T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola
Exercise 9, November 17, 2011, Solutions

1. This problem is to remind us of the basics: the connection between the autocorrelations and the power spectrum through the Fourier transform.

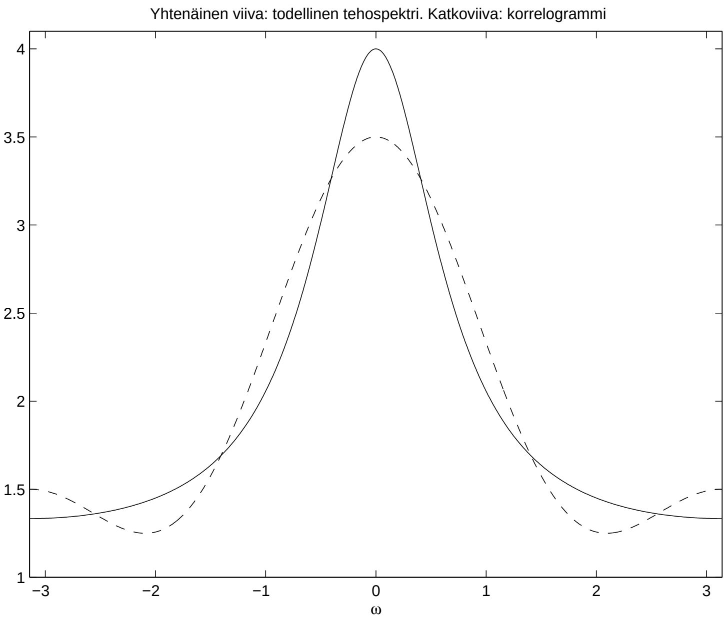
   (a) Start from the definition of the power spectrum:

$$\begin{aligned}
P_x(e^{j\omega}) &= \sum_{l=-\infty}^{\infty} r_x(l)e^{-j\omega l} = 1 + \sum_{l=-\infty}^{\infty} 2^{-|l|}e^{-j\omega l} \\
&= \sum_{l=0}^{\infty} 2^{-l}e^{-j\omega l} + \sum_{l=-\infty}^{0} 2^{l}e^{-j\omega l} = \sum_{l=0}^{\infty}(2e^{j\omega})^{-l} + \sum_{l=0}^{\infty}(2e^{-j\omega})^{-l} \\
&= \frac{1}{1-[2e^{j\omega}]^{-1}} + \frac{1}{1-[2e^{-j\omega}]^{-1}} = \frac{1-1/[2e^{j\omega}]+1-1/[2e^{-j\omega}]}{(1-1/[2e^{j\omega}])(1-1/[2e^{-j\omega}])} \\
&= \frac{2-\frac{1}{2}e^{-j\omega}-\frac{1}{2}e^{j\omega}}{\frac{5}{4}-\frac{1}{2}e^{-j\omega}-\frac{1}{2}e^{j\omega}} = \frac{2-\cos\omega}{5/4-\cos\omega}
\end{aligned}$$

   (b) In the correlogram $L = 2$ is chosen, so that one needs the autocorrelations $r_x(0) = 2, r_x(\pm 1) = 1/2, r_x(\pm 2) = 1/4$. This gives

$$\begin{aligned}
\hat{P}_x(e^{j\omega}) &= \sum_{l=-L}^{L} r_x(l)e^{-j\omega l} = \frac{1}{4}e^{j2\omega} + \frac{1}{2}e^{j\omega} + 2 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \\
&= \frac{1}{2}\cos(2\omega) + \cos(\omega) + 2
\end{aligned}$$

   Note that here we use the true autocorrelations $r_x(k)$, not their estimates $\hat{r}_x(k)$ as in the periodogram. Therefore, the correlogram is not exactly a generalised version of the periodogram. The following figure compares the true power spectrum (—) and its estimate, the correlogram (- - -).



2. Going through this exercise, we get an interpretation of the periodogram as a rough estimate of the powers in the outputs of a filter bank of bandpass filters.

   (a) At the frequency ω_0, the Fourier transform is given by the sum:

   $$X_N(e^{j\omega_0}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega_0 n}.$$

   The convolution as the sum is:

   $$y_{\omega_0}(0) = \sum_{k=-\infty}^{\infty} h_{\omega_0}(0-k)x(k).$$

   These expressions are identical, if the impulse response is chosen as

   $$h_{\omega_0}(k) = \begin{cases} e^{j\omega_0 k} & -N+1 \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

   (b)

   $$H_{\omega_0}(e^{j\omega}) = \sum_k h_{\omega_0}(k)e^{-j\omega k} = \sum_{k=0}^{N-1} e^{-j\omega_0 k}e^{j\omega k} = \sum_{k=0}^{N-1} q^k = \frac{q^N-1}{q-1} = \frac{q^{N/2}(q^{N/2}-q^{-N/2})}{q^{1/2}(q^{1/2}-q^{-1/2})},$$

   with q being the ratio of the adjacent terms in the geometric series $q = e^{j(\omega-\omega_0)}$. On the unit circle, this expression yields sinusoidal terms of the desired form.

   $$H_{\omega_0}\left(e^{j\omega}\right) = e^{j(N-1)(\omega-\omega_0)/2}\frac{\sin(\frac{N(\omega-\omega_0)}{2})}{\sin(\frac{\omega-\omega_0}{2})}$$

   and

   $$|H_{\omega_0}\left(e^{j\omega}\right)| = \left|\frac{\sin(\frac{N(\omega-\omega_0)}{2})}{\sin(\frac{\omega-\omega_0}{2})}\right|.$$

   For small $\Delta\omega = \omega - \omega_0$ the sine function is almost linear and the amplitude response is approximately N. It goes to zero when $\Delta\omega = \pm\frac{2\pi}{N}$ and the numerator is 0. As $\Delta\omega$ deviates further from zero, the denominator deviates more from zero and the amplitude response diminishes strongly as the nominator keeps oscillating in the interval $[-1, 1]$, creating sidelobes. On the unit circle, the denominator grows monotonically and reaches its maximum value 1 when the frequency deviation reaches its maximum value π on the periodic unit circle. $h_{\omega_0}(n)$ thus is a bandpass filter with center frequency ω_0. The passband gets narrower as N increases.
   For the filtered signal $y_{\omega_0}(n) = h_{\omega_0}(n) * x(n)$

   $$\frac{1}{N}\,\mathrm{E}[|y_{\omega_0}(n)|^2] \approx P_x(e^{j\omega_0}),$$

   that is, the power in the filtered signal approximately gives the value of the power spectrum of $x(n)$ at the frequency ω_0. The periodogram can be interpreted as a rough estimate of $\frac{1}{N}\,\mathrm{E}[|y_{\omega_0}(n)|^2]$ as the expression of the periodogram at the frequency ω_0 is $\hat{P}_x(e^{j\omega_0}) = \frac{1}{N}|y_{\omega_0}(N-1)|^2$ (see Hayes p.397 for more details). This gives the periodogram an interpretation as a filter bank that has a separate bandpass filter $h_{\omega_0}(n)$ for each frequency ω_0.

3. In this problem we go through one application of the power spectrum that was mentioned in the lectures.

   (a) Let's assume that the signal arrives in angle θ that is defined as the deviation from the direction of the normal of the sensor plane. Let the signal arriving at the first sensor be

$$s_0(t) = Ae^{j\omega t}$$

   At the m:th sensor the observed signal $s_m(t)$ is the same sinusoid but in different phase. This is because the signal has to travel a longer way to reach the sensor m, and the signal propagates with a finite speed. For the excess distance x travelled we have

$$\sin\theta = \frac{x}{md} \Rightarrow x = md\sin\theta$$

   (plot a figure!) The signal travels this distance in time x/c, with c the known propagation speed of the wave. If the frequency of the sinusoid is ω, during a time interval t the phase changes $t\omega$ radians. During the time x/c required for travelling the excess distance x the phase change is thus k_m radians, with

$$k_m = \omega \cdot \frac{x}{c} = \frac{\omega md\sin\theta}{c}$$

   Using this, we get for the observed signal at the m:th sensor

$$s_m(t) = Ae^{j[\omega t - k_m]} = Ae^{j[\omega t - \frac{\omega md\sin\theta}{c}]}$$

   (b) Fix the time $t = t_0$ and denote $r(m) = s_m(t_0)$. In the expression

$$r(m) = Ae^{j\omega t_0}e^{\frac{-j\omega md\sin\theta}{c}}$$

   the frequency (with respect to the spatial index variable m) is the coefficient in the exponent, i.e.,

$$\frac{\omega d\sin\theta}{c}$$

   The spatial frequency can be determined by estimating the power spectrum of $r(m)$. When ω, d and c, are known, the angle of arrival θ can be found from the estimate.

4. This problem reveals a (possibly unexpected) connection between maximum likelihood estimation and the periodogram.

   The likelihood function is "the probability of the observations once the parameters have been fixed.". In this case, each observation is normally distributed as the noise is Gaussian and there is no other source of randomness in the observations. The expected value of the normal distribution is $Ae^{j\omega n}$ (since the noise has zero mean) and the variance is $\mathrm{Var}(v(n))$. The joint distribution for the whole set of observations can compactly expressed using vector notation

$$\begin{aligned}
\mathbf{x} &= [x(0), \ldots, x(N-1)]^T \\
\mathbf{s} &= A[e^{j0\omega}, \ldots, e^{j(N-1)\omega}]^T \\
\mathbf{v} &= [v(0), \ldots, v(N-1)]^T.
\end{aligned}$$

Then $\mathbf{x} = \mathbf{s}+\mathbf{v}$ and $\mathbf{x} \sim N(\mathbf{s}, \mathbf{R_v})$ with $\mathbf{R_v}$ being the correlation matrix of the noise vector v. As the noise is white, $\mathbf{R_v} = \sigma^2 I$. The likelihood function is the multi-dimensional normal distribution of a complex vector

$$p(\mathbf{x}|A,\omega) = \frac{1}{|\pi \mathbf{R_v}|} \exp\left(-\frac{1}{\sigma^2}(\mathbf{x}-\mathbf{s})^H(\mathbf{x}-\mathbf{s})\right)$$

The parameters A, ω that are to be solved appear solely in the exponent (in the expression for $\mathbf{s}$). Thus the likelihood function attains its maximum when the quantity

$$J = (\mathbf{x}-\mathbf{s})^H(\mathbf{x}-\mathbf{s})$$

is minimised. Let's denote $\mathbf{e} = [e^{j0\omega}, \ldots, e^{j(N-1)\omega}]^T$. Then $\mathbf{s} = A\mathbf{e}$. Let's first consider a constant value of ω. Then the vector $\mathbf{e}$ is constant, and J can be minimised w.r.t. the complex scalar value A:

$$\begin{aligned} J &= (\mathbf{x}-A\mathbf{e})^H(\mathbf{x}-A\mathbf{e}) \\ &= \mathbf{x}^H\mathbf{x} - A\mathbf{x}^H\mathbf{e} - A^*\mathbf{e}^H\mathbf{x} + A^*A\mathbf{e}^H\mathbf{e} \end{aligned}$$

We remember that the real-valued function of the complex variable A can be optimised by differentiating w.r.t. A, regarding the A^* as a constant:

$$\begin{aligned} \frac{\partial J}{\partial A} &= -\mathbf{x}^H\mathbf{e} + A^*\mathbf{e}^H\mathbf{e} = 0 \\ \Rightarrow A^* &= \frac{\mathbf{x}^H\mathbf{e}}{\mathbf{e}^H\mathbf{e}} \\ A &= \frac{\mathbf{e}^H\mathbf{x}}{\mathbf{e}^H\mathbf{e}}. \end{aligned}$$

We note also that

$$\mathbf{e}^H\mathbf{e} = \sum_{n=0}^{N-1} |e^{jn\omega}|^2 = \sum_{n=0}^{N-1} 1 = N$$

Inserting these into the expression for J gives

$$J = \mathbf{x}^H\mathbf{x} - \frac{|\mathbf{x}^H\mathbf{e}|^2}{N}$$

Let's then minimise J w.r.t. ω. $\mathbf{x}$ does not change when ω changes. What remains to be minimised is $-\frac{1}{N}|\mathbf{e}^H\mathbf{x}|^2$, so one can maximise

$$\frac{1}{N}|\mathbf{e}^H\mathbf{x}|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-jn\omega}\right|^2 = \hat{P}_{per}(e^{j\omega})$$

which happens to be the periodogram of the process $x(n)$. The maximum likelihood estimate of the frequency ω is thus given by the maximum of the periodogram.

$$\omega_{ML} = \arg\max_{\omega} \hat{P}_{per}(e^{j\omega})$$

Finally, the phase and amplitude of A can be determined when the optimal frequency ω is known as

$$A_{ML} = \frac{1}{N}\mathbf{e}^H\mathbf{x} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-jn\omega_{ML}} = X_N(e^{j\omega_{ML}})$$

which is the discrete-time Fourier transform of $x(n)$ evaluated at the optimal frequency ω_{ML}.

5. (a)

$$\begin{aligned}
\hat{P}_x(\exp(j\omega)) &= \frac{1}{N}|X(\exp(j\omega)|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\exp(-j\omega n)\right|^2 \\
&= \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\right|^2 \qquad \text{(insert } \omega = 0\text{)} \\
&= \frac{1}{N}\left(\sum_{n=0}^{N-1} x(n)\right)^2 \qquad \text{(the observations are real-valued)}
\end{aligned}$$

(b)

$$\hat{P} = \left(\frac{\sum_n x(n)}{\sqrt{N}}\right)^2 = z^2,$$

where $z = \sum_n x(n)/\sqrt{N}$. Now z is normally distributed, being a linear combination of normally distributed variables. $\mathrm{E}[z] = 0$, as $\mathrm{E}[x(n)] = 0$, and

$$\mathrm{Var}(z) = \sum_n \mathrm{Var}\left(\frac{x(n)}{\sqrt{N}}\right) = \frac{1}{N}\sum_n \mathrm{Var}(x(n)) = \frac{1}{N}N = 1.$$

Hence $z \sim N(0,1)$.

$$\mathrm{E}[\hat{P}] = \mathrm{E}[z^2] = \mathrm{Var}(z) = 1,$$

and the estimator is unbiased.

$$\mathrm{Var}(\hat{P}) = \mathrm{Var}(z^2) = \mathrm{E}[z^4] - \mathrm{E}[z^2]^2 = 3 - 1^2 = 2.$$

This appears to contradict the book and lecture slides, which state that

$$\mathrm{Var}(\hat{P}) = \sigma_v^4 (= 1)$$

The issue is resolved by the fact that here we are considering real-valued noise, while there complex white noise was treated.

(c) The Bartlett method with $L = 1$ is a periodogram of one observation, i.e., the estimator from part (a) with $N = 1$:

$$\hat{P}_x^n(\exp(j0)) = x^2(n)$$

The Bartlett method is to average all these periodograms:

$$\hat{P}(\exp(j0)) = \frac{1}{N}\sum_n \hat{P}_x^n(\exp(j0)) = \frac{1}{N}\sum_n x^2(n).$$

$$\mathrm{E}[\hat{P}] = \frac{1}{N}\sum_n \mathrm{E}[x^2(n)] = \frac{1}{N}\sum_n 1 = \frac{1}{N}N = 1.$$

The estimator is unbiased.

$$\mathrm{Var}(\hat{P}) = \mathrm{Var}(\frac{1}{N}\sum_n x^2(n)) = \frac{1}{N^2}\sum_n \mathrm{Var}(x^2(n)) = \frac{1}{N^2}\sum_n 2 = \frac{2}{N}.$$

The variance of the Bartlett method goes to 0, when $N \to \infty$, unlike for the periodogram. The weakness is that the frequency resolution is significantly reduced.