Discrete Optimization: Homework #5, Ex. #5

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We have to prove that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, $P \neq \emptyset$, contains a line if and only if A does not have full column-rank. We know from linear algebra that A does not have full column-rank if $\ker(A) \neq \mathbf{0}$.

First, we prove $\Rightarrow$:
By hypothesis, we know that P contains a line, i.e. there exits a nonzero $v \in \mathbb{R}^n$ and a $x^* \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$ we have $p \coloneqq x^* + \lambda \cdot v \in P$. We define the rows of $A \in \mathbb{R}^{m \times n}$ by a_j. We need to prove that there exists a nonzero vector $u \in \ker(A)$. We choose $u = v$ and let's prove that $Av = \mathbf{0}$. We'll prove it by contradiction : we suppose that there exists an index $1 \leq i \leq m$ such that $a_i^T * v \neq 0$. We have that, for all λ :

$$a_i^T p = a_i^T x^* + \lambda \cdot a_i^T v \tag{1}$$

If we take $\lambda = \frac{(b_i + \epsilon) - a_i^T * x^*}{a_i^T * v}$ $(\epsilon > 0)$, (1) gives $a_i^T p = b_i + \epsilon > b_i$. This means that $p \notin P$ but this is a contradiction. Thus, A does not have full column-rank.

Then, we prove $\Leftarrow$:
Suppose there exists a nonzero vector $v \in \mathbb{R}^n$ such that $Ax = 0$. Let $x^* \in P$, so $Ax^* \leq b$. We consider the vector $p_\lambda \coloneqq x^* + \lambda \cdot v$, for $\lambda \in \mathbb{R}$. We get :

$$Ap_\lambda = Ax^* + \lambda \cdot Av = Ax^* \leq b$$

Thus, $p_\lambda \in P$ and this is verified for all λ. This implies that P contains a line.