T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 3, September 29, 2011, Solutions

1. (a) Let's write $M(n+1) = M(n) + v(n+1)$ and insert this into the formula of the expected value:

$$\begin{aligned}\mathrm{E}[M(n+1) \mid M(n), \dots] &= \mathrm{E}[M(n) \mid M(n), \dots] + \mathrm{E}[v(n+1) \mid M(n), \dots] \\ &= M(n) + 0 = M(n).\end{aligned}$$

(b) Similar insertion as in the part (a) is performed, resulting in

$$\begin{aligned}\mathrm{Var}[M(n+1) \mid M(n), \dots] &= \mathrm{Var}[M(n) + v(n+1) \mid M(n), \dots] \\ &= \mathrm{Var}[v(n+1) \mid M(n), \dots] = \sigma^2.\end{aligned}$$

In the above we have used the fact that the conditional value of the random variable X is a constant if the condition includes X itself, i.e. $E[X \mid X] = X$ and $\mathrm{Var}[X \mid X] = 0$.

(c) Write $M(n+k) = M(n) + \sum_{i=1}^{k} v(n+i)$ and apply the linearity of the expectation operator:

$$\begin{aligned}\mathrm{E}[M(n+k) \mid M(n), \dots] &= \mathrm{E}[M(n) \mid M(n), \dots] + \sum_{i=1}^{k} \mathrm{E}[v(n+i) \mid M(n), \dots] \\ &= M(n) + \sum_{i=1}^{k} 0 = M(n)\end{aligned}$$

(d) Similarly

$$\begin{aligned}\mathrm{Var}[M(n+k) \mid M(n), \dots] &= \mathrm{Var}\left[M(n) + \sum_{i=1}^{k} v(n+i) \,\middle|\, M(n), \dots \right] \\ &= \sum_{i=1}^{k} \mathrm{Var}[v(n+i) \mid M(n), \dots] = k\sigma^2\end{aligned}$$

since the variance of a sum of uncorrelated variables is the sum of their variances.

2. In this case the principle of orthogonality translates into the estimation error $e(n) = v(n) - \hat{v}(n)$ being orthogonal to all the noise values $v(n-1), \dots, v(n-L)$. Expanding these conditions gives

$$\mathrm{E}\left[\left(v(n) - \sum_{k=1}^{L} a_k v(n-k) \right) v^*(n-l) \right] = 0, \quad l = 1, 2, \dots, L$$

For any value of l in the range, this results in the condition

$$\mathrm{E}[v(n)v^*(n-l)] - \sum_{k=1}^{L} a_k \,\mathrm{E}[v(n-k)v^*(n-l)] = -a_l\sigma^2 = 0$$

As $\sigma^2 > 0$, this leads to $a_l = 0$ for all $l = 1, 2, \dots, L$. That is, all the coefficients in the estimator have zero values, as expected. One property of the noise is the impossibility of linear prediction.

3. The first two parts of the problem deal with the MA(q) process where noise is filtered with an MA(q) filter.

   (a) The difference equation for the MA(q) process is

   $$x(n) = \sum_{l=0}^{q} b_q(l)v(n-l)$$

   where $v(n)$ is zero-mean white noise with unit variance. Then

   $$\mathrm{E}[x(n+k) \mid F_n] = \sum_{l=0}^{q} b_q(l)\,\mathrm{E}[v(n+k-l) \mid F_n].$$

   Of the expectations $\mathrm{E}[v(n+k-l) \mid F_n]$, all those are zero for which $n+k-l > n$, i.e. $k > l$. In this case the expectation is taken from a future noise term, given that past noise and process values are known. But noise is independent of the past values. The conditional expectation thus equals the unconditional one, which for the noise is zero. In contrast, when $n+k-l \leq n$, one deals with already observed noise values, and thus $\mathrm{E}[v(n+k-l) \mid F_n] = v(n+k-l)$. Combining the results, one obtains

   $$\mathrm{E}[x(n+k) \mid F_n] = \sum_{l=k}^{q} b_q(l)v(n+k-l).$$

   One can note that the conditional expectation is zero when the prediction step length k exceeds the model order q.

   (b) The conditional variance is given by

   $$\mathrm{Var}[x(n+k) \mid F_n] = \sum_{l=0}^{q} b_q^2(l)\,\mathrm{Var}[v(n+k-l) \mid F_n]$$

   because the noise values at different time instants are uncorrelated. The variance $\mathrm{Var}(v(n+k-l) \mid F_n) = 0$ when $n+k-l \leq n$, i.e. $k \leq l$, because the already observed noise terms are constants and have no variance. When $n+k-l > n$, one once again deals with future noise values that are independent of the observed noise values included in F_n. Thus the variances equal to one and we have the combined result

   $$\mathrm{Var}[x(n+k) \mid F_n] = \sum_{l=0}^{\min\{k-1,q\}} b_q^2(l), \quad k \geq 1$$

   (c) AR(p): $x(n) + \sum_{l=1}^{p} a(l)x(n-l) = v(n)$

   $$\mathrm{E}[x(n+1) \mid F_n] = -\sum_{l=1}^{p} a(l)\,\mathrm{E}[x(n+1-l)] = -\sum_{l=1}^{p} a(l)x(n+1-l)$$

   as the values in the past have already been observed and are thus included in the condition F_n. The variance

   $$\mathrm{Var}[x(n+1) \mid F_n] = \mathrm{Var}\left[-\sum_{l=1}^{p} a(l)x(n+1-l) \,\middle|\, F_n\right] + \mathrm{Var}[v(n+1) \mid F_n] = 0 + \sigma_v^2 = 1$$

   because the already observed values of x have no variance.

4. The sinusoidal signal can be written in terms of two complex exponentials as

$$
\begin{aligned}
x(n) &= \frac{1}{2j} A e^{j\phi} \exp(jn\omega) - \frac{1}{2j} A e^{-j\phi} \exp(-jn\omega) + v(n) \\
&= B \exp(jn\omega) + B^* \exp(-jn\omega) + v(n)
\end{aligned}
$$

where $B = \frac{1}{2j} A e^{j\phi}$. This converts the model that is non-linear in its parameters into a linear model. Now the orthogonality principle can be applied. To this end, the model is written using vector notation:

$$
\mathbf{x} = B\mathbf{e}_1 + B^*\mathbf{e}_2 + \mathbf{v}
$$

with

$$
\begin{aligned}
\mathbf{x} &= [x(0)\, x(1)\, x(2)]^T \\
\mathbf{v} &= [v(0)\, v(1)\, v(2)]^T \\
\mathbf{e}_1 &= [1\ \exp(j\omega)\ \exp(2j\omega)]^T \\
\mathbf{e}_2 &= [1\ \exp(-j\omega)\ \exp(-2j\omega)]^T
\end{aligned}
$$

Now the orthogonality principle gives the two conditions

$$
\begin{aligned}
\mathbf{e}_1^H \mathbf{v} &= \mathbf{e}_1^H (\mathbf{x} - B\mathbf{e}_1 - B^*\mathbf{e}_2) = 0 \\
\mathbf{e}_2^H \mathbf{v} &= \mathbf{e}_2^H (\mathbf{x} - B\mathbf{e}_1 - B^*\mathbf{e}_2) = 0.
\end{aligned}
$$

These can be manipulated to form

$$
\begin{aligned}
\mathbf{e}_1^H \mathbf{x} &= B\mathbf{e}_1^H \mathbf{e}_1 + B^*\mathbf{e}_1^H \mathbf{e}_2 \\
\mathbf{e}_2^H \mathbf{x} &= B\mathbf{e}_2^H \mathbf{e}_1 + B^*\mathbf{e}_2^H \mathbf{e}_2.
\end{aligned}
$$

Noticing regularity in the form of the equations, one can collect some repeating terms into vectors and matrices

$$
\begin{aligned}
\mathbf{b} &= [B\, B^*]^T \\
\mathbf{E} &= [\mathbf{e}_1\, \mathbf{e}_2],
\end{aligned}
$$

resulting in the conditions to be expressed as one matrix equation that can be solved in a straightforward fashion

$$
\begin{aligned}
\mathbf{E}^H \mathbf{x} &= \mathbf{E}^H \mathbf{E} \mathbf{b} \\
\mathbf{b} &= (\mathbf{E}^H \mathbf{E})^{-1} \mathbf{E}^H \mathbf{x}.
\end{aligned}
$$

Here we have assumed that ω is not an integer multiple of π, resulting in the colums of $\mathbf{E}$ being linearly independent. The structure of $\mathbf{E}$ ensures that the second element is the complex conjugate of the first element in the resulting $\mathbf{b}$. The solution (A, ϕ) is uniquely determined by $\mathbf{b}$ if the amplitude A is assumed to be non-negative (and the phase $\phi \in (-\pi, \pi)$).

5. The given ARMA process has the z-plane representation

$$X(z) = M(z)V(z),$$

$M(z)$ can be solved by z-transforming the difference equation of the process:

$$\left(1 - \frac{1}{2}z^{-1}\right)X(z) = \left(1 - \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2}\right)V(z),$$

resulting in

$$M(z) = \frac{X(z)}{V(z)} = \frac{1 - \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{1}{2}z^{-1}}$$

Epxanding $\frac{1}{1-\frac{1}{2}z^{-1}}$ to its Taylor series:

$$\begin{aligned}
&= \left(1 - \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2}\right)\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k z^{-k} \\
&= \sum_{k=0}^{\infty}\left[\left(\frac{1}{2}\right)^k z^{-k} - \frac{1}{3}\left(\frac{1}{2}\right)^k z^{-(k+1)} + \frac{1}{4}\left(\frac{1}{2}\right)^k z^{-(k+2)}\right] \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k z^{-k} - \sum_{k=0}^{\infty}\frac{1}{3}\left(\frac{1}{2}\right)^k z^{-(k+1)} + \sum_{k=0}^{\infty}\frac{1}{4}\left(\frac{1}{2}\right)^k z^{-(k+2)} \\
&= 1 + \frac{1}{2}z^{-1} + \sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^k z^{-k} - \frac{1}{3}z^{-1} - \sum_{k=1}^{\infty}\frac{1}{3}\left(\frac{1}{2}\right)^k z^{-(k+1)} + \sum_{k=0}^{\infty}\frac{1}{4}\left(\frac{1}{2}\right)^k z^{-(k+2)}
\end{aligned}$$

Simply shifting the summation indices:

$$\begin{aligned}
&= 1 + \frac{1}{2}z^{-1} + \sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^k z^{-k} - \frac{1}{3}z^{-1} - \sum_{k=2}^{\infty}\frac{1}{3}\left(\frac{1}{2}\right)^{k-1} z^{-k} + \sum_{k=2}^{\infty}\frac{1}{4}\left(\frac{1}{2}\right)^{k-2} z^{-k} \\
&= 1 + \left(\frac{1}{2} - \frac{1}{3}\right)z^{-1} + \sum_{k=2}^{\infty}\left(\left(\frac{1}{2}\right)^2 - \frac{1}{3}\times\frac{1}{2} + \frac{1}{4}\right)\left(\frac{1}{2}\right)^{k-2} z^{-k} \\
&= 1 + \frac{1}{6}z^{-1} + \sum_{k=2}^{\infty}\frac{1}{3}\left(\frac{1}{2}\right)^{k-2} z^{-k}
\end{aligned}$$

So $m_0 = 1$, $m_1 = \frac{1}{6}$, and $m_k = \frac{1}{3}\left(\frac{1}{2}\right)^{k-2}$ for all $k \geq 2$ in the resulting MA(∞)-process.