T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola
Exercise 11, December 8, 2011

1. The convergence of the steepest descent adaptive filter is related to the *condition number* χ of the autocorrelation matrix $\mathbf{R}_x$. This number can be bounded in terms of the power spectrum $P_x(e^{j\omega})$ of the process as follows:

$$\chi = \frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{\max_\omega P_x(e^{j\omega})}{\min_\omega P_x(e^{j\omega})}$$

   (a) Use this inequality to bound the condition number of the autocorrelation matrix for the moving average process

$$x(n) = v(n) + \alpha v(n-1)$$

   where $v(n)$ is unit variance white noise.

   (b) Repeat part (a) for the autoregressive process

$$x(n) = \alpha x(n-1) + v(n)$$

   where $|\alpha| < 1$ and $v(n)$ is unit variance white noise.

2. The LMS algorithm is obtained by estimating $\mathrm{E}(e(n)\mathbf{x}^*(n))$ in the gradient method by a pointwise value $e(n)\mathbf{x}^*(n)$.

   (a) Compute the update in the gradient method when we wish to minimize the *absolute error* $\mathrm{E}(|e(n)|)$ instead of the squared error $\mathrm{E}(|e(n)|^2)$. Assume that the parameters $w(k)$ and the process $x(n)$ are real valued.

   (b) Replace the expectation with a pointwise value. What is the resulting method?

3. The LMS adaptive filter minimizes the instantaneous squared error

$$\xi(n) = |e(n)|^2$$

   Consider the modified functional

$$\xi'(n) = |e(n)|^2 + \beta \mathbf{w}_n^H \mathbf{w}_n$$

   where $\beta > 0$.

   (a) Derive the LMS coefficient update equation for $\mathbf{w}_n$ that minimizes $\xi'(n)$.

   (b) Determine the condition on the step size μ that will ensure that $\mathbf{w}_n$ converges to the mean.

   (c) If μ is small enough so that $\mathbf{w}_n$ converges in the mean, what does $\mathbf{w}_n$ converge to?

4. The process $x(n)$ is defined so that the observation $x(n)$ is normally distributed with mean 0 and variance $0.5 \cdot (\text{sgn}\{x(n-1)\} + 3)$, i.e.

$$x(n) \sim \begin{cases} N(0,1) & \text{if } x(n-1) \geq 0 \\ N(0,2) & \text{otherwise} \end{cases}$$

   (a) Calculate the unconditional and conditional (one step ahead) expectations and variances of the observation $x(n)$.
   Hint: the unconditional variance can calculated from the formula $\text{Var}(y) = \text{E}[\text{Var}(y|z)] + \text{Var}(\text{E}[y|z])$ with an appropriate choice of z.

   (b) Is the process $x(n)$ a WSS process?

5. (Bonus point exercise)

   Consider a process with autocorrelations

$$r_x(k) = a^{|k|}$$

   where $|a| < 1$, $a \in \mathbb{R}$.

   (a) Determine the eigenvalues and eigenvectors of the autocorrelation matrix of size 2×2.

   (b) For the autocorrelation matrix of size $p \times p$, find the asymptotic value of the largest eigenvalue λ_{max} as $p \to \infty$.
   Hint: as p grows, $\lambda_{max} \to \max P_x(\exp(j\omega))$, i.e., it is sufficient to find the maximum of the power spectrum.

   (c) Find, as a function of α, the largest step size μ for convergence in the mean of the LMS algorithm. What can you say about the convergence if $|a|$ is very close to 1?