T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 8, November 10, 2011

1. We observe a signal $x(n) = d(n) + v(n)$ where $d(n)$ and $v(n)$ are not correlated with each other and $v(n)$ is zero mean noise. The values of $v(n)$ are correlated. The autocorrelations are $r_d(k) = 2 \cdot (0.8)^{|k|}$ and $r_v(k) = 2 \cdot (0.5)^{|k|}$. Find $H(z)$ of a non-causal IIR Wiener filter when the desired signal is $d(n)$.

2. We observe $x(n)$ and use a FIR Wiener filter to predict m steps into the future, i.e., the desired signal is $d(n) = x(n + m)$, $m > 0$.

   (a) Write the Wiener-Hopf equations.

   (b) Compute the minimum of the prediction error as a function of m and write it using the autocorrelations of $x(n)$. If m is changed, what do you have to compute anew?

3. We observe a signal

$$x(n) = g(n) * d(n) + v(n)$$

   where the LSI system $g(n) = (0.9)^n u(n)$. The step function $u(n) = 1$ when $n \geq 0$ and $u(n) = 0$ when $n < 0$. We know that $r_d(k) = 2 \cdot (0.5)^{|k|}$ and $v(n)$ is zero-mean noise with variance 1. The noise $v(n)$ does not correlate with the target signal $d(n)$. Compute the optimal noncausal IIR Wiener filter $H(z)$ for estimating $d(n)$.

   Hint 1: the solution to a noncausal IIR Wiener filter was given in the lecture.

   Hint 2: if $x(n) \rightarrow H(z) \rightarrow y(n)$ then $r_{yx}(k) = h(k) * r_x(k)$.

4. Consider the process

$$x(n) = d(n) + v(n)$$

   where $d(n)$ is the desired process but only $x(n)$ is observed. Here $x(n)$ and $d(n)$ are WSS processes and $r_{dx}(k) < \infty$ does not depend on the time instant n, and $v(n)$ is white noise that is not correlated with $d(n)$. Moreover, $\mathrm{E}[x(n)] = \mathrm{E}[d(n)] = 0$, $\mathrm{Var}(d(n)) = 1$ and $\mathrm{Var}(v(n)) = 100$.

   (a) Find a filter

$$\hat{d}(n) = w(0)x(n) + w(1)x(n-1)$$

   so that the MSE of the prediction is $\mathrm{E}[|d(n) - \hat{d}(n)|^2] = 1$.

   (b) Assume additionally that $r_d(1) = 0$. Determine the optimal Wiener filter

$$\hat{d}(n) = w(0)x(n) + w(1)x(n-1)$$

   and calculate its MSE. Verify that the MSE is smaller than in part (a).

   (c) What is the MSE in part (b) if the filter $\hat{d}(n) = x(n)$ is used?

5. (Bonus point exercise)

Consider the process

$$x(n) = d(n) + v(n)$$

where $d(n)$ is the desired signel and $v(n)$ noise uncorrelated with $d(n)$.

$$\mathrm{E}[d(n)] = \mathrm{E}[v(n)] = 0, \qquad \mathrm{Var}[v(n)] = \sigma_v^2 = 1,$$

and the autocorrelation of $d(n)$ is $r_d(k) = 0.375(-0.2)^{|k|}$. All signals are real valued.

(a) Find the optimal Wiener filter $\hat{d}(n) = w(1)x(n) + w(2)x(n-1)$.

(b) For the Wiener filter in (a), what is the MSE, i.e. $\mathrm{E}(|d(n) - \hat{d}(n)|^2)$?

(c) What is the MSE if the process is not filtered at all, i.e. $\hat{d}(n) = x(n)$?