T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 5, October 13, 2011

1. Let
$$y(n) = x(n+k) - x(n-k)$$
where k is a constant integer and $x(n)$ is a zero mean wide-sense stationary stochastic process with the spectral density $P_x(\exp(j\omega))$ and autocorrelation sequence $r_x(0), r_x(1), \ldots$.

   (a) Determine the autocorrelation sequence $r_y(m)$ of y.

   (b) Show that the spectral density (power spectrum) of y is
$$P_y(\exp(j\omega)) = 4P_x(\exp(j\omega))\sin^2(k\omega).$$

2. The input of the first order discrete-time filter
$$y(n) = -x(n) + 0.5x(n-1)$$
is a zero mean white noise sequence with variance $\sigma^2 = 1$. Thus the process is a MA(1) process.

   (a) Compute the power spectrum of the filter output $y(n)$.

   (b) Determine the autocorrelation function of $y(n)$ using the inverse transform of the power spectrum.

   Hints: $r_y(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_y(e^{j\omega})e^{jm\omega}d\omega$
   $\int_{-\pi}^{\pi} e^{jm\omega}d\omega =$either 0 or 2π.

3. Consider a second order autoregressive model whose difference equation is
$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 v(n)$$
Here $y(n)$ is the output of the system at time n and $v(n)$ is an input signal which is white noise with mean zero and variance 1.

   (a) Determine the conditions of the system being wide sense stationary.

   (b) Is the system WSS when $a_1 = -0.1$ and $a_2 = -0.8$?

4. Suppose we are given a linear shift-invariant system having a system function
$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$
that is excited by zero mean WSS signal $x(n)$ with an autocorrelation sequence
$$r_x(k) = \left(\frac{1}{2}\right)^{|k|}$$
Let $y(n)$ be the output process, $y(n) = x(n) * h(n)$.

(a) Find the power spectrum, $P_y(z)$, of $y(n)$.

(b) Find the autocorrelation sequence, $r_y(k)$, of $y(n)$.

(c) Find the cross-correlation, $r_{xy}(k)$, between $x(n)$ and $y(n)$.

(d) Find the *cross-power spectral density*, $P_{xy}(z)$, which is the z-transform of the cross-correlation $r_{xy}(k)$.

5. (Bonus point exercise)

Consider the real-valued MA(2) process

$$x(n) = b(0)v(n) + b(1)v(n-1) + b(2)v(n-2) \ .$$

(a) Find the variance of the process $x(n)$ when $\mathrm{Var}(v(n)) = 1$ and the coefficients are $b(0) = 1.0$, $b(1) = 0.7$, and $b(2) = 0.2$

(b) Represent the process $x(n)$ as filtering of white noise with an LSI filter with the impulse response $h(n)$. Is the result still a MA process if $x(n)$ is filtered with $h(n)$ once more to get a process $y(n)$? If yes, what happens to the order of the process?

(c) Find the variance of the process $y(n)$ that results from filtering $x(n)$ with $h(n)$.

Hint: The formula $r_y(k) = r_x(k) * h(k) * h^*(-k)$ may be useful in the solution of the problem.