T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 5, October 13, 2011, Solutions

1. (a)

$$\begin{aligned}
r_y(m) &\triangleq \mathrm{E}[y(n+m)y(n)] \\
&= \mathrm{E}\{[x(n+k+m) - x(n+m-k)][x(n+k) - x(n-k)]\} \\
&= \mathrm{E}[x(n+m+k)x(n+k)] - \mathrm{E}[x(n+m+k)x(n-k)] \\
&\quad -\mathrm{E}[x(n+m-k)x(n+k)] + \mathrm{E}[x(n+m-k)x(n-k)] \\
&= \underline{2r_x(m) - r_x(m-2k) - r_x(m+2k)}.
\end{aligned}$$

   (b) The power spectrum is obtained from the previously calculated autocorrelation via the Fourier transform

$$P_y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} r_y(m)e^{-j\omega m}$$

   or more easily by using the transfer function H as

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega}).$$

   The transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(z^k - z^{-k})X(z)}{X(z)} = z^k - z^{-k},$$

   using the Fourier notation

$$\begin{aligned}
H(e^{j\omega}) &= e^{jk\omega} - e^{-jk\omega} \\
&= 2j\sin(k\omega) \\
\Rightarrow \quad |H(e^{j\omega})|^2 &= 4\sin^2(k\omega) \\
\Rightarrow \quad P_y(e^{j\omega}) &= \underline{4P_x(e^{j\omega})\sin^2(k\omega)}.
\end{aligned}$$

2. The transfer function of the system described by $y(n) = -x(n) + 0.5x(n-1)$ is

$$H(z) = -1 + 0.5z^{-1}.$$

   The signal with power spectrum $P_x(e^{j\omega}) = 1$ is filtered through the system.

   (a) The power spectrum of the output is

$$\begin{aligned}
P_y(e^{j\omega}) &= |H(e^{j\omega})|^2 P_x(e^{j\omega}) = (-1 + 0.5e^{-j\omega})(-1 + 0.5e^{j\omega})P_x(e^{j\omega}) \\
&= [1 - 0.5(e^{j\omega} + e^{-j\omega}) + 0.25] \\
&= \underline{1.25 - \cos\omega}.
\end{aligned}$$

   (b) The autocorrelation sequence of the output can be calculated as the inverse Fourier transform

$$\begin{aligned}
r_y(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_y(e^{j\omega})e^{jm\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1.25e^{jm\omega} - 0.5e^{j(m+1)\omega} - 0.5e^{j(m-1)\omega})\,d\omega.
\end{aligned}$$

Because

$$\int_{-\pi}^{\pi} e^{jm\omega}\, d\omega = \begin{cases} 2\pi, & m = 0 \\ 0, & m = \pm 1, \pm 2, \ldots \end{cases},$$

we have

$$\underline{r_y(m) = \begin{cases} 1.25, & m = 0 \\ -0.5, & m = \pm 1 \\ 0, & \text{otherwise} \end{cases}}.$$

3. The autoregressive model is $y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 v(n)$, $\mathrm{E}[v(n)] = 0$, $\mathrm{E}[v^2(n)] = 1$. Perform the z-transform

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = b_0 V(z),$$

resulting in the transfer function

$$\begin{aligned} H(z) &= \frac{Y(z)}{V(z)} = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} \\ &= \frac{b_0 z^2}{z^2 + a_1 z + a_2}. \end{aligned}$$

(a) The system is WSS iff it is stable $\Leftrightarrow$ the poles p_1, p_2 of $H(z)$ are inside the unit circle $\Leftrightarrow |p_1|, |p_2| < 1$. Solving for p_1, p_2:

$$p_1, p_2 = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_2}\right).$$

The system is stable when $\max\{|p_1|, |p_2|\} < 1$

(b) Now $a_1 = -0.1$ and $a_2 = -0.8$, so the poles are at

$$p_1, p_2 = \frac{1}{2}\left(0.1 \pm \sqrt{0.01 + 4 \cdot 0.8}\right).$$

$p_1 \approx 0.9458$ and $p_2 \approx -0.8458$, implying the system is stable and also WSS.

4. (a) The power spectrum of $y(n)$ is found by

$$P_y(z) = P_x(z) H(z) H^*(1/z^*)$$

so first we must find the power spectrum of $x(n)$:

$$\begin{aligned} P_x(z) &= \sum_{k=-\infty}^{\infty} r_x(k) z^{-k} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{k} - 1 \\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{1}{2}z} - 1 = \frac{1 - \frac{1}{2}z^{-1} + 1 - \frac{1}{2}z - (1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)} \\ &= \frac{3}{4} \cdot \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)} \end{aligned}$$

Then,

$$\begin{aligned} P_y(z) &= P_x(z) H(z) H^*(1/z^*) = \frac{3}{4} \cdot \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)} \cdot \frac{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{3}z)} \\ &= \frac{3}{4} \cdot \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{3}z)} \end{aligned}$$

(b) (The power spectrum could be directly identified as an AR(1)-process, e.g., by looking at equation 3.118 of Hayes' book)
Using partial fractions, the power spectrum can be expanded as

$$\begin{aligned}
P_y(z) &= \frac{3}{4}\cdot\frac{1}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{3}z)} = -\frac{9}{4}\cdot\frac{z^{-1}}{(1-\frac{1}{3}z^{-1})(1-3z^{-1})} \\
&= -\frac{9}{4}\left(-\frac{3}{8}\cdot\frac{1}{1-\frac{1}{3}z^{-1}}+\frac{3}{8}\cdot\frac{1}{1-3z^{-1}}\right) \\
&= \frac{27}{32}\left(\frac{1}{1-\frac{1}{3}z^{-1}}-\frac{1}{1-3z^{-1}}\right)
\end{aligned}$$

Then

$$\begin{aligned}
r_y(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_y(e^{j\omega})e^{jk\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3}{4}\cdot\frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{3}e^{j\omega})}e^{jk\omega}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{27}{32}\left(\frac{1}{1-\frac{1}{3}e^{-j\omega}}-\frac{1}{1-3e^{-j\omega}}\right)e^{jk\omega}\,d\omega
\end{aligned}$$

Expanding to power series (note that in the second expansion the magnitude is > 1):

$$\begin{aligned}
&= \frac{27}{32}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\omega}\right)^{n}+\sum_{n=1}^{\infty}\left(3e^{-j\omega}\right)^{-n}\right)e^{jk\omega}\,d\omega \\
&= \frac{27}{32}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{|n|}e^{-j\omega n}\right)e^{jk\omega}\,d\omega \\
&= \frac{27}{32}\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{|n|}\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega(k-n)}\,d\omega}_{=\delta(k-n)} = \frac{27}{32}\left(\frac{1}{3}\right)^{|k|}
\end{aligned}$$

(c) First, writing $H(z)$ as

$$H(z) = \frac{3}{2}-\frac{1}{2}\cdot\frac{1}{1-\frac{1}{3}z^{-1}}$$

by using partial fractions, we can see that the unit response is

$$h(n) = \frac{3}{2}\delta(n)-\frac{1}{2}\left(\frac{1}{3}\right)^{n}u(n)$$

Using this, the cross-correlation $r_{xy}(k)$ can be calculated starting from the definition

$$\begin{aligned}
r_{xy}(k) =& r_{xy}(n+k,n) = \mathrm{E}[x(n+k)y(n)^*] = \mathrm{E}\left[x(n+k)\sum_{m=-\infty}^{\infty}h(m)^*x(n-m)^*\right] \\
=& \sum_{m=-\infty}^{\infty}h(m)^*\underbrace{\mathrm{E}\left[x(n+k)x(n-m)^*\right]}_{r_x(k+m)} = \sum_{m=-\infty}^{\infty}\left(\frac{3}{2}\delta(m)-\frac{1}{2}\left(\frac{1}{3}\right)^{m}u(m)\right)\left(\frac{1}{2}\right)^{|k+m|} \\
=& \frac{3}{2}\left(\frac{1}{2}\right)^{|k|}-\sum_{m=0}^{\infty}\frac{1}{2}\left(\frac{1}{3}\right)^{m}\left(\frac{1}{2}\right)^{|k+m|}
\end{aligned}$$

Now there's two cases, first assume $k \leq 0$

$$
\begin{aligned}
r_{xy}(k) &= \frac{3}{2}\left(\frac{1}{2}\right)^{-k} - \sum_{m=0}^{-k}\frac{1}{2}\left(\frac{1}{3}\right)^{m}\left(\frac{1}{2}\right)^{|k+m|} - \sum_{m=-k+1}^{\infty}\frac{1}{2}\left(\frac{1}{3}\right)^{m}\left(\frac{1}{2}\right)^{|k+m|} \\
&= \frac{3}{2}\left(\frac{1}{2}\right)^{k} - \sum_{m=0}^{-k}\frac{1}{2}\left(\frac{1}{3}\right)^{m}\left(\frac{1}{2}\right)^{-k-m} - \sum_{m=-k+1}^{\infty}\frac{1}{2}\left(\frac{1}{3}\right)^{m}\left(\frac{1}{2}\right)^{k+m} \\
&= \frac{3}{2}\left(\frac{1}{2}\right)^{-k} - \frac{1}{2}\left(\frac{1}{2}\right)^{-k}\sum_{m=0}^{-k}\left(\frac{2}{3}\right)^{m} - \left(\frac{1}{2}\right)^{k+1}\sum_{m=-k+1}^{\infty}\left(\frac{1}{6}\right)^{m} \\
&= \frac{3}{2}\left(\frac{1}{2}\right)^{-k} - \frac{1}{2}\left(\frac{1}{2}\right)^{-k}\frac{1-(\frac{2}{3})^{-k+1}}{1-\frac{2}{3}} - \left(\frac{1}{2}\right)^{k+1}6^{k-1}\frac{1}{1-\frac{1}{6}} \\
&= \frac{3}{2}\left(\frac{1}{2}\right)^{-k} - \frac{3}{2}\left(\frac{1}{2}\right)^{-k}\left(1-\left(\frac{2}{3}\right)^{-k+1}\right) - \left(\frac{1}{2}\right)^{k+1}6^{k-1}\frac{6}{5} \\
&= \frac{3}{2}\left(\frac{1}{2}\right)^{-k} - \frac{3}{2}\left(\frac{1}{2}\right)^{-k} + \left(\frac{1}{3}\right)^{-k} - \frac{1}{10}\left(\frac{1}{3}\right)^{-k} \\
&= \frac{9}{10}\left(\frac{1}{3}\right)^{-k}
\end{aligned}
$$

And then for $k > 0$

$$
\begin{aligned}
r_{xy}(k) &= \frac{3}{2}\left(\frac{1}{2}\right)^{k} - \sum_{m=0}^{\infty}\frac{1}{2}\left(\frac{1}{3}\right)^{m}\left(\frac{1}{2}\right)^{k+m} = \frac{3}{2}\left(\frac{1}{2}\right)^{k} - \left(\frac{1}{2}\right)^{k+1}\sum_{m=0}^{\infty}\left(\frac{1}{6}\right)^{m} \\
&= \frac{3}{2}\left(\frac{1}{2}\right)^{k} - \left(\frac{1}{2}\right)^{k+1}\frac{1}{1-\frac{1}{6}} = \frac{3}{2}\left(\frac{1}{2}\right)^{k} - \left(\frac{1}{2}\right)^{k+1}\frac{6}{5} \\
&= \frac{3}{2}\left(\frac{1}{2}\right)^{k} - \frac{3}{5}\left(\frac{1}{2}\right)^{k} = \frac{3}{2}\left(\frac{1}{2}\right)^{k} - \frac{3}{5}\left(\frac{1}{2}\right)^{k} = \frac{9}{10}\left(\frac{1}{2}\right)^{k}
\end{aligned}
$$

The complete cross-correlation can thus be written as

$$
r_{xy}(k) = \frac{9}{10}\left(\left(\frac{1}{2}\right)^{k}u(k) + \left(\frac{1}{3}\right)^{-k}u(-k) - \delta(k)\right)
$$

(d) The *cross-power spectral density*, $P_{xy}(z)$, is found by taking the z-transform of the cross-correlation $r_{xy}(k)$.

$$
\begin{aligned}
P_{xy}(z) &= \frac{9}{10}\left(\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z} - 1\right) = \frac{9}{10}\left(\frac{1-\frac{1}{3}z+1-\frac{1}{2}z^{-1}-1+\frac{1}{3}z+\frac{1}{2}z^{-1}-\frac{1}{6}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z)}\right) \\
&= \frac{9}{10}\left(\frac{\frac{5}{6}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z)}\right) = \frac{3}{4}\frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z)}
\end{aligned}
$$

Here we can recognise that $P_{xy}(z) = P_x(z)H^*(1/z^*)$, which can be shown to hold in general by noting that $r_{xy}(k) = r_x(k) * h(-k)$ and taking the z-transform.

5. (a) The variance of the sum can be found as the sum of the separate variances, since the terms are uncorrelated

$$\mathrm{Var}(x(n)) = \sum_{l=0}^{2} b^2(l)\,\mathrm{Var}(v(n-l)) = \sum_{l=0}^{2} b^2(l) = 1 + 0.49 + 0.04 = 1.53$$

(b) Considering $x(n)$ as the output of filtering white noise, we have $x(n) = h(n) * v(n)$, where $h(n)$ is the impulse response of the filter. Comparing this form to

$$x(n) = b(0)v(n) + b(1)v(n-1) + b(2)v(n-2)\ .$$

We identify that $h(k) = b(k)$ for $k = \{0, 1, 2\}$, and 0 otherwise, or

$$h(n) = b(0)\delta(n) + b(1)\delta(n-1) + b(2)\delta(n-2)$$

The process is $y(n)$ was defined as filtering $x(n)$ again by the same filter, so using the associativity of convolution:

$$y(n) = h(n) * x(n) = h(n) * h(n) * v(n) = h_2(n) * v(n)$$

where $h_2(n) = h(n) * h(n)$. As only the first three values of $h(n)$ are non-zero, by studying the expression

$$h_2(n) = h(n) * h(n) = \sum_{m=-\infty}^{\infty} h(m)h(n-m)$$

we can see that $h_2(n)$ is non-zero only for $0 \le n \le 4$. Hence $y(n)$ can be modeled as an MA(4)-process.

(c) We can find the variance by considering the autocorrelations. First, find the autocorrelation of $x(n)$

$$r_x(k) = \delta(k) * h(k) * h^*(-k) = h(k) * h(-k) = \sum_{m=-\infty}^{\infty} h(m)h(-(k-m)).$$

From this
$r_x(0) = h(0)^2 + h(1)^2 + h(2)^2 = 1.53$,
$r_x(\pm 1) = h(0)h(1) + h(1)h(2) = 0.84$ and
$r_x(\pm 2) = h(0)h(2) = 0.2$.
The rest are 0.
By the same idea, the autocorrelation for $y(n)$ is

$$r_y(k) = r_x(k) * h(k) * h(-k) = r_x(k) * r_x(k) = \sum_{m=-\infty}^{\infty} r_x(m)r_x(k-m).$$

By substituting $k = 0$

$$\mathrm{Var}\,(y(n)) = r_y(0) = \sum_l r_x(l)r_x(-l) = \sum_l r_x(l)^2 = 1.53^2 + 2 \cdot 0.84^2 + 2 \cdot 0.2^2 \approx 3.83.$$