A fast algorithm for the canonical representative of a Bell inequality

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January 3, 2014

In a scenario with n parties, our inequality is written using coefficients $\beta(abc...|xyz...)$:

$$\sum_{abc...xyz...} \beta(abc...|xyz...)P(abc...|xyz...) \leqslant u. \tag{1}$$

A vector $\vec{\beta}$ corresponding to $\beta(abc...|xyz...)$ can be written by enumerating the coefficients of $\beta(abc...|xyz...)$. The enumeration is done by increasing the indices in this order: $a, x, b, y, c, z....$

The lexicographic order for such vectors is given by the following:

$$u <_{\text{lex}} v \Leftrightarrow \exists k \text{ s.t. } u_j = v_j, \forall j < k \text{ and } u_k < v_k. \tag{2}$$

We define the *canonical* representative of an inequality β as the representative $\beta' = \beta^f$ under all permutations $f \in F$ such that β' is lexicographically minimal. The group F is the permutation group of all relabelings of outputs, inputs and parties acting on inequalities and probability distributions.

Our algorithm fixes the role of the first party (corresponding to a, x). So when the first party can be exchanged with other parties, all such permutations have to considered independently. With this restriction, we only consider the following permutations:

1. local permutations of outputs and inputs for the first party,

2. local permutations of outputs and inputs for the remaining parties,

3. permutations of $(n-1)$ parties excluding the first.

We now consider the canonical ordering of indices $i \rightarrow (a, x)$ and $j \rightarrow (b, y, c, z, ...)$, with $i = 1, ...,$ I and $j = 1, ..., J$, such that the k-th coefficient of $\vec{\beta}$ is given by $k = (j-1)I + i$. We write this k-th coefficient represented by the pair (i, j) as β_{ij}.

The class 1. of permutations above affects the index i and the other two classes affect the index j. All allowed permutations can be represented by an element g of the group G which acts on the index i and an element h of the group H acting on j, such that:

$$\beta_{ij}^{(g,h)} = \beta_{i^{g^{-1}} j^{h^{-1}}}. \tag{3}$$

As shown in Fig. 1, the object β_{ij} can be viewed as a matrix, with G acting on its rows and H acting on its columns. We write the columns of the matrix β_{ij} as vectors $\vec{c}_j = (\beta_{1j}, ...\beta_{Ij})$. We remark that the first coefficients of the vector $\vec{\beta}$ are given by the vector $\vec{c}_1$.

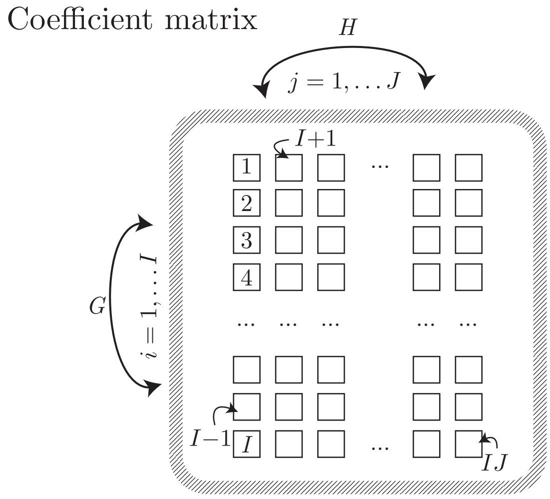

Figure 1. The coefficient matrix β_{ij} for $\beta(abc...|xyz...)$, showing the role of the row index i (enumerating the indices a, x) and the column index j (enumerating all other indices). The indices inside the squares represent the elements of the vector $\vec{\beta}$, enumerating the coefficients of $\beta(abc...|xyz...)$.

First level of the algorithm: canonical $\vec{c}_1'$

We start our algorithm with the following parameters:

- the current column γ, initialized to $\gamma = 1$ at the start,
- a set of canonical candidates ℓ^γ, initialized with the original inequality β: $\ell^\gamma \leftarrow \{\beta\}$, with the decomposition $\beta = (\vec{c}_1, ...\vec{c}_J)$,
- the row permutation group R^γ, initialized with $R^\gamma \leftarrow G$,
- the column permutation group $C^\gamma \leftarrow H$,
- the column permutation subgroup $S^\gamma \subseteq C^\gamma$ which stabilizes γ,
- a set of transversals $\alpha_1, ...\alpha_m \in C^\gamma$ such that C^γ is the disjoint union $C^\gamma = S^\gamma\alpha_1 \cup ... \cup S^\gamma\alpha_m$, with $m = |C^\gamma|/|S^\gamma|$.

The idea is to find all permutations $g^\gamma \in R^\gamma$ and an index $k \in \{1, ..., m\}$ such that for the candidate β, the configuration $\beta' = \beta^{(g^\gamma, \alpha_k)}$ has a vector $\vec{c}_\gamma'$ lexicographically minimal (here $\gamma = 1$ of course, corresponding to the vector of the first coefficients of β'). To do so, we observe that:

$$\vec{c}_\gamma' = \left(\beta_{\gamma^{\alpha_k^{-1}}}\right)^{g^\gamma}. \tag{4}$$

To find such candidates, we enumerate all indices $j = \gamma^{\alpha_k^{-1}}$ for $k = 1, ..., m$, and look at the minimal lexicographic representative $\vec{c}_j^{\,\min} = (\vec{c}_j)^{g_j}$ of the vector $\vec{c}_j$ under the row permutation group R^γ, with $g_j \in R^\gamma$. The canonical vector $\vec{c}_\gamma'$ is then the lexicographic minimum of those $\vec{c}_j^{\,\min}$: $\vec{c}_\gamma' = \min_j \vec{c}_j^{\,\min}$.

We keep the pairs (k, g_j) for which the resulting $\vec{c}_{j = \gamma^{\alpha_k^{-1}}} = \vec{c}_\gamma'$, and populate the set $\ell^{\gamma+1}$ of candidates for the next step with those: $\ell^{\gamma+1} = \left\{\beta^{(g_j, \alpha_k)}\right\}_{j,k \text{ s.t. } \vec{c}_j = \vec{c}_\gamma'}$.

Subsequent columns

The algorithm will then be applied to the next column $\gamma + 1$, with the following updates:

- the set $\ell^{\gamma+1}$ is given by the previous step,

- the row permutation group $R^{\gamma+1}$ is the subgroup $R^{\gamma+1} \subseteq R^{\gamma}$ leaving $\vec{c}_{\gamma}'$ invariant, because we want the next steps of the algorithms to keep the previous $\vec{c}_{\gamma}'$ invariant,

- the column permutation group $C^{\gamma+1} \leftarrow S^{\gamma}$ must only permute columns with indices $\geqslant \gamma+1$,

- the group $S^{\gamma+1} \subseteq C^{\gamma+1}$ stabilizes $\gamma+1$ and the transversals are readily computed.

The algorithm is then performed as above, with the only change that the set $\ell^{\gamma+1}$ can have more than one candidate; the new candidates are generated by looping over the candidates in $\ell^{\gamma+1}$ and over $k=1, \ldots, m$, and selecting the best overall $\vec{c}_{\gamma+1}'$.

When the algorithm terminates at the last column $\gamma = J$, the set ℓ^{J+1} contains the minimal lexicographic representative $\beta' = (\vec{c}_1', \ldots, \vec{c}_J')$.