EE 417 Introduction to Computer Vision / EE 569 3D vision

Image Formation and Camera Calibration

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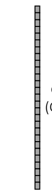
Let's say we have a sensor...



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... and an object we like to photograph

What would an image taken like this look like?

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Bare-sensor imaging

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Bare-sensor imaging

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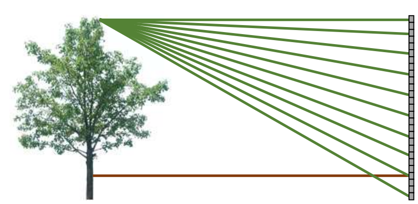
Bare-sensor imaging



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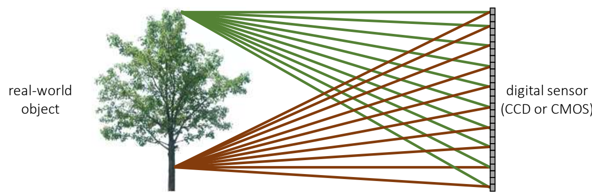
Bare-sensor imaging



All scene points contribute to all sensor pixels

What does the image on the sensor look like?

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Bare-sensor imaging

All scene points contribute to all sensor pixels

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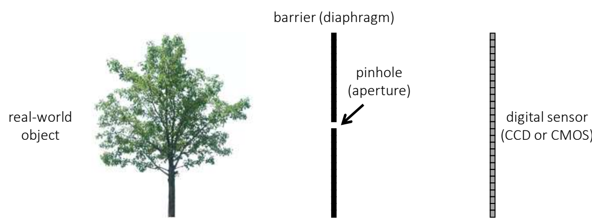
Let's add something to this scene



What would an image taken like this look like?

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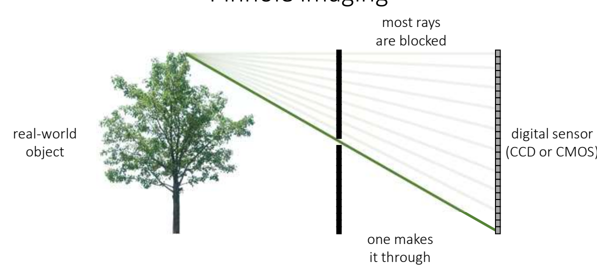
Pinhole imaging



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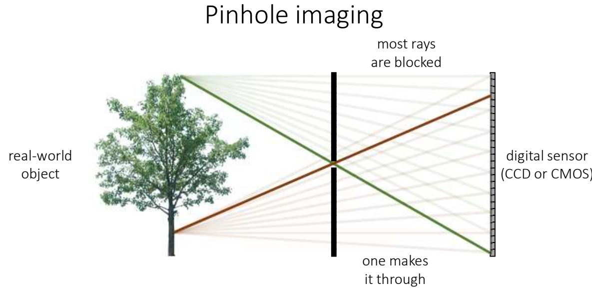
Pinhole imaging



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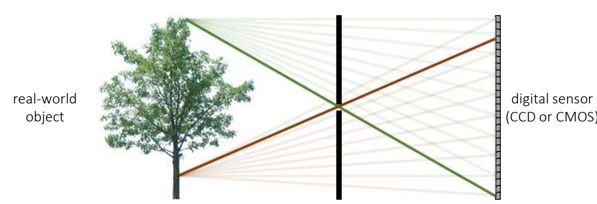
Pinhole imaging



Each scene point contributes to only one sensor pixel

What does the image on the sensor look like?

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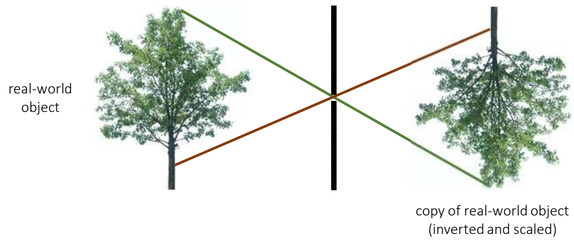
Pinhole imaging



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Pinhole camera

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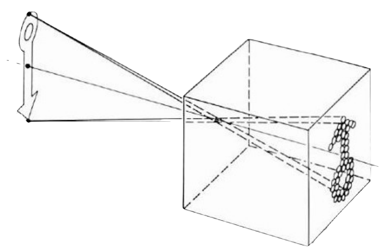
Pinhole camera a.k.a. camera obscura



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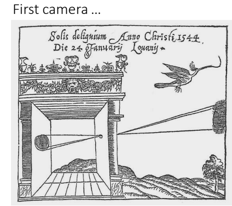
Pinhole camera a.k.a. camera obscura

First mention ...

Chinese philosopher Mozi (470 to 390 BC)

First camera ...



Greek philosopher Aristotle (384 to 322 BC)

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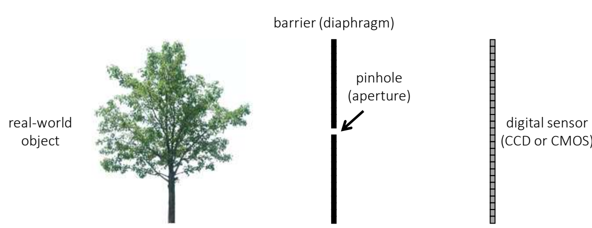
Pinhole camera terms



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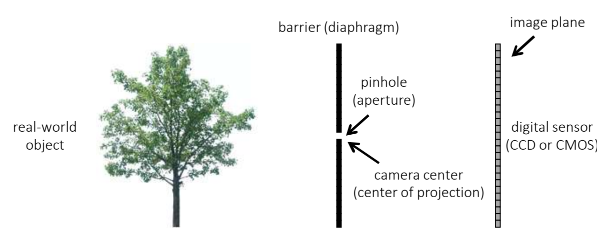
Pinhole camera terms



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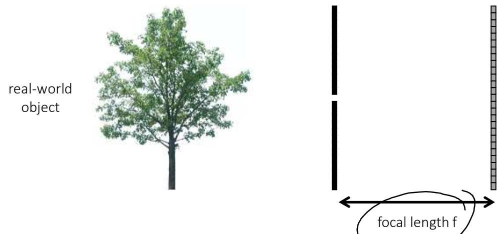
Focal length



distance from the aperture to the sensor

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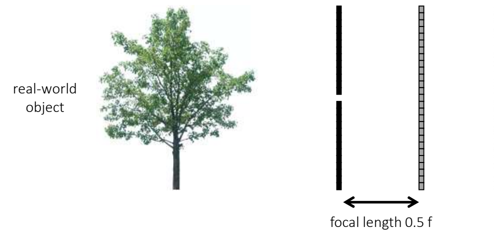
Focal length

What happens as we change the focal length?



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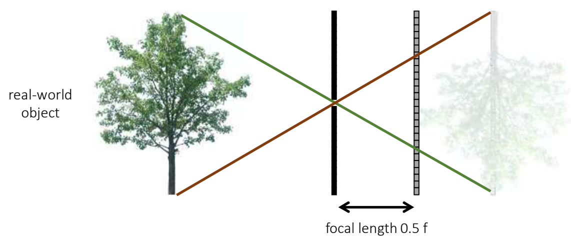
Focal length

What happens as we change the focal length?



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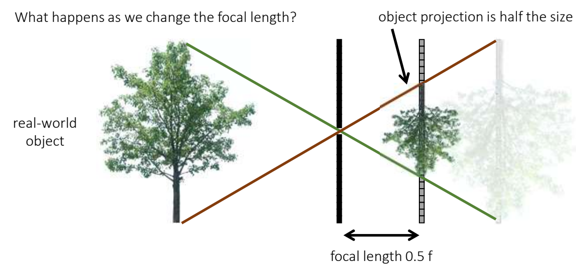
Focal length

What happens as we change the focal length?



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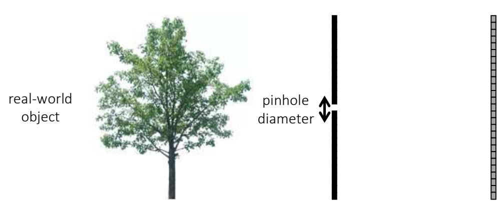
Pinhole size



Ideal pinhole has infinitesimally small size
- In practice that is impossible.

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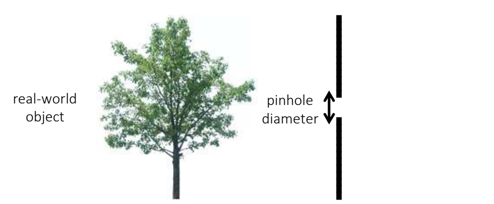
Pinhole size

What happens as we change the pinhole diameter?



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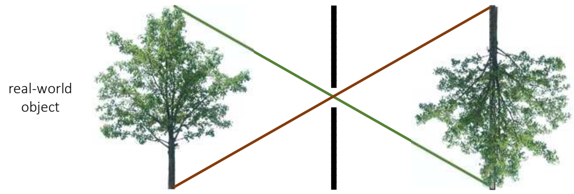
Pinhole size

What happens as we change the pinhole diameter?



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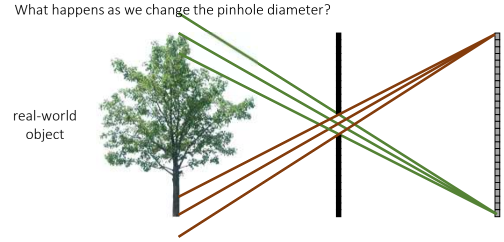
Pinhole size

What happens as we change the pinhole diameter?



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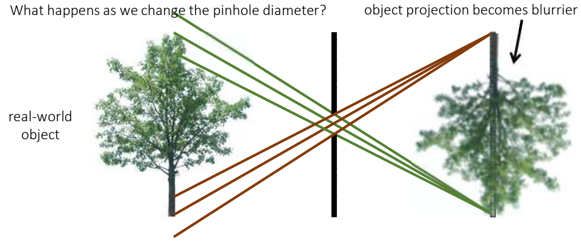
Pinhole size

What happens as we change the pinhole diameter?

object projection becomes blurrier



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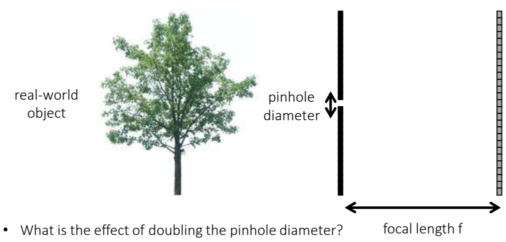
What about light efficiency?



- What is the effect of doubling the pinhole diameter?
- What is the effect of doubling the focal length?

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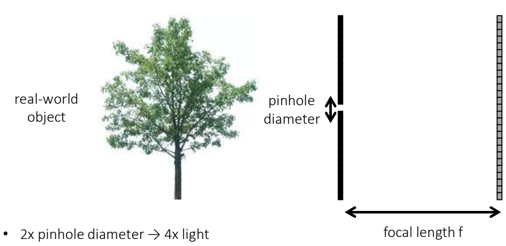
What about light efficiency?



- 2x pinhole diameter → 4x light
- 2x focal length → ¼x light

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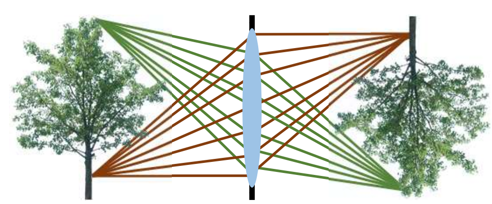
The lens camera



Lenses map “bundles” of rays from points on the scene to the sensor.

How does this mapping work exactly?

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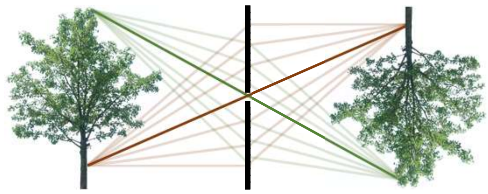
The pinhole camera



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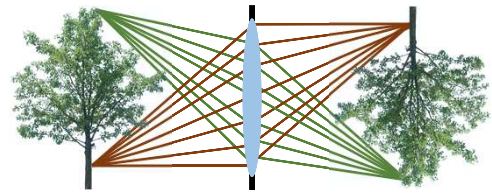
The lens camera



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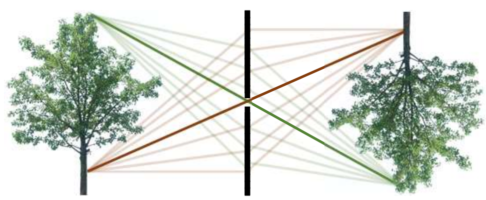
The pinhole camera



Central rays propagate in the same way for both models!

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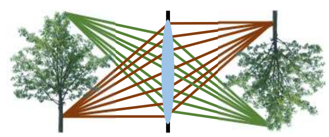
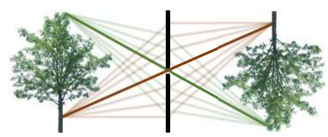
Describing both lens and pinhole cameras





We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.

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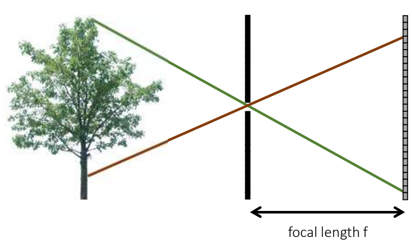
Important difference: focal length

In a pinhole camera, focal length is distance between aperture and sensor



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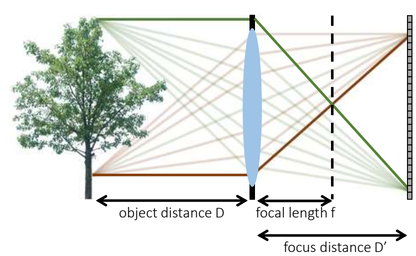
Important difference: focal length

In a lens camera, focal length is distance where parallel rays intersect



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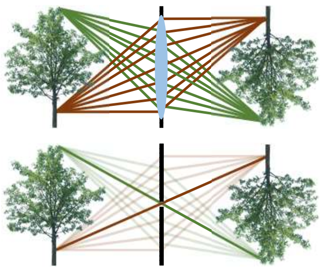
Describing both lens and pinhole cameras



We can derive properties and descriptions that hold for both camera models if:
- We use only central rays.
- We assume the lens camera is in focus.
- We assume that the focus distance of the lens camera is equal to the focal length of the pinhole camera.

Remember: *focal length* f refers to different things for lens and pinhole cameras.
- In this lecture, we use it to refer to the aperture-sensor distance, as in the pinhole camera case.

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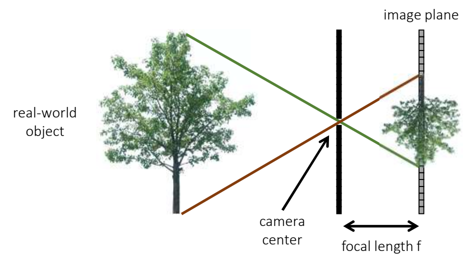
The pinhole camera



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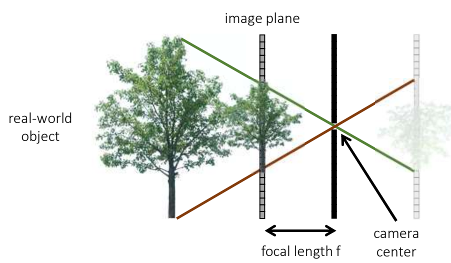
The (rearranged) pinhole camera



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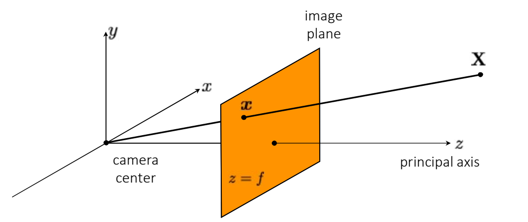
The (rearranged) pinhole camera



What is the equation for image coordinate **x** in terms of **X**?

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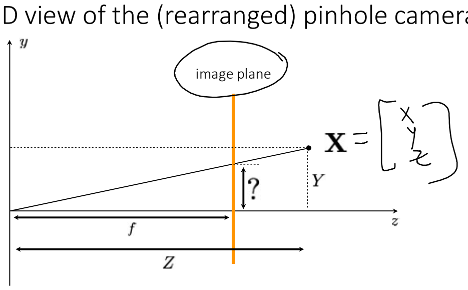
The 2D view of the (rearranged) pinhole camera



What is the equation for image coordinate **x** in terms of **X**?

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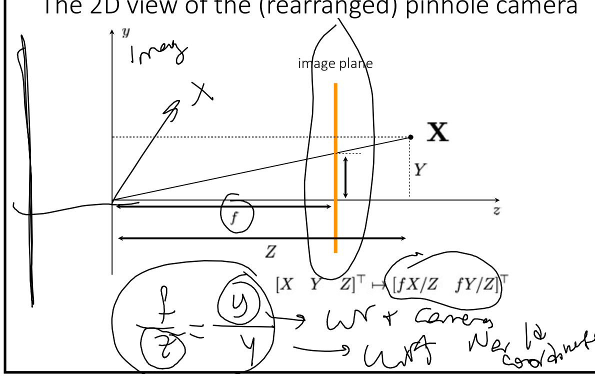
The 2D view of the (rearranged) pinhole camera



$[X \quad Y \quad Z]^\top \mapsto [fX/Z \quad fY/Z]^\top$

$\frac{f}{Z} = \frac{y}{Y}$ → wrt camera → wrt World coordinate

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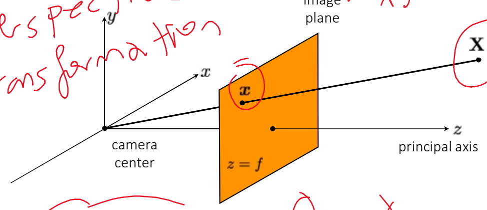
Orthographic projection depth Z of the object

The (rearranged) pinhole camera

$x = X'$, $y = Y'$

Perspective Transformation



$$[X \quad Y \quad Z]^\top \mapsto [fX/Z \quad fY/Z]^\top$$

division by Z is non-linear

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Trick: add one more coordinate:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous image (2D) coordinates — homogeneous scene (3D) coordinates

Converting *from* homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w) \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Homogenous coordinates invariant under scale

Inhomogeneous = heterogeneous = world coordinate

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Homogeneous coordinates

heterogeneous coordinates — homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

add a 1 here

- Represent 2D point with a 3D vector

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Homogeneous coordinates

heterogeneous coordinates — homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} ax \\ ay \\ a \end{bmatrix}$$

- Represent 2D point with a 3D vector
- 3D vectors are only defined up to scale

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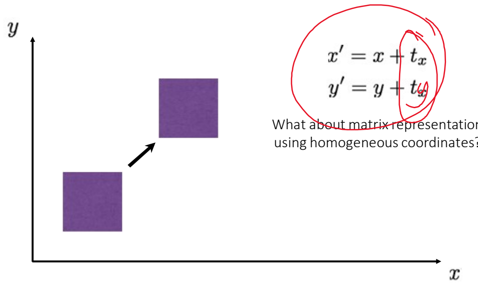
2D translation



$$x' = x + t_x$$
$$y' = y + t_y$$

What about matrix representation using homogeneous coordinates?

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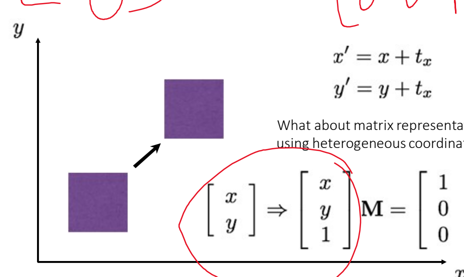
$$\begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D translation



$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation using heterogeneous coordinates?

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

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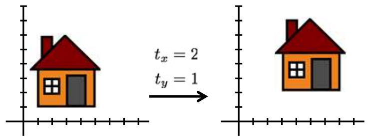
2D translation using homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



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Homogeneous coordinates

Conversion:

•heterogeneous → homogeneous

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

•homogeneous → heterogeneous

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

•scale invariance

$$\begin{bmatrix} x & y & w \end{bmatrix}^\top = \lambda \begin{bmatrix} x & y & w \end{bmatrix}^\top$$

Special points:

- point at infinity

$$\begin{bmatrix} x & y & 0 \end{bmatrix}$$

- undefined

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

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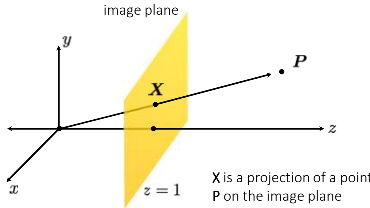
Projective geometry

image point in pixel coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

⇕

image point in homogeneous coordinates $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$



X is a projection of a point **P** on the image plane

What does scaling **X** correspond to?

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2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

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2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

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2D transformations in homogenous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

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2D transformations in Homogenous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

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Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = ? ? ? **p**

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Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = translation(t_x,t_y) rotation(θ) scale(s,s) **p**

Does the multiplication order matter?

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Classification of 2D transformations

Translation: $\begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$

Dof is 2

How many degrees of freedom?

(or free) // # of independent variables

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Classification of 2D transformations

Euclidean (rigid):
rotation + translation
$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$

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Classification of 2D transformations

Euclidean (rigid):
rotation + translation
$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$

tx ty

How many degrees of freedom?

Dof = 3

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Classification of 2D transformations

which other matrix values will change if this increases?

Euclidean (rigid): rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

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Classification of 2D transformations

what will happen to the image if this increases?

Euclidean (rigid): rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

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Classification of 2D transformations

what will happen to the image if this increases?

Euclidean (rigid): rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

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Classification of 2D transformations

Similarity: uniform scaling + rotation + translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Are there any values that are related?

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Classification of 2D transformations

multiply these four by scale **s**

Similarity: uniform scaling + rotation + translation

$$\begin{bmatrix} \boxed{\begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix}} & \begin{matrix} r_3 \\ r_6 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

How many degrees of freedom?

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Classification of 2D transformations

what will happen to the image if this increases?

Similarity: uniform scaling + rotation + translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

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What is camera calibration?

- The process of estimating camera parameters
- The 3D world coordinates are projected on the 2D image plane (film)
- The relationship between the world coordinates and image coordinates is defined by the camera parameters

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Why do we need to calibrate cameras?

- To estimate the 3D geometry of the world → geometric calibration
- To correct imaging artefacts caused by imperfect lenses
- Examples include
  - Stereo reconstruction
  - Multiview reconstruction
  - Single view measurements such as the height of a person
  - Lens distortion correction

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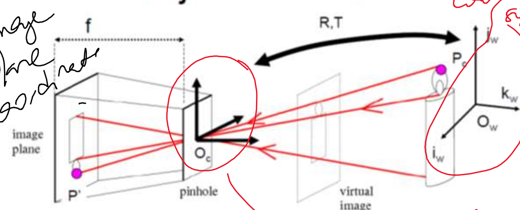
Projective camera

image plane coordinate

world coordinate system



Camera coordinate system

- For convenience, the image plane is usually represented in front so that the image preserves the same orientation (i.e. not flipped)

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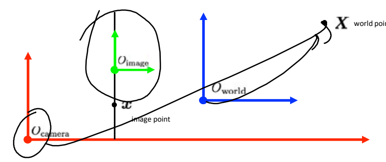
Generalizing the camera matrix

In general, there are *three*, generally different, coordinate systems.



We need to know the transformations between them.

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Geometric Camera calibration

- Made up of 2 transformations:
  - From some (arbitrary) world coordinate system to the camera's 3D coordinate system. ***Extrinisic parameters*** *(camera pose)*
  - From the 3D coordinates in the camera frame to the 2D image plane via projection. ***Intrinisic paramters***

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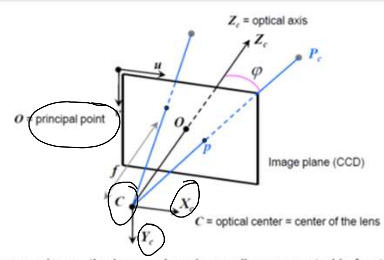
Perspective Camera



- For convenience, the image plane is usually represented in front so that the image preserves the same orientation (i.e. not flipped)
- Note: **a camera does not measure distances but angles!**
  ⇨ a camera is a "bearing sensor"

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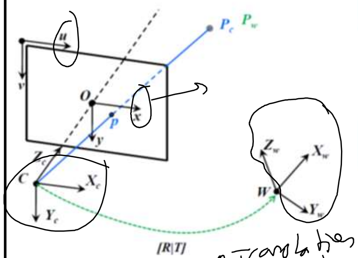
Perspective Projection: from world to pixel coords



Find pixel coordinates ***(u,v)*** of point P_w in the world frame:

0. Convert world point P_w to camera point P_c

Find pixel coordinates ***(u,v)*** of point P_c in the camera frame:

1. Convert P_c to image-plane coordinates ***(x,y)***

2. Convert P_c to (discretised) pixel coordinates ***(u,v)***

↑ Translation of the world wrt camera

R T

↳ wrt ~~Rco~~ rotation of the world coordinate wrt camera

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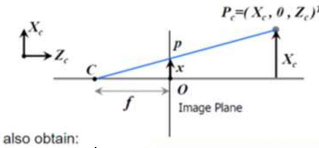
Perspective Projection (1)

From the Camera frame to the image plane

pixel size Δx Δy

- The 3D camera point $P_c = (X_c, 0, Z_c)^T$ projects to point $p = (x, y)$ onto the image plane
- Analysing similar triangles:

$$\frac{x}{f} = \frac{X_c}{Z_c} \Rightarrow x = \frac{fX_c}{Z_c}$$



- So, for the 3D case, we can also obtain:

$$\frac{y}{f} = \frac{Y_c}{Z_c} \Rightarrow y = \frac{fY_c}{Z_c}$$

$u = \frac{x}{\Delta x}$

$v = \frac{y}{\Delta y}$

x y

1. Convert P_c to image-plane coordinates ***(x,y)***

2. Convert P_c to (discretised) pixel coordinates ***(u,v)***

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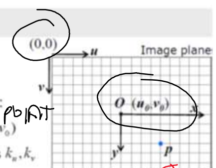
Perspective Projection (2)

From the Camera frame to pixel coordinates



- Convert ***p***, from the local image plane coords ***(x,y)*** to the pixel coords ***(u,v)*** we have to take account for:
  - Pixel coords of the camera ~~optical center~~ Principal point $O = (u_0, v_0)$
  - Scale factors for the pixel-size in both dimensions k_u, k_v

So:

$$u = u_0 + k_u x \Rightarrow u = u_0 + \frac{k_u f X_c}{Z_c}$$

$$v = v_0 + k_v y \Rightarrow v = v_0 + \frac{k_v f Y_c}{Z_c}$$

$u - u_0 = \frac{x}{\Delta x} = \frac{f}{\Delta x}\frac{X_c}{Z_c}$

$k_u = \frac{1}{\Delta x}$ $k_v = \frac{1}{\Delta y}$

- Use **Homogeneous Coordinates** for linear mapping from 3D to 2D, by introducing an extra element (scale):

$$p = \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \tilde{p} = \begin{pmatrix} \lambda u \\ \lambda v \\ \lambda \end{pmatrix}$$ and similarly for the world coordinates. Note, usually $\lambda = 1$

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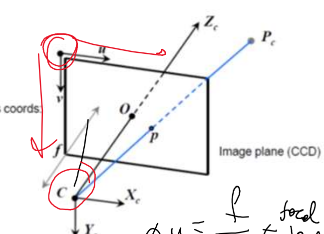
Perspective Projection (3)

- So:

$$u = u_0 + \frac{k_u f X_c}{Z_c}$$

$$v = v_0 + \frac{k_v f Y_c}{Z_c}$$



Expressed in matrix form & Homogenerous coords:

$$\begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} k_u f & 0 & u_0 \\ 0 & k_v f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}$$

Or alternatively

$$\begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = K \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}$$

Focal length in ***u***-direction

Focal length in ***v***-direction

"Calibration matrix"
or
"Matrix of Intrinsic Parameters"
internal

$\alpha u = \frac{f}{\Delta x}$ ← focal length in terms of pixel

DoF of $\underline{K} = 4$

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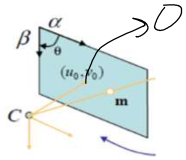
$$K = \begin{bmatrix} \alpha & s & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ → skew

K has 5 degrees of freedom!



→ O

$s = \alpha \cot \theta$ Axis skew causes shear distortion

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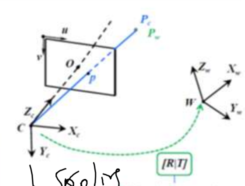
Perspective Projection (4)

From the Camera frame to the World frame

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

in homogeneous coords

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} R & T \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$



$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}$$

↓ Scaling factor

$[R\ T]$ external parameters

Projection Matrix

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \boxed{K[R|T]} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$\underbrace{K[R|T]}_{M}$

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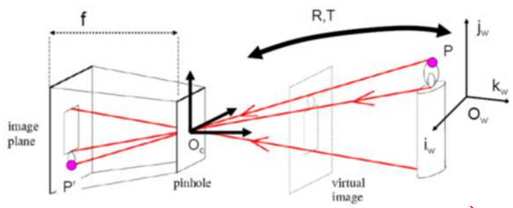

$$P' = MP_w$$

Projection matrix

$$= \boxed{K}\boxed{\begin{bmatrix} R & T \end{bmatrix}}P_w$$

Internal parameters

External parameters

Dof of $\underline{M}$

$= 11$

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Calibration

- How to determine **M**

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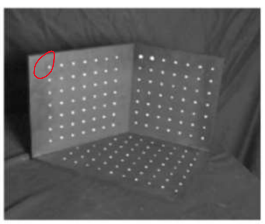
Calibration using a reference object

- Place a known object in the scene
  - identify correspondence between image and scene
  - compute mapping from scene to image



Issues

- must know geometry very accurately
- must know 3D->2D correspondence

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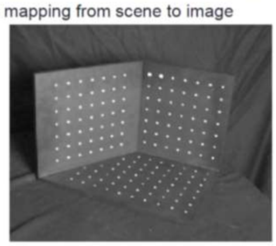
Estimating the projection matrix

- Place a known object in the scene
  - identify correspondence between image and scene
  - compute mapping from scene to image



$$\begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

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- Projective Camera Matrix:

$$p = K[R \quad t]P = MP$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Only up to a scale, so 11 DOFs.

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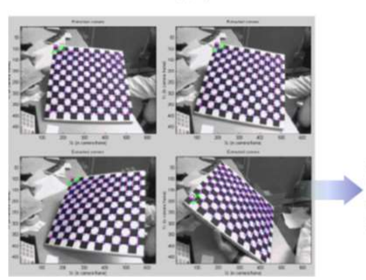
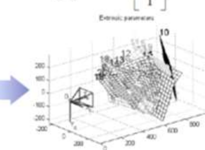
Camera Calibration

- Use our camera model to interpret the projection from world to image plane
- Using known correspondences of $p \Leftrightarrow P$, we can compute the unknown parameters K, R, T by applying the perspective projection equation
- … so associate known, physical distances in the world to pixel-distances in image



Projection Matrix

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \boxed{K[R|T]} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$



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Direct linear calibration - homogeneous

$$\begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} u_i &= \frac{m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + m_{23}} \\ v_i &= \frac{m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + m_{23}} \end{aligned}\right\}$$

$$u_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + m_{23}) = m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}$$
$$v_i(m_{20}X_i + m_{21}Y_i + m_{22}Z_i + m_{23}) = m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}$$

P_i^T $\quad -u_i \{ P_i^T$

One pair of equations for each point

$$\begin{bmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -u_iX_i & -u_iY_i & -u_iZ_i & -u_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -v_iX_i & -v_iY_i & -v_iZ_i & -v_i \end{bmatrix} \begin{bmatrix} m_{00} \\ m_{01} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \\ m_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Direct linear calibration - homogeneous

$$\underbrace{\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix}}_{\mathbf{A}\ 2n \times 12} \underbrace{\begin{bmatrix} m_{00} \\ m_{10} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \\ m_{23} \end{bmatrix}}_{\mathbf{m}\ 12} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}\ 2n}$$

This is a homogenous set of equations.

$\underline{A}_{2n\times 12} \cdot \vec{m}_{12\times 1} = \vec{0}_{12\times 1}$ $\quad$ n # of corresp

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Homogeneous M x N Linear Systems

M=number of equations
N=number of unknown

$$A \; x = 0$$

Rectangular system (M>N)

- 0 is always a solution
- To find non-zero solution

  Minimize $|Ax|^2$

  under the constraint $|x|^2 = 1$

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Direct linear calibration - homogeneous

$$\underbrace{\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix}}_{\mathbf{A}\ 2n \times 12} \underbrace{\begin{bmatrix} m_{00} \\ m_{10} \\ m_{02} \\ m_{03} \\ m_{10} \\ m_{11} \\ m_{12} \\ m_{13} \\ m_{20} \\ m_{21} \\ m_{22} \\ m_{23} \end{bmatrix}}_{\mathbf{m}\ 12} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}\ 2n\times 1}$$

rank(A^TA) = rank(A^T) = rank(A)

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This is a homogenous set of equations.

When over constrained, defines a least squares problem

- minimize $\|\mathbf{Am}\|$
  - Since **m** is only defined up to scale, solve for unit vector $\mathbf{m}^*$
  - Solution: $\mathbf{m}^*$ = eigenvector of $\mathbf{A}^T\mathbf{A}$ with *smallest* eigenvalue
  - Works with 6 or more points

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Camera Calibration

- We know that : $\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K[R|T] \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$
- So there are 11 values to estimate: (the overall scale doesn't matter, so e.g. m_{34} could be set to 1) $\quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$
- Each observed point gives us a pair of equations:

$$u_i = \frac{\lambda u_i}{\lambda} = \frac{m_{11}X_i + m_{12}Y_i + m_{13}Z_i + m_{14}}{m_{31} + m_{32} + m_{33} + m_{34}}$$
$$v_i = \frac{\lambda v_i}{\lambda} = \frac{m_{21}X_i + m_{22}Y_i + m_{23}Z_i + m_{24}}{m_{31} + m_{32} + m_{33} + m_{34}}$$

- To estimate 11 unknowns, we need **at least 6** points to calibrate the camera ⇨ solved using linear least squares

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Direct linear calibration - inhomogeneous

- Another approach: 1 in lower r.h. corner for 11 d.o.f

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \simeq \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Now "regular" least squares since there is a non-variable term in the equations:

$$u_i = \frac{m_{00}X_i + m_{01}Y_i + m_{02}Z_i + m_{03}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$
$$v_i = \frac{m_{10}X_i + m_{11}Y_i + m_{12}Z_i + m_{13}}{m_{20}X_i + m_{21}Y_i + m_{22}Z_i + 1}$$

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$$X_i q_{11} + Y_i q_{12} + Z_i q_{13} + q_{14} - u_i X_i q_{31} - u_i Y_i q_{32} - u_i Z_i q_{33} = u_i$$
$$X_i q_{21} + Y_i q_{22} + Z_i q_{23} + q_{24} - v_i X_i q_{31} - v_i Y_i q_{32} - v_i Z_i q_{33} = v_i$$

How many equations do we need to solve this?

How many points do we need to solve this?

$$\underbrace{\begin{bmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -u_iX_i & -u_iY_i & -u_iZ_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -v_iX_i & -v_iY_i & -v_iZ_i \end{bmatrix}}_{\text{2N x 11 Matrix}} \underbrace{\begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \\ q_{21} \\ q_{22} \\ q_{23} \\ q_{24} \\ q_{31} \\ q_{32} \\ q_{33} \end{bmatrix}}_{\text{11 vector of unknowns}} = \underbrace{\begin{bmatrix} u_i \\ v_i \end{bmatrix}}_{\text{2N vector}}$$

$\vec{m}$

$B \vec{m} = \vec{p}_i$

$\vec{m} = (B^T B)^{-1} B^T P_i$

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Solving for q

- Since every point gives two equations, we need at least 6 non-coplanar points to solve $A\mathbf{q} = \mathbf{b}$
- We can solve this using linear least squares

$$A\mathbf{q} = \mathbf{b}$$
$$\Rightarrow A^T A\mathbf{q} = A^T\mathbf{b}$$
$$\Rightarrow \mathbf{q} = (A^T A)^{-1} A^T\mathbf{b}$$

- Can be solved in one line of Matlab $\mathbf{q} = A\backslash\mathbf{b}$;
- For a unique solution, $A^T A$ must be non-singular i.e. $\mathrm{rank}(A^T A)$ or $\mathrm{rank}(A)$ must be 11. Need $2N \geq 11,\ N \geq 6$ non-coplanar points.

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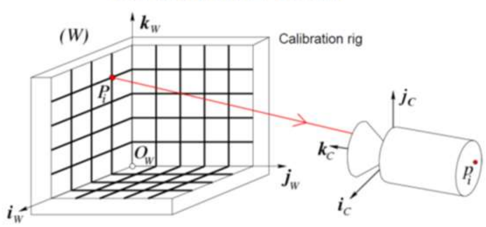
Calibration Problem



- $P_1 \ldots P_n$ with known positions in $[O_w, i_w, j_w, k_w]$
- $p_1, \ldots p_n$ known positions in the image

Goal: compute intrinsic and extrinsic parameters

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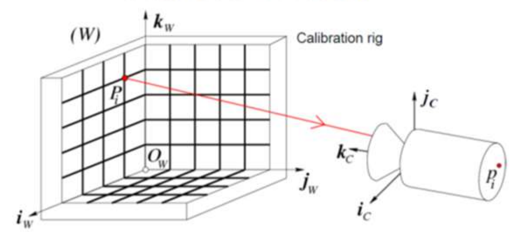
Calibration Problem



How many correspondences do we need?

- P has 11 unknown
- We need 11 equations
- 6 correspondences would do it

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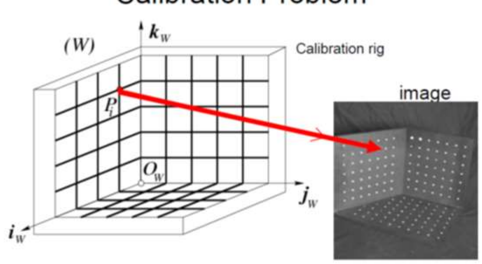
Calibration Problem



In practice: user may need to look at the image and select the n>=6 correspondences

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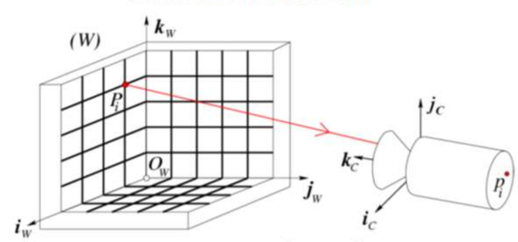
Calibration Problem



$$P_i \to M\,P_i \to p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \dfrac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix} \qquad M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

in pixels

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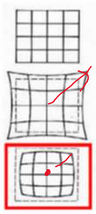
Radial Distortion

- Caused by imperfect lenses
- Deviations are most noticeable for rays that pass through the edge of the lens



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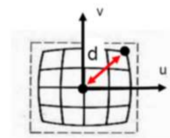
Radial Distortion



$$\begin{bmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{M}\,\mathrm{P}_i \rightarrow \begin{bmatrix} u_i \\ v_i \end{bmatrix} = p_i$$

$$M = \begin{bmatrix} m_1^T \\ m_2^T \\ m_3^T \end{bmatrix}$$

$$d^2 = a\,u^2 + b\,v^2 + c\,u\,v$$

To model radial behavior

$$\lambda = 1 \pm \underbrace{\sum_{p=1}^{3} \boxed{\kappa_p} d^{2p}}_{\text{Polynomial function}}$$

Distortion coefficient

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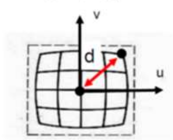
Radial Distortion

Estimating m_1 and m_2…

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \dfrac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \dfrac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$

How to do that?

Hint:



$$\frac{u_i}{v_i} = \text{slope}$$

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Radial Distortion

Estimating m_1 and m_2…

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \dfrac{\mathbf{m}_1^T P_i}{\mathbf{m}_3^T P_i} \\ \dfrac{\mathbf{m}_2^T P_i}{\mathbf{m}_3^T P_i} \end{bmatrix} \qquad \frac{u_i}{v_i} = \frac{\dfrac{(\mathbf{m}_1^T P_i)}{\cancel{(\mathbf{m}_3^T P_i)}}}{\dfrac{(\mathbf{m}_2\,P_i)}{\cancel{(\mathbf{m}_3\,P_i)}}} = \frac{\mathbf{m}_1\,P_i}{\mathbf{m}_2\,P_i}$$

$$\begin{cases} v_1(\mathbf{m}_1\,P_1) - u_1(\mathbf{m}_2\,P_1) = 0 \\ v_i(\mathbf{m}_1\,P_i) - u_i(\mathbf{m}_2\,P_i) = 0 \\ \vdots \\ v_n(\mathbf{m}_1\,P_n) - u_n(\mathbf{m}_2\,P_n) = 0 \end{cases} \qquad Q\,\mathbf{n} = 0 \qquad \mathbf{n} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix}$$

Tsai technique [87]

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Radial Distortion

Once that $\mathbf{m}_1$ and $\mathbf{m}_2$ are estimated…

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \dfrac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \dfrac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$

$\mathbf{m}_3$ is non linear function of $\mathbf{m}_1$, $\mathbf{m}_2$, λ

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General Calibration Problem

$$X = f(P)$$

f( ) is nonlinear

measurement — X; parameter — P

-Newton Method
-Levenberg-Marquardt Algorithm

- Iterative, starts from initial solution
- May be slow if initial solution far from real solution
- Estimated solution may be function of the initial solution
- Newton requires the computation of J, H
- Levenberg-Marquardt doesn't require the computation of H

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