T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 9, November 17, 2011

1. The signal $x(n)$ has the autocorrelation function $r_x(l) = 2^{-|l|} + \delta(l)$, where $\delta(l) = 1$ when $l = 0$ and $\delta(l) = 0$ otherwise.

   (a) Compute the power spectrum for the signal $x(n)$.

   (b) *Correlogram* is an estimate for the power spectrum. In the correlogram the values of the correlation function up to delay L are used:

   $$\hat{P}_x(e^{j\omega}) = \sum_{l=-L}^{L} r_x(l) e^{-j\omega l}$$

   Thus it can be thought of as a generalization of the periodogram (in the periodogram, $L = N-1$). Compute the correlogram of $x(n)$ with $L = 2$ using correlation function values $r_x(0), r_x(\pm 1), r_x(\pm 2)$.

2. The periodogram can be written as

   $$\hat{P}_x(e^{j\omega}) = \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2$$

   where

   $$X_N(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}.$$

   That is, the periodogram can be calculated using the discrete-time Fourier transform of the observed data.

   (a) Write X_N in a form which shows that its value for a frequency ω_0 is obtained by filtering the observed signal $x(n)$ with some LSI-filter $h_{\omega_0}(n)$ and taking the value $y_{\omega_0}(0)$ from the filtered signal. In other words, determine the impulse response $h_{\omega_0}(n)$ so that for the frequency ω_0,

   $$y_{\omega_0}(0) = [h_{\omega_0}(n) * x(n)]|_{n=0} = X_N(e^{j\omega_0}).$$

   (b) Write the frequency response $H_{\omega_0}(e^{j\omega})$ of the filter $h_{\omega_0}(n)$ in a form that makes it easy to consider the amplitude response $|H_{\omega_0}(e^{j\omega})|$ for different frequencies ω.

   Hint: Try to write $H_{\omega_0}(e^{j\omega})$ so that it contains terms of the form $e^{j\omega_k} - e^{-j\omega_k} = 2j \sin \omega_k$ where ω_k is some frequency.

3. (Beamforming: estimation of the direction of arrival.) Assume that we have M sensors measuring a signal. The sensors are aligned in a row with a distance d from each other. A complex sinusoid with a known frequency ω arrives from a very distant source. We may assume that the source is so far away that the sinusoid forms a planar wave, and the signal arrives in all sensors from the same direction θ. The signal has a constant, finite, speed c.

   (a) Compute the signal arriving at each sensor, as a function of the direction of arrival θ.

   (b) Form a discrete signal by having each sensor measure the signal simultaneously at a fixed time t. The index is the number of the sensor. What is the spatial frequency (the frequency corresponding to the spatial index variable) of this signal? How would you solve θ?

4. (Demonstration.) Assume that we observe a sinusoid in noise:

$$x(n) = A\exp(j\omega n) + v(n), \quad n = 0, 1, \ldots, N-1$$

   where A is a complex constant and $v(n)$ is white Gaussian noise. Estimate the frequency ω, the phase ϕ and the amplitude $|A|$ using the maximum likelihood method.

5. (Bonus point exercise)

   Let $x(n),\ n = 0, 1, \ldots, N-1$ be real-valued zero mean normally distributed white noise with unit variance (i.e., $x(n) \sim N(0, 1)$ for all n).

   (a) Write the periodogram at the frequency $\omega = 0$ as a function of the observations.

   (b) Calculate the mean and variance of the periodogram of part (a), that is $\mathrm{E}[\hat{P}_x(e^{0 \cdot j})]$ and $\mathrm{Var}(\hat{P}_x(e^{0 \cdot j}))$. NB: Many results in the lecture slides concern complex white noise, while real-valued white noise behaves slightly differently.

   Hints:

   - $\mathrm{Var}(z) = \mathrm{E}[z^2] - \mathrm{E}[z]^2$.
   - If $z \sim N(0, 1)$ then $\mathrm{E}[z^4] = 3$.

   (c) Write the estimator given by the Bartlett method at the frequency $\omega = 0$ when the observations are partioned into N subsequences with length $L = 1$. Calculate the mean and the variance of the estimator at the frequency $\omega = 0$.