n-dimensional Vector

- An *n*-dimensional vector *v* is denoted as follows:

$$v = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \leftarrow \text{column vector}$$

- The transpose v^T is denoted as follows:

$$v^T = [\, x_1 \; x_2 \cdots x_n \,] \leftarrow \text{row vector}$$

1

Inner (or dot) product → Scalar

- Given $v^T = (x_1, x_2, \dots, x_n)$ and $w^T = (y_1, y_2, \dots, y_n)$, their dot product defined as follows:

$$\vec{v}^T . \vec{w} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \quad \text{(scalar)}$$

or $$v . w = [\, x_1 \; x_2 \cdots x_n \,] \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} = \vec{v}^T \vec{w}$$

2

Outer product:

$$\vec{v}\,\vec{w}^T = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} [\, w_1 \; w_2 \quad w_m \,]$$

$$= \begin{bmatrix} v_1 w_1 & & v_1 w_m \\ v_2 w_1 & & \\ v_n w_1 & & v_n w_m \end{bmatrix} n \times m$$

3

Orthogonal / Orthonormal vectors

- A set of vectors $x_1, x_2, \dots, x_n$ is *orthogonal* if

$$\|\vec{X}\|^2 = \vec{X}^T . \vec{X}$$

$$x_i^T x_j = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- A set of vectors $x_1, x_2, \dots, x_n$ is *orthonormal* if

$\vec{x}_i^T \vec{x}_j = 0$ for $i \neq j$

$\|X_i\|^2 = 1$

$$x_i^T x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

4

Linear combinations

- A vector v is a linear combination of the vectors $v_1, \dots, v_k$:

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

  where $c_1, \dots, c_k$ are scalars

- Example: any vector in R^3 can be expressed as a linear combinations of the unit vectors $i = (1, 0, 0)$, $j = (0, 1, 0)$, and $k = (0, 0, 1)$

$$v = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

5

Space spanning

do they need to be linearly independent?

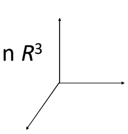
- A set of vectors $S=(v_1, v_2, \dots, v_k)$ *span* some space *W* if every vector in *W* can be written as a linear combination of the vectors in *S*

for all $\vec{w} \in W$

$$\vec{w} = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

-Example: the vectors *i*, *j*, and *k* span R^3

$$v = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$



6

Linear dependence

- A set of vectors v_1, ..., v_k are *linearly dependent* if <u>at least</u> one of them is a linear combination of the others.

$$v_j = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

(i.e., v_j does not appear at the right side)

7

Linear independence

- A set of vectors v_1, ..., v_k is *linearly independent* if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \Rightarrow \underline{c_1 = c_2 = \cdots = c_k = 0}$$

Example:

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Longrightarrow \text{linearly independent}$$

Let $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$, then $\begin{bmatrix} -c_1 + c_2 \\ c_1 + c_2 \\ -c_1 + (-c_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This can only be true if $\underline{c_1 = c_2 = 0}$

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Remark: $\vec{0}$ is always linearly dependent to all vectors in a vector space

$$\vec{X} + (-\vec{X}) = \vec{0}$$

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Vector basis

- A set of vectors (v_1, ..., v_k) is said to be a *basis* for a vector space W if
  - (1) (v_1, ..., v_k) are <u>linearly independent</u>
  - (2) (v_1, ..., v_k) <u>span W</u>

$\dim(W) = \#$ of basis $= k$

- Standard bases:

R^2: $i = (1, 0)$, $j = (0, 1)$

R^3: $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

R^n: $(1,0,\ldots,0)$ $(0,1,\ldots,0)$, ..., $(0,0,\ldots,1)$

$e_i = [0 \ldots 1 \ldots 0]^T$ (i-th position)

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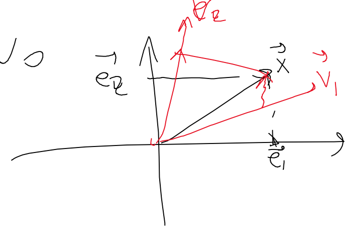
Q Is the basis unique?

A No



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Q. Are orthogonal vectors (orthonormal) linearly independent?

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Q $\{\vec{v}_1 \quad \vec{v}_k\}$ basis a vector space

$\vec{v}_i^T \vec{v}_j = k\delta_{ij} = \begin{cases} k \text{ for } i \neq j \\ 0 \text{ for } i = j \end{cases}$

$c_1 \vec{v}_1 + \quad + c_k \vec{v}_k = \vec{0}$

$c_1 \underset{\neq 0}{v_1^T v_1} + \underbrace{\sim + c_k v_1^T \vec{v}_k}_{0} = \vec{0}$

$c_1 k = 0 \Rightarrow c_1 = 0 = c_2 = \quad c_k$

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Some Definitions

An $m \times n$ (read "m by n") ***matrix***, denoted by **A**, is a rectangular array of entries or elements (numbers, or symbols representing numbers) enclosed typically by square brackets, where m is the number of rows and n the number of columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

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$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$ identity matrix (3)

diagonal matrix $\underline{A} = \begin{bmatrix} \cdot & & 0 \\ & \cdot & \\ 0 & & \cdot \end{bmatrix}$

Definitions (Con't)

- **A** is ***square*** if $m = n$.
- **A** is ***diagonal*** if all off-diagonal elements are 0, and not all diagonal elements are 0.
- **A** is the ***identity matrix*** (**I**) if it is diagonal and all diagonal elements are 1.
- **A** is the ***zero*** or ***null matrix*** (**0**) if all its elements are 0.
- The ***trace*** of **A** equals the sum of the elements along its main diagonal.
- Two matrices **A** and **B** are ***equal*** iff the have the same number of rows and columns, and $a_{ij} = b_{ij}$.

if and only if

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$\text{trace}(\underline{A}) = \text{tr}(\underline{A}) = \sum A_{ii}$

$\underline{A}^T = [A_{ji}]$

Transpose of matrix $\underline{A}$

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$\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow$ Symmetric since $\underline{A}^T = \underline{A}$

Definitions (Con't)

- The ***transpose*** $\mathbf{A}^T$ of an $m \times n$ matrix **A** is an $n \times m$ matrix obtained by interchanging the rows and columns of **A**.
- A square matrix for which $\mathbf{A}^T = \mathbf{A}$ is said to be ***symmetric***.
- Any matrix X for which **XA=I** and **AX=I** is called the ***inverse*** of **A**.
- Let c be a real or complex number (called a ***scalar***). The ***scalar multiple*** of c and matrix **A**, denoted c**A**, is obtained by multiplying every elements of **A** by c. If $c = -1$, the scalar multiple is called the ***negative*** of **A**.

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$\underline{X} \triangleq \underline{A}^{-1}$ $\quad \underline{X}\,\underline{A} = \underline{A}\,\underline{X} = \underline{I}$

inverse of a matrix

(only square matrices)

if exists

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Matrix Operations

$(A)(B) = C = [c_{ij}]$, A: $m \times n$, B: $n \times p$, C: $m \times p$

- Matrix addition/subtraction
  - Matrices must be of same size.
- Matrix multiplication

$$\underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix}} \underset{q \times p}{\begin{bmatrix} b_{11} & b_{12} & . & b_{1p} \\ b_{21} & b_{22} & . & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{q1} & b_{q2} & . & b_{qp} \end{bmatrix}} = \underset{m \times p}{\begin{bmatrix} c_{11} & c_{12} & . & c_{1p} \\ c_{21} & c_{22} & . & c_{2p} \\ \dots & \dots & c_{ij} & \dots \\ c_{m1} & c_{m2} & . & c_{mp} \end{bmatrix}}$$

Condition: n = q $\quad c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$ $\quad \boxed{AB \neq BA}$

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$$A = \begin{bmatrix} r_1^T \\ r_2^T \\ r_m^T \end{bmatrix} \qquad B = [\vec{c}_1 \ \vec{c}_2 \ \vec{c}_n]$$

$$A \cdot B = \begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vec{r}_m^T \end{bmatrix} [\vec{c}_1 \ \vec{c}_2 \ \vec{c}_n] = \begin{bmatrix} r_1^T c_1 & r_2^T c_1 & r_m^T c_1 \\ r_m^T c_n & r_m^T c_n & r_m^T c_n \end{bmatrix}$$

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Identity Matrix

$$AI = IA = A, \text{ where } I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & . & 1 \end{bmatrix}$$

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Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix}, A^T = \begin{bmatrix} a_{11} & a_{21} & . & a_{m1} \\ a_{12} & a_{22} & . & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & . & a_{mn} \end{bmatrix}$$

Property: $(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1} A^{-1}$

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Symmetric Matrices

$$A = A^T \ (a_{ij} = a_{ji})$$

Example: $\begin{bmatrix} 4 & 5 & -3 \\ 5 & 7 & 2 \\ -3 & 2 & 10 \end{bmatrix}$

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Determinants

2 x 2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

3 x 3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

n x n

$$det(A) = \sum_{j=1}^{m} (-1)^{j+k} a_{jk} det(A_{jk}), \text{ for any } k: 1 \le k \le m$$

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Matrix Inverse

- The inverse A^{-1} of a matrix A has the property:

  $$AA^{-1}=A^{-1}A=I$$

- A^{-1} exists only if $det(A) \neq 0$ or full rank
- Terminology
  - **Singular matrix:** A^{-1} does not exist
  - **Ill-conditioned matrix:** A is close to being singular

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Matrix Inverse (cont'd)

$det(A\,B)=det(A)\,det(B)$

$det(I)=1$

- Properties of the inverse:

$$det(A^{-1}) = \frac{1}{det(A)}$$

$$(AB)^{-1}=B^{-1}A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

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Matrix trace

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

properties:

$$tr(A^T) = tr(A)$$

$$tr(A \pm B) = tr(A) \pm tr(B)$$

$$tr(AB) = tr(BA)$$

$$(\text{in general, } tr(AB) \neq tr(A)tr(B))$$

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Rank of matrix

- Equal to the dimension of the largest square sub-matrix of A that has a non-zero determinant.

Example: $\begin{bmatrix} 4 & 5 & 2 & 14 \\ 3 & 9 & 6 & 21 \\ 8 & 10 & 7 & 28 \\ 1 & 2 & 9 & 5 \end{bmatrix}$ has rank 3

$$det(A) = 0, \text{ but } det(\begin{bmatrix} 4 & 5 & 2 \\ 3 & 9 & 6 \\ 8 & 10 & 7 \end{bmatrix}) = 63 \neq 0$$

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Rank of matrix (cont'd)

- **Alternative definition:** the maximum number of linearly independent columns (or rows) of A.

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \Rightarrow c_1 = c_2 = \cdots = c_k = 0$$

Example: $1\begin{bmatrix} 4 \\ 3 \\ 8 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 9 \\ 10 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \\ 7 \\ 9 \end{bmatrix} - 1\begin{bmatrix} 14 \\ 21 \\ 28 \\ 5 \end{bmatrix} = 0$ Therefore, rank is not 4 !

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$A_{m \times n}$ $\quad rank(A) \leq min(m,n)$

$v_1 \quad v_2 \quad v_3$

Q Are $\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix} \right\}$ linearly independent?

A. No! $v_1, v_2, v_3 \in \mathbb{R}^2 \Rightarrow dim(\mathbb{R}^2) = 2$

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2D transformations

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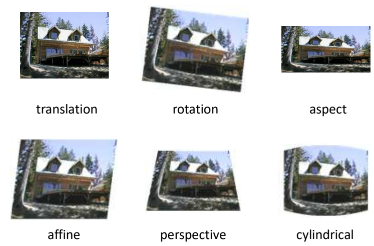
2D transformations



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2D planar transformations

y

x

33

2D planar transformations

y

Scale

How would you implement scaling?

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

x

34

2D planar transformations

y

Scale

$$x' = ax$$
$$y' = by$$

What's the effect of using different scale factors?

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

x

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2D planar transformations

y

Scale

$$x' = ax$$
$$y' = by$$

matrix representation of scaling:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix S}} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

x

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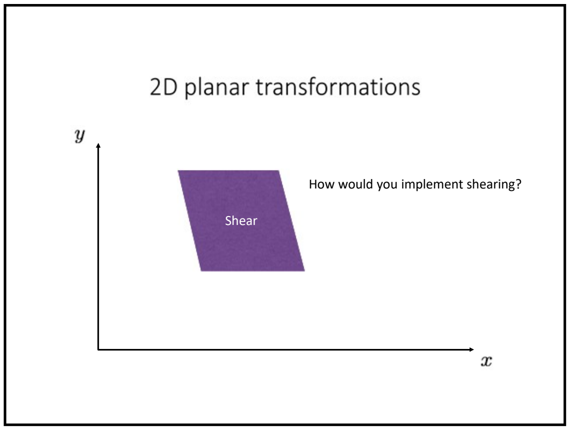

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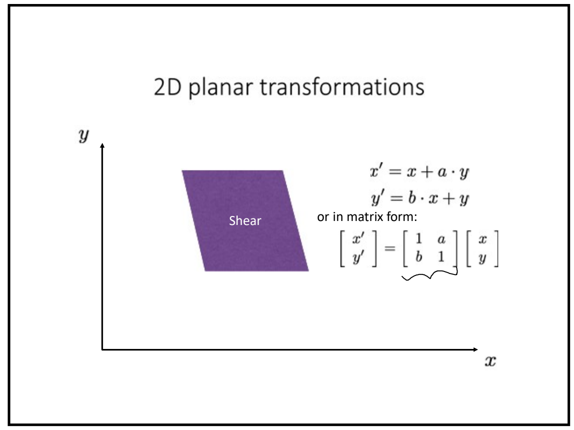

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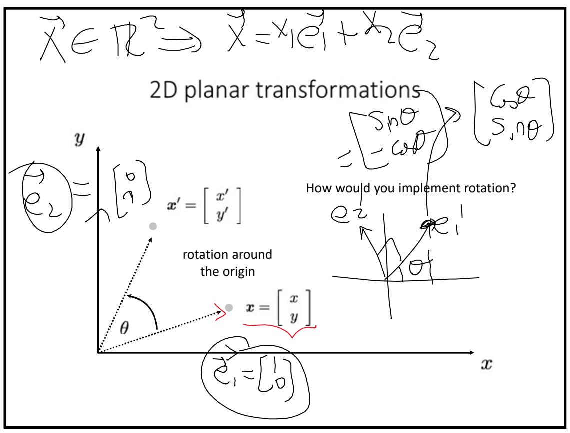

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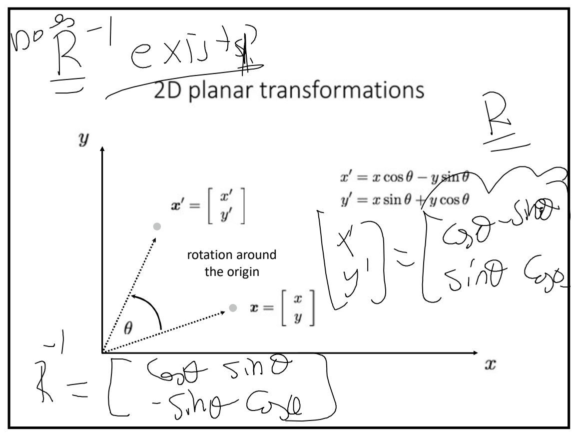

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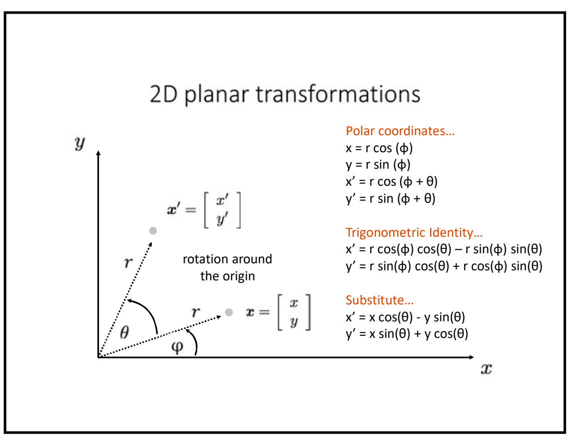

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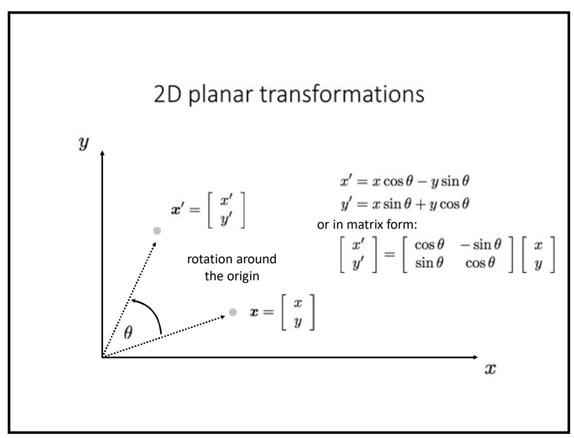

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2D planar and linear transformations

$$\boldsymbol{x}' = f(\boldsymbol{x}; p)$$

$$\downarrow$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

parameters p — point $\boldsymbol{x}$

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2D planar and linear transformations

Scale $\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$	Flip across y $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Rotate $\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Flip across origin $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
Shear $\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$	Identity $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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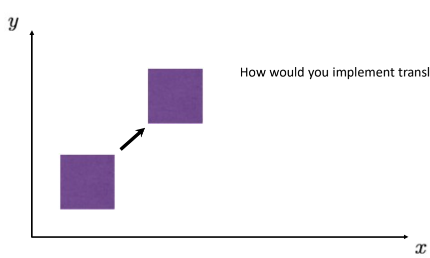
2D translation



How would you implement translation?

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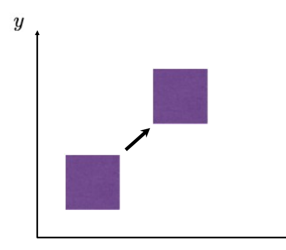
2D translation



$$x' = x + t_x$$
$$y' = y + t_y$$

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2-D Rotation

- This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- 3-D is more complicated
  - Need to specify an *axis of rotation*
  - Simple cases: rotation about X, Y, Z axes

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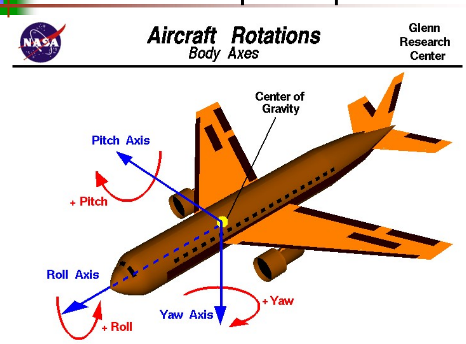
Rotation example: airplane



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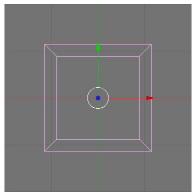
3-D Rotation

- *What does the 3-D rotation matrix look like for a rotation about the Z-axis?*
  - Build it coordinate-by-coordinate



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- 2-D rotation from last slide: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

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3-D Rotation

- *What does the 3-D rotation matrix look like for a rotation about the Y-axis?*
  - Build it coordinate-by-coordinate

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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3-D Rotation

- *What does the 3-D rotation matrix look like for a rotation about the X-axis?*
  - Build it coordinate-by-coordinate

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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$\underline{\underline{R}} \cdot \underline{R^T} = I \Rightarrow \underline{R^{-1}} = \underline{R^T} \Leftarrow$ Unitary Transform

Rotation Matrices

- Rotation matrix is *orthogonal*
  - Columns/rows are orthonormal and therefore linearly independent
- The inverse of an orthogonal matrix is just its transpose:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}^{T} = \begin{bmatrix} a & d & h \\ b & e & i \\ c & f & j \end{bmatrix}$$

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Eigenvalues and Eigenvectors

- The vector **v** is an eigenvector of matrix *A* and **λ** is an eigenvalue of *A* if:

A is square

$$\underline{A}\vec{v} = \lambda \vec{v} \quad \text{(assume non-zero v)}$$

- **Interpretation:** the linear transformation implied by *A* cannot change the direction of the eigenvectors v, only their magnitude.

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$(A - \lambda I)\vec{v} = \vec{0}$

Computing λ and v

- To find the eigenvalues λ of a matrix *A*, find the roots of the *characteristic polynomial*:

$$det(A - \lambda I) = 0$$

Example:

$$A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix} \Rightarrow det\left(\begin{bmatrix} 5-\lambda & -2 \\ 6 & -2-\lambda \end{bmatrix}\right) = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0 \text{ or } \lambda_1 = 1,\ \lambda_2 = 2$$

$$Av = \lambda v \qquad v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

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Properties

- Eigenvalues and eigenvectors are only defined for square matrices (i.e., $m = n$)
- Eigenvectors are not unique (e.g., if v is an eigenvector, so is kv)
- Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then:

$$\prod_i \lambda_i = det(A)$$

if $\lambda = 0$ is an eigenvalue, then the matrix is not invertible

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Properties (cont'd)

If A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the corresponding eigevectors $v_1, v_2, \ldots v_n$ form a basis:

(1) linearly independent

(2) span R^n

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Matrix diagonalization

- Given A, find P such that $P^{-1}AP$ is diagonal (i.e., P diagonalizes A)

modal matrix

- Take $P = [v_1\ v_2 \ldots v_n]$, where $v_1, v_2, \ldots v_n$ are the eigenvectors of A:

$$Av = \lambda v \Rightarrow AP = P\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix} \quad \text{or} \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix}$$

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$$AP = A[\vec{v}_1 \cdots \vec{v}_n]$$
$$= [A\vec{v}_1\ A\vec{v}_2 \cdots A\vec{v}_n]$$
$$= [\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2 \cdots \lambda_n\vec{v}_n]$$
$$= \underbrace{[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]}_{P}\underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}$$
$$AP = P\Lambda \Rightarrow \boxed{A = P\Lambda P^{-1}}$$

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Matrix diagonalization (cont'd)

$$AP = P\Lambda \Rightarrow A = P\Lambda P^{-1}$$

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\boxed{\Lambda = P^{-1}AP}$$

$$\lambda_1 = 0,\ \lambda_2 = 2,\ v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

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Matrix decomposition

- Let us assume that A is diagonalizable, then:

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix} \longrightarrow A = P\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix}P^{-1}$$

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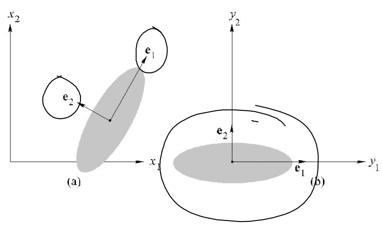
Eigenvalues & Eigenvectors (Con't)



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Decomposition of symmetric matrices

- The eigenvalues of symmetric matrices are all real.
- The eigenvectors corresponding to distinct eigenvalues are orthogonal.

$$A = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix} P^{-1} \xrightarrow{P^{-1}=P^T} A=PDP^T=\sum_{i=1}^{n} \lambda_i v_i v_i^T$$

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