Aalto University

T61.3040

Parametric methods for estimating the power spectrum and Frequency Estimation

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Diagram of content of the final part of the course

Noisy signal,
Noise Reduction

Wiener filtering:
IIR, FIR

Wiener-Hopf equations
(solving Wiener filter)

Adaptive filtering:
- gradient model
- LMS-algorithm

non stationary signals

Power spectrum estimation:
- Periodogram
- Variations: Bartlett, Welch, Blackman-Tukey
- Minimum variance method
- Maximum entropy method

Frequency estimation by/in sine signal models:
- A direct solution
- Pisarenko method
- MUSIC method

Estimation of variance:
ARCH models

Today

Power spectrum estimation:

- Periodogram
- Variations: Bartlett, Welch, Blackman-Tukey
- Minimum variance method
- Maximum entropy method

Frequency estimation by/in sine signal models:

- A direct solution
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- Periodogram and other Fourier methods: the process is assumed to be WSS and ergodic in the autocorrelation
- Periodogram resolution is of the order $1/N$, when N observations are available
- In this case, for example, two sine signal whose frequency difference is less than $1/N$ are not distinguished in the periodogram
- In addition, the variance of the periodogram is large

- In parametric methods, the estimated numbers of “quantities” does not grow when the number of observations N increases
- If the process $x(n)$ is assumed to have a parametric model then often we can calculated the power spectrum model as a function of parameters
- Especially using sine-signal model provides efficient methods for sine signal spectrum estimation
- Let’s first consider two parametric methods, which are not derived from the process model

Minimum variance method

- We want to calculate the power of process $x(n)$ for frequency ω_i ie $P_x(\exp(j\omega_i))$
- Filtered $x(n)$ by bandpass filter $g_i(n)$, whose passband is $\omega_i \pm \Delta/2$
- Response is $y_i(n) = x(n) * g_i(n)$ the power spectrum is

$$P_{y_i} = P_x |G_i|^2$$

Minimum variance method

- The total power of the filtered process $y_i(n)$ is approximately

$$\mathrm{E}(|y_i(n)|^2) = \frac{1}{2\pi}\int_{\omega_i-\Delta/2}^{\omega_i+\Delta/2} P_x(\exp(j\omega))d\omega$$
$$\approx P_x(\exp(j\omega_i))\frac{\Delta}{2\pi}$$

- This provides a power spectrum value

$$\hat{P}_x(\exp(j\omega_i)) = \frac{\mathrm{E}(|y_i(n)|^2)}{\Delta/2\pi}$$

Minimum variance method

- Assume that $g_i(n)$ is FIR filter with p+1 coefficients
- Minimum variance method:
  - select $g_i(n)$ so that the power $\mathrm{E}(|y_i(n)|^2)$ as small as possible with condition $|G_i(\exp(j\omega_i))| = 1$
  - The MV estimate of Power spectrum is $\mathrm{E}(|y_i(n)|^2)/(\Delta/2\pi)$
  - We calculate the passband bandwidth Δ so that the results are correct for a white noise

Minimum variance method

- Let's define

$$g_i = [g_i(0), g_i(1), \ldots, g_i(p)]^T$$

$$e_i = [1, \exp(j\omega_i), \exp(j2\omega_i), \ldots, \exp(jp\omega_i)]^T$$

- This optimization problem must be solved

$$\min_{g_i} \mathrm{E}\left(|\, y_i(n)\,|^2\right) = \min_{g_i} g_i^H R_x g_i, \quad g_i^H e_i = 1$$

Minimum variance method

- Solution (derived in the book) is

$$g_i = \frac{R_x^{-1} e_i}{e_i^H R_x^{-1} e_i}$$

$$\min_{g_i} \mathrm{E}(|y_i(n)|^2) = \frac{1}{e_i^H R_x^{-1} e_i}$$

- The lower formula is the necessary power for the filtered signal

Minimum variance method

- For white noise the

$$\hat{P}_x = \frac{\mathrm{E}(|y_i(n)|^2)}{\Delta/2\pi} = \frac{2\pi/\Delta}{e_i^H R_x^{-1} e_i} = \frac{\sigma^2 2\pi/\Delta}{e_i^H e_i} = \frac{\sigma^2 2\pi/\Delta}{p+1}$$

- Required: $\hat{P}_x = P_x = \sigma^2$ then $2\pi/\Delta = p+1$
- Using this in the general Minimum Variance solution

$$\hat{P}_{MV}(\exp(j\omega_i)) = \frac{E(|y_i(n)|^2)}{\Delta/2\pi} = \frac{p+1}{e_i^H R_x^{-1} e_i}$$

Maximum entropy method

- In the periodogram we estimate autocorrelation,

$$\hat{r}_x(k),\ |k| < N$$

- The remaining autocorrelations were set to zero
- Unknown variables set equal to zero is not necessarily optimal
- Another way: selecting $\hat{r}_x(k),\ |k| \geq N$ so that the obtained autocorrelation sequence corresponds a random process

Maximum entropy method

- Measuring randomness by means of entropy (not discussed further in this course)
- It can be shown that for a normally distributed WSS process $x(n)$, the entropy is maximized by the estimates $\hat{r}_x(k),\ |k| > p$ obtained from the AR(p)-process
- YW equations can be solved when values are known

Maximum entropy method

$$r_x(0), \ldots, r_x(p):$$

$$\begin{bmatrix} r_x(0) & r_x^*(1) & \ldots & r_x^*(p) \\ r_x(1) & r_x(0) & \ldots & r_x^*(p-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(p) & r_x(p-1) & \ldots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} |b(0)|^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Autocorrelation $r_x(p+1)$, $r_x(p+2)$, ... will be obtained by solving the parameters $a_p(k)$ from YW equations

Maximum entropy method

- The power spectrum estimate of the process $x(n)$ is

$$\hat{P}_{mem}(\exp(j\omega)) = \frac{|b(0)|^2}{\left|1+\sum_{k=1}^{p} a_p(k)\exp(-jk\omega)\right|^2}$$

  - This is also the power spectrum of AR(p)-process
  - justification is more general than the assumption AR-model
  - the power spectrum of a process that is as random as possible given that part of the autocorrelation is known

Maximum entropy method

- The above analysis was based on the assumption that the estimated autocorrelation values are correct
- As previously, the autocorrelation is estimated from the observations, and this can be done for example by autocorrelation or covariance methods
- Covariance methods generally provide better resolution than the autocorrelation method
- Autocorrelation method has the advantage that R_x has a Toeplitz matrix structure, which can be used when solving the parameters of the AR process

Maximum entropy method

- Example: estimate the power spectrum of a AR (4)-process with different methods. the system generating the process has poles at the points

$$z_{1,2} = 0.98\exp(\pm j(0.2\pi))$$
$$z_{3,4} = 0.98\exp(\pm j(0.3\pi))$$

- $N = 128$ samples.
- Demo: mem.R

Frequency Estimation

- Consider a complex sine signal model

$$x(n) = \sum_{i=1}^{p} A_i \exp(jn\omega_i) + v(n), \quad A_i = |A_i| \exp(j\phi_i)$$

- ω_i frequencies and amplitudes $|A_i|$ are unknown constants
- Phases ϕ_i are uncorrelated and uniformly distributed in the interval $[-\pi, \pi)$

Frequency Estimation

- The process $x(n)$ has the power spectrum

$$P_x(\exp(j\omega)) = \sum_{i=1}^{p} 2\pi \,|\, A_i \,|^2 \, \delta(\omega - \omega_i) + \sigma^2$$

- In the power spectrum, there is some impulse δ at ω_i which is the frequency of the sine signal
- Elsewhere, the value of power spectrum is the noise variance σ^2

Frequency Estimation

- Frequencies ω_i could be estimated by peaks in the periodogram
- This is not necessarily a good solution because of the poor resolution in the periodogram
- Process model provides significantly better methods in the case of sine signal

Frequency Estimation

- One sine signal with noise: $$x(n) = A_1 \exp(jn\omega_1) + v(n)$$ $$= s(n) + v(n), \;\; \mathrm{var}(v(n)) = \sigma_v^2$$

- Consider M successive values of the processes
- at time n, the observations of the sine sequence $s(n)$ form a vector

$$\underline{s}(n) = A_1[e^{jn\omega_1}, e^{j(n-1)\omega_1}, \ldots, e^{j(n-M+1)\omega_1}]^T$$

Frequency Estimation

- At the time $n + 1$ we obtain

$$\underline{s}(n+1) = A_1[e^{j(n+1)\omega_1}, e^{jn\omega_1}, \ldots, e^{j(n-M+2)\omega_1}]^T$$
$$= \exp(j\omega_1)\underline{s}(n)$$

- Vectors $s(n)$ and $s(n+1)$ are in the same direction
- The direction of the vector depends on the frequency ω_i

Frequency Estimation

- Needed in the future signal vector,

$$e_i = \left[1, e^{j\omega_i}, e^{j2\omega_i}, \ldots, e^{j(M-1)\omega_i}\right]^T$$

- Which can be thought as a sine signal with frequency ω_i and complex amplitude $A = 1$
- In the case of one sine, this vector defines the subspace in which there is the noise-free part of the the process
- (the time index is the other way around as in the example above)

Frequency Estimation

- The process $x(n)$ has autocorrelation $r_x(k)$

$$r_x(k) = P_1 \exp(jk\omega_1) + \sigma_v^2 \delta(k)$$
$$= r_s(k) + r_n(k), \quad P_1 = |A_1|^2$$

- The autocorrelation matrix R_x can be presented as the

$$R_x = R_s + R_n$$

- In other words, the noisy sine signal autocorrelation can be separated into the signal and noise autocorrelations

Frequency Estimation

- Using the signal vector, we can write

$$R_s = |A_1|^2 e_1 e_1^H$$

- This is easy to verify by calculating the element

$$\begin{aligned} R_s[k,l] &= |A_1|^2 [e_1]_k [e_1^*]_l \\ &= |A_1|^2 \exp(j(k-l)\omega_1) \\ &= r_s(k-l) \end{aligned}$$

Frequency Estimation

- Properties of the matrix R_s (one sine):

1. matrix R_s has one nonzero eigenvalue $\lambda_1^s = MP_1$
2. the corresponding eigenvector v_1 is e_1
3. eigenvectors v_1, ... v_M are orthogonal with e_1
4. corresponding eigenvalues $\lambda_2^s, \ldots, \lambda_M^s$ are zero

Frequency Estimation

- R_x has the same eigenvectors than R_s:

$$R_x v_i = (R_s + \sigma_v^2 I) v_i = (\lambda_i^s + \sigma_v^2) v_i$$

- At the same time, we obtained the eigenvalues

$$\lambda_i = \lambda_i^s + \sigma_v^2$$

- So is the largest eigenvalue is $\lambda_{max} = MP_1 + \sigma_v^2$
- And all other eigenvalues are σ_v^2

Frequency Estimation

- Eigenvalues and vectors contain the necessary information to estimate the power P_1, the variance σ_v^2 and frequency ω_1:

1. Calculate the eigenvalues and eigenvectors of the matrix R_x. The largest eigenvalue is $MP_1 + \sigma_v^2$ and the rest are σ_v^2
2. Noise variance is $\sigma_v^2 = \lambda_{min}$ and the sine signal power is

$$P_1 = \frac{1}{M}(\lambda_{max} - \lambda_{min})$$

3. Sine signal frequency can be solved from the eigenvector v_{max} corresponding to the largest eigenvalue, which is

$$v_{max} \propto [1, \exp(j\omega_1), \ldots, \exp(j(M-1)\omega_1)]^T$$

Frequency Estimation

- Two sine signals:

$$x(n) = A_1 \exp(jn\omega_1) + A_2 \exp(jn\omega_2) + v(n)$$

- The autocorrelation function is

$$r_x(k) = P_1 \exp(jk\omega_1) + P_2 \exp(jk\omega_2) + \sigma_v^2 \delta(k)$$

- Autocorrelation matrix can be written as the sum

$$R_x = P_1 e_1 e_1^H + P_2 e_2 e_2^H + \sigma_v^2 I$$

Frequency Estimation

- By assembling the vectors e_i and powers P_i as matrices

$$E = [e_1, e_2], \quad P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

- We can write

$$\begin{aligned} R_x &= EPE^H + \sigma_v^2 I \\ &= R_s + R_n \end{aligned}$$

- Where

$$R_s = EPE^H$$

Frequency Estimation

- Matrix R_x has M a eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_M \geq 0$$

- Since the matrix R_s has only two nonzero eigenvalues, λ_1 and λ_2 are greater than σ_v^2
- Other eigenvalues are $\lambda_3 = \cdots = \lambda_M = \sigma_v^2$

Frequency Estimation

- The two largest eigenvalues correspond to signal eigenvectors v_1 and v_2 which span a signal subspace
- Noise eigenvectors v_3 ... v_M span a M-2 dimensional noise subspace
- Spaces are orthogonal because the R_x is a Hermitian matrix, i.e. its eigenvectors form a orthonormal set

Frequency Estimation

- Is v_1 parallel to e_1, such as the case of one sine?
- Generally not: However, e_1 and e_2 are in the signal subspace spanned by v_1 and v_2
- It follows that e_1 and e_2 are orthogonal to the eigenvectors v_3 ... v_M

Frequency Estimation

- Let's check that the e_1 and e_2 are in the signal subspace
- Because the signal subspace is spanned by the orthogonal vectors v_1 and v_2, then subspace dimension is 2
- Because $R_s v_i = \lambda_i v_i$ so

$$\begin{aligned} v_i &= \frac{1}{\lambda_i} P_1 e_1 (e_1^H v_i) + \frac{1}{\lambda_i} P_2 e_2 (e_2^H v_i) \\ &= c_1 e_1 + c_2 e_2,\ i = 1,2 \end{aligned}$$

- So both of the vectors v_1, v_2 can be represented as a linear combination of vectors e_1, e_2

Frequency Estimation

- Then the vectors v_1, v_2 are in the subspace spanned by the signal vectors
- But this subspace dimension is also 2, so it must be the same subspace which is spanned by vectors v_1, v_2
- It also follows that e_1, e_2 are orthogonal to the noise eigenvectors

Frequency Estimation

- The general case, with *p* sine signal:

$$x(n) = \sum_{i=1}^{p} A_i \exp(jn\omega_i) + v(n)$$

$$r_x(k) = \sum_{i=1}^{p} P_i \exp(jk\omega_i) + \sigma_v^2 \delta(k)$$

$$R_x = R_s + R_n = \sum_{i=1}^{p} P_i e_i e_i^H + \sigma_v^2 I$$

$$R_x = EPE^H + \sigma_v^2 I$$

Frequency Estimation

- Changing the number of sines from 2 to p there is no surprise:
- R_x eigenvectors $v_1, \ldots, v_p$ span the signal subspace which includes all signal vectorst $e_1, \ldots, e_p$

  - Eigenvalues $\lambda_1, \ldots, \lambda_p > \sigma_v^2$

  - The rest of the eigenvalues $\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_M = \sigma_v^2$

  - signal vectors $e_1, \ldots, e_p$ are orthogonal to the noise subspace spanned by $v_{p+1}, v_{p+2} \ldots v_M$

Pisarenko method

- we assume that the number p of sine signals is known (can also be estimated from the observations)
- We estimate R_x which has size $p + 1 \times p + 1$
- Noise-subspace is then spanned by one vector v_{p+1}
- All signal vectors are orthogonal to the vector v_{p+1}

Pisarenko method

- This provide the Pisarenko method. Calculate the pseudo-spectrum

$$\hat{P}_{PHD}(\exp(j\omega)) = \frac{1}{|e^H v_{p+1}|^2}$$

- Where $e = [1, \exp(j\omega), \ldots, \exp(jp\omega)]^T$

- The term $|e^H v_{p+1}|^2$ measures the projection of the signal vector e to the noise subspace

Pisarenko method

- When $w = w_i$ for some i, the projection is zero
- Otherwise the projection has a positive absolute value
- In the pseudo-spectrum, there appears peaks at the right frequencies, because we calculate the inverse (1/value) of the projection
- So the pseudo-spectrum indicates the frequencies of the sine signals, but it is not an estimate of the actual power spectrum

MUSIC method

- Pisarenko, one noise eigenvector
- MUSIC (multiple sinusoid classification): Several noise eigenvectors
- Correlation matrix has size $M \times M$, where $M > p$
- where there are M-p noise eigenvector $v_{p+1}, \ldots, v_M$
- which are all orthogonal to the signal vectors $e_1, \ldots, e_p$

MUSIC method

- This provides the MUSIC estimator

$$\hat{P}_{MU}(\exp(j\omega)) = \frac{1}{\sum_{i=p+1}^{M} |e^H v_i|^2}$$

- Now, in the denominator is measured the projection of the signal vector onto a *M-p* dimensional noise subspace
- Again, with the "real" frequencies, the projection is theoretically zero

Amplitude and phase estimation

- Pisarenko and MUSIC provide a pseudo-spectrum, from which can be estimated we can estimate sine signal frequency
- Signal model $x(n) = \sum_{i=1}^{p} A_i \exp(jn\omega_i) + v(n)$
  where the frequencies are assumed to be known

Amplitude and phase estimation

- We obtain the system of equations

$$\begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \end{bmatrix} = \begin{bmatrix} e^{jn\omega_1} & \ldots & e^{jn\omega_p} \\ e^{j(n-1)\omega_1} & \ldots & e^{j(n-1)\omega_p} \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix} + \begin{bmatrix} v(n) \\ v(n-1) \\ \vdots \end{bmatrix}$$

$$x = EA + v$$

- Minimize $\|x - EA\|^2 = \|v\|^2$

  which corresponds to the overdetermined solution to a system of linear equations

Amplitude and phase estimation

- Because the frequencies have been solved then the matrix *E* is known
- The result is a vector, whose components are $A_i = |A_i| \exp(j\phi_i)$
- i.e., the amplitude is the modulus of A_i and the phase is A_i's phase
- Demo: music.R