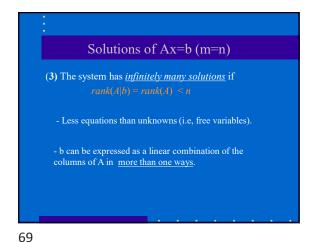


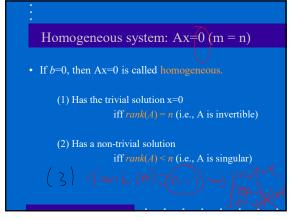
68

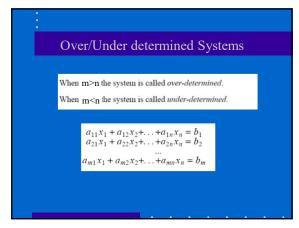
Image Processing Fundamentals

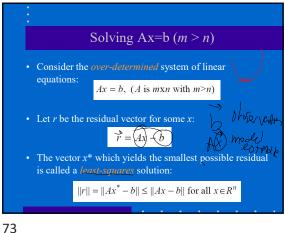


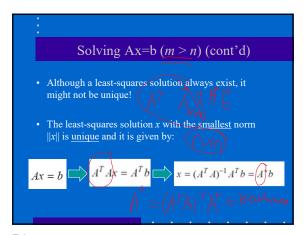
Solutions of Ax=b (m=n) The following statements are equivalent: (a) rank(A|b) = rank(A) = n(c) $det(A) \neq 0$ (d) b has a unique expansion in the column space of A a_{22}

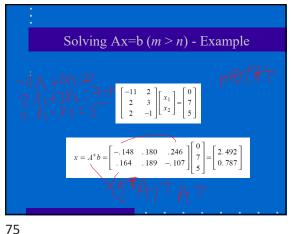
70

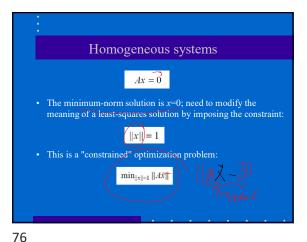


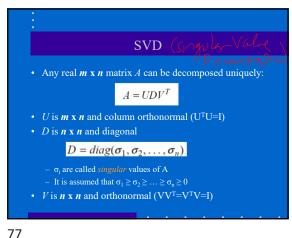


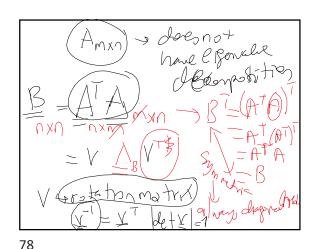


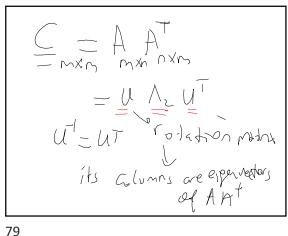


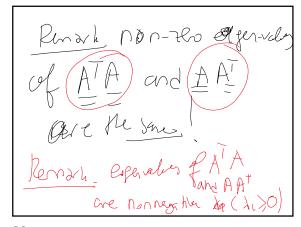


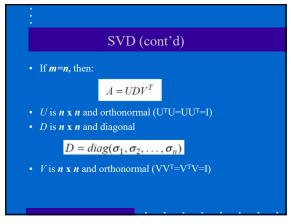






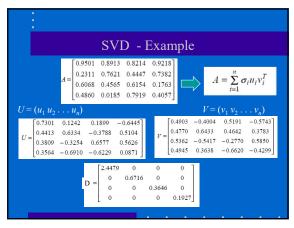


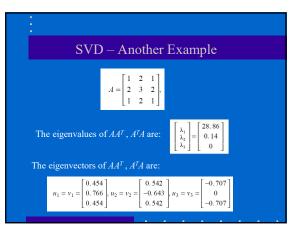




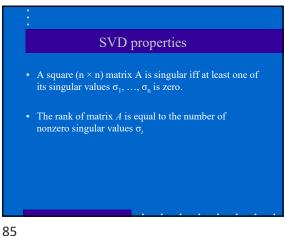
SVD (cont'd) • The columns of U are eigenvectors of AAT $AA^T = UDV^TVDU^T = UD^2U^T$ • The columns of V are eigenvectors of A^TA $A^T A = VDU^T UDV^T = VD^2 V^T$ • If λ_i is an eigenvalue of A^TA (or AA^T), then $\lambda_i = \sigma_i^2$

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Computing A⁻¹ using SVD • If A is a $n \times n$ nonsingular matrix, then its inverse can be computed as follows: $A^{-1} = VD^{-1}U^{T}$ $A = UDV^T$ $A^{-1} = V \left[diag(\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_n^{-1}) \right] U^T$ $(U^TU=UU^T=I \text{ or } U^T=U^{-1} \text{ and } V^TV=VV^T=I \text{ or } V^T=V^{-1})$

Least–squares Solution of Homogeneous Equations

Derivation II — SVD Let A = USV^T, where U is m × n orthonormal, S is n × n diagonal with descending order, and V^T is n × n also orthonormal. From orthonormality of U, V follows that ||USV^Th|| = ||SV^Th|| and ||V^Th|| = ||h||. Substitute y = V^Th. Now, we minimize ||Sy|| subject to ||y|| = 1. Remember that S is diagonal and the elements are sorted descendently. Than, it is clear that y = [0,0,...,1]^T. From substitution we know that h = Vy from which follows that sought h is the last column of the matrix V.

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- To find an extreme (the sought \mathbf{h}) we must solve $\frac{\partial}{\partial \mathbf{h}} (\mathbf{h}^{\top} \mathbf{A}^{\top} \mathbf{A} \mathbf{h} + \lambda (1 \mathbf{h}^{\top} \mathbf{h})) = 0$.
- We derive: 2A^TAh − 2λh = 0.
- After some manipulation we end up with: $(\mathbb{A}^\top \mathbb{A} \lambda \mathbb{E}) \mathbf{h} = 0$ which is the characteristic equation. Hence, we know that \mathbf{h} is an eigenvector of $(\mathbb{A}^\top \mathbb{A})$ and λ is an eigenvalue.
- The least-squares error is $e = \mathbf{h}^{\top} \mathbf{A}^{\top} \mathbf{A} \mathbf{h} = \mathbf{h}^{\top} \lambda \mathbf{h}$.
- The error will be minimal for \(\lambda = \min_i \lambda_i\) and the sought solution is then the eigenvector of the matrix (A^TA) corresponding to the smallest eigenvalue.

Homogeneous systems (cont'd)
 The min_{|x|=1}||Ax|| solution for homogeneous systems is not always unique.
 Special case: rank(A) = n - 1 (m ≥ n - 1, σ_n=0)
 Solution: x = av_n (a is a constant)
 (v_n is the last column of V; the one corresponding to the smallest σ)