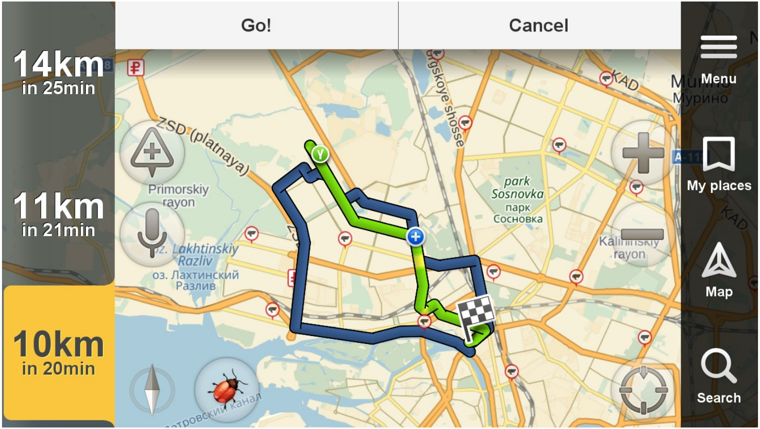
FASTEST ROUTE IN A GRAPH

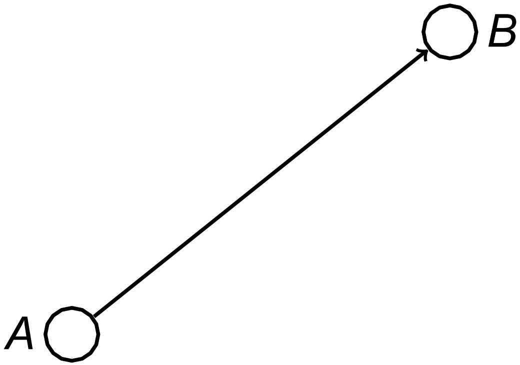
Michael Levin

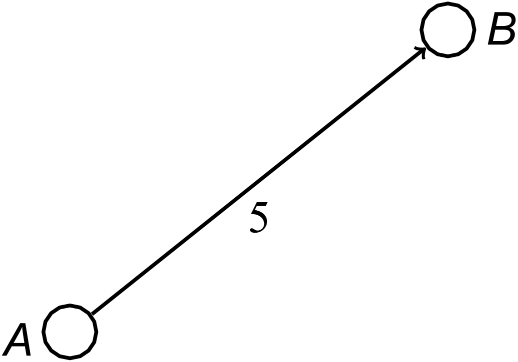
Outline

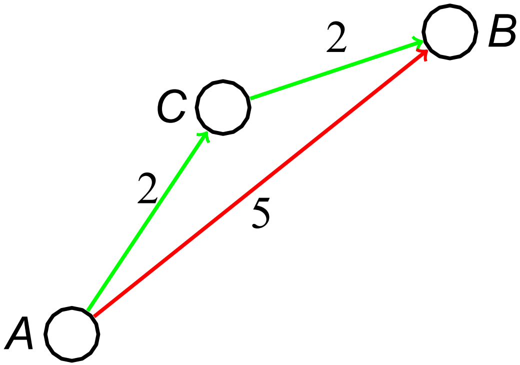
Fastest Route

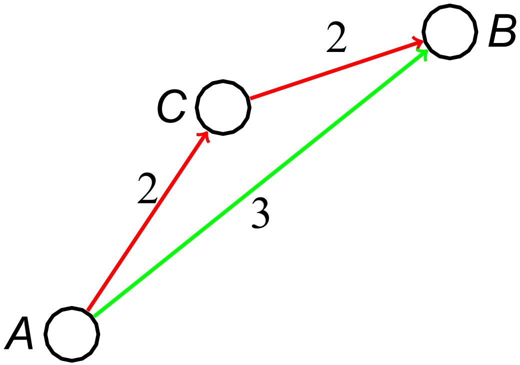
What is the fastest route to get home from work?




A
B


A
B
5

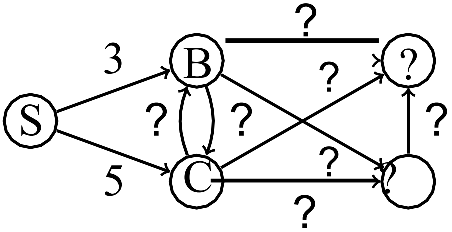

A
B
C
2
2
5



2
B
C
2
3
A

Intuition

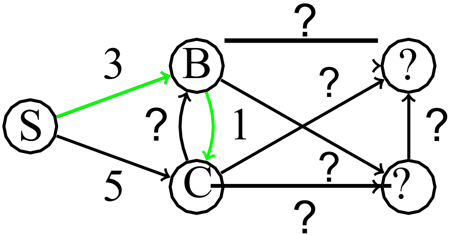
- Assume that we stay at S and observe two outgoing edges:



- Can we be sure that the distance from S to C is 5?

Intuition

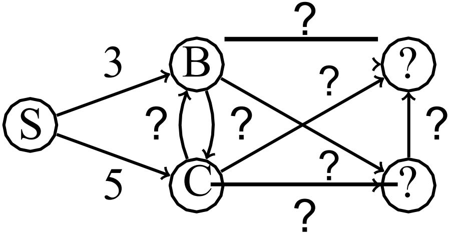
- Can we be sure that the distance from S to C is 5?



- No, because the weight of the edge (B, C) might be equal to, say, 1.

Intuition

- Can we be sure that the distance from S to B is 3?



- Yes, because there are no negative weight edges.

Outline

1. Fastest Route
2. Naive Algorithm
3. Dijkstra's Algorithm

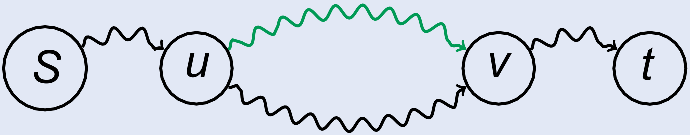
Optimal substructure

Observation

Any subpath of an optimal path is also optimal.

Proof

Consider an optimal path from S to t and two vertices u and v on this path. If there were a shorter path from u to v we would get a shorter path from S to t.



Corollary

If $S \to \ldots \to u \to t$ is a shortest path from S to t, then

$$d(S, t) = d(S, u) + w(u, t)$$

Edge relaxation

- `dist`[v] will be an upper bound on the actual distance from S to v.
- The edge relaxation procedure for an edge (u, v) just checks whether going from S to v through u improves the current value of `dist`[v].

Relax$((u, v) \in E)$

if $dist[v] > dist[u] + w(u, v)$:

$dist[v] \leftarrow dist[u] + w(u, v)$

$prev[v] \leftarrow u$

Naive approach

Naive(G, S)

for all $u \in V$:

  $dist[u] \leftarrow \infty$

  $prev[u] \leftarrow nil$

$dist[S] \leftarrow 0$

do:

  relax all the edges

while at least one *dist* changes

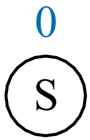
Correct distances

Lemma

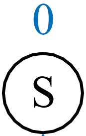
After the call to `Naive` algorithm all the distances are set correctly.

Outline

Intuition

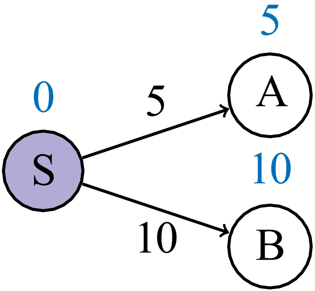


Intuition



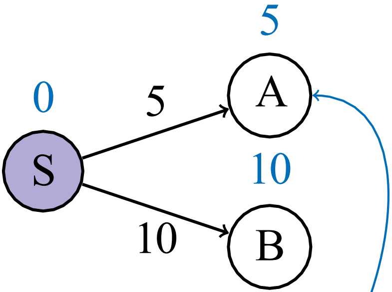
initially, we only know the distance to S

Intuition



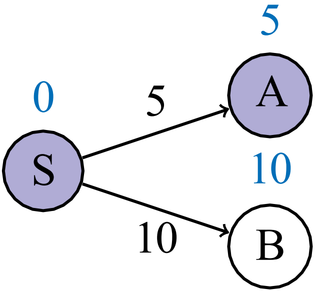
let's relax all the edges from S

Intuition



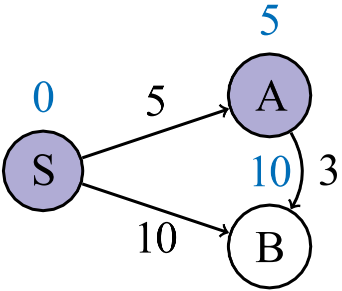
we now know the distance for A

Intuition



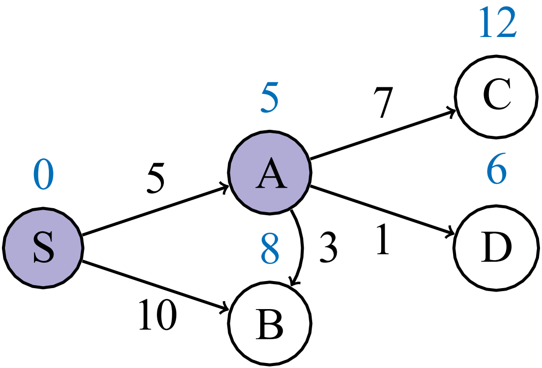
now, let's relax all the edges from A

Intuition



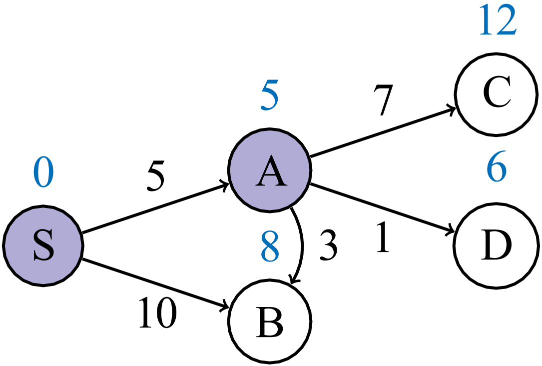
we discover an edge (A, B) of weight 3 that updates dist[B]

Intuition



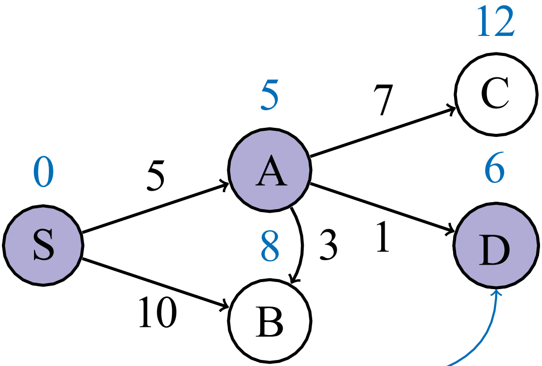
we also discover a few more outgoing edges

Intuition



what is the next vertex for which we already know the correct distance?

Intuition

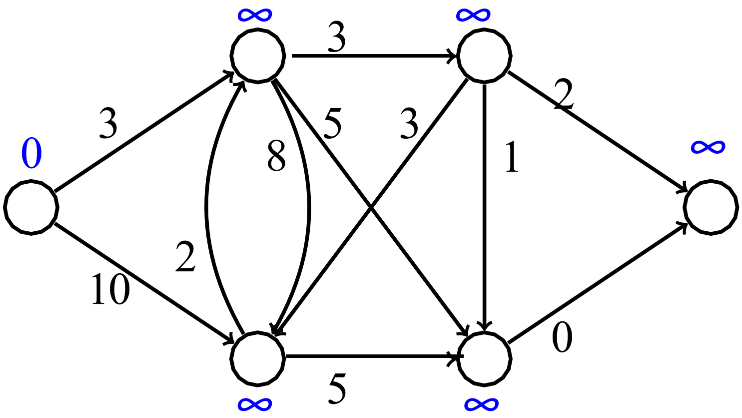


it is *D*

Main ideas of Dijkstra's Algorithm

- We maintain a set R of vertices for which `dist` is already set correctly (known region).
- The first vertex added to R is S.
- On each iteration we take a vertex outside of R with the minimal `dist`-`value`, add it to R, and relax all its outgoing edges.

Example



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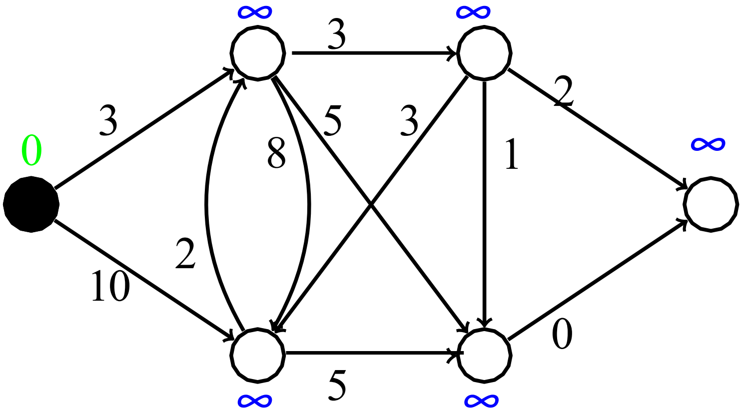

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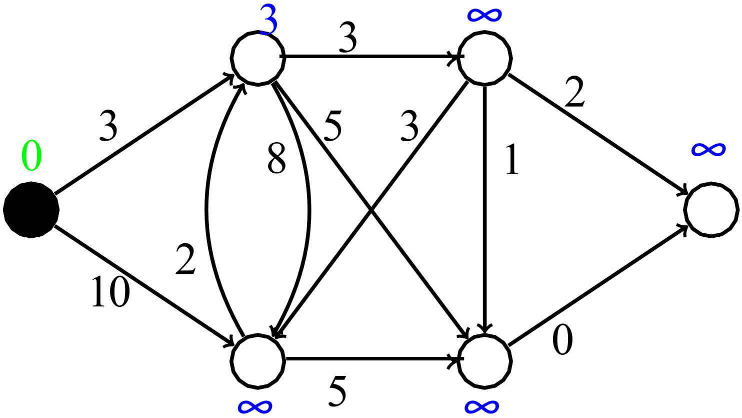

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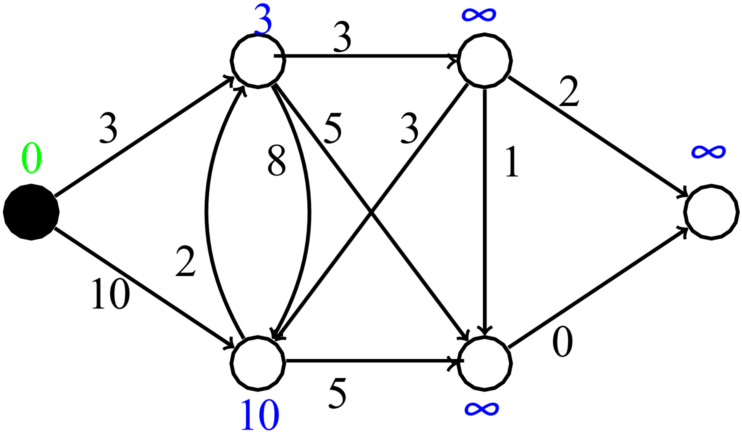

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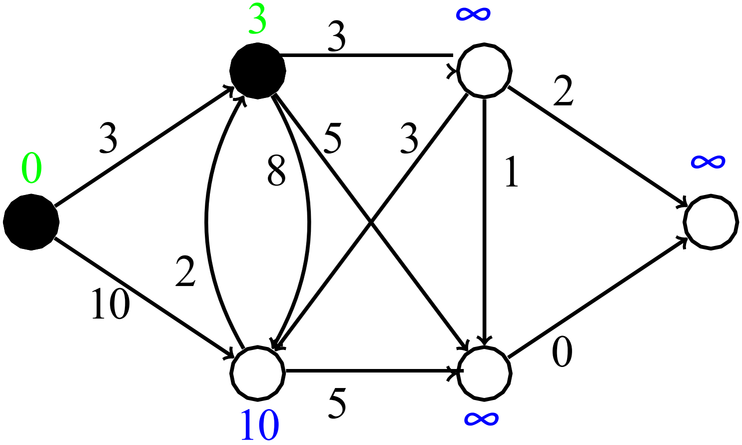

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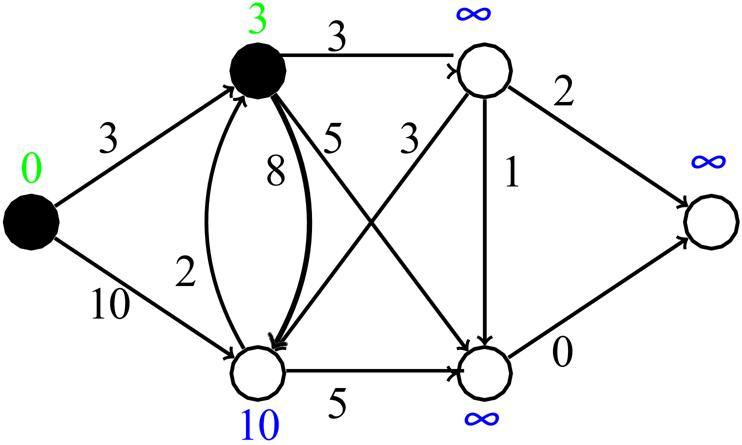

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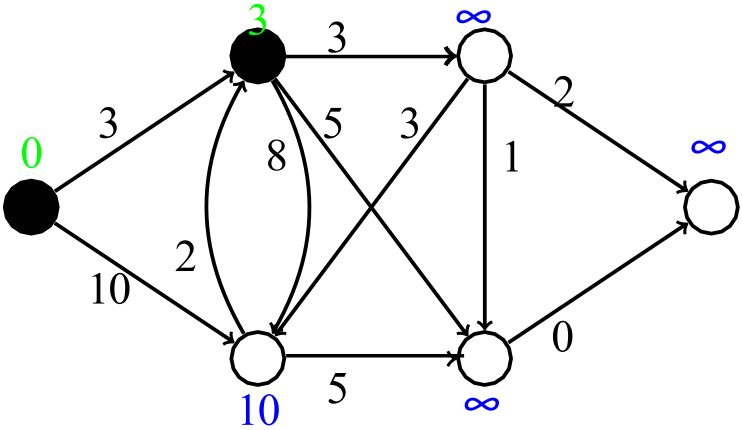

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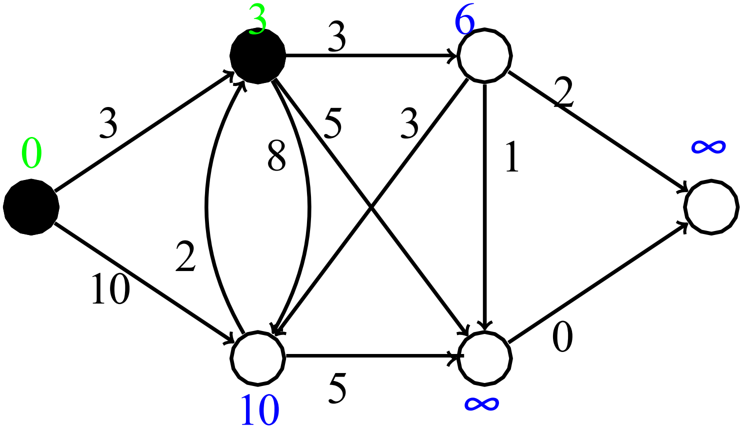

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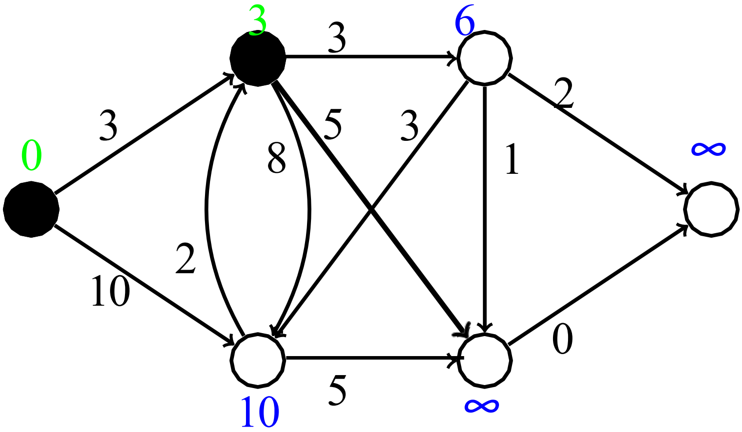

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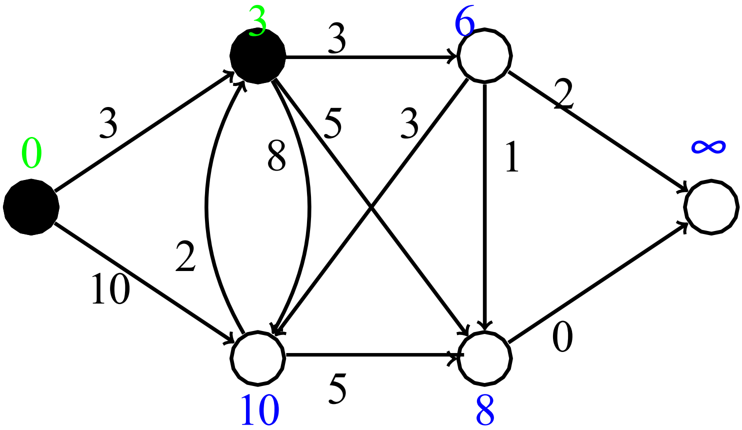

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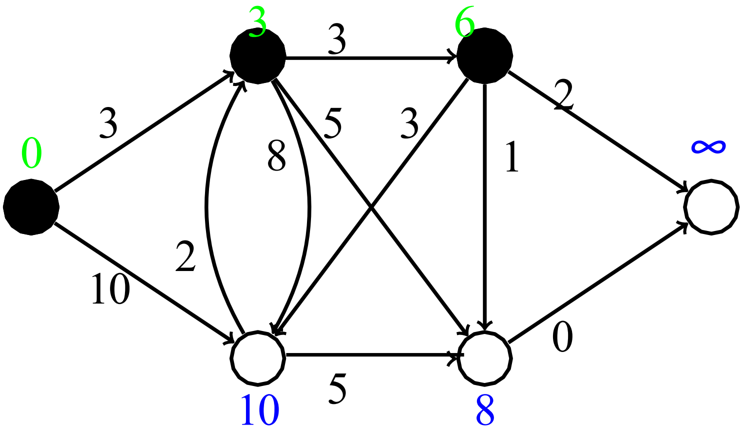

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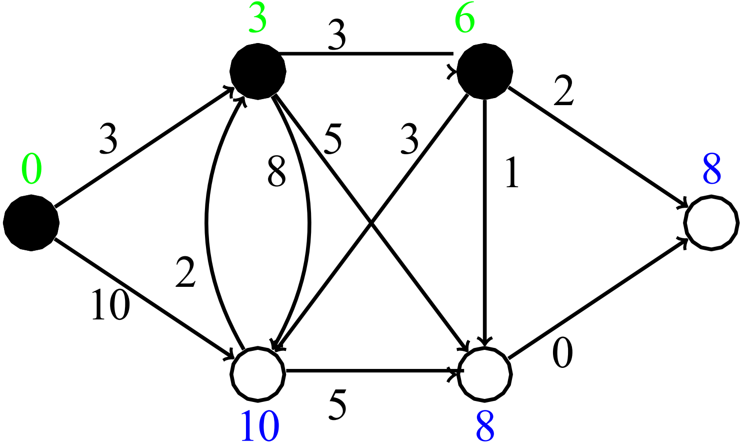

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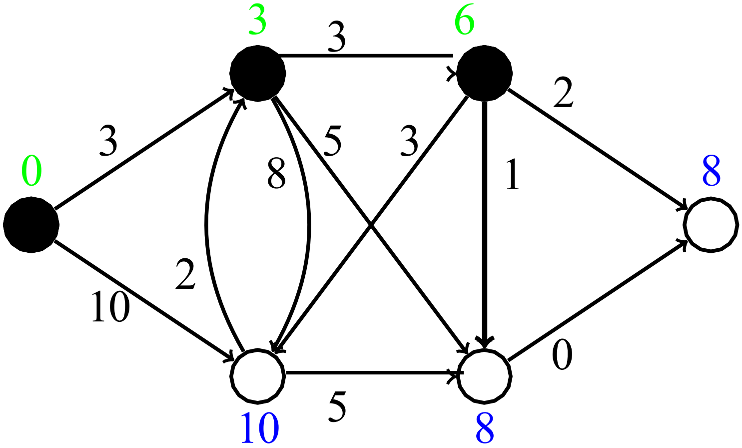

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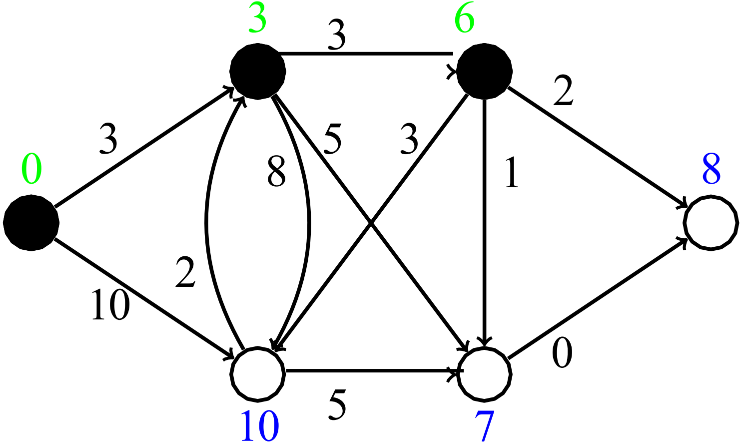

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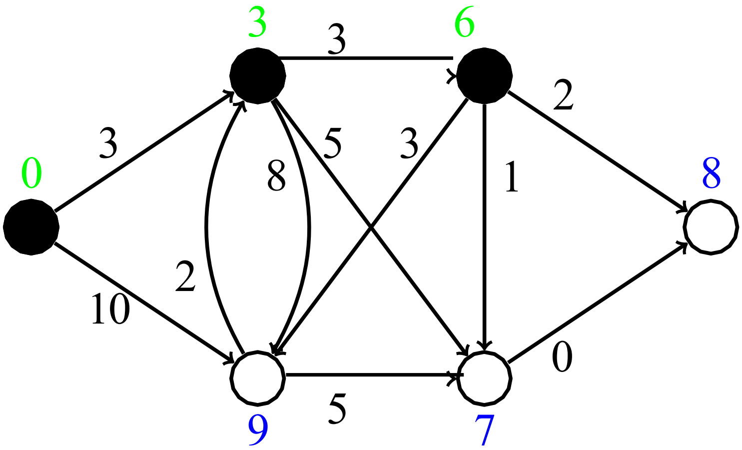

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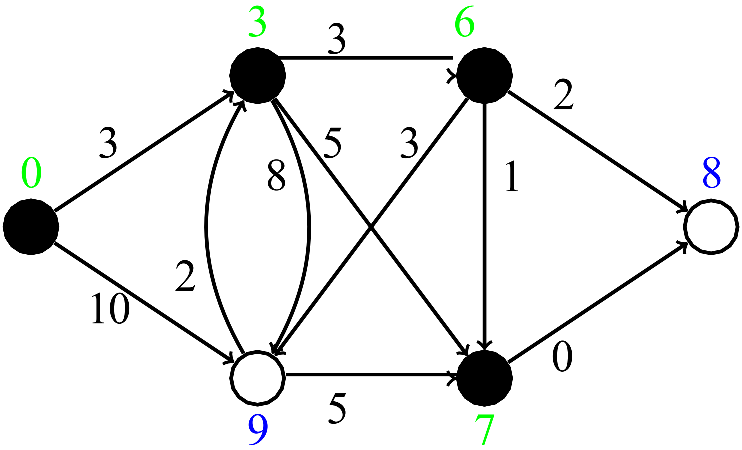

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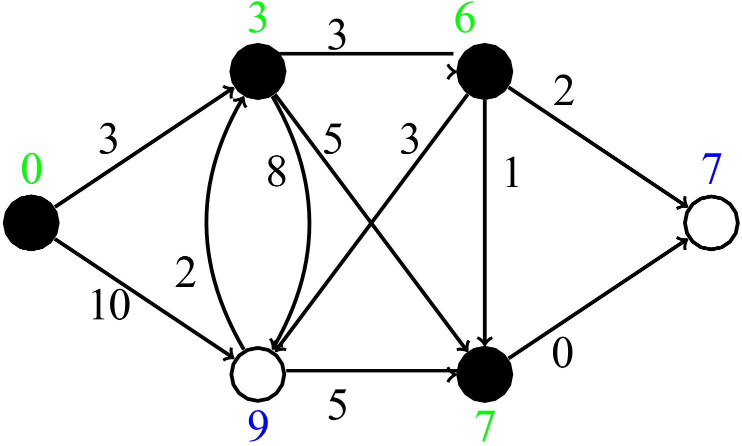

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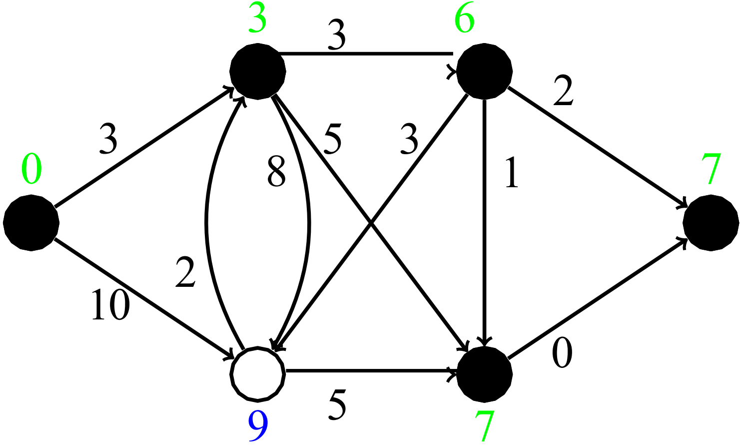

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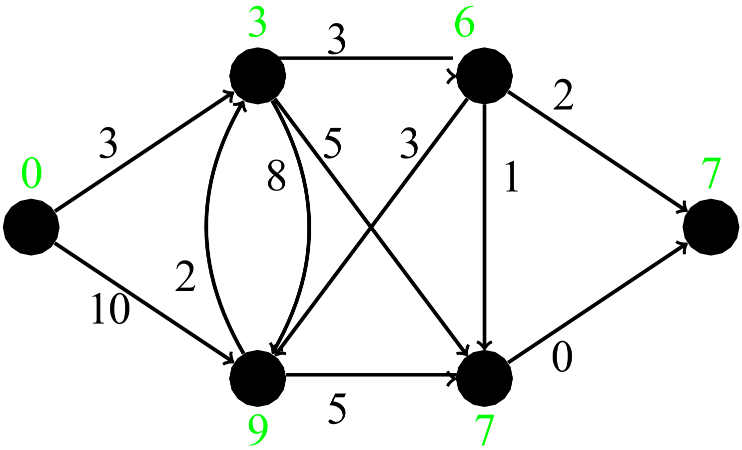

Pseudocode

Dijkstra(G, S)

```
for all u ∈ V:
   dist[u] ← ∞, prev[u] ← nil
dist[S] ← 0
H ← MakeQueue(V) {dist-values as keys}
while H is not empty:
   u ← ExtractMin(H)
   for all (u, v) ∈ E:
      if dist[v] > dist[u] + w(u, v):
         dist[v] ← dist[u] + w(u, v)
         prev[v] ← u
         ChangePriority(H, v, dist[v])
```

Running time

Total running time:

$T(\text{MakeQueue}) + |V| . T(\text{ExtractMin}) + |E| . T(\text{ChangePriority})$

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$T(\text{MakeQueue}) + |V|.T(\text{ExtractMin}) + |E|.T(\text{ChangePriority})$

Priority queue implementations:

- array:

$O(|V| + |V|^2 + |E|) = O(|V|^2)$

Running time

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Priority queue implementations:

- array:

$O(\ |V| + |V|^2 + |E|\ ) = O(|V|^2)$

- binary heap:

  $O(\ |V| + |V| \log |V| + |E| \log |V|\ )$
  $= O((|V| + |E|) \log |V|)$

Conclusion

- Can find the minimum time to get from work to home
- Can find the fastest route from work to home
- Works for any graph with non-negative edge weights
- Works in $O(|V|^2)$ or $O((|V|+|E|) \log |V|)$ depending on the implementation