T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola
Exercise 3, September 29, 2011

1. Assume that observations $v(n),\ n = 0, 1, 2, \ldots$ are normally distributed with expectation zero and variance σ^2. Moreover, the observations on different time instants are independent. A *sum process* is defined as the sum $M(n) = \sum_{i=0}^{n} v(i)$.

   (a) Check that $\mathrm{E}[M(n+1)|M(n), M(n-1), \ldots] = M(n)$.

   (b) Determine $\mathrm{Var}[M(n+1)|M(n), M(n-1), \ldots]$.

   (c) Find $\mathrm{E}[M(n+k)|M(n), M(n-1), \ldots]$, where k is a positive integer.

   (d) Find $\mathrm{Var}[M(n+k)|M(n), M(n-1), \ldots]$, where k is a positive integer.

2. Let $v(n)$ be a noise process, i.e., for each $v(n)$ the expected value is zero and the variance is σ^2. In addition, values of the noise on different time instants are not correlated. We try to predict $v(n)$ in a linear manner using earlier values. We get an estimator $\hat{v}(n) = \sum_{k=1}^{L} a_k v(n-k)$.
   Using the principle of orthogonality, calculate the coefficients a_k that minimize the MSE.

3. (a) Given a MA(q) model, calculate the prediction $\mathrm{E}[x(n+k)|F_n]$ given by the model, where F_n represents all the information at time n, i.e. the observations $x(n), x(n-1), \ldots$ and noise $v(n), v(n-1), \ldots$

   (b) Calculate also the conditional variance $\mathrm{Var}[x(n+k)|F_n]$.

   (c) For an AR(p) process, calculate $\mathrm{E}[x(n+1)|F_n]$ and $\mathrm{Var}[x(n+1)|F_n]$.

   The noise is normally distributed with mean zero and variance 1.

4. You have observed the values $x(0), x(1), x(2)$ from the real-valued sinusoidal signal

$$x(n) = A\sin(n\omega + \phi) + v(n),$$

   where $v(n)$ is the noise term. Assume that you know the frequency ω. Find the amplitude A and the phase ϕ that minimize the squared error

$$\sum_{n=0}^{2} [x(n) - A\sin(n\omega + \phi)]^2 \ .$$

   Hints:

   - Write the sinusoid as the sum of two complex exponentials.
   - Use the principle of orthogonality.
   - The inner product for complex valued vectors is $\mathbf{x}^H\mathbf{y}$ where $\mathbf{x}^H = (\mathbf{x}^T)^*$ ($\mathbf{x}$ is transposed, and then the complex conjugate is taken for each element).

5. (Bonus point exercise)
Convert the ARMA process

$$x(n) - \frac{1}{2}x(n-1) = v(n) - \frac{1}{3}v(n-1) + \frac{1}{4}v(n-2)$$

into a MA process. Hint: use polynomials of z^{-1} and the fact that $X(z) = M(z)V(z)$ where $M(z) = m_0 + m_1 z^{-1} + m_2 z^{-2} + \dots$ is the MA representation of the process and find the coefficients m_i.