \theta_{i2} \beta_{02} - \frac{1}{2\sigma^2} \beta_{02}^2}$$

$$\propto e^{-n_{i2} e^{\theta_{i2}} + \theta_{i2} (y_{i2} - \frac{1}{2\sigma^2} \theta_{i2} + \frac{1}{\sigma^2} \beta_{02})}$$

This is a distribution we do not recognize, so we need to use Metropolis update.

After running the code, I got some pretty bad results. Attempting to fix the results by using simpler versions of my derivations

$$V2: e^{-\kappa_{iz} \beta \exp(\theta_{iz}) + Y_{iz} \theta_{iz}} \times e^{-\frac{1}{2\sigma^2} (\theta_{iz} - \beta_{0z})^2}$$

Full conditionals for β_{02}

- β_{02} shows up in prior of θ_{iz} or mean, and in its own density.
- Note that we have G_7 θ_{iz} for an age group d. so we need a joint likelihood of these θ_{iz} .

• So, $\ell = \prod_{i=1}^{N=G_7} \frac{1}{(2\pi\sigma_x^2)^{\frac{1}{2}}} e^{-\frac{1}{2\sigma_x^2} (\theta_{iz} - \beta_{02})^2}$

$$= \left(\frac{1}{2\pi\sigma_x^2} \right)^{-\frac{N}{2}} e^{-\frac{1}{2\sigma_x^2} \sum_i (\theta_{iz} - \beta_{02})^2}$$

Don't need

$$P(\beta_{02} | \bar{t}_2^2) = \frac{1}{\sqrt{2\pi\tau_2^2}} e^{-\frac{1}{2\tau_2^2} \beta_{02}^2}$$

Don't need

• Therefore:

$$P(\beta_{02} | \cdot) \propto e^{-\frac{1}{2\sigma_x^2} \sum_i (\theta_{iz} - \beta_{02})^2 - \frac{1}{2\tau_2^2} \beta_{02}^2}$$

Working with exponential from ℓ :

$$\sum_i (\theta_{id} - \beta_{0d})^2 = \sum_i (\theta_{id}^2 - 2\theta_{id}\beta_{0d} + \beta_{0d}^2) = \\ = \sum_i \theta_{id}^2 - 2\beta_{0d} \sum_i \theta_{id} + n\beta_{0d}^2.$$

So, the whole contribution from ℓ can be written as

$$e^{-\frac{1}{2\sigma_d^2} \left(\sum \theta_{id}^2 - 2\beta_{0d} \sum \theta_{id} + n\beta_{0d}^2 \right)} = \\ = e^{-\frac{\sum \theta_{id}^2}{2\sigma_d^2} + \frac{\beta_{0d} \sum \theta_{id}}{\sigma_d^2} - \frac{n\beta_{0d}^2}{2\sigma_d^2}}$$

Return to $p(\beta_{0d} | \cdot)$

$$p(\beta_{0d} | \cdot) \propto e^{\left[-\frac{\sum \theta_{id}^2}{2\sigma_d^2} + \frac{\beta_{0d} \sum \theta_{id}}{\sigma_d^2} - \frac{n\beta_{0d}^2}{2\sigma_d^2} - \frac{1}{2\sigma_d^2} \beta_{0d}^2 \right]}$$

can remove
n.c. w/ β_0

$$\propto e^{\frac{\beta_{0d} \sum \theta_{id}}{\sigma_d^2} - \frac{n\beta_{0d}^2}{2\sigma_d^2} - \frac{1}{2\sigma_d^2} \beta_{0d}^2} =$$

$$= e^{\beta_{0d} \left(\frac{\sum \theta_{id}}{\sigma_d^2} \right) - \beta_{0d}^2 \left(\frac{n}{2\sigma_d^2} - \frac{1}{2\sigma_d^2} \right)}$$

Stop here for now

Attempt v2 to fix the code / results.

Using simple version of the full constraint

$$P(\beta_{02} | \cdot) \propto e^{-\frac{1}{2\sigma_2^2} \sum_i (\theta_{02} - \beta_{02})^2 - \frac{1}{2T_2} \beta_{02}^2}$$

Full conditional for θ_2^2 .

θ_2^2 prior is C

θ_2^2 shows up in ℓ of θ_{iz} for fixed z

$$p(\theta_2^2 | \cdot) \propto (2\pi\theta_2^2)^{-\frac{n}{2}} e^{-\frac{1}{\theta_2^2} \sum (\theta_{iz} - \beta_{0z})^2} \times (\theta_2^2)^{-\alpha-1} e^{-\frac{b}{\theta_2^2}}$$

$$\propto (\theta_2^2)^{-\alpha-1-\frac{n}{2}} \times e^{-\frac{1}{\theta_2^2} \left[\frac{1}{2} \sum (\theta_{iz} - \beta_{0z})^2 + b \right]}$$

$$= (\theta_2^2)^{-\frac{(n+2)}{2}-1} \times e^{-\underbrace{\left[\frac{1}{2} \sum (\theta_{iz} - \beta_{0z})^2 + b \right]}_{\text{param. } \beta^*} / \theta_2^2}$$

as expected we obtained an Inverse Gamma

Problem 3

• Update ratio for β_{02}

$$\cdot P(\beta_{02} | \cdot) \propto e^{-\frac{1}{2\theta_2^2} \sum_i (\theta_{i2} - \beta_{02})^2 - \frac{1}{2\bar{\epsilon}_2^2} \beta_{02}^2}$$

• we have chosen a symmetric candidate density for β_{02} , ratio will not involve $q(\cdot | \cdot)$.

$$r(\beta_{02}^*) = \frac{e^{-\frac{1}{2\theta_2^2} \sum_i (\theta_{i2} - \beta_{02}^*)^2 - \frac{1}{2\bar{\epsilon}_2^2} \beta_{02}^{*2}}}{e^{-\frac{1}{2\theta_2^2} \sum_i (\theta_{i2} - \hat{\beta}_{02})^2 - \frac{1}{2\bar{\epsilon}_2^2} \hat{\beta}_{02}^2}} = \\ = e^{\exp \left(-\frac{1}{2\theta_2^2} \left[\sum_i (\theta_{i2} - \beta_{02}^*)^2 - \sum_i (\theta_{i2} - \hat{\beta}_{02})^2 \right] \right)} \times \\ \exp \left(-\frac{1}{2\bar{\epsilon}_2^2} [\beta_{02}^{*2} - \hat{\beta}_{02}^2] \right).$$

On the log scale:

$$= -\frac{1}{2\theta_2^2} \left\{ \sum_i (\theta_{i2} - \beta_{02}^*)^2 - \sum_i (\theta_{i2} - \hat{\beta}_{02})^2 \right\} -$$

This is fine for $\sqrt{2}$

$$= -\frac{1}{2\bar{\epsilon}_2^2} \left\{ (\beta_{02}^*)^2 - (\hat{\beta}_{02})^2 \right\}.$$

• Update for θ_{id}

Do not use

$$p(\theta_{id} | \cdot) \propto e^{-n_{id} e^{\theta_{id}}} + \theta_{id} \left(Y_{id} - \frac{1}{2\sigma_2^2} \theta_{id} + \frac{1}{\sigma_2^2} \beta_{id} \right)$$

Having chosen Normal of symmetric density for θ_{id} , we can just take ratios of $p(\cdot)$

$$r(\theta_{id}^*) = \frac{e^{-n_{id} e^{\theta_{id}^*}} + \theta_{id}^* \left(Y_{id} - \frac{1}{2\sigma_2^2} \theta_{id}^* + \frac{1}{\sigma_2^2} \beta_{id} \right)}{e^{-n_{id} e^{\theta_{id}'}} + \theta_{id}' \left(Y_{id} - \frac{1}{2\sigma_2^2} \theta_{id}' + \frac{1}{\sigma_2^2} \beta_{id} \right)}$$

$$= e^{-n_{id} [e^{\theta_{id}^*} - e^{\theta_{id}'}]}$$

$$\exp(Y_{id} (\theta_{id}^* - \theta_{id}')) - \frac{1}{2\sigma_2^2} [\theta_{id}^{*2} - \theta_{id}'^2] + \frac{\beta_{id}}{2\sigma_2^2} [\theta_{id}^* - \theta_{id}']$$

On the log scale:

$$-n_{id} [e^{\theta_{id}^*} - e^{\theta_{id}'}] +$$

$$Y_{id} (\theta_{id}^* - \theta_{id}') - \frac{1}{2\sigma_2^2} [\theta_{id}^{*2} - \theta_{id}'^2] + \frac{\beta_{id}}{2\sigma_2^2} [\theta_{id}^* - \theta_{id}']$$

θ_{id} update v2:

$$e^{-u_{id} \exp(\theta_{id}^*) + y_{id} \dot{\theta}_{id}} e^{-\frac{1}{2\sigma_d^2} (\theta_{id}^* - \beta_{0d})^2}$$

$$e^{-u_{id} \exp(\theta_{id}^*) + y_{id} \dot{\theta}_{id}} e^{-\frac{1}{2\sigma_d^2} (\theta_{id}^* - \beta_{0d})^2} \approx$$

$$= \exp(-u_{id} [\exp(\theta_{id}^*) - \exp(\theta_{id}^*)]) \times$$

$$\exp(y_{id} (\theta_{id}^* - \theta_{id}^*)) \times$$

$$\exp\left(-\frac{1}{2\sigma_d^2} \left[(\theta_{id}^* - \beta_{0d})^2 - (\theta_{id}^* - \beta_{0d})^2 \right]\right)$$