Discrete Optimization: Homework #3, Ex. #6

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If x^* is feasible, we will call $A_{x^*}x \leq b_{x^*}$ the subsystem of $Ax \leq b$ that is satisfied by x^* with equality. We can notice that if $rank(A_{x^*}) = n$, then x^* is an extreme point since the unique solution of $A_{x^*}x = b_{x^*}$ is x^* (Theorem 3.1). Suppose now that $rank(A_{x^*}) < n$. Similarly to the proof of Thm 3.2, we claim that if x^* is feasible, then there exists a y^* with $c^T y^* > c^T x^*$ and $rank(A_{y^*}) > rank(A_{x^*})$. If we prove this claim, it will imply that in at most n steps we will find an extreme point.
To prove this, let $d \neq 0 \in \mathbb{R}^n$ be a vector with $A_{x^*}d = 0$. If $c^T d < 0$, we can switch it to $-d$. As $c^T d > 0$, we consider the points $x^* + \lambda d$ with $\lambda > 0$ and let $\lambda_{max} = max\{\lambda$ such that $x^* + \lambda d$ is feasible$\}$. $Rank(A_{(x^*+\lambda d)}) > rank(A_{x^*})$ and $c^T(x^* + \lambda d) > c^T x^*$ so the claim is verified.
We now prove the claim in the case where $c^T d = 0$. If $c^T d = 0$, then $Ad \neq 0$ because $rank(A) = n$. Let λ_{max} be $max\{\lambda \geq 0 : A(x^* \pm \lambda d) \leq b\}$. Then one can choose $y^* = x^* + \lambda_{max} d$ or $y^* = x^* - \lambda_{max} d$, one of which has to satisfy the condition of the claim.

This means that we can use this process at most n times to find an x such that $Ax = b$ which would be an extreme point. Finding d is solving a system of n equations, which can be done in polynomial time as seen during the lecture. We only consider all non-zero components of d because if $d_i = 0$ $: \mid \frac{(b-A_{x^*}x^*)_i}{d_i} \mid \longrightarrow \infty$. Finding $\lambda_{max} = min\{\frac{(b-A_{x^*}x^*)_i}{d_i}\}$ can be done in polynomial time. In the case where $c^T d = 0$, then $\lambda_{max} = min\{\frac{(b-Ax^*)_i}{d_i}\}$ which can be easily calculated in polynomial time.