T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 1, September 15, 2011

The first problem set does not contain a bonus point exercise.

1. A square matrix $\mathbf{U}$ is *unitary* if $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|$ for all vectors $\mathbf{x}$. Here the norm of the vector $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{|x_1|^2 + |x_2|^2 + \ldots} = \sqrt{\mathbf{x}^H\mathbf{x}}$.
   (a) The vectors $\mathbf{x}_i$, $i = 1, \ldots, N$ are mutually orthonormal if $\mathbf{x}_i^H\mathbf{x}_j = \delta_{ij}$, where
   $$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$
   Are the columns of an unitary matrix orthonormal?
   (b) Let $\mathbf{U}_1$ and $\mathbf{U}_2$ be unitary. Is $\mathbf{U}_1\mathbf{U}_2$ also unitary?
   (c) Is the inverse of an unitary matrix $\mathbf{U}$ unitary?

2. A matrix $\mathbf{A}$ is *Hermitian* if $\mathbf{A} = \mathbf{A}^H$, and *positive definite* if $\mathbf{x}^H\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. A Hermitian conjugate of a matrix is formed by transposing the matrix and then replacing the elements by their complex conjugates.

   Let $\mathbf{A}$ be Hermitian and positive definite.
   (a) Let λ be an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ the corresponding eigenvector, i.e. $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Show that λ is real and positive.
   (b) For any Hermitian matrix $\mathbf{A}$ a spectral decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$ can be written. Here $\mathbf{U}$ is unitary and $\mathbf{\Lambda}$ is a real valued diagonal matrix. Show that the eigenvectors of $\mathbf{A}$ can be chosen to be mutually orthogonal.

3. Let $\mathbf{B}$ be a square matrix and $\mathbf{D} = \mathbf{B}^H\mathbf{B}$. What can you say about the definiteness of $\mathbf{D}$?

4. Consider the discrete filter with difference equation $y(n) - ay(n-1) = b_1x(n) + b_2x(n-1)$ where $x(n)$ is input and $y(n)$ is output.
   (a) One solution of the difference equation is $y(n) = h(n) * x(n)$. Determine this $h(n)$ or, alternatively, its z-transform $H(z)$.
   (b) Show that the system defined by the solution in (a) is linear and shift-invariant (LSI).
   (c) Let $a = 0.9$, $b_1 = 1$ and $b_2 = 0$. Is the system stable?
   (d) (demo) What are the other solutions of the difference equation? Do they define LSI systems?

5. During this course we often need to find the optima of real valued functions that depend on complex valued parameters. Differentiation is not as straightforward as in the case of real valued parameters.
   (a) What is the derivative of az^2 with respect to z?
   (b) What is the derivative of z^* with respect to z?
   (c) How to find the optimum of $|z - 1|^2$?