1 $1 \mid\mid \sum w_j C_j$

1.1 Alternative Formulation 1

The notation implies that we have a single machine and the objective is to minimize a weighted sum of the completion times.

Parameters:
l = $C_{max} - 1$

Decision variables:
$x_{jt} = 1$ if job j starts at time t, 0 otherwise

Model

$$\begin{aligned} \min & \sum_{j=1}^{n} \sum_{t=0}^{l} x_{jt} \times (t + p_j) \times w_j \\ \text{s.t.} \quad & \sum_{t=0}^{l} x_{jt} = 1 \quad \forall j \\ & \sum_{j=1}^{n} \sum_{s=max\{0,t-p_j\}}^{max\{0,t-1\}} x_{js} \leq 1 \quad \forall t \end{aligned} \tag{1}$$

1.2 Alternative Formulation 2

Decision variables:
$x_{jk} = 1$ if job j precedes job k, 0 otherwise

Note that the completion of time of job j is $C_j = \sum_{k=1}^{n} P_k \times x_{kj} + p_j$

Model

$$\begin{aligned} \min & \sum_{j=1}^{n} [\sum_{k=1}^{n} x_{kj} \times p_j] \times w_j \\ \text{s.t.} \quad & x_{kj} + x_{jk} = 1 \quad \forall (j,k), j \neq k \\ & x_{jj} = 0 \quad \forall j \\ & x_{kj} + x_{lk} + x_{jl} \geq 1 \quad \forall (j,k,l), \quad j \neq k \neq l \\ & x_{jk} \in \{0,1\} \quad \forall (j,k) \end{aligned} \tag{2}$$

2 $Q_m \mid r_j, M_j, prec, d_j \mid \theta_1 C_{max} + \theta_2 \sum w_j T_j$

The notation implies that there are m machines in parallel each with a different speed v_i. Jobs have release dates r_j and due dates d_j. Each job j can only processed by machines in the set M_j. There are precedence constraints between jobs. The objective is to minimize a composite objective that consists of the makespan and and a weighted sum of tardiness.

Parameters

1. p_{ij} processing time of job j on machine i
2. v_i speed of machine i
3. r_j, d_j release date and due date of job
4. M_j set of machines that can process job j
5. l: max time
6. w_j: weight of job j

Decision Variables

1. T_j tardiness of job j
2. C_{max}: makespan
3. x_{ijt}: 1 if job j assigned to machine i at time t, 0 otherwise

Model

$$
\begin{aligned}
\min \quad & \theta_1 C_{max} + \theta_2 \sum_{j=1}^{n} w_j T_j \\
\text{s.t.} \quad & C_{max} \geq \sum_{i=1}^{m} \sum_{t=0}^{l} x_{ijt} \times (t + \frac{p_{ij}}{v_i}) \quad \forall j \\
& T_j \geq 0 \quad \forall j \\
& T_j \geq \sum_{i=1}^{m} \sum_{t=0}^{l} x_{ijt} \times (t + \frac{p_{ij}}{v_i}) - d_j \quad \forall j \\
& \sum_{i=1}^{m} \sum_{t=0}^{l} x_{ijt} \times t \geq r_j \quad \forall j \\
& \sum_{i=1}^{m} \sum_{t=0}^{l} x_{ijt} = 1 \quad \forall j \\
& \sum_{t=0}^{l} x_{ijt} = 0 \quad \forall (i,j) \quad s.t. \quad i \not\in M_j \\
& \sum_{j=1}^{n} \sum_{s=max\{0,t-\frac{p_{ij}}{v_i}\}}^{max\{0,t-1\}} x_{ijs} \leq 1 \quad \forall (i,t) \\
& \sum_{i=1}^{m} \sum_{t=0}^{l} x_{ijt} \times (t + \frac{p_{ij}}{v_i}) \leq \sum_{i=1}^{m} \sum_{t=0}^{l} x_{ikt} \times t \quad \forall (j,k) \quad s.t. j \quad precedes \quad k
\end{aligned}
\tag{3}
$$

3 Scheduling with custom scenario

3.1 Scenario

1. Multiple jobs with precedence constraints
2. Jobs belong to categories and can be ran together in parallel if they belong to the same category at no additional cost
3. Continuous time
4. Objective is to minimize makespan

3.2 Mathematical Model

Let there be n jobs and c categories. Let a_j be the parameter that denotes the category that job j belongs to. Let p_j be the duration of job j. Let z_{ij} be the binary parameter that denotes whether job i has to come before job j due to precedence constraints. Finally, let d_j be the due date of job j.

Decision variables:

1. C_{max}: makespan
2. s_j: start time of job j
3. e_j: end time of job j
4. y_{ij}: no overlap constraint

Model:

$$
\begin{aligned}
\min \quad & C_{max} \\
\text{s.t.} \quad & C_{max} \geq e_j \quad \forall j \\
& e_j = s_j + p_j \quad \forall j \\
& e_j \leq d_j \quad \forall j \\
& e_i \leq s_j \quad \forall (i,j) \quad s.t. \quad z_{ij} = 1 \\
& e_i \leq s_j + M \times y_{ij} \quad \forall (i,j) \quad s.t. \quad i < j, \quad z_{ij} = 0 \quad a_j = a_i \\
& e_i \leq s_j + M \times (1 - y_{ij}) \quad \forall (i,j) \quad s.t. \quad i < j, \quad z_{ij} = 0, \quad a_j = a_i \\
& y_{ij} \in \{0, 1\} \\
& s_j \geq 0 \quad \forall j \\
& e_j \geq 0 \quad \forall j
\end{aligned}
\tag{4}
$$

4 Modeling Extensions

4.1 Modeling the number of machine startups

Let x_t be the binary decision variable denoting whether the machine is active or not at time t. Let s_t be the binary decision variable denoting whether the machine is started up at time t. Then the following three inequalities derive s_t from x_t

$$x_t \geq s_t \tag{5}$$

$$1 - x_{t-1} \geq s_t \tag{6}$$

$$x_t - x_{t-1} \leq s_t \tag{7}$$