Constructive representation theory and applications to causal structures Part II: representations*

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*. Download software for MATLAB/Octave at github.com/replab/replab

Representations

Ingredients

- A compact/finite group G

- A finite dimensional vector space $\mathbb{R}^d$ or $\mathbb{C}^d$

- Representation: morphism written $g \mapsto \rho_g$ of type

real $\rho\colon G \to \overset{\text{orthogonal}}{\mathcal{O}(d)}$ complex $\rho\colon G \to \overset{\text{unitary}}{\mathcal{U}(d)}$

$$\text{such that} \qquad \rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}.$$

(RepLAB limitations: compact groups and orthogonal/unitary representations)

Finitely generated group

$$G = \langle g_1, \ldots, g_n \rangle$$

and every $g \in G$ is a finite product of $\{g_1, \ldots, g_n\}$: $\Rightarrow$ specify $\rho_{g_1}, \ldots, \rho_{g_n}$

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Example

Generator $\in S_4$ Representation on $\vec{P} \in \mathbb{R}^4$

$\pi_{\text{input}} = [3, 4, 1, 2]$ $\rho_{\pi_{\text{input}}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$\pi_{\text{output 1}} = [2, 1, 3, 4]$ $\rho_{\pi_{\text{output 1}}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Explicit representations

Finitely generated group

$$G = \langle g_1, \ldots, g_n \rangle$$

and every $g \in G$ is a finite product of $\{g_1, \ldots, g_n\}$: $\Rightarrow$ specify $\rho_{g_1}, \ldots, \rho_{g_n}$

Example

Generator $\in S_4$ | Representation on $\vec{P} \in \mathbb{R}^4$ | Another rep. on $\mathbb{R}^4$

$\pi_{\text{input}} = [3, 4, 1, 2]$

$$\rho_{\pi_{\text{input}}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\sigma_{\pi_{\text{input}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\pi_{\text{output 1}} = [2, 1, 3, 4]$

$$\rho_{\pi_{\text{output 1}}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{\pi_{\text{output1}}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Canonical representations for a permutation group $G \subseteq S_n$

- Natural representation using permutation matrices on $\mathbb{R}^n$

- Representation on tensor space for given d, permuting tensor indices

$$\vec{v} \in \underbrace{\mathbb{R}^d \otimes \cdots \otimes \otimes \mathbb{R}^d}_{n \text{ times}}, \qquad v_{i_1 i_2 \ldots i_n} \text{ with } i_1, \ldots, i_n \in \{1, \ldots, d\}$$

… for a signed permutation group $G \subseteq B_n$

- Natural representation using generalized permutation matrices on $\mathbb{R}^n$

Combining representations (of the same group)

Given G and two representations

$$\rho\colon G \to \begin{cases} \mathcal{O}(d_1) \\ \mathcal{U}(d_1) \end{cases}, \qquad \sigma\colon G \to \begin{cases} \mathcal{O}(d_2) \\ \mathcal{U}(d_2) \end{cases},$$

- Construct the *direct sum* $\rho \oplus \sigma$ (neutral: empty representation 0)

  of dimension $d_1 + d_2$
- Construct the *tensor product* $\rho \otimes \sigma$ (neutral: trivial rep. 1)

  of dimension $d_1 \cdot d_2$
- If ρ is real, complexify ρ by using the map $\mathbb{R} \to \mathbb{C}$
- If ρ is complex, construct the conjugate $\overline{\rho}$

Example

States invariant under joint unitary transformation

Compact unitary group $\mathcal{U}(2)$

Its "identity" representation $\rho : \mathcal{U}(2) \to \mathcal{U}(2)$ acts on $\mathbb{C}^2$

State invariant under $\rho \otimes \rho$

$$|\psi^-\rangle \propto |01\rangle - |10\rangle$$

Choi state of channels invariant under unitary transformations

The Choi state has transposition on input subsystem

State invariant under $\bar{\rho} \otimes \rho$

$$|\varphi^+\rangle \propto |00\rangle + |11\rangle$$

Decomposing representations

- A compact/finite group G
- A finite dimensional vector space $\mathbb{K}^d$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$
- Representation $\rho\colon G \to \begin{cases} \mathcal{O}(d) \\ \mathcal{U}(d) \end{cases}$

An *invariant subspace* $V = \langle \vec{v}_1, \ldots, \vec{v}_m \rangle \subseteq \mathbb{K}^d$: $\quad \forall \vec{v} \in V, g \in G, \qquad \rho_g \vec{v} \in V$

A *subrepresentation* ρ' is the restriction of ρ to an invariant subspace V.

$$\rho'_g = \mathcal{V} \rho_g \mathcal{V}^\dagger, \qquad \mathcal{V} = \begin{pmatrix} \vec{v}_1 & \ldots & \vec{v}_m \end{pmatrix}^\dagger,$$

where the vectors $\vec{v}_i$ are orthonormal.

Note that $\mathcal{V}$ gives the basis of V with row vectors, *beware the conjugate transpose.*

Decomposing representations

- A compact/finite group G with representation $\rho\colon G \to \begin{cases} \mathcal{O}(d) \\ \mathcal{U}(d) \end{cases}$
- A finite dimensional vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$$\mathbb{K}^d = V^1 \oplus V^2 \oplus V^3 \oplus \cdots \oplus V^N$$

where each of the V^i is invariant, and contains no nontrivial subrepresentation = irreducible representations.

Then

- Recognize equivalent representations and regroup them into isotypic components
- Real irreducible representations are of three types: real, complex, quaternion

  Recognize and put in “canonical” form

Examples

See `https://github.com/replab/replab/blob/master/jupyter/QCS_part2_Representations.ipynb`