

Exercise 5

We have a WSS process with the autocorrelation sequence

$$r_x(k) = a^{|k|}$$

$$|a| < 1, a \in \mathbb{R}$$

A)

The 2×2 autocorrelation matrix is

$$R_x = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

and the eigenvalues and vectors are given by the following equations:

$$R_x \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+a \\ 1+a \end{bmatrix} = (1+a) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$R_x \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-a \\ a-1 \end{bmatrix} = (1-a) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

As the eigenvectors are usually constrained to have unit length, we get:

$$\lambda_1 = 1 + a$$

$$\lambda_2 = 1 - a$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

B)

A property of the eigenvalues of the autocorrelation called the *eigenvalue extremal property* states that the eigenvalues are bounded by the minimum and maximum of the power-spectrum. It can apparently also be shown that the largest eigenvalue approaches the maximum of the power spectrum as the size of the autocorrelation matrix approaches infinity. In other words, the asymptotic value of λ_{max} as $p \rightarrow \infty$ is the maximum of the power spectrum.

For the power spectrum we get:

$$\begin{aligned}
 P(e^{i\omega}) &= \sum_{k=-\infty}^{\infty} a^{|k|} e^{-ik\omega} \\
 &= \sum_{k=0}^{\infty} (ae^{-i\omega})^k + \sum_{k=0}^{\infty} (ae^{i\omega})^k - 1 \\
 &= \frac{1}{1 - ae^{-i\omega}} + \frac{1}{1 - ae^{i\omega}} - 1 \\
 &= \frac{1 - a^2}{1 - 2a \cos(\omega) + a^2}
 \end{aligned}$$

so that the maximum is when $\cos(\omega) = 1$ & $a \geq 0$ or $\cos(\omega) = -1$ & $a < 0$.
Now the maximum can be written as

$$\max P(e^{i\omega}) = \lim_{p \rightarrow \infty} \lambda_{max} \equiv \lambda_{max} = \frac{1 + |a|}{1 - |a|}$$

c)

In *Hayes* it is proved that the LMS algorithm converges in the mean whenever the step size μ satisfies the condition

$$0 < \mu < \frac{2}{\lambda_{max}} = \frac{2(1 - |a|)}{1 + |a|} = \mu_{max}$$

We can now see that as $|a| \rightarrow 1$, $\mu_{max} \rightarrow 0$ and the step size must be made smaller and smaller for the algorithm to converge. The closer $|a|$ is to unity the more the original signal resembles a constant signal, the the more the power spectrum looks like a delta function and the more ill-conditioned the matrix R_x becomes.