Discrete Optimization: Homework #9, Ex. #2

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We will show this by induction on the dimension n. Since the polyhedron P is bounded, then P is a polytope. So one can write P as $\mathrm{conv}(X)$ with $X \subseteq \mathbb{R}^n$ such that X is finite. Thus, X is the set of all vertices of P.

If $n = 2$, then there exist three vertices that are affinely independent since P is a full-dimensional polytope.

Let $n > 2 :$ we suppose that the result is correct for n and we will show it for $n+1$. Since P is bounded and full-dimensional, there exists a hyperplane $H_i = \{x \in \mathbb{R}^n : A_i x = b_i\}$ (where A_i is the i-th row of A) such that its intersection with P is not empty. The intersection of this hyperplane and the polyhedron P can be seen as a full-dimensional and bounded polyhedron P' in $\mathbb{R}^{n-1}$, but we know by the induction hypothesis that there exist $v_1, ..., v_n$ that are affinely independent vertices of P'. These vertices seen in $\mathbb{R}^n$ are also vertices of P since they satisfy with equality the equation of the hyperplane. By the hint and because P is bounded and full-dimensional, there exists a vertex v_{n+1} that is not contained in the hyperplane H_i. This implies that the vertices $v_1, ..., v_{n+1}$ are affinely independant.