T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 4, October 6, 2011

1. Let $x(n)$ be a stationary random process with zero mean and autocorrelation $r_x(k)$. Form the process $y(n)$ as follows:

$$y(n) = x(n) + f(n)$$

where $f(n)$ is a known deterministic sequence. Find the mean $m_y(n)$ and autocorrelation $r_y(k, l)$ of the process $y(n)$. Is the process $y(n)$ wide-sense stationary?

2. Compute the autocorrelation sequence $r_x(0), r_x(1), \ldots$ for the following processes. Assume $\mathrm{E}[v(n)] = 0$ and $\mathrm{Var}[v(n)] = 1$.

   (a) The MA(2) process $x(n) = 1.0v(n) + 0.5v(n-1) + 0.25v(n-2)$

   (b) The AR(1) process $x(n) = -0.9x(n-1) + 2.0v(n)$

3. Consider the discrete process

$$x(n) = A\cos(\omega_0 n + \theta) + v(n),$$

which consists of a sinusoid and additive white noise. The amplitude A and the frequency ω_0 are constants and the phase θ is uniformly distributed on the interval $[0, 2\pi]$. The white noise $v(n)$ has zero mean and variance σ^2. Form a sample vector $\mathbf{x}(n) = [x(n), x(n-1)]^T$ using two successive samples.

   (a) Compute the theoretical autocorrelation matrix $\mathbf{R}_x = \mathrm{E}[\mathbf{x}(n)\mathbf{x}^T(n)]$.

   (b) Compute the eigenvalues of $\mathbf{R}_x$.

   (c) (Demo) Draw a graph, which shows the dependence of the eigenvalues on the signal-to-noise ratio $A^2/2\sigma^2$.

4. (Demo) What conditions must the elements of the matrix

$$\mathbf{R} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfy so that it could be the autocorrelation matrix of a discrete-time stochastic process? Assume that $\mathrm{E}(x(n)) = 0$, the process has a finite variance and that $r_x(k, l)$ only depends on $k - l$. Is the matrix positive semidefinite?

5. (Bonus point exercise)

   Let $v(n)$ be zero mean Gaussian white noise with unit variance.

   (a) Consider the random process $x(n) = x(n-1) + v(n)$, $n \geq 0$, with initial value $x(-1) = 0$. Show that this process has a constant mean. Then show that the time average

$$\hat{m}(N) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$$

does not converge to this expected value in the sense required by the definition of ergodicity in mean.

Hint 1: Write the estimator $\hat{m}(N)$ in terms of of the noise $v(n)$.
Hint 2: $\sum_{k=1}^{N} k^2 = \frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{1}{6}N$.

(b) Now consider the random process

$$y(n) = 0.8y(n-1) + v(n)$$

Show that this process is ergodic in the mean, given that $\mathrm{E}[y(0)] = 0$, using the ergodicity theorem(s).