T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 4, October 6, 2011, Solutions

1. Mean:

$$m_y(n) = \mathrm{E}[y(n)] = \mathrm{E}[x(n) + f(n)] = f(n)$$

Autocorrelation:

$$\begin{aligned} r_y(k,l) &= \mathrm{E}[y(k)y^*(l)] = \mathrm{E}[(x(n) + f(n))(x^*(n) + f^*(n))] \\ &= \mathrm{E}[x(k)x^*(l)] + \mathrm{E}[x(k)f^*(l)] + \mathrm{E}[f(k)x^*(l)] + \mathrm{E}[f(k)f^*(l)] \\ &= r_x(k-l) + f(k)f^*(l) \end{aligned}$$

The process $y(n)$ is wide-sense stationary if and only if $f(n)$ is constant.

2. (a) We use the fact that $\mathrm{E}[v(n)v(m)] = 0$ if $n \neq m$, and $\mathrm{E}[v(n)v(n)] = \mathrm{Var}(v(n)) = 1$ (that is, $\mathrm{E}[v(n)v(m)] = \delta(n-m)$)

$$\begin{aligned} r_x(0) &= \mathrm{E}[|x(n)|^2] = \mathrm{E}[(1.0v(n) + 0.5v(n-1) + 0.25v(n-2))^2] \\ &= 1^2 + 0.5^2 + 0.25^2 = 1.3125 \\ r_x(1) &= \mathrm{E}[x(n)x^*(n-1)] = 1 \times 0.5 + 0.5 \times 0.25 = 0.625 \\ r_x(2) &= \mathrm{E}[x(n)x^*(n-2)] = 1 \times 0.25 = 0.25 \\ r_x(3) &= 0 \\ r_x(k) &= 0 \qquad \forall k \text{ s.t. } |k| \geq 3 \end{aligned}$$

Also, $r_x(-1) = r_x^*(1) = 0.625$ and $r_x(-2) = 0.25$.

(b)

$$r_x(0) = \mathrm{E}[x(n)x^*(n)] = \mathrm{E}[(-0.9x(n-1))^2] + \mathrm{E}[(2v(n))^2] = 0.81r_x(0) + 4$$

From this we can solve that $r_x(0) = 4/0.19 \approx 21.05$

$$\begin{aligned} r_x(1) &= \mathrm{E}[x(n)x^*(n-1)] = \mathrm{E}[-0.9|x(n-1)|^2 + 2v(n)x^*(n-1)] = -0.9r_x(0) \\ r_x(2) &= \mathrm{E}[x(n)x^*(n-2)] = \mathrm{E}[-0.9x(n-1)x^*(n-2) + 2v(n)x^*(n-2)] \\ &= -0.9r_x(1) = (-0.9)^2 r_x(0) \\ &\vdots \\ r_x(k) &= (-0.9)^{|k|} r_x(0) \qquad \forall k \in \mathbb{Z} \end{aligned}$$

3.

$$x(n) = A\cos(\omega_0 n + \theta) + v(n).$$

$v(n)$ is white noise: $r_v(k) = \sigma^2\delta(k)$. θ is uniformly distributed on the interval $[0, 2\pi]$, corresponding to the density function

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

(a) The autocorrelation:

$$\begin{aligned} r_x(k) &= \mathrm{E}[x(n)x(n+k)] \\ &= A^2\ \mathrm{E}[\cos(\omega_0 n+\theta)\cos(\omega_0(n+k)+\theta)] + A\ \mathrm{E}[\cos(\omega_0 n+\theta)]\,\underbrace{\mathrm{E}[v(n+k)]}_{=0} + \\ &\quad + A\ \mathrm{E}[\cos(\omega_0(n+k)+\theta)]\,\underbrace{\mathrm{E}[v(n)]}_{=0} + \underbrace{\mathrm{E}[v(n)v(n+k)]}_{=r_v(k)}. \end{aligned}$$

Because

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x+y)+\cos(x-y)],$$

so

$$\begin{aligned} r_x(k) =& \frac{A^2}{2}\ \mathrm{E}[\cos(2\omega_0 n+\omega_0 k+2\theta)+\cos(\omega_0 k)] + \sigma^2\delta(k) \\ =& \frac{A^2}{2}\int_{-\infty}^{\infty}\cos(2\omega_0 n+\omega_0 k+2\theta)f(\theta)\,d\theta + \frac{A^2}{2}\cos(\omega_0 k)+\sigma^2\delta(k) \\ =& \underbrace{\frac{A^2}{2}\frac{1}{2\pi}\int_0^{2\pi}\cos(2\omega_0 n+\omega_0 k+2\theta)d\theta}_{=0} + \frac{A^2}{2}\cos(\omega_0 k)+\sigma^2\delta(k) \\ =& \frac{A^2}{2}\cos(\omega_0 k)+\sigma^2\delta(k) \\ \Rightarrow \mathbf{R}_x =& \begin{bmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{bmatrix} = \begin{bmatrix} \frac{A^2}{2}+\sigma^2 & \frac{A^2}{2}\cos(\omega_0) \\ \frac{A^2}{2}\cos(\omega_0) & \frac{A^2}{2}+\sigma^2 \end{bmatrix} \end{aligned}$$

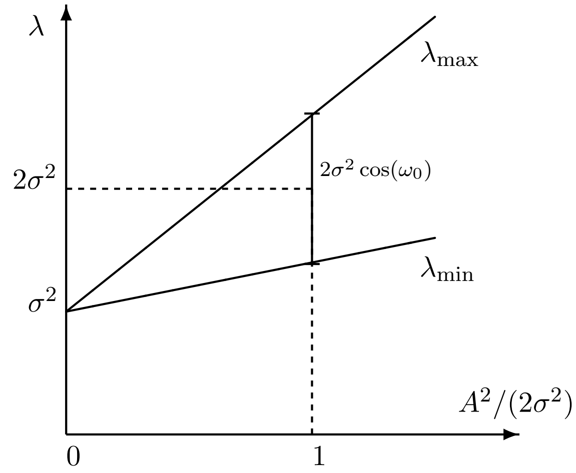
(b) The eigenvalues:

$$\begin{aligned} |\mathbf{R}_x - \lambda\mathbf{I}| = \left(\frac{A^2}{2}+\sigma^2-\lambda\right)^2 - \frac{A^4}{4}\cos^2(\omega_0) = 0 \\ \frac{A^2}{2}+\sigma^2-\lambda = \pm\frac{A^2}{2}\cos(\omega_0) \\ \lambda =& \sigma^2 + \frac{A^2}{2} \pm \frac{A^2}{2}\cos(\omega_0) \\ =& \underline{\sigma^2\left[1+\frac{A^2}{2\sigma^2}(1\pm\cos(\omega_0))\right]}. \end{aligned}$$

The eigenvalues of the autocorrelation matrix indicate the type of components the time series signal consists of. Often in real life the observed signal is the sum of couple of signals and noise. For generality, we consider here complex sinusoidal signals (a real sinusoid can always expressed with two complex sinusoids). If N of the eigenvalues of the autocorrelation matrix are rather large and the rest M approximately equal but clearly smaller, close to zero, one can infer that the signal concicts of N actual complex sinusoids and noise. The nonzero eigenvalues indicate the power in each sinusoid, and the eigenvalues near zero the noise variance.

Additional information on the number of signals: Mati Wax and Thomas Kailath, “Detection of Signals by Information Theoretic Criteria”, IEEE Transactions on Acoustics, Speech and Signal Processing, Vol. ASSP-33, No. 2, April 1985, pp. 387–392.

(c)



Here $\cos(\omega_0) = 0.6$, i.e., $\omega_0 \approx 0.9273$.

4. Semidefiniteness:
   Assume that $\mathbf{R}$ is an autocorrelation matrix, i.e. $\mathbf{R} = E[\mathbf{x}\mathbf{x}^H]$. Then

$$\begin{aligned} \mathbf{y}^H \mathbf{R} \mathbf{y} &= \mathbf{y}^H E[\mathbf{x}\mathbf{x}^H]\mathbf{y} = E[\mathbf{y}^H \mathbf{x}\mathbf{x}^H \mathbf{y}] \\ &= E[(\mathbf{x}^H \mathbf{y})^H (\mathbf{x}^H \mathbf{y})] = E[|\mathbf{x}^H \mathbf{y}|^2] \geq 0 \end{aligned}$$

   for all vectors $\mathbf{y} \neq \mathbf{0}$. An autocorrelation matrix is thus always positive semidefinite.
   Because $r_x(k,l)$ is dependent just on the difference $k-l$, one can write

$$\mathbf{R} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{bmatrix}.$$

   (1) $\mathbf{R}$ is a Toetplitz matrix, that is all its diagonals have constant values $\Rightarrow \underline{a = d}$. In general: $n \times n$ matrix A is Toeplitz if $a_{ij} = a_{i+1,j+1}$ for all $i < n$ and $j < n$.

   (2) $\mathbf{R}$ is hermitian $\Rightarrow \underline{c = b^*}$ and $\underline{a \in \mathbb{R}}$ (the latter condition can be also deduced from a being the variance of the zero mean process.)

   (3) $\underline{a = d = r_x(0) \geq 0}$. The equality implies zero variance and thus the zero process.

   (4) $\mathbf{R}$ is positive semidefinite $\Rightarrow$ the eigenvalues are greater or equal to zero:

$$|\mathbf{R} - \lambda \mathbf{I}| = \begin{vmatrix} a-\lambda & b \\ b^* & a-\lambda \end{vmatrix} = (a-\lambda)^2 - |b|^2 = 0$$

$$\Rightarrow \lambda_1 = a + |b|, \lambda_2 = a - |b|$$

$$\lambda_1 \geq 0, \lambda_2 \geq 0 \Rightarrow \underline{a \geq |b|}.$$

   (5) the autocorrelations are always finite , that is $\underline{a, |b| < \infty}$.

5. (a) The process $x(n)$ can be written as

$$x(n) = \sum_{k=0}^{n} v(k)$$

   and we deduce that $\mathrm{E}[x(n)] = 0$. The time average can also be represented only using the noise terms:

$$\hat{m}(N) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) = \frac{1}{N} \sum_{k=1}^{N} k v(N-k)$$

A WSS process $x(n)$ is ergodic in the mean, if

$$\lim_{N\to\infty} \mathrm{E}\left[|\hat{m}_x(N) - m_x|^2\right] = 0,$$

where m_x is the mean of the process, and $\hat{m}_x(N)$ the time average over N samples. Here $m_x = 0$, so

$$\mathrm{E}\left[|\hat{m}_x(N) - m_x|^2\right] = \mathrm{E}\left[|\hat{m}_x(N)|^2\right] = \mathrm{E}\left[\hat{m}_x(N)^2\right].$$

This is also the variance of the time average, since it is zero mean (as it is a weighted sum of noise terms). We can apply the rules for variance of a linear combination of independent random variables:

$$\mathrm{Var}(\hat{m}_x(N)) = \frac{1}{N^2}\sum_{k=1}^{N} k^2 \,\mathrm{Var}(v(N-k)) = \frac{1}{N^2}\sum_{k=1}^{N} k^2$$

Using the provided hint we get that

$$= N^{-2}\left(\frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{1}{6}N\right) = \frac{1}{3}N + \frac{1}{2} + \frac{1}{6N}$$

This expression does not go to zero when N increases, i.e., the process is not ergodic in the mean.

(b) Let us first show that the process is zero mean for all $n \geq 0$:

$$\mathrm{E}[y(n)] = \mathrm{E}[0.8y(n-1) + v(n)] = 0.8\,\mathrm{E}[y(n-1)] = 0.8^n\,\mathrm{E}[y(0)] = 0$$

The ergodicity theorems deal with autocovariance $c_y(k)$. For the zero mean process $y(n)$ this is the same as the autocorrelation $r_y(k) = \mathrm{E}[y(k)y^*(0)]$.
According to Mean Ergodic Theorem 2[1], a sufficient condition for the process being ergodic in the mean is that $c_y(0) < \infty$ and

$$\lim_{k\to\infty} c_y(k) = 0.$$

As in exercise 2, we have that

$$r_y(0) = \mathrm{E}[y(n)y^*(n)] = \mathrm{E}[(0.8y(n-1))^2] + \mathrm{E}[(v(n))^2] = 0.64r_y(0) + 1$$

from which we can solve $r_y(0) = 1/0.36 \approx 2.778$, and

$$\begin{aligned}
r_y(1) &= \mathrm{E}[y(n)y^*(n-1)] = \mathrm{E}[0.8|y(n-1)|^2 + v(n)y^*(n-1)] = 0.8r_y(0)\\
&\vdots\\
r_y(k) &= 0.8^k r_y(0) \qquad k > 0\\
r_y(-k) &= 0.8^k r_y(0) \qquad k > 0
\end{aligned}$$

We see that $c_y(0) = r_y(0) \approx 2.778 < \infty$ and

$$\lim_{k\to\infty} c_y(k) = \lim_{k\to\infty} r_y(k) = \lim_{k\to\infty} 0.8^k r_y(0) = 0.$$

By Mean Ergodic Theorem 2, the process $y(x)$ is thus ergodic in the mean.

[1] The theorem assumes that the process is WSS. The exercise problem as stated leaves open the possibility that $y(n)$ would be unstable in the sense that the variance is infinite or grows without bound as $n \to -\infty$. We need to assume something like $\sup_n \mathrm{Var}(y(n)) < \infty$, which would ensure that $y(n)$ is WSS.