Can a bipartite inequality be made symmetric?

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Abstract

We give a procedure to find whether a bipartite inequality has a form which is invariant under the permutation of parties.

Let $I = I(ab|xy)$ and $u \in \mathbb{R}$ describe a bipartite inequality:

$$\sum_{abxy} I(ab|xy) P(ab|xy) \leqslant u. \tag{1}$$

By considering subsets of parties, fully symmetric multipartite inequalities can also be recognized. The case of, e.g., cyclic permutations is unsolved.

We define the equivalence relation $I \sim J$ when $I = J + \text{signaling terms}$. We assume that Alice and Bob have the same number of inputs, and that the number of outputs for the inputs $x = 1, 2, \ldots$ and $y = 1, 2, \ldots$ is the same for Alice and Bob.

Let G be the permutation group acting on outputs and inputs of a single party. We write G_{A} the local permutation group for Alice, acting on pairs (a, x), and G_{B} the local permutation group for Bob, acting on pairs (b, y). The groups $G_{\mathrm{A}}, G_{\mathrm{B}}$ include relabelings of outputs and inputs. Let e be the identity element for these groups.

The action of $g_{\mathrm{A}} \in G_{\mathrm{A}}$ on $I \rightarrow I^{g_A}$ is described by:

$$I^{g_A}(a'b|x'y) = I(ab|xy), \qquad (a', x') = (a, x)^{g_{\mathrm{A}}}, \tag{2}$$

and the same for $g_{\mathrm{B}} \in G_{\mathrm{B}}$.

Let $H = \{e, \pi\}$ be the permutation group of parties, with e the identity and π the swapping of Alice and Bob, such that $I^{\pi}(ba|yx) = I(ab|xy)$.

Then any relabeling of the outputs, inputs and parties of I can be described by $r = [g_{\mathrm{A}}, g_{\mathrm{B}}, h]$, with $g_{\mathrm{A}} \in G_{\mathrm{A}}$, $g_{\mathrm{B}} \in G_{\mathrm{B}}$, $h \in H$, and the action $I^{[g_{\mathrm{A}}, g_{\mathrm{B}}, h]} = ((I^{g_{\mathrm{A}}})^{g_{\mathrm{B}}})^h$.

Let R be the group of all these relabelings.

Definition 1. *The inequality I is said to be symmetric if it is invariant under permutation π of Alice and Bob: $I^{\pi} \sim I$.*

We are now interested in the following: is there any $g_{\mathrm{A}} \in G_{\mathrm{A}}$ and $g_{\mathrm{B}} \in G_{\mathrm{B}}$ so that $I^{[g_{\mathrm{A}}, g_{\mathrm{B}}, e]} = \left(I^{[g_{\mathrm{A}}, g_{\mathrm{B}}, e]}\right)^{[e, e, \pi]}$?

To answer this question, we will look at the symmetry subgroup leaving I invariant.

Definition 2. *The* symmetry subgroup S_I *of the inequality I is the set of all elements $r \in R$ leaving I invariant:*

$$S_I = \{r \in R | I^r \sim I\}. \tag{3}$$

The group axioms can be verified easily for S_I.

The generators of the symmetry subgroup S_I can be split in two types: those of the form $[g_{\mathrm{A}}, g_{\mathrm{B}}, e]$ who do not permute the parties, and those of the form $[g_{\mathrm{A}}, g_{\mathrm{B}}, \pi]$ who permute the parties.

Lemma 3. *It is always possible to find generators for S_I so that at most one generator permutes the parties.*

Proof. If the set $\{s_1, s_2, s_3, ...\}$ generates S_I and, say, s_1 and s_2 both permute the parties, it is sufficient to replace $s_2' \to s_2 s_1$ which permutes the parties twice, and thus does not permute them. This operation can be repeated until at most one generator permutes the parties. □

Now comes the main proposition:

Proposition 4. *To determine if the inequality I has a symmetric form, it is sufficient to look at the generators of S_I, after using Lemma 3. If there is no generator which permutes the parties, the inequality cannot be put into a symmetric form. If there is a single generator of the form $s = [g_1, g_2, \pi]$, then we test for the existence of single party relabeling $g \in G$ such that the inequality is invariant under $s' = [g^{-1} g_2, g\, g_1, e]$.*

If and only if there exists such g, the inequality has a symmetric form.

Note: most of the time, the generator $s = [g_1, g_2 = g_1^{-1}, \pi]$ which has the trivial solution $g = g_1^{-1}$.

Examples

For the following examples, the structure of the symmetry group has been computed using `Faacets`.

CHSH form given by Ito. — Let us take the form of CHSH given in Avis&Ito list:

$$\langle A_1 B_1 \rangle + \langle A_2 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_2 \rangle \leqslant 2, \tag{4}$$

with the following generators for its symmetry group:

- output permutations: $A_1(1,2)\, A_2(1,2)\, B_1(1,2)\, B_2(1,2)$,
- output and input permutations: $A_1(1,2)\, B(1,2)$ and $B_2(1,2)\, A(1,2)$,
- party permuting generator: $A(1,2)\, B(1,2)\, (A, B)$.

Notation:

- A_x(cycle) applies the permutation given by *cycle* to the outputs of Alice for the given input x,
- A(cycle) applies the permutation given by *cycle* to the inputs of Alice (the last two notations apply also to Bob),
- (A, B) permutes the parties.

Coming back to Proposition 4, we look for an element g such that our inequality is invariant under $[g^{-1} A(1,2), gB(1,2), e]$. This element is $g =$"permutation of inputs $(1,2)$", which gives:

$$\langle A_1 B_1 \rangle + \langle A_2 B_1 \rangle + \langle A_1 B_2 \rangle - \langle A_2 B_2 \rangle \leqslant 2. \tag{5}$$

Another inequality by Ito. —

The inequality is:

$$
\begin{aligned}
5\langle A_1 B_1\rangle + 4\langle A_2 B_1\rangle - 3\langle A_3 B_1\rangle - 3\langle A_4 B_1\rangle \\
+6\langle A_5 B_1\rangle + 6\langle A_6 B_1\rangle - 1\langle A_7 B_1\rangle - 3\langle A_1 B_2\rangle - 2\langle A_2 B_2\rangle \\
+2\langle A_3 B_2\rangle + 2\langle A_4 B_2\rangle + 6\langle A_5 B_2\rangle - 3\langle A_8 B_2\rangle - 3\langle A_1 B_3\rangle \\
-2\langle A_2 B_3\rangle + 2\langle A_3 B_3\rangle + 2\langle A_4 B_3\rangle + 6\langle A_6 B_3\rangle + 3\langle A_8 B_3\rangle + \langle A_1 B_4\rangle + 2\langle A_2 B_4\rangle \\
+\langle A_3 B_4\rangle + \langle A_4 B_4\rangle + \langle A_7 B_4\rangle - 2\langle A_1 B_5\rangle + 2\langle A_2 B_5\rangle + \langle A_1 B_6\rangle + \langle A_3 B_6\rangle \\
+\langle A_1 B_7\rangle + \langle A_4 B_7\rangle - 1\langle A_3 B_8\rangle + \langle A_4 B_8\rangle &\leqslant 34,
\end{aligned}
$$

with the generators:

- output permutations: $A_1(1,2)\,A_2(1,2)\,...\,A_8(1,2)\;B_1(1,2)\;B_2(1,2)...B_8(1,2)$, not surprising because the inequality is a correlation inequality with no marginal terms,

- output and input permutations: $A_8(1,2)\,A(5,6)\,B(2,3)$.

As the symmetry group of the inequality does not have a generator permuting the parties, the inequality does not have a symmetric form.

The inequality $I3322$ given by Ito. —

$$
\begin{aligned}
\langle A_1\rangle - \langle B_1\rangle + \langle A_2\rangle - \langle B_2\rangle \\
+\langle A_1 B_1\rangle + \langle A_1 B_2\rangle - \langle A_1 B_3\rangle \\
+\langle A_2 B_1\rangle + \langle A_2 B_2\rangle + \langle A_2 B_3\rangle \\
-\langle A_3 B_1\rangle + \langle A_3 B_2\rangle &\leqslant 4,
\end{aligned}
$$

with the generators:

- input and output permutations: $A_3(1,2)\,B(1,2)$ and $B_3(1,2)\,A(1,2)$,

- party permuting generator: $A_1(1,2)\,A_2(1,2)\,B_1(1,2)\,B_2(1,2)\,A(1,2)\,B(1,2)\,(A,B)$.

Here, $g_1 = A_1(1,2)\,A_2(1,2)\,A(1,2)$ and $g_2 = B_1(1,2)\,B_2(1,2)\,B(1,2)$. We notice that, as single party relabelings, $g_1 = g_2$ and $g_1 g_1 = g_2 g_2 = e$.

Then $g =$ "permute the output for inputs 1 and 2, and permute the inputs 1 and 2", which finds the following symmetric form:

$$
\begin{aligned}
-\langle A_2\rangle - \langle B_1\rangle - \langle A_1\rangle - \langle B_2\rangle \\
-\langle A_2 B_1\rangle - \langle A_2 B_2\rangle + \langle A_2 B_3\rangle \\
-\langle A_1 B_1\rangle - \langle A_1 B_2\rangle - \langle A_1 B_3\rangle \\
-\langle A_3 B_1\rangle + \langle A_3 B_2\rangle &\leqslant 4.
\end{aligned}
$$

Proof of the proposition

To prove the proposition, we need the following:

Lemma 5. *Let I be an inequality which has a symmetric form. Then there is a $g \in G$ such that $I^{[g,e,e]}$ is symmetric.*

Proof.

- let $J = I^{[g_1,g_2,\pi]}$ be a symmetric form. Then $J = J^{[e,e,\pi]} = \left(I^{[g_1,g_2,\pi]}\right)^{[e,e,\pi]} = I^{[g_2,g_1,e]}$ is also a symmetric form which leads to the second case below.

– let $J = I^{[g_1, g_2, e]}$ be a symmetric form. Then $J' = J^{\left[g_2^{-1}, g_2^{-1}, e\right]}$ is also a symmetric form, and $J' = \left(I^{[g_1, g_2, e]}\right)^{\left[g_2^{-1}, g_2^{-1}, e\right]} = I^{\left[g_1 g_2^{-1}, e, e\right]}$, which proves the lemma. □

Lemma 6. *If I has a symmetric form, then there exists a $s_I \in S_I$ with $s_I = [g^{-1}, g, \pi]$.*

Proof. Let $J = I^{[g, e, e]}$ be a symmetric form of I by the previous Lemma. Then:

$$J^{[e, e, \pi]} = \left(I^{[g^{-1}, e, e]}\right)^{[e, e, \pi]} = I^{[g^{-1}, e, e]} = J, \tag{6}$$

and $\left(I^{[g^{-1}, e, \pi]}\right)^{[g, e, e]} = \left(I^{[g^{-1}, e, e]}\right)^{[g, e, e]}$ which implies $I = I^{[g^{-1}, g, \pi]}$, and thus $[g^{-1}, g, \pi] \in S_I$. □

Let us now prove the main proposition:

Proof. If I can be put into a symmetric form, there exists a $[g^{-1}, g, \pi] \in S_I$ by Lemma 6. If indeed, our party-permuting generator is of that form, the case is solved.

If the party-permuting generator is of the general form $[g_1, g_2, \pi]$ and the inequality has a symmetric form, there exists $[j, k, e] \in S_I$ such that $[g_1, g_2, \pi] = [g^{-1}, g, \pi]\,[j, k, e] = [g^{-1}k, g j, \pi]$. This can be seen because the subgroup of such $[j, k, e]$ is normal.

If there exists such $[j, k, e]$, then $[j, k, e] = [g^{-1}, g, \pi][g_1, g_2, \pi] = [g^{-1}g_2, g g_1, e]$, which proves the proposition.

If no such element exists, then there is no $[g^{-1}, g, \pi] \in S_I$, and by Lemma 6, the inequality does not have a symmetric form.

We do not know of a faster algorithm than brute enumeration of either elements $g \in G$ or $[j, k, e] \in S_I$. Luckily, most of the time the orders of either G or $|S_I|$ are small.

Note: this factorization is probably related to the Zappa–Szép or bicrossed product, or subset product of subgroups. □