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Source: *The Annals of Mathematical Statistics*, Vol. 30, No. 1 (Mar., 1959), pp. 165-168

Published by: Institute of Mathematical Statistics

Stable URL: <https://www.jstor.org/stable/2237130>

Accessed: 15-02-2020 02:30 UTC

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## CONSENSUS OF SUBJECTIVE PROBABILITIES: THE PARI-MUTUEL METHOD

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A certain probability space is contemplated by a group of  $m$  individuals, each of whom endows it with his own subjective probability distribution. Suppose, now, that we wish to form a distribution which represents, in some sense, a consensus of those individual distributions. Various possibilities suggest themselves: the average, the convolution—but wait. There actually exists a popular institution which, theoretically at least, does perform just such an aggregation of personal probabilities. We refer to the pari-mutuel method of betting on horse races. In this system the final “track’s odds” on a given horse are proportional to the amount bet on the horse. We shall here investigate the type of consensus given by this mechanism, which turns out to be quite different from any of the obvious aggregation schemes that might occur to one.

In formulating the pari-mutuel model we assume the  $m$  individuals involved are bettors, labeled  $B_1, \dots, B_m$ , concerned with a race involving  $n$  horses, labeled  $H_1, \dots, H_n$ . We assume further that each  $B_i$ , after careful study of the form sheets, the condition of the track, and other relevant material, has arrived at an estimate of the relative merits of each of the  $H_j$ ’s which he expresses in quantitative terms. Specifically, we are given an  $m \times n$  *subjective probability matrix*  $P = (p_{ij})$  where  $p_{ij}$  is the probability, in the opinion of  $B_i$ , that  $H_j$  will win the race.

Having determined his subjective probability distribution,  $B_i$  will now bet the amount  $b_i$ , a fixed positive number called  $B_i$ ’s *budget*, in a way which maximizes his *subjective expectation*. This means, of course, that  $B_i$  will not necessarily bet the whole amount  $b_i$  on that  $H_j$  for which  $p_{ij}$  is largest. In general,  $B_i$  will “bet the odds,” that is, he will wait until the final track odds, or more conveniently, track probabilities, are announced. If these are  $\pi_1, \dots, \pi_n$ , he will examine the ratios  $p_{ij}/\pi_j$  and distribute  $b_i$  among those  $H_j$  for which this ratio is a maximum. We shall refer to this course of action as  $B_i$ ’s *strategy*.

A technical difficulty is immediately apparent. We have already stated that the final track probabilities  $\pi_1, \dots, \pi_n$  are proportional to the amounts bet on  $H_1, \dots, H_n$ , respectively (this is true whether or not the track retains a percentage). Thus, in practice, the  $\pi_j$ ’s are not known until each  $B_i$  has made his bet. On the other hand,  $B_i$  must know  $\pi_1, \dots, \pi_n$  before he can determine his bet. There is, therefore, a serious question as to whether there exist final track probabilities and individuals’ bets compatible both with the bettors’ strategies and the pari-mutuel principle. It is the purpose of this note to show that such probabilities and bets do exist and that the probabilities are in fact unique,

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Received June 16, 1958.

thus giving a well-defined notion of consensus. Of course, the "influence" of  $B_i$  on the consensus will depend on his budget  $b_i$ , the case of equal influence being, by definition, that of equal budgets.

It will be convenient to choose the unit of money so that  $\sum_{i=1}^m b_i = 1$ . We shall also assume that each column of the matrix  $P$  contains at least one positive entry. If this were not so then, say,  $p_{ij} = 0$  for all  $i$  and none of the  $B_i$ 's would bet on  $H_j$  under any circumstances. We could then eliminate  $H_j$  from consideration entirely.

We shall now arithmetize the conditions which must be satisfied under the pari-mutuel system. Let  $\beta_{ij}$  be the amount which  $B_i$  bets on  $H_j$ . These must satisfy the *budget relation*.

$$(1) \quad \sum_{j=1}^n \beta_{ij} = b_i.$$

Next, the *pari-mutuel* condition requires that

$$(2) \quad \sum_{i=1}^m \beta_{ij} = \pi_j,$$

which is simply the statement that the final track probability  $\pi_j$  is proportional to the total amount bet on  $H_j$ . Equality holds here because of the normalization of the monetary unit. (We are using Greek letters to represent unknowns, Latin letters for the given constants of the problem.)

Finally, we must express the fact that each  $B_i$  is maximizing his expectation. The reader will easily verify that the condition is the following:

$$(3) \quad \text{if } \mu_i = \max_s \frac{p_{is}}{\pi_s} \text{ and } \beta_{ij} > 0, \text{ then } \mu_i = \frac{p_{ij}}{\pi_j},$$

which states that  $B_i$  bets only on those  $H_j$ 's for which his expectation is a maximum.

Nonnegative numbers  $\pi_j$  and  $\beta_{ij}$  which satisfy (1), (2) and (3) are called *equilibrium probabilities* and *bets*. Their existence can be proved by means of fixed-point theorems. We prefer, however, to prove existence in an elementary manner using a variational method which seems to be of interest in itself. We define a function  $\phi$  and show that the variables which maximize it correspond to a solution of (1), (2) and (3).

The function  $\phi$  has  $mn$  arguments  $\xi_{ij}$  and is defined by the rule:

$$\phi(\xi_{11}, \dots, \xi_{mn}) = \sum_{i=1}^m b_i \log \sum_{j=1}^n p_{ij} \xi_{ij},$$

the variables  $\xi_{ij}$  being restricted to the domain  $D$  defined by:

$$(4) \quad \xi_{ij} \geq 0, \quad \text{for all } i, j,$$

$$(5) \quad \sum_{i=1}^m \xi_{ij} = 1, \quad \text{for all } j.$$

We shall come back and discuss the meaning of the function  $\phi$  after we have shown its relation to the pari-mutuel problem.

If we include minus infinity in the range of  $\phi$ , then  $\phi$  is continuous on the compact set  $D$ , hence attains a maximum at some point  $(\bar{\xi}_{11}, \dots, \bar{\xi}_{mn})$  of  $D$ . At this maximum the term  $\sum_{j=1}^n p_{ij} \xi_{ij}$  is positive for every  $i$  (otherwise  $\phi$  would be minus infinity, which is clearly not its maximum value). The partial derivatives of  $\phi$  at the maximum are given by

$$\frac{\partial \phi}{\partial \bar{\xi}_{ij}} = \frac{b_i p_{ij}}{\sum_s p_{is} \bar{\xi}_{is}}.$$

We now assert:

**EXISTENCE THEOREM.** *A set of equilibrium probabilities  $\pi_j$  and bets  $\beta_{ij}$  are given by*

$$(6) \quad \pi_j = \max_i \frac{\partial \phi}{\partial \bar{\xi}_{ij}} = \max_i \frac{b_i p_{ij}}{\sum_s p_{is} \bar{\xi}_{is}}$$

$$(7) \quad \beta_{ij} = \bar{\xi}_{ij} \pi_j.$$

**PROOF.** We must show that the numbers  $\pi_j$ ,  $\beta_{ij}$  satisfy (1), (2) and (3). The *pari-mutuel* condition (2) follows at once upon summing (7) on  $i$  and using condition (5) on the  $\bar{\xi}_{ij}$ .

The verification of (1) and (3) depends on the fact that

$$(8) \quad \text{if } \bar{\xi}_{ij} > 0, \text{ then } \pi_j = \frac{\partial \phi}{\partial \bar{\xi}_{ij}}.$$

To see this, suppose (8) is false and for some  $i, j$  we have  $\bar{\xi}_{ij} > 0$  and  $\pi_j > \partial \phi / \partial \bar{\xi}_{ij}$ . By definition of  $\pi_j$  we have  $\pi_j = \partial \phi / \partial \bar{\xi}_{kj} > \partial \phi / \partial \bar{\xi}_{ij}$  for some index  $k$ . Which means that by slightly decreasing  $\bar{\xi}_{ij}$  and increasing  $\bar{\xi}_{kj}$  by the same amount (which would not violate (4) or (5)), we could increase the value of  $\phi$ , which is impossible since we are already at a maximum. Thus (8) is established.

We next verify condition (1). From (7) and (8) we have

$$\beta_{ij} = \bar{\xi}_{ij} \pi_j = \bar{\xi}_{ij} \frac{b_i p_{ij}}{\sum_s p_{is} \bar{\xi}_{is}}.$$

Summing the above on  $j$ ,

$$\sum_j \beta_{ij} = b_i \frac{\sum_j p_{ij} \bar{\xi}_{ij}}{\sum_s p_{is} \bar{\xi}_{is}} = b_i.$$

Finally, we must prove (3). Since we assumed that none of the columns of the matrix  $P$  is identically zero, we know that each  $\pi_j$  is positive. Thus from (6) we have

$$(9) \quad \frac{p_{ij}}{\pi_j} \leq \frac{\sum_s p_{is} \bar{\xi}_{is}}{b_i}$$

and  $\mu_i$ , as defined in (3), is  $(1/b_i) \sum_i p_{ii} \bar{\xi}_{ii}$ . We see, then, from (7) and (8) that if  $\beta_{ij}$  is positive,  $\bar{\xi}_{ij}$  is positive and hence  $\mu_i = p_{ij}/\pi_j$ , thus showing that (3) holds. (The fact that the  $\pi_j$ 's sum to 1 is, of course, a consequence of (1) and (2) and the normalization of the  $b_i$ 's.) Q.E.D.

The function  $\phi$  can be interpreted as follows. From (7) we have  $\sum_j p_{ij} \bar{\xi}_{ij} = \sum_j \beta_{ij} (p_{ij}/\pi_j)$ , which is exactly the subjective expectation of  $B_i$  when he bets  $\beta_{ij}$  on  $H_j$  with track probabilities  $\pi_1, \dots, \pi_n$ . Thus, at equilibrium the bettors, as a group, maximize a weighted sum of logarithms of subjective expectations, the weights being the bettors' budgets. As noted previously, equilibrium probabilities turn out to be unique, although equilibrium bets need not be unique. Furthermore, not every collection of  $\beta_{ij}$ 's, obtained by having each  $B_i$  act according to his strategy at equilibrium probabilities, need be equilibrium bets.

As a final result we show

UNIQUENESS THEOREM. *Equilibrium probabilities are unique.*

PROOF. Let  $\pi_1, \dots, \pi_n$  and  $\bar{\pi}_1, \dots, \bar{\pi}_n$  be equilibrium probabilities, let  $\beta_{ij}, \bar{\beta}_{ij}$  be corresponding bets, and  $\mu_i, \bar{\mu}_i$  as defined in (3). Then for all  $i, j, k$  we have:

$$\beta_{ij} \mu_i \pi_j = \beta_{ij} p_{ij} \leq \beta_{ij} \bar{\mu}_i \bar{\pi}_j$$

$$\bar{\beta}_{ik} \bar{\mu}_i \bar{\pi}_k = \bar{\beta}_{ik} p_{ik} \leq \bar{\beta}_{ik} \mu_i \pi_k$$

whence, since  $\mu_i, \bar{\mu}_i, \pi_j, \pi_k$  are positive,  $\beta_{ij} \bar{\beta}_{ik} (\bar{\pi}_k/\pi_k) \leq \beta_{ij} \bar{\beta}_{ik} \bar{\pi}_j/\pi_j$ . Summing on  $j, k$  we get:  $b_i \sum_k \bar{\beta}_{ik} (\bar{\pi}_k/\pi_k) \leq b_i \sum_j \beta_{ij} \bar{\pi}_j/\pi_j$ ; dividing by  $b_i$  and summing on  $i$ :  $\sum_k (\bar{\pi}_k \bar{\pi}_k/\pi_k) \leq \sum_j \bar{\pi}_j = 1$ .

Let  $x_k = \bar{\pi}_k/\sqrt{\pi_k}$ ,  $y_k = \sqrt{\pi_k}$ . From the Cauchy-Schwartz inequality we have:

$$(\sum x_k y_k)^2 = (\sum \bar{\pi}_k)^2 = 1 \leq (\sum x_k^2) (\sum y_k^2) = \left( \sum \frac{\bar{\pi}_k \bar{\pi}_k}{\pi_k} \right) (\sum \pi_k) \leq 1.$$

Thus the vectors  $(x_1, \dots, x_n), (y_1, \dots, y_n)$  are dependent and  $\bar{\pi}_k = \mu \pi_k$ . But  $\sum \bar{\pi}_k = \sum \pi_k = 1$ , hence  $\mu = 1$  and  $\pi_k = \bar{\pi}_k$ , proving uniqueness.

The referee has suggested the following instructive example which indicates the somewhat "pathological" nature of the pari-mutuel consensus. In the case of two bettors with equal budgets if the first bettor's subjective probability distribution on two horses is  $(\frac{1}{2}, \frac{1}{2})$ , then the equilibrium probabilities will be  $(\frac{1}{2}, \frac{1}{2})$  regardless of the subjective probabilities of the second bettor, as the reader will easily verify.