Discrete Optimization: Homework #4, Ex. #6

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Let $X = \{a_1, ..., a_t\}$ be a finite set of $\mathbb{R}^n$, we will prove that cone(X) is closed and convex.

By Caratheodory's theorem, $\forall x \in \text{cone}(X)$ there exists $\widetilde{X} = \{a'_1, ..., a'_k\} \subseteq X$ such that $x \in \widetilde{X}$ and the vectors $\{a'_i\}$ in $\widetilde{X}$ are linearly independent. Let A be the matrix with a'^T_i $(i = 1, ..., k)$ on the k-first rows and for the last $n-k$ rows of A can be obtained by the Gram-Schmidt process from the first k rows. A is invertible since it is a $n \times n$ matrix and all the rows are linearly independent.
By exercise #2, cone$(\widetilde{X}) = \{x \in \mathbb{R}^n : a'^{-1}_i x \geq 0, i = 1, ..., k; a'^{-1}_j x = 0, j = k+1, ..., n\}$. Since cone$(\widetilde{X})$ is the finite intersection of the pre-image by a linear application of a closed set (either $[0, \infty[$ or $\{0\}$), it is also closed.
There is a finite number of $\widetilde{X}$ since X is finite. So, we can enumerate all such $\widetilde{X}$. We will prove that cone$(X) = \bigcup_{i=1}^{m} \text{cone}(\widetilde{X_i})$.
By construction, $\forall x \in \text{cone}(X)$ $\exists j \in \{1, ..., m\}$ such that $x \in \text{cone}(\widetilde{X_j})$. On the other way, if $x \in \bigcup_{i=1}^{m} \text{cone}(\widetilde{X_i})$, there exists $j \in \{1, ..., m\}$ such that $x \in \text{cone}(\widetilde{X_j})$. By definition of a cone-hull and since every vectors of $\widetilde{X}$ are also in cone(X) ($\widetilde{X_j} \subseteq X$), this implies $x \in \text{cone}(X)$.
That's why, cone(X) is closed because it is equal to a finite union of closed sets.

$\forall x, y \in \text{cone}(X)$, $\forall \lambda \in [0, 1]$ we need to show $\lambda x + (1-\lambda) y \in \text{cone}(X)$.
$x = \sum_{i=1}^{t} \alpha_i a_i$ and $y = \sum_{i=1}^{t} \beta_i a_i$ since $x, y \in \text{cone}(X)$.
$\lambda x + (1-\lambda) y = \sum_{i=1}^{t} (\lambda \alpha_i + (1-\lambda)\beta_i) a_i$, we can replace $\lambda \alpha_i + (1-\lambda)\beta_i$ by γ_i. Since $\lambda \in [0, 1]$ and $\alpha_i, \beta_i \in \mathbb{R}_{\geq 0}$, $\gamma_i \geq 0$. This implies that cone(X) is convex :

$$\lambda x + (1-\lambda) y = \sum_{i=1}^{t} \gamma_i a_i \in cone(X)$$