Continuous-Time Stochastic Processes: Continuous Paths

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Stochastic processes: Basic definitions

For a fixed $\omega \in \Omega$, the set

$$X(\omega) := \{X_t(\omega)\}_{t \in I} = \{X(t, \omega)\}_{t \in I}$$

is called a sample path. If the index set I in this definition of the stochastic process is an interval $I \subset \mathbb{R}$, then it is called a continuous-time stochastic process.

Markov process

An $\mathbb{R}^d$-valued stochastic process $\{(X_t, F_t)\}_{t \in I}$ on a probability space $(\Omega, F, \mathbb{P})$ is called a Markov process with initial distribution ν if we have

$$\mathbb{P}(X_0 \in A) = \nu(A) \quad \forall A \in B(\mathbb{R}^d),$$
$$\mathbb{P}(X_t \in A | F_s) = \mathbb{P}(X_t \in A | X_s) \quad \forall A \in B(\mathbb{R}^d), \quad t \geq s$$

meaning that the distribution of future values of X only depends on the past via the present value X_t.

Monte Carlo and stochastic processes

There are some facts that we have to consider first:

- Do we want to imitate the real process as well as possible when simulating a stochastic process?
- The elements $X_t, t \in I$ of a stochastic process are usually not independent.
- The index set I can be noncountable.

Up to here, we are mainly interested in calculating expected values by the Monte Carlo method.
Let $X = \{X_t, t \in I\}$ be a stochastic process and let $g(X) = g(X_t(\omega), t \in I)$ be a functional on the path of this stochastic process.
If we are able to simulate

$$X_i(\omega) = \{X_{t,i}(\omega), t \in I\}$$

of the path of the stochastic process X independently, then we can define the (crude) Monte Carlo method for stochastic processes:

$$\text{Approximate} \quad \mathbb{E}(g(X)) \quad \text{by the arithmetic mean} \quad \frac{1}{N}\sum_{i=1}^{N} g(X_i(\omega)).$$

We only have to to be able to simulate independent replications of paths of a stochastic process to apply the crude Monte Carlo method since we have to use a functional g as otherwise talking of an expectation would make no sense.

The main different types of functionals:

- If the functional $g(x)$ only depends on the value of the stochastic process X at a particular time T for a real-valued function $h(.)$ such that

$$g(X) = h(X_T),$$

then we only have to know the distribution of the stochastic process at time T. If this distribution is explicitly known, the Monte Carlo simulation reduces to a simple one of ordinary random variables, and there is no additional complexity due to the fact that X_T is the result of a stochastic process.

- If we have

$$g(X) = h(X_{t_1}, \dots, X_{t_1})$$

  for a real valued function $h(.)$, we now have to simulate realizations of the vectors $(X_{t_1}, \dots, X_{t_n})$ where the components X_{t_i} are not independent.
- If the functional $g(x)$ cannot be reduced to one of the two cases, one is often not able to determine the distribution of $g(X)$ and thus we have to use suitable approximation methods.

Simulation of a discrete-time stochastic process

Algorithm 1 Simulation of a discrete-time stochastic process with independent increments

Let $\{X_t\}_{t\in[1,n]}$ be a discrete-time stochastic process with independent increments. Let $\mathbb{P}_k$ be the distribution of the k-th incerement $X_k - X_k - 1$.
Set $X_0(\omega) = 0$.
Simulate random numbers $Y_k(\omega), k = 1, \ldots, n$ with $Y_k \sim \mathbb{P}_k$.
Set $X_k(\omega) = X_{k-1}(\omega) + Y_k(\omega), k = 1, \ldots, n$.

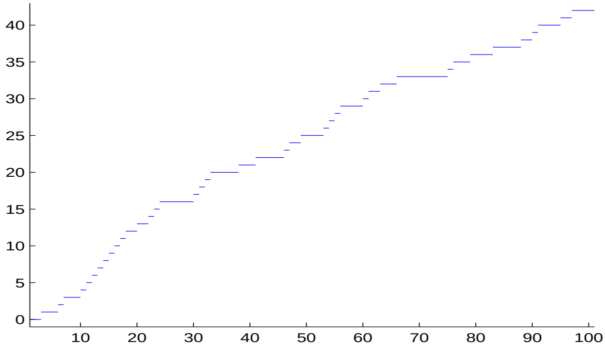

Figure: A random walk by a given Bernoulli distribution

Continuous time stochastic process

```
clear all, close all
T = 1; N = 20; dt = T/N; x = zeros(1,N); x_0=0;
mu=82; Sigma=4.8;
u=rand(N,1);
z=norminv(u,mu,Sigma);
x(1)=x_0+z(1);
for j = 2:N
   x(j) = x(j-1) + z(j); % next
end
x
scatter([0:dt:T],[0,x]) % W(0) = 0
xlabel('t', 'FontSize', 12), ylabel('X(t)', 'FontSize', 12)
```

Figure: Continuous paths by a given normal distribution

Variance reduction for stochastic processes

- Control variate techniques: approximating the functional $g(X)$ by a functional $h(X)$ which is simpler to compute than using a different process as a control variate. For computing the expectation of $h(.)$, the unconditional mean control variate would be a possible variance reducing approximation method.
- Stratified sampling: stratify the joint distribution of $(X_{t_1}, \ldots, X_{t_n})$ if $\mathbb{E}(g(X))$ depends strongly on the distribution of the underlying stochastic process X at some particular times $t_1, \ldots, t_n$.
- Importance sampling: apply importance sampling method for variance reduction when $g(X)$ is only nonzero if the process X does not leave a specified area on a given time interval.

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Brownian Motion

By a standard one-dimensional Brownian motion on $[0, T]$, we mean a stochastic process $\{W_t\}_{t \geq 0}$ with the following properties:
(i) $W_0 = 0$ $\mathbb{P}$-a.s.
(ii) $W_t - W_s \sim N(0, t-s)$ for $0 \leq s < t$.
(iii) $W_t - W_s$ is independent of $W_u - W_r$ for $0 \leq r \leq u \leq s < t$.
An n-dimensional Brownian motion is the $\mathbb{R}^n$-valued process

$$W(t) = (W_1(t), \ldots, W_n(t))$$

with components W_i being independent one-dimensional Brownian motions.

Brownian motion can be associated with its natural filtration

$$F_t^W := \sigma\{W_s | 0 \leq s \leq t\}, \quad t \in [0, \infty). \tag{1}$$

$$F_t := \sigma\{F_t^W \cup N | N \in F, \mathbb{P}(N) = 0\}, \quad t \in [0, \infty) \tag{2}$$

is called as Brownian filtration.

Correlated Brownian Motion

Assume that we are given a two-dimensional, independent Brownian motion $(W_1(t), W_2(t))$. Then, we can obtain a two-dimensional Brownian motion $(\tilde{W}_1(t), \tilde{W}_2(t))$ with

$$Corr(\tilde{W}_1(t), \tilde{W}_2(t)) = \rho$$

by setting

$$\tilde{W}_1(t) = W_1(t), \quad \tilde{W}_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2(t). \tag{3}$$

Cholesky decomposition

We can generate an n-dimensional Brownian motion $\tilde{W}(t)$ with a given positive definite covariance matrix Σ, then by using its Cholesky decomposition

$$\Sigma = LL' \tag{4}$$

and setting

$$\tilde{W}(t) = LW(t). \tag{5}$$

Theorem

(i) A one-dimensional Brownian motion W_t is a martingale.
(ii) A Brownian motion with drift μ and volatility σ with $\mu, \sigma \in \mathbb{R}$,

$$X_t := \mu t + \sigma W_t, \quad t > 0,$$

is a martingale if and only if $\mu = 0$, a super-martingale if and only if $\mu \leq 0$, and a sub-martingale is and only if $\mu \geq 0$.

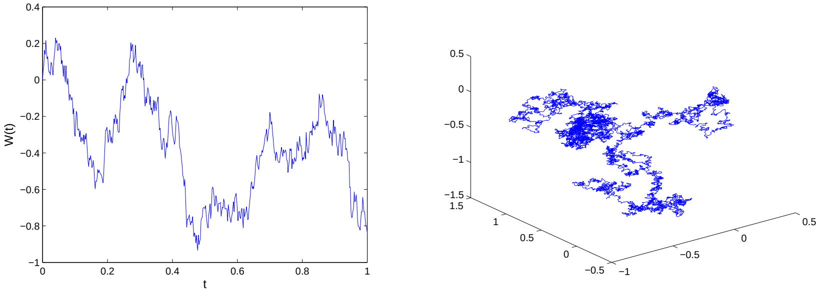

Figure: (a) A path of a Brownian motion for $N = 500$, (b) three dimensional Brownian motion for $N = 10^4$.

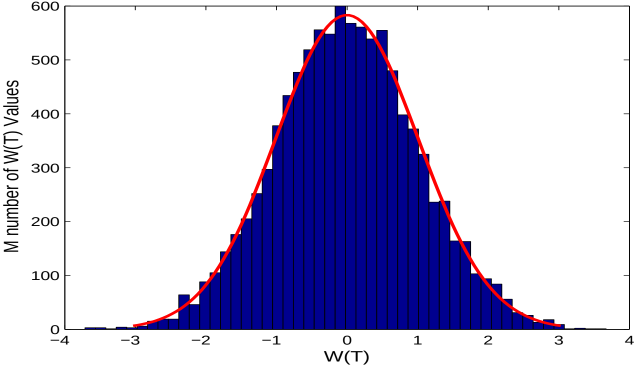

Figure: Histogram corresponding to 1000 paths

Let $(S, \mathcal{B}(S))$ be a metric space with metric ρ and the Borel-σ-field $\mathcal{B}(S)$ over S. Let further $\mathbb{P}, \mathbb{P}_n, n \in \mathbb{N}$ be probability measures on $(S, \mathcal{B}(S))$. Then we say that the sequence $\mathbb{P}_n$ converges weakly towards $\mathbb{P}$ if for each continuous and bounded real valued function f on S we have

$$\int_S f dP_n \xrightarrow{n \to \infty} \int_S f dP.$$

Weak convergence

Definition Let $X_n = \{X_n(t)\}_{t\in[0,1]}$ be a sequence of continuous stochastic process. We then say that X_n converges weakly towards the continuous process X if we have

$$\mathbb{E}(f(X_n)) \xrightarrow{n\to\infty} \mathbb{E}(f(X))$$

for all $f \in C(C[0,1], \mathbb{R})$.
Weak convergence of the stochastic processes means weak convergence of the underlying probability distributions $\mathbb{P}_n \to \mathbb{P}$.

The usual convergence in distribution of $\mathbb{R}^k$-valued random variables is implied by the convergence in distribution of the corresponding stochastic processes. The converse is in general not true.

So, it is not obvious that if we choose a sequence I_n of partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0, 1]$, then our linear interpolation based-simulation approach of the Brownian X_n will indeed convergence in distribution towards the Brownian motion $W_n = \{W_n(t)\}_{t \in [0,1]}$.

The next theorem implies that this approach indeed leads to the desired weak convergence.

Donsker's theorem

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with $E(\xi_i) = 0$, $0 < Var(\xi_i) = \sigma^2 < \infty$. Let

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} \xi_n. \tag{6}$$

We construct a sequence X_n of stochastic processes by

$$X_n(t,\omega) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}(\omega) + (nt - [nt]) \frac{1}{\sigma\sqrt{n}} \xi_{[nt]+1}(\omega) \tag{7}$$

for $t \in [0,1]$, $n \in \mathbb{N}$. Then this sequence converges weakly towards the one-dimensional Brownian motion $\{W(t)\}_{t\in[0,1]}$ such that we have

$$X_n \xrightarrow{n \to \infty} W \quad \text{in distribution.}$$

Corollary

Let $X_t = \mu \cdot t + \sigma W_t$, $t \geq 0$ be a Brownian motion with drift μ and volatility σ. We then have

$$\lim_{t\to\infty} \frac{X_t}{t} = \mu \quad \mathbb{P} - \text{a.s.} \tag{8}$$

This corollary means that in a Brownian motion with drift, the drift asymptotically dominates the fluctuations of the Brownian motion.

Definition

Let **W** be standard Brownian motion process which is restricted to the interval $[0,1]$. Then a Brownian bridge is a stochastic process $\mathbf{X} = \{X_t : t \in [0,1]\}$ with state space $\mathbb{R}$ that satisfies the following properties:

- $X_0 = 0$ and $X_1 = 0$ (each with probability 1)
- **X** is a Gaussian process. (i.e. $X_{t_1,\ldots,t_k} = (X_{t_1},\ldots,X_{t_k})$)
- $\mathbb{E}(X_t)$=0 for $t \in [0,1]$.
- $cov(X_s, X_t) = min\{s,t\} - st$ for $s,t \in [0,1]$.
- With probability 1, $t \to X_t$ is continuous on $[0,1]$.

Brownian Bridge Construction

A standard Brownian motion process $\mathbf{W} = \{W_t : t \in [0, \infty)\}$ is a continuous Gaussian process with $W_0 = 0, \mathbb{E}(W_t) = 0$ for $t \in [0, \infty)$ and $cov(W_s, W_t) = min\{s, t\}$ for $s, t \in [0, \infty)$. Construction is as following:

- Suppose that $\mathbf{W} = \{W_t : t \in [0, \infty)\}$ is a standard Brownian motion, and let $X_t = W_t - tW_t$ for $t \in [0, 1]$. Then $\mathbf{X} = \{X_t : t \in [0, 1]\}$ is a Brownian bridge.

Proof.

- Note that $X_0 = W_0 = 0$ and $X_1 = W_1 - W_1 = 0$.
- Linear combinations of the variables in $\mathbf{X}$ reduce to linear combinations of the variables in $\mathbf{W}$ and hence have normal distributions. Thus $\mathbf{X}$ is a Gaussian process.
- $\mathbb{E}(X_t) = \mathbb{E}(W_t) - t\mathbb{E}(W_1) = 0$ for $t \in [0, 1]$.
- $cov(X_s, X_t) = cov(W_s - sW_1, W_t - tW_1) = min\{s, t\} - st$ for $s, t \in [0, 1]$.
- $t \to X_t$ is continuous on $[0, 1]$ since $t \to W_t$ is continuous on $[0, 1]$.

□

General Brownian Bridge

Definition (General Brownian Bridge)

Let $\{W_t\}_{t\in[0,T]}$ be one-dimensional Brownian motion, let $a, b \in \mathbb{R}$ be two real numbers. Then, the process

$$B_t^{a,b} = a\frac{T-t}{T} + b\frac{t}{T} + \left(W_t - \frac{t}{T}W_T\right), t \in [0,T]$$

is called as **Brownian bridge** from a to b.

Obviously, the process $B_t^{a,b}$ starts in a at time $t = 0$ and ends in b at time T. Then a Brownian bridge from a to b satisfies

$$B_t^{a,b} \sim \mathcal{N}(a + \frac{t}{T}(b-a), t - \frac{t^2}{T}).$$

General Brownian Bridge

The Brownian bridge process $\mathbf{X} = \{X_t : t \in [0, T]\}$ from a to b is characterized by the following properties:

- $X_0 = a$ and $X_1 = b$ (each with probability 1).
- $\mathbf{X}$ is a Gaussian process.
- $\mathbb{E}(X_t) = a\frac{T-t}{T} + b\frac{t}{T}$ for $t \in [0, T]$.
- $cov(X_s, X_t) = min\{s, t\} - st$ for $s, t \in [0, T]$.
- With probability 1, $t \to X_t$ is continuous on $[0, T]$.

If we consider log-linear model for a stock price as $ln(P_i(t)) = ln(p_i) + \widetilde{b_i}t +' randomness'$, Brownian motion and bridge are appropriate for 'randomness'.

Algorithm for Wiener Process

We can simulate Brownian motion path by dividing the interval $[0, T]$ into a grid such as $0 = t_1 < t_2 < \ldots < t_{N-1} = T$ with $t_{i+1} - t_i = \triangle t$. Then we set $i = 1$ and $W(0) = W(t_1) = 0$ and iterate the following algorithm.

- Generate a (new) random number z from the standard Gaussian distribution.
- $i = i + 1$.
- Set $W(t_i) = W(t_{i-1}) + z \cdot \sqrt{\triangle t}$.
- If $i \leq N$, iterate from step 1.

R Code of Algorithm for Wiener Process

```
set.seed (123)
N <- 100 # number of end-points of the grid including T
T <- 1 # length of the interval [0,T] in time units
delta <- T/N # time increment
W <- numeric(N+1) # initialization of the vector W
t <- seq (0,T, length =N+1)
 for (i in 2:(N+1))
   W[i] <- W[i-1] + rnorm (1) * sqrt (delta)
   plot (t,W, type ="l", main ="Wiener process",
         ylim =c(-1,1))
```

Wiener Process

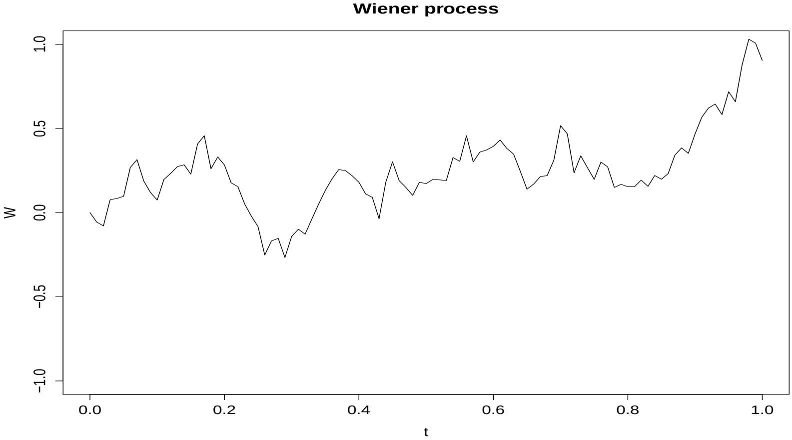

Figure: Wiener Process Path

Brownian Motion as the Limit of a Random Walk

Brownian motion can be also seen as the limit of a random walk. Given a sequence of iid random variables $X_1, X_2, \dots, X_n$, taking only two values $+1$ and -1 with equal probability and considering the partial sum,

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Then, as $n \to \infty$, $P\left(\frac{s_{[nt]}}{\sqrt{n}} < x\right) \to P(W(t) < x)$, where $[x]$ is the integer part of the real number x. This result is a refinement of the central limit theorem and in our case we have $S_n/\sqrt{n} \to N(0,1)$.

R Code of Algorithm for Brownian Motion as Random Walk

```
set.seed(123)
n <- 10 # far from the CLT, number of random variables
T <- 1
t <- seq (0,T,length =100)
#runif generates n random numbers from the uniform
#distribution in (0, 1) and it transforms these into a
#sequence of zeros and ones, then TRUE or FALSE
S <- cumsum (2*( runif (n ) >0.5) -1) #maps 0 to -1 and 1 to 1,calculate S_n
W <- sapply (t, function (x) ifelse (n*x >0,S[n*x],0)) #select from S
W <- as.numeric(W)/ sqrt (n)
plot (t,W, type ="l",ylim =c( -1 ,1))
n <- 100 # closer to the CLT
S <- cumsum (2*( runif(n) >0.5) -1)
W <- sapply (t, function (x) ifelse (n*x >0,S[n*x],0))
W <- as.numeric(W)/ sqrt (n)
lines (t,W, lty =2)
n <- 1000 # quite close to the limit
S <- cumsum (2*( runif(n) >0.5) -1)
W <- sapply (t, function (x) ifelse (n*x >0,S[n*x],0))
W <- as.numeric(W)/ sqrt(n)
lines (t,W, lty =3)
```

Brownian Motion as Random Walk

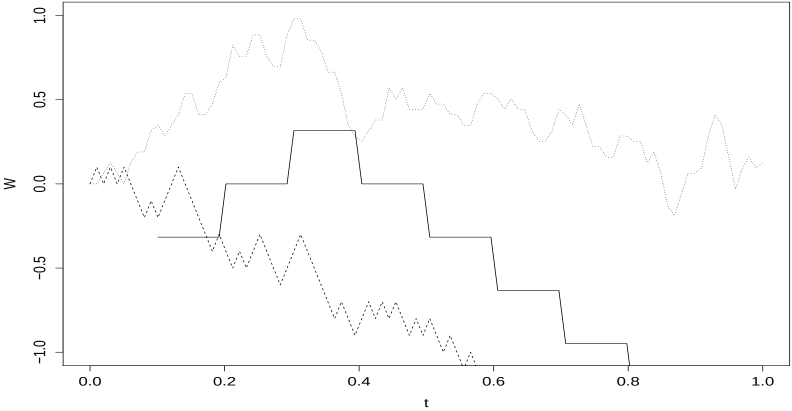

Figure: Path of the Wiener process as the limit of a random walk; continuous line $n = 10$, dashed line $n = 100$, dotted line $n = 1000$.

Brownian Motion as $L^2[0,T]$ Expansion

Here $L^2([0,T])$ expansion of random processes in terms of a sequence of independent random variables and coefficients is used.It is a series of only countably many terms and is useful for representing a process on some fixed interval $[0,T]$. L^2 is the space of functions from $[0,T]$ to $\mathbb{R}$ defined as

$$L^2 = \{f : [0,T] \to \mathbb{R} : \|f\|_2 < \infty\},$$

where $\|f\|_2 = \left(\int_0^T |f(t)|^2 \, dt\right)^{1/2}$. Then we have

$$W(t,\omega) = \sum_{i=0}^{\infty} Z_i(\omega)\phi_i(t), 0 \leq t \leq T,$$

with $\phi_i(t) = \frac{2\sqrt{2T}}{(2i+1)\pi} sin\left(\frac{(2i+1)\pi t}{2T}\right)$. The function ϕ_i form a basis of orthogonal functions and Z_i a sequence of i.i.d. Gaussian random variables.

R Code of Algorithm for Brownian Motion as $L^2[0,T]$ Expansion

```
set.seed (123)
phi <- function (i,t,T){
   (2*sqrt(2*T))/((2 *i +1)*pi)*sin(((2*i+1)*pi*t)/(2*T))
   }
T <- 1 # length of time interval
N <- 100 # number of points
t <- seq (0,T,length=N+1) # time increment vector
W <- numeric(N+1) # initialization of vector W
n <- 10 # number of random variables
Z <- rnorm(n) # random numbers distributed standard normal
for (i in 2:(N+1))
   W[i] <- sum (Z*sapply (1:n, function(x) phi(x,t[i],T)))
plot (t,W, type ="l",ylim =c( -1 ,1))
n <- 50
Z <- rnorm (n)
for (i in 2:(N+1))
     W[i] <- sum (Z*sapply (1:n, function(x) phi(x,t[i],T)))
lines (t,W,lty =2)
n <- 100
Z <- rnorm (n)
for (i in 2:(N+1))
    W[i] <- sum (Z*sapply (1:n, function(x) phi(x,t[i],T)))
lines (t,W,lty =3)
```

Brownian Motion as $L^2[0,T]$ Expansion

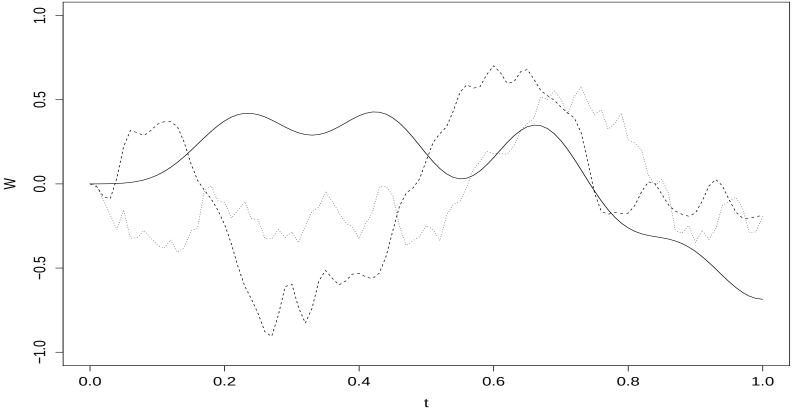

Figure: Path of the Wiener process as L^2 space; continuous line $n = 10$, dashed line $n = 50$, dotted line $n = 100$.

Geometric Brownian Motion

Process has independent multiplicative increments and it is generally used in finance to model the dynamics of assets. It is a function of the standard Brownian motion

$$S(t) = x \exp\left\{\left(r - \frac{\sigma^2}{2}\right) + \sigma W(t)\right\}, t > 0,$$

with $S(0) = x, x \in \mathbb{R}$ is the initial value $\sigma > 0$ and r are volatility and interest rate. For trajectory of W and time vector t one can get the path. Equivalently one can simulate the increments of S. For $Z \sim N(0,1)$,

$$S(t + \triangle t) = S(t) \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\triangle t + \sigma\sqrt{\triangle t}Z\right\}.$$

Then generalized geometric Brownian motion, starting from x at time s is,

$$Z_{s,x}(t) = x \exp\left\{\left(r - \frac{\sigma^2}{2}\right)(t - s) + \sigma\left(W(t) - W(s)\right)\right\}, t \geq s.$$

R Code for Geometric Brownian Motion

```
set.seed(123)
r <- 1
sigma <-0.5
x <- 10 # start point
N <- 100 # number of end points of the grid including T
T <- 1 # length of the interval [0,T] in time units
Delta <- T/N # time increment
W <- numeric(N+1) # initialization of the vector W
t <- seq(0,T,length=N+1) # time vector
for (i in 2:(N+1))
  W[i] <- W[i-1]+rnorm(1)*sqrt(Delta) # Wiener process
S <- x*exp((r-sigma^2/2)*t+sigma*W)
plot(t,S,type="l",main="Geometric Brownian Motion")
```

Geometric Brownian Motion

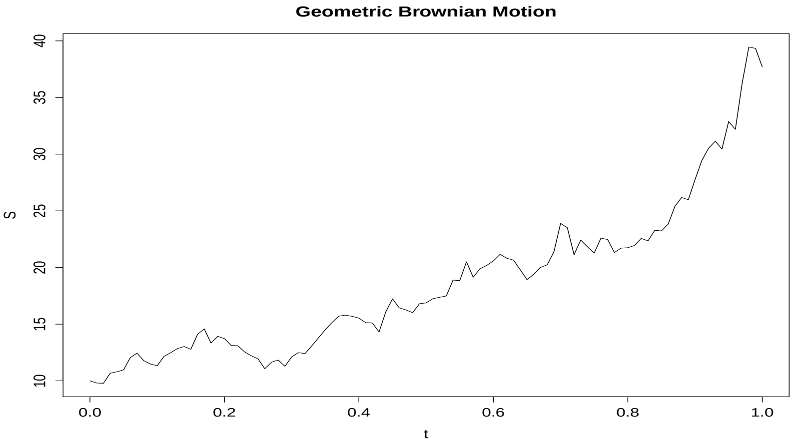

Figure: A trajectory of the geometric Brownian motion obtained from the simulation of the path of the Wiener process.

Brownian Bridge

There is another usage of Wiener process that is called Brownian bridge, which is Brownian motion starting at a at time t_0 and passing through some point b at time T, $T > t_0$.

$$W_{t_0,a}^{T,b}(t) = a + W(t - t_0) - \frac{t - t_0}{T - t_0} \cdot \left(W(T - t_0) - b + a\right).$$

R Code for Brownian Bridge

```
set.seed (123)
N <- 100 # number of end points of the grid including T
T <- 1 # length of the interval [0 ,T] in time units
Delta <- T/N # time increment
W <- numeric(N+1) # initialization of the vector W
t <- seq(0,T,length=N+1) # time vector
for (i in 2:(N+1))
  W[i] <- W[i-1]+rnorm(1)*sqrt(Delta)
a <- 0 # starting point
b <- -1 #end point
BB <- a+W-t/T*(W[N+1]-b+a) #Brownian bridge
plot (t,BB , type ="l")
abline (h=-1, lty =3)
```

Brownian Bridge

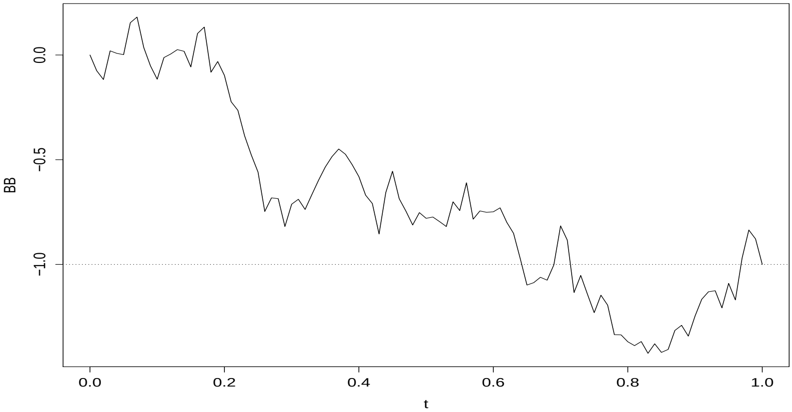

Figure: A simulated trajectory of the Brownian bridge starting at a at time 0 and terminating its run at $b = -1$ at time T.

SDE Package-BM Function

Trajectory of $W_{t_0,x} = \{W(t), t_0 \leq t \leq T | W(t_0 = x\}$.

```
BM <- function (x=0, t0 =0, T=1, N =100){
  if(T <= t0) stop ("wrong times")
  dt <- (T-t0)/N #time increment
  t <- seq(t0 ,T, length=N+1) #time vector
  X <- ts(cumsum(c(x,rnorm(N)*sqrt(dt))),
          start=t0, deltat=dt)
  return (invisible(X))
}
```

SDE Package-BBridge Function

Trajectory of $W_{t_0,x}^{T,y}(t) = \{W(t), t_0 \leq t \leq T | W(t_0 = x, W(T) = y\}$.

```
BBridge <- function (x=0, y=0, t0 =0, T=1, N =100){
  if(T <= t0) stop ("wrong times")
  dt <- (T-t0)/N
  t <- seq(t0, T, length=N+1)
  X <- c(0,cumsum(rnorm(N)*sqrt(dt)))
  BB <- x+X-(t-t0)/(T-t0)*(X[N+1]-y+x)
  X <- ts(BB, start=t0 , deltat=dt)
  return(invisible(X))
}
```

SDE Package-GBM Function

Trajectory of geometric Brownian motion.

```
# x = starting point at time 0
# r = interest rate
# sigma = square root of the volatility
GBM <- function (x, r=0, sigma , T=1, N=100){
  tmp <- BM(T=T,N=N)
  S <- x*exp((r-sigma^2/2)*time(tmp)+sigma*as.numeric(tmp))
  X <- ts(S, start=0, deltat=1/N)
  return(invisible(X))
}
```

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Itô Integral

- Too many rapid changes in random processes make paths non-integrable, i.e. P-almost all paths of the Brownian motion $\{W_t\}_{t\in[0,\infty)}$ are nowhere differentiable.
- A stochastic process $\{X_t\}_{t\in[0,T]}$ is called a *simple process* if there exist real numbers $0 = t_0 < t_1 < \ldots < t_p = T$, $p \in \mathbb{N}$, and bounded random variables $\Phi_i : \Omega \to \mathbb{R}, i = 0, 1, \ldots, p$, with Φ_0 and Φ_i are $\mathcal{F}_0$ and $\mathcal{F}_{t_{i-1}}$ measurable such that for each $\omega \in \Omega$,

$$X_t(\omega) = X(t,\omega) = \Phi_0(\omega) \cdot 1_{\{0\}}(t) + \sum_{i=1}^{p} \Phi_i(\omega) \cdot 1_{(t_{i-1},t_i]}(t).$$

Itô Integral

Definition (Stochastic Integral)

For a simple process $\{X_t\}_{t\in[0,T]}$ the *stochastic integral* $I.(X)$ for $t \in (t_k, t_{k+1}]$ is defined according to

$$I_t(X) := \int_0^t X_s dW_s := \sum_{1\leq i\leq k} \Phi_i \left(W_{t_i} - W_{t_{i-1}}\right) + \Phi_{k+1} \left(W_t - W_{t_k}\right),$$

or more generally for $t \in [0, T]$:

$$I_t(X) := \int_0^t X_s dW_s := \sum_{1\leq i\leq k} \Phi_i \left(W_{t_i\wedge t} - W_{t_{i-1}\wedge t}\right).$$

Itô Approximating

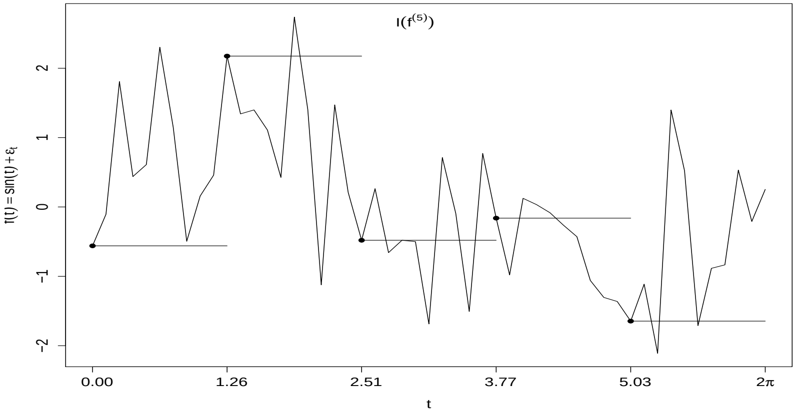

Figure: The simple (piecewise constant) process $f^{(5)}(t)$ approximating $f(t) = sin(t) + \epsilon_t$ used to construct Itô integrals. Note that $f^{(n)}(t)$ is defined as right continuous.

Itô Integral

Theorem (Elementary properties of the Stochastic Integral)

Let $X := \{X_t\}_{t\in[0,T]}$ *be a simple process. Then we have*

- $\{I_t(X)\}_{t\in[0,T]}$ *is a continuous martingale with respect to* $\{\mathcal{F}_t\} t \in [0,T]$. *In particular, we have* $\mathbb{E}(I_t(X)) = 0$ *for all* $t \in [0,T]$.
- $\mathbb{E}\left[\left(\int_0^t X_s dW_s\right)^2\right] = \mathbb{E}\left(\int_0^t X_s^2 ds\right)$ *for* $t \in [0,T]$. *(Itô Isometry)*
- $\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t X_s dW_s\right|\right)^2 \le 4\cdot\mathbb{E}\left(\int_0^T X_s^2 ds\right)$.

1-Dimensional Itô Formula

Let W be a one-dimensional Brownian motion, X a real-valued Itô process with representation

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2-function. Then, for all $t \geq 0$ we have

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s) dX_s + \frac{1}{2} \cdot \int_0^t f^{''}(X_s) d\left\langle X \right\rangle_s$$

$$= f(X_0) + \int_0^t \left(f^{'}(X_s) \cdot K_s + \frac{1}{2} \cdot f^{''}(X_s) \cdot H_s^2 \right) ds + \int_0^t f^{'}(X_s) H_s dW_s, \mathbb{P}\text{−}a.$$

1-Dimensional Itô Formula

In other representation of Itô formula is

$$df(X_t) = f^{'}(X_t)dX_t + \frac{1}{2} \cdot f^{''}(X_t)d\left\langle X\right\rangle_t .$$

Ex. Let $W_0 = 0$, and choose $X_t = W_t$ and $f(t,x) = \frac{1}{2}x^2$. Then $Y_t = \frac{1}{2}W_t^2$ and by Itô formula

$$df\left(\frac{1}{2}W_t^2\right) = df\left(dY_t\right) = W_t dW_t + \frac{1}{2}(dW_t)^2$$

where $(dW_t)^2 = d_t$. Thus $\frac{1}{2}W_t^2 = \int_0^t W_s dW_s + \frac{1}{2}t$. Nonvanishing quadratic variation of W_t term is t.

Multi-Dimensional Itô Formula

Let $X(t) = (X_1(t), \dots, X_n(t))$ be an n-dimensional Itô process with

$$X_i(t) = X_i(0) + \int_0^t K_i(s)ds + \sum_{j=1}^{m} \int_0^t H_{ij}(s)dW_j(s), i = 1, \dots, n,$$

and $W(t)$ an m-dimensional Brownian motion. Let further $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a $C^{1,2}$-function, i.e. f is continuous, continuously differentiable with respect to the first variable (time) and twice continuously differentiable with respect to the last n variables (space). We then have

$$f\left(t, X_1(t), \dots, X_n(t)\right) = f\left(0, X_1(0), \dots, X_n(0)\right) + \int_0^t f_t\left(s, X_1(s), \dots, X_n(s)\right) ds$$

$$+ \sum_{i=1}^{n} \int_0^t f_{x_i}\left(s, X_1(s), \dots, X_n(s)\right) dX_i(s) + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t f_{x_i, x_j}\left(s, X_1(s), \dots, X_n(s)\right) d\left\langle X_i, X_j \right\rangle_s .$$

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