Exercise 5

A)

Taking the expectation of the process definition gives (and knowing that $v(n)$ is white with zero mean and unit variance):

$$\mathrm{E}\left[x(n)\right] = \mathrm{E}\left[x(n) - 1\right] = \cdots = \mathrm{E}\left[x(-1)\right] = 0 = m_x.$$

If the process is ergodic in the mean, the time average should converge to the true mean m_x. Let's first note the following:

$$x(-1) = 0$$
$$x(0) = v(0)$$
$$x(1) = v(0) + v(1)$$
$$\vdots$$
$$x(n) = \sum_{k=0}^{n} v(k)$$

Now we can write the time average over N instants as

$$\hat{m}(N) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{n} v(k)$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}(N - n)v(n)$$

The type of convergence required for ergodicity is mean square convergence:

$$\mathrm{E}\left[\left|\hat{m}(n) - m_x\right|^2\right] = \frac{1}{N^2}\mathrm{E}\left[\left|\sum_{n=0}^{N-1}(N - n)v(n)\right|^2\right]$$
$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(N - n)(N - m)\mathrm{E}\left[\left|v(n)v(m)\right|\right]$$
$$= \frac{1}{N^2}\sum_{n=0}^{N-1}(N - n)^2$$
$$= \frac{\frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{1}{6}N}{N^2} \to \infty, \text{ when } N \to \infty$$

Thus $x(n)$ is not ergodic in the mean.

B)

We can rewrite the process as

$$y(n) = 0.8y(n - 1) + v(n) = 0.8^n y(0) + \sum_{k=0}^{n-1} 0.8^k v(n - k)$$

In order for $y(n)$ to be ergodic in the mean the expected value

$$\mathrm{E}\left[y(n)\right] = 0.8^n \mathrm{E}\left[y(0)\right]$$

must be constant. Clearly this happens only if $\mathrm{E}\left[y(0)\right] = 0$, which we now assume. To see if the process is WSS, the autocovariance $c_y(n, n + l)$ (which is equal to the autocorrelation in this case) needs to be a function of only the lag l:

$$c_y(n, n + l) = \mathrm{E}\left[y(n)y^*(n + l)\right] = \mathrm{E}\left[y(n)\left(0.8^l y^*(n) + \sum_{k=0}^{l-1} 0.8^k v^*(n + l - k)\right)\right]$$

Now if we assume that the noise is uncorrelated with the signal (a fair assumption), we get

$$c_y(n, n + l) = 0.8^l \mathrm{E}\left[y(n)y^*(n)\right] = 0.8^l \mathrm{E}\left[\left|y(n)\right|^2\right]$$

which depends only on the lag if the process has constant variance $\mathrm{E}\left[\left|y(n)\right|^2\right] = \sigma_y^2$. I couldn't find any other way to prove that this is the case than noting that the process we're looking at is a MA(1) process which is WSS (assuming it's stable) and thus has to have an autocorrelation function that depends only on the lag.

Now according to one of the ergodicity theorems, a sufficient condition for a WSS process to be ergodic in the mean is that the autocovariance sequence converges to zero as the lag approaches infinity. With the above WSS assumption (and assuming that the process is stable, i.e $\sigma_y^2 < \infty$), we get:

$$\lim_{l\to\infty} c_y(l) = \lim_{l\to\infty} 0.8^l \sigma_y^2 = 0$$

Thus $y(n)$ is ergodic in the mean.