

Discrete Optimization: Homework #3, Ex. #6

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If x^* is feasible, we will call $A_{x^*}x \leq b_{x^*}$ the subsystem of $Ax \leq b$ that is satisfied by x^* with equality. We can notice that if $\text{rank}(A_{x^*}) = n$, then x^* is an extreme point since the unique solution of $A_{x^*}x = b_{x^*}$ is x^* . Suppose now that $\text{rank}(A_{x^*}) < n$. Similarly to the proof of Thm 3.2, we claim that if x^* is feasible, then there exists a y^* with $c^T y^* > c^T x^*$ and $\text{rank}(A_{y^*}) > \text{rank}(A_{x^*})$. If we prove this claim, it will imply that in at most n steps we will find an extreme point.

To prove this, let $d \neq 0 \in \mathbb{R}^n$ be a vector with $A_{x^*}d = 0$. If $c^T d < 0$, we can switch it to $-d$. As $c^T d > 0$, we consider the points $x^* + \lambda d$ with $\lambda > 0$ and let $\lambda_{\max} = \max\{\lambda \text{ such that } x^* + \lambda d \text{ is feasible}\}$. $\text{Rank}(A_{(x^* + \lambda d)}) > \text{rank}(A_{x^*})$ and $c^T(x^* + \lambda d) > c^T x^*$ so the claim is verified.

We now prove the claim in the case where $c^T d = 0$. If $c^T d = 0$, then $Ad \neq 0$ because $\text{rank}(A) = n$. Let λ_{\max} be $\max\{\lambda \geq 0 : A(x^* \pm \lambda d) \leq b\}$. Then one can choose $y^* = x^* + \lambda_{\max}d$ or $y^* = x^* - \lambda_{\max}d$, one of which has to satisfy the condition of the claim.

This means that we can use this process at most n times to find an x such that $Ax = b$ which would be an extreme point. Finding d is solving a system of n equations, which can be done in polynomial time as seen during the lecture. We only consider all non-zero components of d because if $d_i = 0$: $\left| \frac{(b - A_{x^*}x^*)_i}{d_i} \right| \longrightarrow \infty$. Finding $\lambda_{\max} = \min\left\{\frac{(b - A_{x^*}x^*)_i}{d_i}\right\}$ can be done in polynomial time. In the case where $c^T d = 0$, then $\lambda_{\max} = \min\left\{\frac{(b - Ax^*)_i}{d_i}\right\}$ which can be easily calculated in polynomial time.