Constructive representation theory and applications to causal structures Part I: groups*

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∗. Download software for MATLAB/Octave at github.com/replab/replab

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Book by Serre / Sym. SOS paper by Gatermann&Parrilo

Goal

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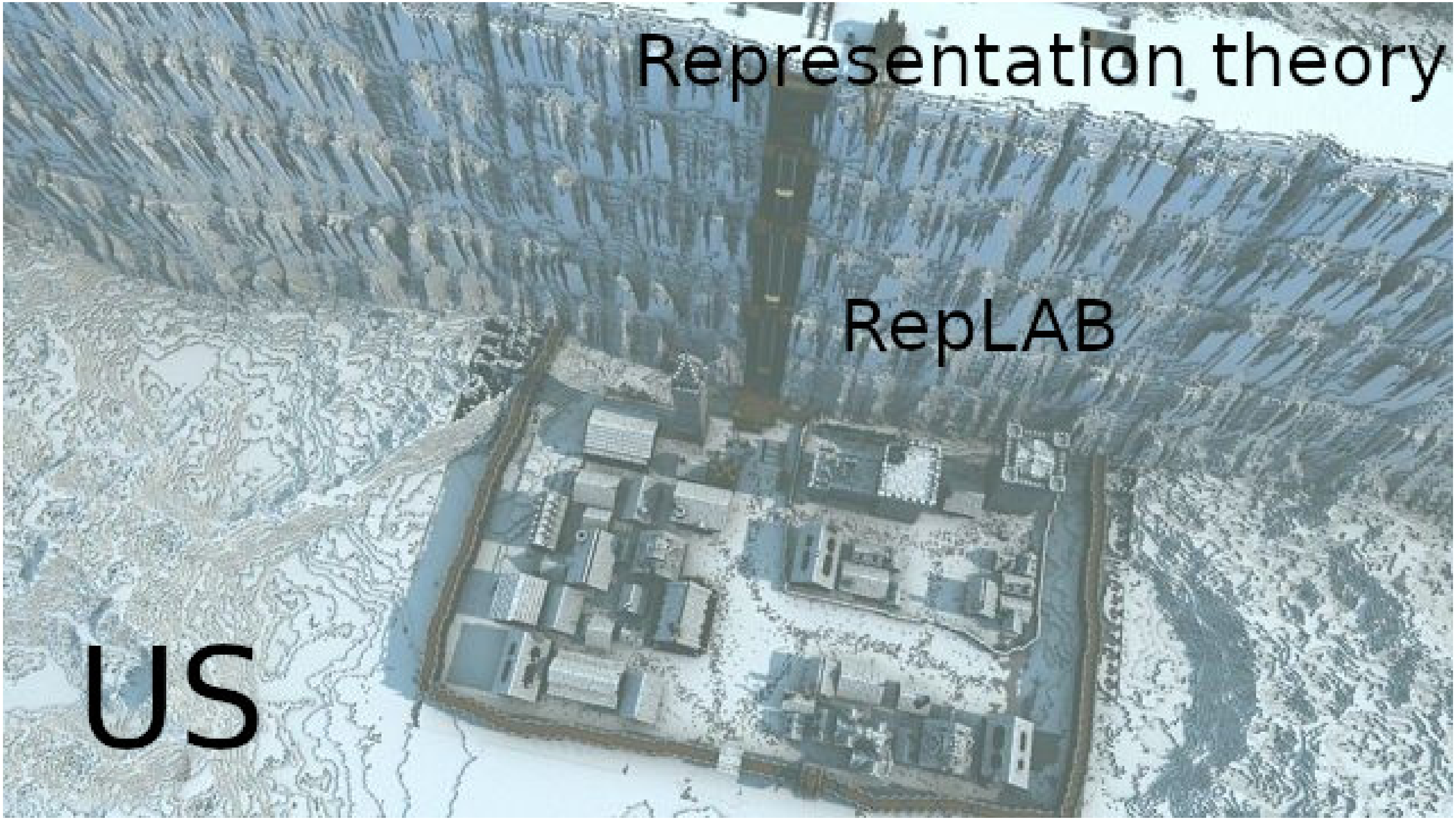

Goal

- Enable causal verification algorithms for larger problems

- Symmetry for symmetry sake

  (in particular: what is a good “API” to construct groups?)

- Teaching tool: “hands-on” representation theory

Outline

1. Groups

   Numerical examples

2. Representations

   Numerical examples

3. RepLAB: black-box representation theory

4. A taste of composition

Groups

A *group* is a set G closed under a binary operation $\cdot$ such that (axioms)

Associativity	$(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c = abc$	$\forall a, b, c \in G$
Identity $1 \in G$	$1 \cdot a = a \cdot 1 = a$	$\forall a \in G$
Inverse a^{-1}	$a \cdot a^{-1} = a^{-1} \cdot a = 1$	$\forall a \in G$

A *compact group* has a Haar measure and can be integrated over (*from which we can sample from uniformly*).

A *finitely generated group* is the closure of a finite set written $\langle g_1, g_2, \ldots, g_N \rangle$, where the g_i are the group generators.

A *finite group* has $|G| < \infty$.

Permutation groups

The symmetric group S_N is the set of bijections $\{1, \ldots, N\} \to \{1, \ldots, N\}$.

In RepLAB, we write $\pi \in S_N$ by its images:

$$\pi = [\pi(1), \pi(2), \ldots, \pi(N)].$$

Composition $\sigma = \pi\, \rho$ for $\pi\,, \rho \in S_N$ is such that:

$$\sigma(i) = \pi(\rho(i)) = (\pi \circ \rho)(i), \qquad i = 1, \ldots, N.$$

(Left action convention in RepLAB as in physics. Alternative $i^\sigma = i^{\pi\rho} = (i^\pi)^\rho$ *in CGT.)*

Permutation groups

Example: two measurements $x = 1, 2$ with two outcomes $a = 1, 2$ each

$$\vec{P} = (P_{a|x}) = \left(\overbrace{P_{1|1}, P_{2|1}, P_{1|2}, P_{2|2}}^{1,2,3,4} \right) \in \mathbb{R}^4$$

What are the "physical" permutations $\subseteq S_4$?

$$\pi_{\text{input}} = [3, 4, 1, 2],$$

$$\pi_{\text{output 1}} = [2, 1, 3, 4], \qquad \pi_{\text{output 2}} = [1, 2, 4, 3].$$

$$G_{\text{Alice}} = \langle \pi_{\text{input}}, \pi_{\text{output 1}}, \pi_{\text{output 2}} \rangle.$$

Order of G_{Alice}? Are all generators necessary?

Signed permutation group

The signed symmetric group (hyperoctahedral group) B_N is the set of bijections

$$\{-N, \ldots, -1, 1, \ldots, N\} \to \{-N, \ldots, -1, 1, \ldots, N\}$$

such that $\sigma(-i) = -\sigma(i)$. In RepLAB, we write $\pi \in B_N$ by its images:

$$\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(N)].$$

Composition $\sigma = \pi\,\rho$ for $\pi, \rho \in B_N$ is such that:

$$\sigma(i) = \pi(\rho(i)) = (\pi \circ \rho)(i), \qquad i = -N, \ldots, -1, 1, \ldots, N.$$

Example: CHSH

Action on variables

$$\{-B_1, -B_0, -A_1, -A_0, A_0, A_1, B_0, B_1\} \approx \{-4, \ldots, -1, 1, \ldots, 4\}$$

for the CHSH expression

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$$

Symmetries:

$$\begin{aligned} A_i &\leftrightarrow B_i \\ A_x \to -A_x, \quad B_y &\to -B_y \\ A_0 \leftrightarrow A_1, \quad B_1 &\to -B_1 \end{aligned} \qquad \begin{aligned} \sigma_{\text{parties}} &= [3, 4, 1, 2] \\ \sigma_{\text{flip}} &= [-1, -2, -3, -4] \\ \sigma_{\text{other}} &= [2, 1, 3, -4] \end{aligned}$$

What is the order of $G_{\text{CHSH}} = \langle \sigma_{\text{parties}}, \sigma_{\text{flip}}, \sigma_{\text{other}} \rangle$?

Presentations

Presentation of a group (for ex. in Groupprops wiki):

$$D_8 = \left\langle \underbrace{x\,,a}_{\text{Generators}} \middle| \overbrace{a^4 = x^2 = 1, x\,a\,x^{-1} = a^{-1}}^{\text{Relations}} \right\rangle$$

(here a is a 90° rotation and x a reflection)

We do not support presentations in RepLAB.

$\Rightarrow$ find an isomorphic permutation group (use GAP System if necessary)

(Outer) group constructions

- **Direct product**

  Given groups $G = \{g_i\}_i$ and $H = \{h_j\}_j$, construct $G \times H = \{(g_i, h_j)\}_{ij}$

  $$\text{Factors commute:} \qquad (g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2)$$

- **Semidirect product**

  Given $H = \{h_i\}_i$, $N = \{\nu_j\}_j$, and $\varphi \colon H \to (N \to N)$, construct $H \ltimes N$

  $$(h_1, \nu_1) \cdot (h_2, \nu_2) = (h_1 h_2, \varphi_{h_2^{-1}}(\nu_1)\nu_2)$$

  Convention: $(1, \varphi_h(\nu)) = (h, 1) \cdot (1, \nu) \cdot (h^{-1}, 1)$ *is left conjugation.*

(Outer) group constructions

- **Wreath product** of group A by S_n, written $W = S_n \wr A$

$$H = S_n \qquad\qquad N = A^n = A_1 \times A_2 \times \cdots \times A_n$$

$$W = H \ltimes N = S_n \ltimes (A_1 \times \cdots \times A_n)$$

$$W \ni w = (h, (a_1, \ldots, a_n)), \qquad \varphi_h((a_1, \ldots, a_n)) = (a_{h^{-1}(1)}, \ldots, a_{h^{-1}(n)})$$

Describes n components that can be permuted, each with own symmetry group A.

(Note: notation reversed compared to Wikipedia for internal consistency)

Example

Alice with two measurements settings ($x = 1, 2$) with two outcomes each ($a = 1, 2$).

Outcome relabelings $\quad S_2 \times S_2 = \{(\sigma_1, \sigma_2) : \sigma_1, \sigma_2 \in S_2\}$

Setting relabelings $\quad S_2$

Symmetry group $\quad S_2 \wr S_2 = S_2 \ltimes (S_2 \times S_2)$

As permutation on $\vec{P}$:	As wreath product element:
$[2, 1, 3, 4]$	$([1, 2], ([2, 1], [1, 2]))$
$[1, 2, 4, 3]$	$([1, 2], ([1, 2], [2, 1]))$
$[3, 4, 1, 2]$	$([2, 1], ([1, 2], [1, 2]))$

Examples

See `https://github.com/replab/replab/blob/master/jupyter/QCS_part1_Groups.ipynb`