# Constructive representation theory and applications to causal structures Part I: groups*

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*. Download software for MATLAB/Octave at github.com/replab/replab

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Book by Serre / Sym. SOS paper by Gatermann&Parrilo

# Goal

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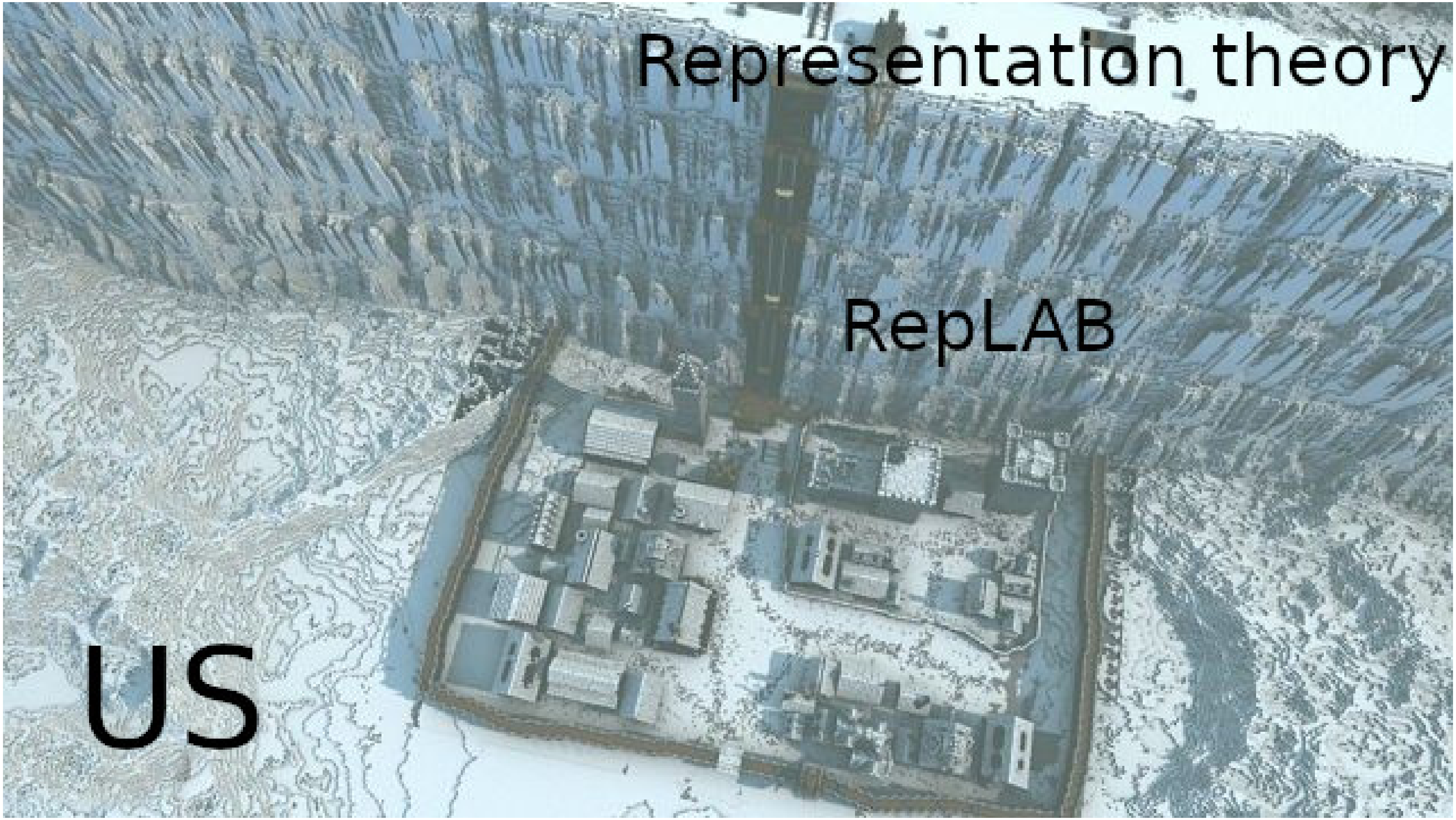

# Goals



- Enable causal verification algorithms for larger problems

- Symmetry for symmetry sake

  (what is a good “API” to construct groups?)

- Teaching tool: “hands-on” representation theory

# Outline

# Groups



A *group* is a set $G$ closed under a binary operation $\cdot$ such that (axioms)

| | | |
|---|---|---|
| Associativity | $(a\cdot b)\cdot c = a\cdot(b\cdot c) = a\cdot b\cdot c = abc$ | $\forall a, b, c \in G$ |
| Identity $1 \in G$ | $1\cdot a = a\cdot 1 = a$ | $\forall a \in G$ |
| Inverse $a^{-1}$ | $a\cdot a^{-1} = a^{-1}\cdot a = 1$ | $\forall a \in G$ |

A *compact group* has a Haar measure and can be integrated over
(*from which we can sample from uniformly*).

A *finitely generated group* is the closure of a finite set written $\langle g_1, g_2, \ldots, g_N\rangle$,
where the $g_i$ are the group generators.

A *finite group* has $|G| < \infty$.

# Permutation groups



The symmetric group $S_N$ is the set of bijections $\{1, \ldots, N\} \to \{1, \ldots, N\}$.

In RepLAB, we write $\pi \in S_N$ by its images:

$$\pi = [\pi(1), \pi(2), \ldots, \pi(N)].$$

Composition $\sigma = \pi\,\rho$ for $\pi\,, \rho \in S_N$ is such that:

$$\sigma(i) = \pi(\rho(i)) = (\pi \circ \rho)(i), \qquad i = 1, \ldots, N.$$

*(Left action convention in RepLAB as in physics. Alternative $i^\sigma = i^{\pi\rho} = (i^\pi)^\rho$ in CGT.)*

Example: two measurements $x = 1, 2$ with two outcomes $a = 1, 2$ each

$$\vec{P} = (P_{a|x}) = \left( \overbrace{P_{1|1}, P_{2|1}, P_{1|2}, P_{2|2}}^{1,2,3,4} \right) \in \mathbb{R}^4$$

What are the "physical" permutations $\subseteq S_4$ ?

$$\pi_{\text{input}} = [3, 4, 1, 2],$$

$$\pi_{\text{output 1}} = [2, 1, 3, 4], \qquad \pi_{\text{output 2}} = [1, 2, 4, 3].$$

$$G_{\text{Alice}} = \langle \pi_{\text{input}}, \pi_{\text{output 1}}, \pi_{\text{output 2}} \rangle.$$

Order of $G_{\text{Alice}}$? Are all generators necessary?

# Signed permutation group



The signed symmetric group (hyperoctahedral group) $B_N$ is the set of bijections

$$\{-N, ..., -1, 1, ..., N\} \to \{-N, ..., -1, 1, ..., N\}$$

such that $\sigma(-i) = -\sigma(i)$. In RepLAB, we write $\pi \in B_N$ by its images:

$$\sigma = [\sigma(1), \sigma(2), ..., \sigma(N)].$$

Composition $\sigma = \pi\,\rho$ for $\pi, \rho \in B_N$ is such that:

$$\sigma(i) = \pi(\rho(i)) = (\pi \circ \rho)(i), \qquad i = -N, ..., -1, 1, ..., N.$$

# Example: CHSH



Action on variables

$$\{-B_1, -B_0, -A_1, -A_0, A_0, A_1, B_0, B_1\} \approx \{-4, \ldots, -1, 1, \ldots, 4\}$$

for the CHSH expression

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$$

Symmetries:

$$\begin{aligned} A_i &\leftrightarrow B_i \\ A_x \to -A_x, &\quad B_y \to -B_y \\ A_0 \leftrightarrow A_1, &\quad B_1 \to -B_1 \end{aligned} \qquad \begin{aligned} \sigma_{\text{parties}} &= [3, 4, 1, 2] \\ \sigma_{\text{flip}} &= [-1, -2, -3, -4] \\ \sigma_{\text{other}} &= [2, 1, 3, -4] \end{aligned}$$

What is the order of $G_{\text{CHSH}} = \langle \sigma_{\text{parties}}, \sigma_{\text{flip}}, \sigma_{\text{other}} \rangle$ ?

Presentation of a group (for ex. in Groupprops wiki):

$$D_8 = \left\langle \underbrace{x\,,a}_{\text{Generators}} \,\middle|\, \overbrace{a^4 = x^2 = 1, x\,a\,x^{-1} = a^{-1}}^{\text{Relations}} \right\rangle$$

(here $a$ is a $90^\circ$ rotation and $x$ a reflection)

No support for presentations in RepLAB.

$\Rightarrow$ find an isomorphic permutation group (use GAP System if necessary)

# (Outer) group constructions



- **Direct product**

  Given groups $G = \{g_i\}_i$ and $H = \{h_j\}_j$, construct $G \times H = \{(g_i, h_j)\}_{ij}$

  $$\text{Factors commute:} \qquad (g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2)$$

- **Semidirect product**

  Given $H = \{h_i\}_i$, $N = \{\nu_j\}_j$, and $\varphi\colon H \to (N \to N)$, construct $H \ltimes N$

  $$(h_1, \nu_1) \cdot (h_2, \nu_2) = (h_1 h_2, \varphi_{h_2^{-1}}(\nu_1)\nu_2)$$

*Convention:* $(1, \varphi_h(\nu)) = (h, 1) \cdot (1, \nu) \cdot (h^{-1}, 1)$ *is left conjugation.*

# (Outer) group constructions

- **Wreath product** of group $A$ by $S_n$, written $W = S_n \wr A$

$H = S_n$ $\qquad$ $N = A^n = A_1 \times A_2 \times \cdots \times A_n$

$$W = H \ltimes N = S_n \ltimes (A_1 \times \cdots \times A_n)$$

$$W \ni w = (h, (a_1, \ldots, a_n)), \qquad \varphi_h((a_1, \ldots, a_n)) = (a_{h^{-1}(1)}, \ldots, a_{h^{-1}(n)})$$

Describes $n$ components that can be permuted, each with own symmetry group $A$.

*(Note: notation reversed compared to Wikipedia for internal consistency)*

# Example



Alice with two measurements settings ($x=1,2$) with two outcomes each ($a=1,2$).

Outcome relabelings $\qquad S_2 \times S_2 = \{(\sigma_1, \sigma_2) \colon \sigma_1, \sigma_2 \in S_2\}$

Setting relabelings $\qquad S_2$

Symmetry group $\qquad S_2 \wr S_2 = S_2 \ltimes (S_2 \times S_2)$

| As permutation on $\vec{P}$: | As wreath product element: |
|---|---|
| $[2,1,3,4]$ | $([1,2],([2,1],[1,2]))$ |
| $[1,2,4,3]$ | $([1,2],([1,2],[2,1]))$ |
| $[3,4,1,2]$ | $([2,1],([1,2],[1,2]))$ |

# Examples



See `https://github.com/replab/replab/blob/master/jupyter/QCS_part1_Groups.ipynb`