T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 8, November 10, 2011, Solutions

1. For an optimal noncausal IIR-filter, the Wiener-Hopf equations can be written as

$$h(k) * r_x(k) = r_{dx}(k).$$

The z-transform of this yields

$$H(z) = \frac{P_{dx}(z)}{P_x(z)}$$

The desired signal $d(n)$ is uncorrelated with $v(n)$. Consequently, $r_x(k) = r_d(k) + r_v(k)$ and $r_{dx}(k) = r_d(k)$. Then

$$\begin{aligned} r_x(k) &= 2 \cdot ((0.8)^{|k|} + (0.5)^{|k|}) \\ r_{dx}(k) &= 2 \cdot (0.8)^{|k|}. \end{aligned}$$

Compute $P_x(z)$ by z-transforming $r_x(k)$:

$$P_x(z) = \sum_k r_x(k) z^{-k} = 2 \sum_k (0.8)^{|k|} z^{-k} + 2 \sum_k (0.5)^{|k|} z^{-k}.$$

Here we can apply the general result

$$\begin{aligned} \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k} &= 1 + \sum_{k=1}^{\infty} (a/z)^k + \sum_{k=1}^{\infty} (az)^k \\ &= 1 + \frac{a/z}{1 - a/z} + \frac{az}{1 - az} \\ &= \frac{1 - a^2}{(1 - az^{-1})(1 - az)}. \end{aligned}$$

Inserting this results in

$$P_x(z) = \frac{2(1 - (0.8)^2)}{(1 - 0.8z^{-1})(1 - 0.8z)} + \frac{2(1 - (0.5)^2)}{(1 - 0.5z^{-1})(1 - 0.5z)}.$$

Similarly

$$P_{dx}(z) = 2 \sum_k (0.8)^{|k|} z^{-k} = \frac{2(1 - (0.8)^2)}{(1 - 0.8z^{-1})(1 - 0.8z)}.$$

Then

$$\begin{aligned} H(z) = \frac{P_{dx}(z)}{P_x(z)} &= \frac{0.72}{(1 - 0.8z^{-1})(1 - 0.8z)} \left(\frac{0.72}{(1 - 0.8z^{-1})(1 - 0.8z)} + \frac{1.5}{(1 - 0.5z^{-1})(1 - 0.5z)} \right)^{-1} \\ &= \frac{0.72(1 - 0.5z^{-1})(1 - 0.5z)}{0.72(1 - 0.5z^{-1})(1 - 0.5z) + 1.5(1 - 0.8z^{-1})(1 - 0.8z)} \\ &= \frac{0.72(1 - 0.5z^{-1})(1 - 0.5z)}{3.36 - 1.56z^{-1} - 1.56z} \end{aligned}$$

Factorizing the denominator this expression attains the form

$$H(z) = \frac{0.72(1 - 0.5z^{-1})(1 - 0.5z)}{2.30(1 - 0.68z^{-1})(1 - 0.68z)}$$

2. (a) The general form of the Wiener-Hopf equations in the case of a FIR filter are

$$\mathbf{R}_x \mathbf{w} = \mathbf{r}_{dx}.$$

In the lectures the case of one-step-ahead prediction $d(n) = x(n+1)$ was discussed, and in that case we had the result

$$r_{dx}(k) = r_x(k+1).$$

Because now $d(n) = x(n+m)$, we have

$$r_{dx}(k) = \mathrm{E}[d(n)x^*(n-k)] = \mathrm{E}[x(n+m)x^*(n-k)] = r_x(k+m)$$

With this result, the WH-equations can be written as

$$\mathbf{R}_x \mathbf{w} = \mathbf{r}_x(m),$$

where $\mathbf{r}_x(m) = [r_x(m), r_x(m+1), \ldots, r_x(m+p-1)]^T$.

(b) The prediction error is $e(n) = d(n) - \hat{d}(n) = x(n+m) - \sum_{k=0}^{p-1} w(k)x(n-k)$. Applying the principle of orthogonality results in

$$\begin{aligned}
\mathrm{E}(|e(n)|^2) &= \mathrm{E}\left(e(n)\left[x(n+m) - \sum_{k=0}^{p-1} w(n)x(n-k)\right]^*\right) \\
&= \mathrm{E}(e(n)x^*(n+m)) \\
&= \mathrm{E}\left(\left[x(n+m) - \sum_{k=0}^{p-1} w(k)x(n-k)\right]x^*(n+m)\right) \\
&= r_x(0) - \sum_{k=0}^{p-1} w(k)r_x^*(k+m) \\
&= r_x(0) - \mathbf{r}_x(m)^H \mathbf{w}
\end{aligned}$$

As $\mathbf{w} = \mathbf{R}_x^{-1}\mathbf{r}_x(m)$, we get

$$\mathrm{E}(|e(n)|^2) = r_x(0) - \mathbf{r}_x(m)^H \mathbf{R}_x^{-1} \mathbf{r}_x(m)$$

Here only the vector $\mathbf{r}_x(m) = [r_x(m), r_x(m+1), \ldots, r_x(m+p-1)]^T$ depends on the value of m. Evaluation of the prediction error for a set of different m is therefore relatively lightweight computationally, as it is enough to invert $\mathbf{R}_x$ only once.

3. The WH-solution is generally

$$H(z) = \frac{P_{dx}(z)}{P_x(z)}.$$

The power spectrum of the observed signal $x(n) = g(n) * d(n) + v(n)$ is the sum of power spectrum of the signal $g(n) * d(n)$ and power spectrum of the noise because the noise does not correlate with $g(n) * d(n)$. So we have

$$P_x(z) = G(z)G^*(1/z^*)P_d(z) + P_v(z)$$

$G(z)$ can be computed by z-transforming $g(n) = (0.9)^n u(n)$:

$$G(z) = \sum_{k=-\infty}^{\infty} g(k)z^{-k} = \sum_{k=0}^{\infty}(0.9)^k z^{-k} = \frac{1}{1-0.9z^{-1}}$$

(the sum of a geometric series). Correspondingly $G^*(1/z^*) = 1/(1-0.9z)$. $P_v = 1$ as the noise is white. P_d is found by z-transforming $r_d(k) = 2 * (0.5)^{|k|}$. From the solution of problem 1 we have $\sum_{k=-\infty}^{\infty} a^{|k|}z^{-k} = (1-a^2)/[(1-az^{-1})(1-az)]$, so that

$$P_d(z) = \frac{2(1-0.25)}{(1-0.5z^{-1})(1-0.5z)}.$$

Finally, we need P_{dx}. Because of the uncorrelatedness of the noise, $P_{dx} = P_{dy}$ with $y(n) = g(n) * d(n)$. According to the given hint, the cross-correlation between the input and output of a filter is given by the convolution of the impulse response of the filter and the autocorrelation of the input. Now we identify d as the input, g as the impulse response and u as the output, resulting in

$$r_{yd}(k) = g(k) * r_d(k).$$

In order to arrive at the cross power spectrum P_{dy} through z-transform we would need the cross-correlation r_{dy}. It is obtained from r_{yd} by taking the complex conjugate and changing the sign of the lag index, that is

$$r_{dy}(k) = r_{yd}^*(-k) = g^*(-k) * r_d(k)$$

(by using $r_d^*(-k) = r_d(k)$). z-transforming this (recalling that $g^*(k)$ becomes $G^*(z^*)$ and thus $g^*(-k)$ becomes $G^*(1/z^*)$) results in the power spectrum

$$P_{dx} = P_{dy} = G^*(1/z^*)P_d(z)$$

This yields the result

$$\begin{aligned}
H(z) &= \frac{G^*(1/z^*)P_d(z)}{G(z)G^*(1/z^*)P_d(z) + P_v(z)} \\
&= \frac{\frac{1}{1-0.9z} \cdot \frac{1.5}{(1-0.5z^{-1})(1-0.5z)}}{\frac{1}{1-0.9z^{-1}} \cdot \frac{1}{1-0.9z} \cdot \frac{1.5}{(1-0.5z^{-1})(1-0.5z)} + 1} \\
&= \frac{1.5(1-0.9z^{-1})}{1.5 + (1-0.9z^{-1})(1-0.9z)(1-0.5z^{-1})(1-0.5z)} \\
&= \frac{1.5(1-0.9z^{-1})}{4.6625 + 0.45z^{-2} + 0.45z^2 - 2.03z^{-1} - 2.03z} \\
&= 0.402\frac{1-0.9z^{-1}}{(1-az^{-1})(1-a^*z^{-1})(1-az)(1-a^*z)}
\end{aligned}$$

where $a \approx 0.243 + 0.248j$ is one root of the denominator (as the coefficients are real, a^* is also a root, and because of the $z \leftrightarrow z^{-1}$ symmetry, also $1/a$ and $1/a^*$ are roots).

4. (a) The trivial solution

$$\hat{d}(n) = 0 \cdot x(n) + 0 \cdot x(n-1) = 0$$

gives the required accuracy, as

$$\mathrm{E}[|d(n) - \hat{d}(n)|^2] = \mathrm{E}[|d(n)|^2] = \mathrm{Var}(d(n)) = 1.$$

The large variance of the noise $v(n)$ does not affect the measure of accuracy, as it is determined by the difference to the "noiseless" target process $d(n)$

(b) The optimal coefficients $w(n)$ are determined by the Winer-Hopf equations

$$\sum_{l=0}^{1} w(l) r_x(k-l) = r_{dx}(k), \quad k = 0, 1.$$

here $r_x(k) = \mathrm{E}[x(n)x^*(n-k)]$ is the autocorrelation of the observed signal, and $r_{dx}(k) = \mathrm{E}[d(n)x^*(n-k)]$ the cross-correlation between the target and observed signals. In matrix form, the equations are

$$\mathbf{R}_x \mathbf{w} = \mathbf{r}_{dx},$$

where

$$\mathbf{R}_x = \begin{bmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{bmatrix}$$

and $\mathbf{r}_{dx} = [r_{dx}(0) \;\; r_{dx}(1)]^T$.
The required correlations are found by noting that the noise $v(n)$ is uncorrelated with the target signal $d(n)$, so

$$r_x = r_d + r_v$$

and

$$r_{dx} = r_d.$$

Using the given information that $d(0) = \mathrm{Var}(d(n)) = 1$ and $d(1) = 0$

$$\mathbf{R}_x = \mathbf{I} + 100\mathbf{I} = 101\mathbf{I}$$

and

$$\mathbf{r}_x = [r_d(0) \;\; r_d(1)]^T = [1 \;\; 0]^T.$$

From the WH-equations we then get that

$$\mathbf{w} = \mathbf{R}_x^{-1} \mathbf{r}_{dx} = \frac{1}{101} \mathbf{r}_{dx} = [1/101 \;\; 0]^T$$

so $w(0) = 1/101$ and $w(1) = 0$.
The MSE is

$$\mathrm{E}[|d(n) - \hat{d}(n)|^2] = r_d(0) - \mathbf{r}_{dx}^H \mathbf{R}_x^{-1} \mathbf{r}_{dx} = 1 - \frac{1}{101}[1 \;\; 0][1 \;\; 0]^T = 100/101 < 1.$$

(c) If no filter is used, i.e., $\hat{d}(n) = x(n)$, the MSE is

$$\mathrm{E}[|d(n) - x(n)|^2] = E[|d(n) - d(n) - v(n)|^2] = r_v(0) = 100.$$

5.

$$r_d(k) = 0.375(-0.2)^{|k|},$$

$$x(n) = d(n) + v(n).$$

$d(n)$ and $v(n)$ are uncorrelated. In addition, the noise was specified as white:

$$r_v(k) = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned} r_x(k) =& \mathrm{E}[x(k)x(0)] = \mathrm{E}\{[d(k) + v(k)][d(0) + v(0)]\} \\ =& \mathrm{E}[d(k)d(0)] + \mathrm{E}[v(k)v(0)] \\ =& r_d(k) + r_v(k), \end{aligned}$$

$$\begin{aligned} r_{dx}(k) =& \mathrm{E}[d(k)x(0)] = \mathrm{E}\{d(k)[d(0) + v(0)]\} \\ =& \mathrm{E}[d(k)d(0)] \\ =& r_d(k). \end{aligned}$$

(a) From the WH-equations $\mathbf{w} = \mathbf{R}_x^{-1}\mathbf{r}_{dx}$, where

$$\mathbf{R}_x = \begin{bmatrix} 1.375 & -0.075 \\ -0.075 & 1.375 \end{bmatrix}, \quad \mathbf{r}_{dx} = \begin{bmatrix} 0.375 \\ -0.075 \end{bmatrix},$$

$$\Rightarrow \mathbf{w} = \begin{bmatrix} 0.2706 \\ -0.0398 \end{bmatrix}.$$

So the optimal filter for the target signal is $\hat{d}(n) = 0.2706x(n) - 0.0398x(n-1)$.

(b) The MSE is given by

$$\varepsilon_{\min} = r_d(0) - \mathbf{r}_{dx}^H\mathbf{w} = 0.2706$$

(c) With no filtering, $\hat{d}(n) = x(n) = d(n) + v(n)$. Then the MSE is

$$\mathrm{E}(|e(n)|^2) = \mathrm{E}(|d(n) - \hat{d}(n)|^2) = \mathrm{E}(|d(n) - d(n) - v(n)|^2) = r_v(0) = \sigma_v^2 = 1$$

This is considerably larger than for the optimally filtered case.