Exercise 5

A)

The periodogram of signal $x(n)$ with N observations can be written as

$$\hat{P}_x(e^{i\omega}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-in\omega}\right|^2$$

and for $\omega = 0$ and $x(n) \in \mathbb{R}$ this gives

$$\hat{P}_x(1) = \frac{1}{N}\left(\sum_{n=0}^{N-1} x(n)\right)^2$$

B)

$$\mathrm{E}\left[\hat{P}_x(1)\right] = \frac{1}{N}\mathrm{E}\left[\left(\sum_{n=0}^{N-1} x(n)\right)^2\right]$$

and because $\mathrm{E}\left[x(n)^2\right] = 1$ and $x(n)$ is white (the cross correlations are zero), we get

$$\mathrm{E}\left[\hat{P}_x(1)\right] = \frac{1}{N}\sum_{n=0}^{N-1}\mathrm{E}\left[x(n)^2\right] = 1$$

For the variance we get

$$\begin{aligned}\mathrm{Var}\left[\hat{P}_x(1)\right] &= \mathrm{E}\left[x(n)^2\right] - \mathrm{E}\left[x(n)\right]^2 \\ &= \frac{1}{N^2}\mathrm{E}\left[\left(\sum_{n=0}^{N-1} x(n)\right)^4\right] - 1\end{aligned}$$

Using the moment factoring theorem we can write

$$\begin{aligned}\mathrm{E}\left[\left(\sum_{n=0}^{N-1} x(n)\right)^4\right] &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\mathrm{E}\left[x(k)x(l)\right]\mathrm{E}\left[x(m)x(n)\right] \\ &+ \sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\mathrm{E}\left[x(k)x(m)\right]\mathrm{E}\left[x(l)x(n)\right] \\ &+ \sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\mathrm{E}\left[x(k)x(n)\right]\mathrm{E}\left[x(l)x(m)\right]\end{aligned}$$

Inspecting these sums reveals that each term (or row in the previous expression) gives N^2, so that we have

$$\mathrm{Var}\left[\hat{P}_x(1)\right] = \frac{3N^2}{N^2} - 1 = 2$$

c)

In the Bartlett's method we divide the observed sequence into N subsequences and then estimate the power spectrum by taking the average of the periodograms of the subsequences. Now our subsequence length is $L = 1$, giving the estimate

$$\hat{P}_x(e^{i\omega}) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)^2$$

that is independent of the phase ω. The expecation is now

$$\mathrm{E}\left[\hat{P}_x(e^{i\omega})\right] = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{E}\left[x(n)^2\right] = 1$$

and the variance

$$\begin{aligned}
\mathrm{Var}\left[\hat{P}_x(e^{i\omega})\right] &= \frac{1}{N^2}\mathrm{E}\left[\left(\sum_{n=0}^{N-1} x(n)^2\right)^2\right] - 1 \\
&= \frac{1}{N^2}\left(\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \mathrm{E}\left[x(m)^2\right]\mathrm{E}\left[x(n)^2\right] + 2\mathrm{E}\left[x(m)x(n)\right]^2\right) - 1 \\
&= \frac{1}{N^2}\left(N^2 + 2N\right) - 1 = \frac{2}{N}
\end{aligned}$$