T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola
Exercise 7, November 3, 2011

1. (a) Write an ARMA(p, q) process $x(n)$ as an AR(1) process $\mathbf{y}(n)$ in vector form. I.e., define vectors $\mathbf{y}(n)$ and $\mathbf{b}$ and the matrix $\mathbf{A}$ appropriately so that

$$x(n) + \sum_{l=1}^{p} a(l)x(n-l) = \sum_{l=0}^{q} b(l)v(n-l).$$

can be written in the form

$$\mathbf{y}(n) = \mathbf{A}\mathbf{y}(n-1) + \mathbf{b}v(n)$$

(b) Compute the conditional expectations and variances for a vector form AR(1) process $\mathbf{y}(n)$ for k steps into the future, i.e., $\mathrm{E}(\mathbf{y}(n+k) \mid \mathbf{y}(n), \dots)$ and $\mathrm{Var}(\mathbf{y}(n+k) \mid \mathbf{y}(n), \dots)$. Hint: assume that the AR(1) process in vector form can be written as a MA(∞) process in vector form as

$$\mathbf{y}(n) = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{b} v(n-k)$$

For a vector, $\mathrm{Var}(\mathbf{y}) = \mathrm{E}[(\mathbf{y}-\mathbf{m})(\mathbf{y}-\mathbf{m})^H]$ where $\mathbf{m} = \mathrm{E}(\mathbf{y})$

2. Consider a signal $x(n) = \rho^n$, $0 \le n \le N-1$. Form an AR(1) model using the covariance method and compute the sum of squares of the error signal. When is the system stable?

3. Consider a signal $(x(0), \dots, x(N)) = (-1, 2, 3, 4, 5, 6)$. Form an AR(2) model using the covariance method. Determine also the coefficients for the time-inversed observations $(6, 5, 4, 3, 2, -1)$ and compare the results.

   Recall that for the term $b(0)$ of an optimal AR model, $\sigma_v^2 |b(0)|^2$ corresponds to the mean squared error of the corresponding linear predictor.

4. (Demo) We try to predict an AR(p) process with a linear predictor of order M. Discuss the estimation error and the predictor coefficients when

   (a) $p > M$.
   (b) $p = M$.
   (c) $p < M$.

5. (Bonus point exercise)

   You observe values $x(0), x(1)$ from a real valued process known to be zero mean. At least one of the values is non-zero. The goal is to fit an AR(1) model to the observations.

   (a) Fit an AR(1) model using the *autocorrelation method* and solve the parameters $a(1)$ and $b(0)$. Is the resulting model stable?
   (b) Fit an AR(1) model using the *covariance method*. Is the model stable?