Exercise 5

Although it was not specified, we assume that $v(n)$ is white.

A)

The variance of $x(n)$ follows easily, as the variance of the sum of uncorrelated variables is the sum of their variances and $\text{var}\,(ax) = a^2\text{var}\,(x)$ if a is constant. We were also told that $\text{var}\,\big(v(n)\big) = 1$. We then get

$$\begin{aligned}\text{var}\,\big(x(n)\big) &= b(0)\text{var}\,\big(v(n)\big) + b(1)\text{var}\,\big(v(n-1)\big) + b(2)\text{var}\,\big(v(n-1)\big) \\ &= b(0)^2 + b(1)^2 + b(2)^2 \\ &= 1.53\end{aligned}$$

B)

The result of filtering an input $v(n)$ with a LSI filter can be computed from the convolution of the filter's impulse response and the input:

$$x(n) = h(n) * v(n)$$

The impulse response of the MA-process is finite and can be easily read from the coefficients. So we get:

$$x(n) = \sum_{m=-\infty}^{\infty} \big(b(0)\delta(n-m) + b(1)\delta(n-1-m) + b(2)\delta(n-2-m)\big)\, v(m)$$

Now if we think of filtering $x(n)$ again with $h(n)$, we could just take the convolution again. However as the convolution operation corresponds to multiplication of z-transforms, we get:

$$Y(z) = H(z)H(z)V(Z) = \Big(b(0) + b(1)z^{-1} + b(2)z^{-2}\Big)^2\, V(z)$$

From here we can see that the result is a MA(4) process.

C)

We can simply calculate the difference equation and then take the variance as in A):

$$\begin{aligned}y(n) &= \sum_{l=0}^{2}\sum_{m=0}^{2} b(l)b(m)v(n-m-l) \\ \Rightarrow \text{var}\,\big(y(n)\big) &= \sum_{l=0}^{2}\sum_{m=0}^{2} b(l)^2b(m)^2 = \sum_{l=0}^{2} b(l)^2 \sum_{m=0}^{2} b(m)^2 \\ &= \text{var}\,\big(x(n)\big)^2 = 2.3409\end{aligned}$$