Aalto University

T61.3040

Wiener Filter(ing)

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Diagram of the content of first part of the course

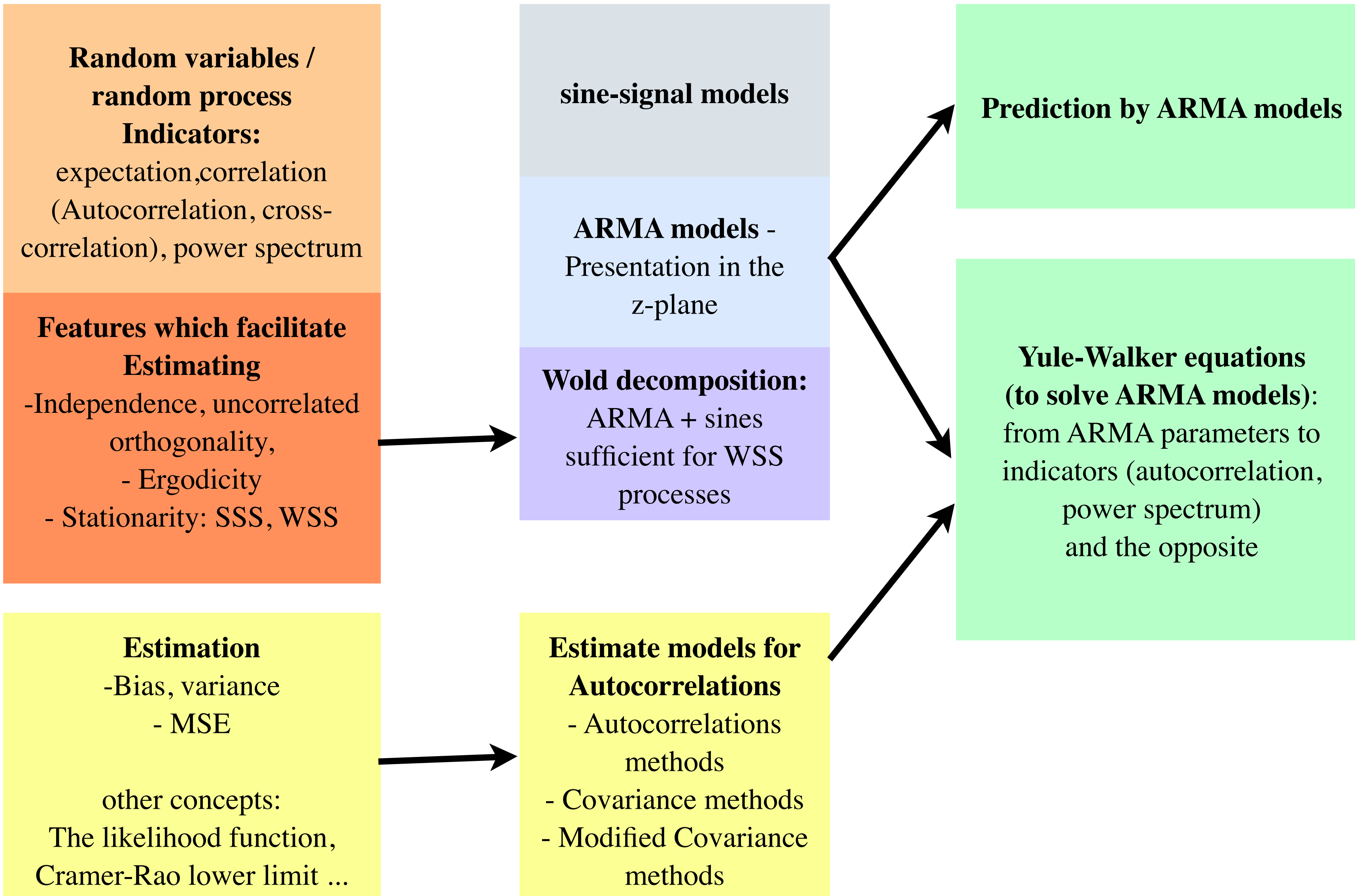
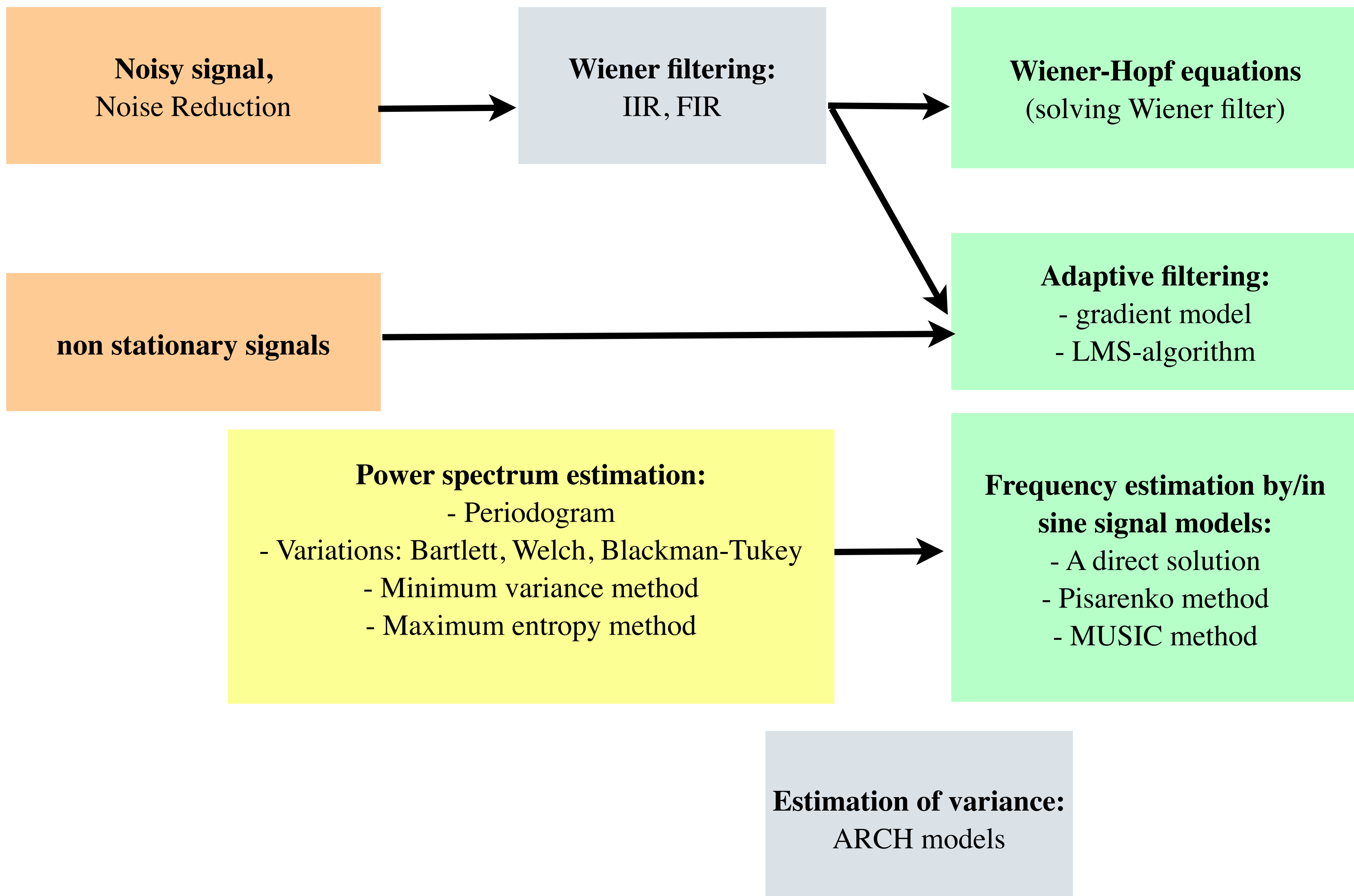


Diagram of content of the final part of the course



Today

Noisy signal,
Noise Reduction

→

Wiener filtering:
IIR, FIR

→

Wiener-Hopf equations
(solving Wiener filter)

Noisy signal

- Earlier, we predicted the process $x(n)$ from the previous values
- We can also predict the process $d(n)$ which is not observed, but it depends on the observed process $x(n)$
- in general, it can be written:

$$x(n) = d(n) + v(n)$$

- by assumptions of the processes and their inter-dependence, then we can estimate $d(n)$ using the observed process $x(n)$

Noisy signal

- Wiener filtering can be interpreted as noise reduction because the interesting process $d(n)$ is observed after addition of the noise $v(n)$
- In this context, the noise does not always mean white noise: the process $v(n)$ can in principle be any kind of noise
- It can be called noise because it contains the part of the detected process that we want to filter out

Noisy signal

Examples:

- Speech signal: the incoming signal $x(n)$ measured by the microphone may contain echoes and noise
- Heart beat measurement, in addition to the heart beats, there is also breathing and sounds from outside of the body
- Image and video signals may contain noise, which reduces the image quality

Wiener Filtering

- The prediction of the process $d(n)$ can be thought by filtering out the noise $v(n)$
- The familiar “pass filtering” from DSP can remove noise, but the best filter choice is not trivial
- Wiener filter is the optimal filter in the sense of MSE, which estimates $d(n)$ from observations

$$x(n) = d(n) + v(n)$$

Wiener Filtering

- Comparing prediction and Wiener filtering
- Linear prediction (AR (p)-process):

$$\hat{x}(n) = \sum_{k=1}^{p} -a(k)x(n-k),$$ we have to know $r_x(k)$

- Linear Wiener filtering:

$$\hat{d}(n) = \sum_{k=0}^{p-1} w(k)x(n-k),$$ we have to know $r_{dx}(k)$

Special cases:

Application	Measured	to estimate
prediction	$x(n) = d(n) + v(n)$	$d(n+k)$, $k>0$
smoothing	$x(n) = d(n) + v(n)$	$d(n-k)$, $k>0$
linear prediction	$x(n) = d(n-1) + v(n)$	$d(n)$
deconvolution	$x(n) = d(n) * g(\mathrm{n}) + v(n)$	$d(n)$

Optimal FIR filter

- First we solve the Wiener filter, which is a finite impulse response (FIR) filter:

$$W(z) = \sum_{n=0}^{p-1} w(n) z^{-n}$$

- Correlations are assumed to be known: $r_x(k)$, $r_d(k)$ and $r_{dx}(k)$
- $x(n)$ and $d(n)$ are WSS processes and $r_{dx}(k) < \infty$ and does not depend on n

Optimal FIR filter

- Wiener filter, where *p* is the order, it is therefore in the time domain

$$\hat{d}(n) = \sum_{l=0}^{p-1} w(l)x(n-l)$$

- Due to the orthogonality principle, we assume that $\hat{d}(n)$ linear combination of values:

$$x(n), x(n-1), \ldots, x(n-p+1)$$

Optimal FIR Filter Solution

- orthogonality principle gives orthogonality conditions

$$
\begin{aligned}
\mathrm{E}(e(n)x^*(n-k)) &= \mathrm{E}(d(n)x^*(n-k)) \\
&\quad - \sum_{l=0}^{p-1} w(l)\,\mathrm{E}(x(n-l)x^*(n-k)) \\
&= 0, \quad k = 0, 1, \ldots, p-1
\end{aligned}
$$

Optimal FIR Filter Solution

- by writing the expectation as correlation, you get the Wiener-Hopf equations:

$$\sum_{l=0}^{p-1} w(l) r_x(k-l) = r_{dx}(k), \quad k = 0, 1, \ldots, p-1$$

Optimal FIR Filter Solution

- Equations in matrix form are $R_x\, w = r_{dx}$

$$\begin{bmatrix} r_x(0) & r_x^*(1) & \dots & r_x^*(p-1) \\ r_x(1) & r_x(0) & \dots & r_x^*(p-2) \\ \vdots & \vdots & & \vdots \\ r_x(p-1) & r_x(p-2) & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(p-1) \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \\ \vdots \\ r_{dx}(p-1) \end{bmatrix}$$

- The solution is $R_x^{-1} r_{dx}$

Optimal FIR Filter Solution

- Previously we obtained parameters minimizing MSE
- Calculating the minimum of the error $\mathrm{E}(|e(n)|^2)$

$$
\begin{aligned}
\mathrm{E}(|e(n)|^2) &= \mathrm{E}(e(n)(d(n) - \hat{d}(n))^*) \\
&= \mathrm{E}(e(n)d^*(n)) \quad \text{orthogonality conditions} \\
&= \mathrm{E}(d(n)d^*(n)) - \sum_{l=0}^{p-1} w(l)\,\mathrm{E}(x(n-l)d^*(n))
\end{aligned}
$$

Optimal FIR Filter Solution

- Writing expectation as correlations and using $R_x^{-1} r_{dx}$ gives results

$$
\begin{aligned}
\mathrm{E}\left(|e(n)|^2 \right) &= r_d(0) - \sum_{l=0}^{p-1} w(l) r_{dx}^*(l) \\
&= r_d(0) - r_{dx}^H w \\
&= r_d(0) - r_{dx}^H R_x^{-1} r_{dx}
\end{aligned}
$$

- Result holds only when *w* is the solution of the optimal Wiener filter

Optimal FIR Filter Solution

- WH-equations provide the solution as a function of correlations $r_x(k)$ and $r_{dx}(k)$
- Autocorrelation $r_x(k)$ can be estimated using observations similarly to solving ARMA processes
- $r_{dx}(k)$ estimation is generally impossible, since $d(n)$ is not detected at any time n
- solution requires additional assumptions
- Let's considered the 4 example cases

Example 1:

- $x(n) = d(n) + v(n)$, where the zero mean $v(n)$ is not correlated with $d(n)$
- often a reasonable assumption, if the cause of the noise can not depend on $d(n)$
- e.g., the speech signal and the air conditioner noise, satisfy previous assumptions

Example 1:

- We calculate $r_x(k)$ and $r_{dx}(k)$ using previous assumptions,
- We obtain

$$\begin{aligned} r_{dx}(k) &= \mathrm{E}\big(d(n)x^*(n-k)\big) \\ &= \mathrm{E}\big(d(n)d^*(n-k)\big) + \mathrm{E}\big(d(n)v^*(n-k)\big) \\ &= \mathrm{E}\big(d(n)d^*(n-k)\big) \quad \text{d and v are not correlated} \\ &= r_d(k) \end{aligned}$$

Example 1:

- In addition, we get

$$
\begin{aligned}
r_x(k) &= \mathrm{E}\left(x(n+k)x^*(n)\right) \\
&= \mathrm{E}\left(d(n+k)d^*(n)\right) + \mathrm{E}\left(v(n+k)v^*(n)\right) \\
&\quad + \mathrm{E}\left(d(n+k)v^*(n)\right) + \mathrm{E}\left(v(n+k)d^*(n)\right) \\
&= r_d(k) + r_v(k) + r_{dv}(k) + r_{vd}(k) \\
&= r_d(k) + r_v(k)
\end{aligned}
$$

- Because $d(n)$ and $v(n)$ are uncorrelated

Example 1:

- We obtain the Wiener-Hopf equations

$$(R_d + R_v)w = r_d$$

where $r_d = \left[r_d(0), r_d(1), \ldots, r_d(p-1)\right]^T$

- If the noise is white noise then we get

$$(R_d + \sigma^2 I)w = r_d$$

Example 1:

- The matrices $R_d + R_v = R_x$ estimated from observations
- Also needed additional assumptions in order to obtain r_d
- How good is a filter if you need to know already r_d ?
  Answer: r_d tells only the statistical properties of $d(n)$.
- We want to estimate by filter the actual values $d(n)$!
- Example: when observing $x(n) = A\sin(\omega n + \phi) + v(n)$

  where $v(n)$ is a Gaussian white noise with variance σ^2

- demo: wfilt.R

Example 2:

- Linear prediction, $d(n) = x(n+1)$
- In WH-equations we have r_{dx} which is now

$$
\begin{aligned}
r_{dx}(k) &= \mathrm{E}\left(d(n)x^*(n-k)\right) \\
&= \mathrm{E}\left(x(n+1)x^*(n-k)\right) \\
&= r_x(k+1)
\end{aligned}
$$

Example 2:

- We get rid of $d(n)$ and get the Wiener-Hopf equations

$$\begin{bmatrix} r_x(0) & r_x^*(1) & \cdots & r_x^*(p-1) \\ r_x(1) & r_x(0) & \cdots & r_x^*(p-2) \\ \vdots & \vdots & & \vdots \\ r_x(p-1) & r_x(p-2) & \cdots & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(p-1) \end{bmatrix} = \begin{bmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(p) \end{bmatrix}$$

Example 2:

- The minimum error is obtained

$$\varepsilon = r_x(0) - \sum_{l=0}^{p-1} w(l) r_x^*(l+1)$$

- These are similar to the YW equations for the AR-model
- Note! signs of the coefficients have changed, because correspondence between the AR model and the Wiener filter

Example 3:

- Noisy signal prediction: $x(n) = y(n) + v(n)$

- The desired signal $d(n) = y(n+1)$,
- Assuming y and v non-correlated: $\hat{d}(n) = w(0)x(n) + \ldots$

$$r_x(k) = \mathrm{E}\left(x(n)x^*(n-k)\right) = r_y(k) + r_v(k) = r_d(k) + r_v(k)$$

$$\begin{aligned} r_{dx}(k) &= \mathrm{E}\left(d(n)x^*(n-k)\right) = \mathrm{E}\left(y(n+1)x^*(n-k)\right) \\ &= \mathrm{E}\left(y(n+1)y^*(n-k)\right) = r_y(k+1) = r_d(k+1) \end{aligned}$$

- The only difference between with the noise-free case is that r_v is added to r_d

Example 3:

- In matrix form, we get the Wiener-Hopf equations:

$$(R_d + R_v)w = r_{dx}$$

Where $r_{dx} = [r_d(1), r_d(2), \ldots, r_d(p)]^T$ as in the noiseless case

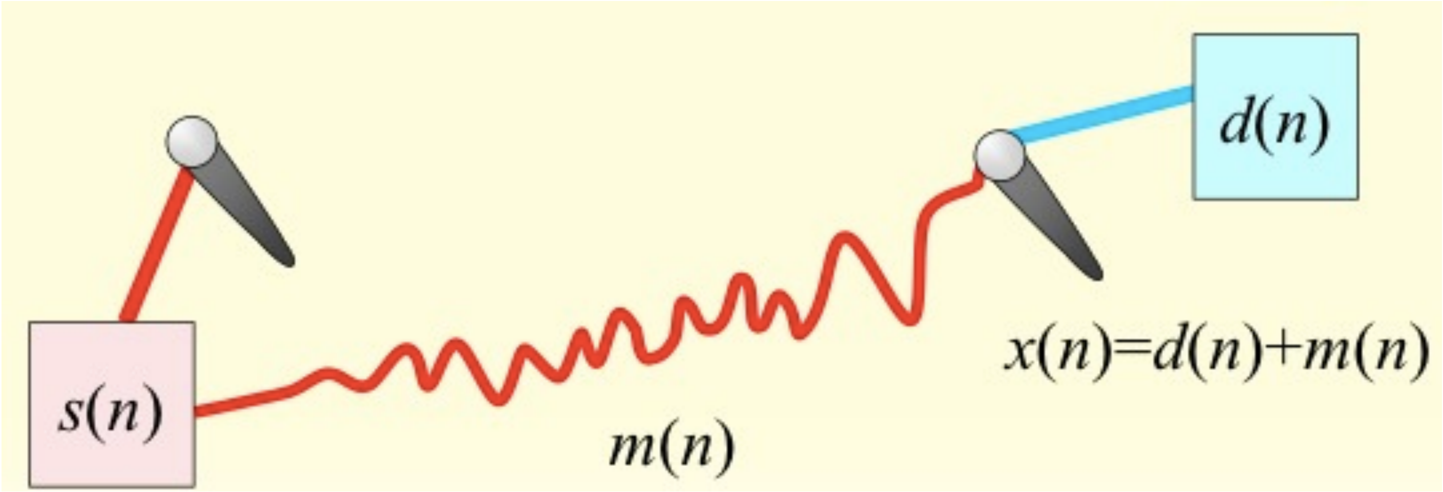
- demo: wpred.R

Example 4:

- Noise reduction using the observed noise
- Measuring by microphone for the speech signal $d(n)$
- Microphone are noisy signal $x(n) = d(n) + \mathrm{m}(n)$
- The objective is to estimate $d(n)$

Example 4:

- in addition we assume that the noise is caused by a noise source $s(n)$
- which can be observed directly
- Noise $m(n)$ entering the microphone is changed along the way
- So m ≠ s, but processes are not uncorrelated



Example 4:

- The problem can be solved by estimating the microphone noise $m(n)$
- If this is done well, $d(n)$ may be obtained by subtracting the estimated $m(n)$ from $x(n)$
- Estimation of microphone noise is possible because we directly observe process $s(n)$

Example 4:

- Now the process $s(n)$ has been observed and the desired process is $m(n)$
- General WH equations: $R_x\, w = r_{dx}$
- Substituting $s \rightarrow x$ and $m \rightarrow d$

  we obtain the WH equations: $R_s\, w = r_{ms}$

- R_s can be estimated directly from observations

Example 4:

- Calculated the cross-correlation is obtained developing:

$$
\begin{aligned}
r_{ms}(k) &= \mathrm{E}\left(m(n)s^*(n-k)\right) \\
&= \mathrm{E}\left(x(n)s^*(n-k)\right) - \mathrm{E}\left(d(n)s^*(n-k)\right) \\
&= r_{xs}(k)
\end{aligned}
$$

s and *d* uncorrelated

- So the $R_{ms} = R_{xs}$ can be estimated from the observations
- $m(n)$ is obtained using the Wiener filter $w = R_s^{-1} R_{xs}$

Example 4:

- Test in practice:

$$d(n) = \sin(n\omega_0 + \phi)$$
$$m(n) = a_1(1)m(n-1) + a_1(2)m(n-2) + g(n)$$
$$s(n) = a_2(1)s(n-1) + a_2(2)s(n-2) + g(n)$$

- AR (2)-processes, $m(n)$ and $s(n)$ are correlated, since $g(n)$ is the same for both process
- demo: noisec.R

Optimal IIR filter

- Causal FIR filter: $h(n) = 0$ when $n<0$ or $n>p-1$
- Generally LSI (linear shift-invariant) filter is a non-causal IIR filter
- Then the impulse response may differ from zero for all values of n
- We solve the non-causal IIR Wiener filter

Optimal IIR filter

- Error signal and the MSE are

$$e(n) = d(n) - \hat{d}(n) = d(n) - \sum_{l=-\infty}^{\infty} h(l)x(n-l)$$

$$\varepsilon = \mathrm{E}(|e(n)|^2)$$

- WH equations follow from orthogonality condition

Optimal IIR filter

- Orthogonality conditions:

$$\mathrm{E}(e(n)x^*(n-k)) = 0; \quad -\infty < k < \infty$$

- Difference to the the FIR case is the infinite number of equations
- Substituting the error signal to the orthogonality conditions, we get

$$\sum_{l=-\infty}^{\infty} h(l) r_x(k-l) = r_{dx}(k); \quad -\infty < k < \infty$$

Optimal IIR filter

- Orthogonality condition is a convolution

$$h(k) * r_x(k) = r_{dx}(k), \quad -\infty < k < \infty$$

- z-transformation of both sides will give

- $$H(z)P_x(z) = P_{dx}(z) \Rightarrow$$ $$H(z) = \frac{P_{dx}(z)}{P_x(z)}$$ (IIR Wiener - Filter)

Optimal IIR filter

- Example: $x(n) = d(n) + v(n)$, $E(v(n)) = 0$ and v, d are not correlated
- Then $r_x(k) = r_d(k) + r_v(k)$ so

$$P_x(z) = P_d(z) + P_v(z)$$

- Cross-correlation is $r_{dx}(k) = r_d(k)$ which gives

$$P_{dx}(z) = P_d(z)$$

Optimal IIR filter

- IIR Wiener filter is

$$H(z) = \frac{P_d(z)}{P_d(z) + P_v(z)}$$

- Interpretation of power changes in frequency omega:

$$— \quad P_d(\exp(j\omega)) << P_v(\exp(j\omega)): \quad H(\exp(j\omega)) \approx 0,$$

Omega is filtered off

$$— \quad P_d(\exp(j\omega)) >> P_v(\exp(j\omega)): \quad H(\exp(j\omega)) \approx 1,$$

Omega's power remains