T-61.3040 Statistical Signal Modeling Lendasse, Miche, Eirola

Exercise 10, December 1, 2011, Solutions

1. Let's find the autocorrelation of the general case of a real sinusoid $s(t) = A\sin(\omega t + \phi) = \frac{A}{2j}\left(e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right)$ by writing the sine in terms of complex exponentials.

$$\begin{aligned}
r_s(k) &= \mathrm{E}[s(t)s^*(t-k)] = \mathrm{E}\left[\frac{A}{2j}\left(e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right)\frac{A^*}{-2j}\left(e^{-j(\omega(t-k)+\phi)} - e^{j(\omega(t-k)+\phi)}\right)\right] \\
&= \frac{|A|^2}{4}\,\mathrm{E}\left[e^{j(\omega t+\phi-\omega t+\omega k-\phi)} - e^{j(\omega t+\phi+\omega t-\omega k+\phi)} - e^{j(-\omega t-\phi-\omega t+\omega k-\phi)} + e^{j(-\omega t-\phi+\omega t-\omega k+\phi)}\right] \\
&= \frac{|A|^2}{4}\,\mathrm{E}\left[e^{j\omega k} - e^{j(2\omega t-\omega k+2\phi)} - e^{j(-2\omega t+\omega k-2\phi)} + e^{-j\omega k}\right] \\
&= \frac{|A|^2}{4}\left(e^{j\omega k} + e^{-j\omega k} - e^{j(2\omega t-\omega k)}\,\mathrm{E}\left[e^{2j\phi}\right] - e^{j(-2\omega t+\omega k)}\,\mathrm{E}\left[e^{-2j\phi)}\right]\right) \\
&= \frac{|A|^2}{4}\left(e^{j\omega k} + e^{-j\omega k}\right) = \frac{|A|^2}{2}\cos(\omega k)
\end{aligned}$$

The expectations are 0, since

$$\mathrm{E}\left[e^{j\phi}\right] = \frac{1}{2\pi}\int_0^{2\pi} e^{j\phi}\,d\phi = \frac{1}{2\pi j}e^{j\phi}\Big|_{\phi=0}^{2\pi} = 0$$

So $r_x(k) = 0.005\cos(0.5k)$ and $r_y(k) = 0.02\cos(0.2k)$.

We still need to evaluate

$$\begin{aligned}
r_z(k) &= \mathrm{E}[(x(t)+y(t))(x^*(t-k)+y^*(t-k))] \\
&= \mathrm{E}[x(t)x^*(t-k)] + \mathrm{E}[y(t)y^*(t-k)] + \mathrm{E}[x(t)y^*(t-k)] + \mathrm{E}[y(t)x^*(t-k)] \\
&= r_x(k) + r_y(k) + r_{xy}(k) + r_{yx}(k)
\end{aligned}$$

The cross-correlations on the last line are zero because x and y are independent and have zero mean. Then

$$r_z(k) = r_x(k) + r_y(k) = 0.005\cos(0.5k) + 0.02\cos(0.2k)$$

2. The easy way:

Split the signal into $s(t)$ and noise $v(t)$. As they are independent from each other, $r_x(k) = r_s(k) + r_v(k)$. The noise is white, so $r_v(k) = \sigma^2\delta(k)$. For the signal:

$$\begin{aligned}
r_s(0) &= \mathrm{E}[Ae^{j\omega t}A^*e^{-j\omega t}] = |A|^2 \\
r_s(1) &= \mathrm{E}[Ae^{j\omega t}A^*e^{-j\omega(t-1)}] = |A|^2e^{j\omega}
\end{aligned}$$

Recall that the only random part in this process is the *phase* of A. Now

$$\begin{cases} r_x(0) = r_s(0) + r_v(0) \\ r_x(1) = r_s(1) + r_v(1) \end{cases} \implies \begin{cases} 2 = |A|^2 + \sigma^2 \\ j = |A|^2e^{j\omega} \end{cases}$$

From the second equation we immediately see that $|A|^2 = 1$ and $\omega = \pi/2$. Then we get from the first equation that $\sigma^2 = 1$.

The more general way:

The $M \times M$ correlation matrix is ($M = 2$)

$$\begin{aligned}\mathbf{R}_x &= \mathbf{R}_s + \mathbf{R}_n \\ &= P_1 \begin{bmatrix} 1 & e^{-j\omega} \\ e^{j\omega} & 1 \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\end{aligned}$$

where the rank of $\mathbf{R}_s$ is 1, i.e. it has only one eigenvalue that is different from zero. That value is MP_1 with P_1 being the power of the sinusoid. Let's solve the eigenvalue equation of the given correlation matrix:

$$\begin{vmatrix} 2-\lambda & -j \\ j & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

This yields the eigenvalues $\lambda_1 = 3, \lambda_2 = 1$. The smaller of the eigenvalues gives directly the noise variance, i.e. $\sigma^2 = 1$. The larger eigenvalue is $\lambda_1 = MP_1 + \sigma^2 = 2P_1 + 1 = 3$, so that the power of the sinusoid is $P_1 = |A|^2 = 1$.

In order to determine the frequency, we need the eigenvector $\mathbf{v} = [v_1, v_2]^T$ that corresponds to the larger eigenvalue λ_1:

$$\begin{aligned}&\mathbf{R}_x\mathbf{v} = 3\mathbf{v} \implies \\ &\begin{cases} 2v_1 - jv_2 = 3v_1 \\ jv_1 + 2v_2 = 3v_2 \end{cases} \implies \\ &jv_1 = v_2 \implies \\ &\mathbf{v} = [v_1, jv_1]^T = v_1[1, j]^T.\end{aligned}$$

On the other hand, the eigenvector $\mathbf{v}$ is of the form $\mathbf{v} = a\mathbf{e}_1$, with $\mathbf{e}_1 = [1 \ \exp(j\omega)]$. This results in the frequency $\omega = \pi/2$ because $\exp(j\pi/2) = \cos(\pi/2) + j\sin(\pi/2) = j$.

If desired, we can scale the eigenvector to unit length. Then $v_1 = 1/\sqrt{2} = M^{-1/2}$. Also in general, the unit length vector $\mathbf{v} = M^{-1/2}\mathbf{e}_1$ (note that the length of the vector $\mathbf{e}_1$ is always $M^{1/2}$).

3. The autocorrelation matrix is (as $x(t)$ is real-valued, c must be real)

$$\mathbf{R}_x = \begin{bmatrix} 1 & c & 0 \\ c & 1 & c \\ 0 & c & 1 \end{bmatrix}$$

Its eigenvalues are 1, $1 - \sqrt{2}c$, and $1 + \sqrt{2}c$, so it is positive semi-definite (and a valid autocorrelation matrix) only for $-\frac{1}{\sqrt{2}} \leq c \leq \frac{1}{\sqrt{2}}$.

(a) The maximum entropy method for power spectrum estimation is equivalent to fitting an all-pole model to the autocorrelations. This is done by solving the Yule-Walker equations:

$$\begin{bmatrix} r_x(0) & r_x^*(1) & r_x^*(2) \\ r_x(1) & r_x(0) & r_x^*(1) \\ r_x^*(2) & r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \epsilon \\ 0 \\ 0 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & c & 0 \\ c & 1 & c \\ 0 & c & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \epsilon \\ 0 \\ 0 \end{bmatrix}$$

Leading to the pair of equations

$$\begin{cases} a_1 + ca_2 = -c \\ a_2 = -ca_1 \end{cases}$$

Substituting a_2 from the second equation into the first gives

$$a_1 - c^2 a_1 = -c$$

and the solution is easily obtained:

$$a_1 = \frac{-c}{1-c^2}, \qquad a_2 = \frac{c^2}{1-c^2}$$

The noise coefficient can be found from the first row of the YW equations

$$\epsilon = 1 + ca_1 = 1 - \frac{c^2}{1-c^2} = \frac{1-2c^2}{1-c^2}$$

The MEM spectrum is then given by

$$\hat{P}_{MEM}(e^{j\omega}) = \frac{\epsilon}{|\mathbf{e}^H \mathbf{a}|^2}$$

where $\mathbf{e} = [1,\, e^{jw},\, e^{j2w}]^T$ and $\mathbf{a} = [1,\, a_1,\, a_2]^T$

$$\begin{aligned}
\hat{P}_{MEM}(e^{j\omega}) &= \frac{\epsilon}{|1 - \frac{c}{1-c^2}e^{-j\omega} + \frac{c^2}{1-c^2}e^{-j2\omega}|^2} \\
&= \frac{\epsilon}{(1 - \frac{c}{1-c^2}e^{j\omega} + \frac{c^2}{1-c^2}e^{j2\omega})(1 - \frac{c}{1-c^2}e^{-j\omega} + \frac{c^2}{1-c^2}e^{-j2\omega})} \\
&= \frac{1-2c^2}{1-c^2}\frac{1}{(1 - \frac{c}{1-c^2}e^{j\omega} + \frac{c^2}{1-c^2}e^{j2\omega})(1 - \frac{c}{1-c^2}e^{-j\omega} + \frac{c^2}{1-c^2}e^{-j2\omega})} \\
&= \frac{(1-2c^2)(1-c^2)}{(1-c^2-ce^{j\omega}+c^2e^{j2\omega})(1-c^2-ce^{-j\omega}+c^2e^{-j2\omega})} \\
&= \frac{(1-2c^2)(1-c^2)}{(1-c^2)^2 + c^2 + c^4 - (c(1-c^2)+c^3)(e^{j\omega}+e^{-j\omega}) + c^2(1-c^2)(e^{j2\omega}+e^{-j2\omega})} \\
&= \frac{(1-2c^2)(1-c^2)}{1-c^2+2c^4 - c(e^{j\omega}+e^{-j\omega}) + c^2(1-c^2)(e^{j2\omega}+e^{-j2\omega})} \\
&= \frac{(1-2c^2)(1-c^2)}{1-c^2+2c^4 - 2c\cos(\omega) + 2c^2(1-c^2)\cos(2\omega)}
\end{aligned}$$

(b) Directly applying the definition of the power spectrum:

$$\hat{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x(k)e^{-jk\omega} = ce^{j\omega} + 1 + ce^{-j\omega} = 1 + 2c\cos(\omega)$$

Note however that extending the autocorrelation function with zeros does not necessarily lead to a valid power spectrum. For instance, in this case if $c > 0.5$, the power estimate would be *negative* for some values of ω.

4. In the minimum variance method we have now $p = 2$ as the autocorrelations up to lag 2 are given. The autocorrelation matrix is

$$\mathbf{R}_x = \begin{bmatrix} 1 & c & 0 \\ c & 1 & c \\ 0 & c & 1 \end{bmatrix}$$

with inverse matrix

$$\mathbf{R}_x^{-1} = 1/(1-2c^2) \begin{bmatrix} 1-c^2 & -c & c^2 \\ -c & 1 & -c \\ c^2 & -c & 1-c^2 \end{bmatrix}$$

The minimum variance power spectrum estimate is

$$\hat{P}_{MV}(e^{j\omega}) = \frac{p+1}{\mathbf{e}^H \mathbf{R}_x^{-1} \mathbf{e}}$$

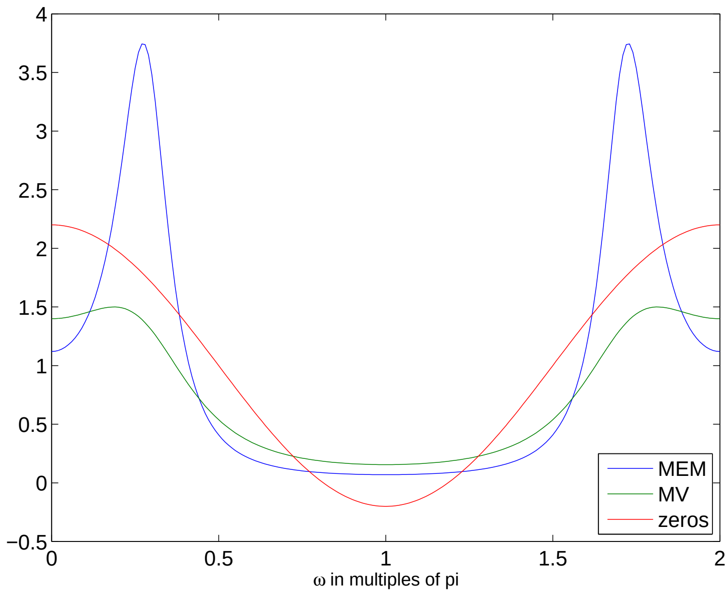
where $\mathbf{e} = [1,\, e^{jw},\, \ldots,\, e^{jpw}]^T$ again and

$$\begin{aligned} \mathbf{e}^H \mathbf{R}_x^{-1} \mathbf{e} &= \frac{1}{1-2c^2}[1 - c^2 - ce^{j\omega} + c^2 e^{j2\omega} - ce^{-j\omega} + 1 - ce^{j\omega} + c^2 e^{-j2\omega} - ce^{-j\omega} + 1 - c^2] \\ &= \frac{1}{1-2c^2}[3 - 2c^2 - 2c(e^{j\omega} + e^{-j\omega}) + c^2(e^{j2\omega} + e^{-j2\omega})] \\ &= \frac{1}{1-2c^2}[3 - 2c^2 - 4c\cos(\omega) + 2c^2\cos(2\omega)] \end{aligned}$$

We obtain

$$\hat{P}_{MV}(e^{j\omega}) = \frac{3(1-2c^2)}{3 - 2c^2 - 4c\cos(\omega) + 2c^2\cos(2\omega)}$$

The following figure shows a comparison of the three estimated power spectra with $c = 0.6$.



5. Separate the autocorrelation matrix into two parts $\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_v = \mathbf{R}_s + \sigma^2\mathbf{I}$. Thus $\mathbf{R}_x$ has the same eigenvectors as $\mathbf{R}_s$, and the eigenvalues are found by adding σ^2 to the eigenvalues of $\mathbf{R}_s$. For the signal part, the autocorrelation matrix is

$$\mathbf{R}_s = \sum_{i=1}^{p} |A_i|^2 \mathbf{e}_i \mathbf{e}_i^H$$

where $\mathbf{e}_i = [1\, e^{j\omega_i}\, e^{2j\omega_i} \ldots e^{(M-1)j\omega_i}]^T$ (see the lecture slides, or Hayes p. 457).

The rank of this matrix is p, so $M - p$ eigenvalues are 0. In order to find the non-zero eigenvalues, we note that the above representation of $\mathbf{R}_s$ as the sum of several outer products resembles the eigen-decomposition of a hermitian matrix (provided the vectors are orthonormal). In other words, if the vectors $\mathbf{e}_i$ are orthogonal, they will be eigenvectors.

Calculate the inner product $\mathbf{e}_i^H \mathbf{e}_k$

$$\mathbf{e}_i^H \mathbf{e}_k = \sum_{l=0}^{M-1} \exp\left(-j\frac{2\pi}{M}li\right) \exp\left(j\frac{2\pi}{M}lk\right) = \sum_{l=0}^{M-1} q^l,$$

where $q = \exp\left(j\frac{2\pi}{M}(k-i)\right)$. Thus

$$\mathbf{e}_i^H \mathbf{e}_k = \frac{1 - \exp\left(j\frac{2\pi}{M}(k-i)M\right)}{1 - \exp\left(j\frac{2\pi}{M}(k-i)\right)} = 0, \text{ for } k \neq i.$$

And if $k = i$,

$$\mathbf{e}_i^H \mathbf{e}_k = M.$$

So the vectors $\mathbf{e}_i$ are orthogonal, and eigenvectors of $\mathbf{R}_s$. To check this, and find the corresponding eigenvalues, calculate the product:

$$\mathbf{R}_s \mathbf{e}_i = \sum_{k=1}^{p} |A_k|^2 \mathbf{e}_k \mathbf{e}_k^H \mathbf{e}_i = |A_i|^2 \mathbf{e}_i \mathbf{e}_i^H \mathbf{e}_i = M|A_i|^2 \mathbf{e}_i$$

The eigenvalue of $\mathbf{R}_s$ corresponding to the vector $\mathbf{e}_i$ is $M|A_i|^2$.

So, the autocorrelation matrix

$$\mathbf{R}_x = \sum_{i=1}^{p} |A_i|^2 \mathbf{e}_i \mathbf{e}_i^H + \sigma^2 \mathbf{I},$$

has the eigenvalues

$$\begin{cases} M|A_i|^2 + \sigma^2, & i \in [1, 2, \ldots, p] \\ \sigma^2, & i \in [p+1, \ldots, M] \end{cases}$$

the σ^2 is an eigenvalue with multiplicity $M-p$. An orthonormal set of $M-p$ eigenvectors $\mathbf{v}_i$ ($i \in [p+1, \ldots, M]$) can be found for this noise subspace. If the signal eigenvectors ($i \in [1, 2, \ldots, p]$) are scaled as $\mathbf{v}_i = \frac{1}{\sqrt{M}}\mathbf{e}_i$, we have the inner products for all eigenvectors as

$$\mathbf{v}_i^H \mathbf{v}_k = \delta_{ik} \qquad \forall i, k \in [1, \ldots, M]$$