PhsHW11

November 22, 2024

[1]:
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# Question 1.1
#|----------------- --------------------
#|
#|
#(~)
#|
#|      ---> (V0+)     ---> (V1+)
#|----------------  -------------------
#     <--(V0-)
```

For $V_0(x,t)$:

$$V_0(x,t) = Re(e^{j\omega t}[V_f^+ e^{-j\beta x} + V_f^- e^{+j\beta x}])$$

where -L x 0

$$V_1(x,t) = Re(e^{j\omega t}[V_t^+ e^{-j\beta x}])$$

where x 0

$$I_0(x,t) = Re(e^{j\omega t}[\frac{V_f^+}{Z_0} e^{-j\beta x} - \frac{V_f^-}{Z_0} e^{+j\beta x}])$$

$$I_1(x,t) = Re(e^{j\omega t}[\frac{V_t^+}{Z_1} e^{-j\beta x}])$$

At x = -L:

$$V_0(-L,t) = V_{in}(t) = Re(V_s e^{j\omega t})$$

$$Re(e^{j\omega t}[V_f^+ e^{j\beta L} + V_f^- e^{-j\beta L}]) = Re(V_s e^{j\omega t})$$

Boundary conditions:

At x = -L:

$$V_f^+ e^{j\beta L} + V_f^- e^{-j\beta L} = V_s$$

At x = 0 (voltage continuity):

$$V_f^+ + V_f^- = V_t^+$$

At x = 0 (current continuity):

$$\frac{V_f^+}{Z_0} - \frac{V_f^-}{Z_0} = \frac{V_t^+}{Z_1}$$

With reflection coefficient:

$$\rho = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

$$V_f^+(e^{j\beta L} + \rho e^{-j\beta L}) = V_s$$

Therefore:

$$V_f^+ = \frac{V_s}{e^{j\beta L} + \rho e^{-j\beta L}}$$

Since $V_f^- = \rho V_f^+$:

$$V_f^- = \frac{\rho V_s}{e^{j\beta L} + \rho e^{-j\beta L}}$$

And from voltage continuity at x = 0:

$$V_t^+ = V_f^+ + V_f^- = V_f^+(1+\rho)$$

Therefore:

$$V_t^+ = \frac{V_s(1+\rho)}{e^{j\beta L} + \rho e^{-j\beta L}}$$

[2]:
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# Question 1.2
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We know:

$$V_f^+ = \frac{V_s}{e^{j\beta_0 L} + \rho e^{-j\beta_0 L}}$$

$$V_f^- = \frac{\rho V_s}{e^{j\beta_0 L} + \rho e^{-j\beta_0 L}}$$

where = 1/2

Substituting these in:

$$V_0(x,0) = Re[V_s(\frac{e^{-j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}} + \frac{\frac{1}{2}e^{+j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}})]$$

$$= Re[V_s(\frac{e^{-j\beta_0 x} + \frac{1}{2}e^{+j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}})]$$

Similarly for second cable (x 0):

$$V_1(x,0) = Re[V_t^+ e^{-j\beta_0 x}]$$

$$= Re[V_s \frac{\frac{3}{2}e^{-j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}}]$$

Since _0L = 4 : $e^{j\beta_0 L} = e^{-j\beta_0 L} = 1$ Therefore:

$$V_0(x,0) = Re[V_s(\frac{2}{3})(e^{-j\beta_0 x} + \frac{1}{2}e^{+j\beta_0 x})]$$

$$V_1(x,0) = Re[V_s e^{-j\beta_0 x}]$$

For -L x 0:

$$V_0(x,0) = V_s \cos(\beta_0 x)$$

For x 0:

$$V_1(x,0) = V_s \cos(\beta_0 x)$$

[3]:
```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
Vs = 1.0
beta0_l0 = 4*np.pi  #  l  = 4

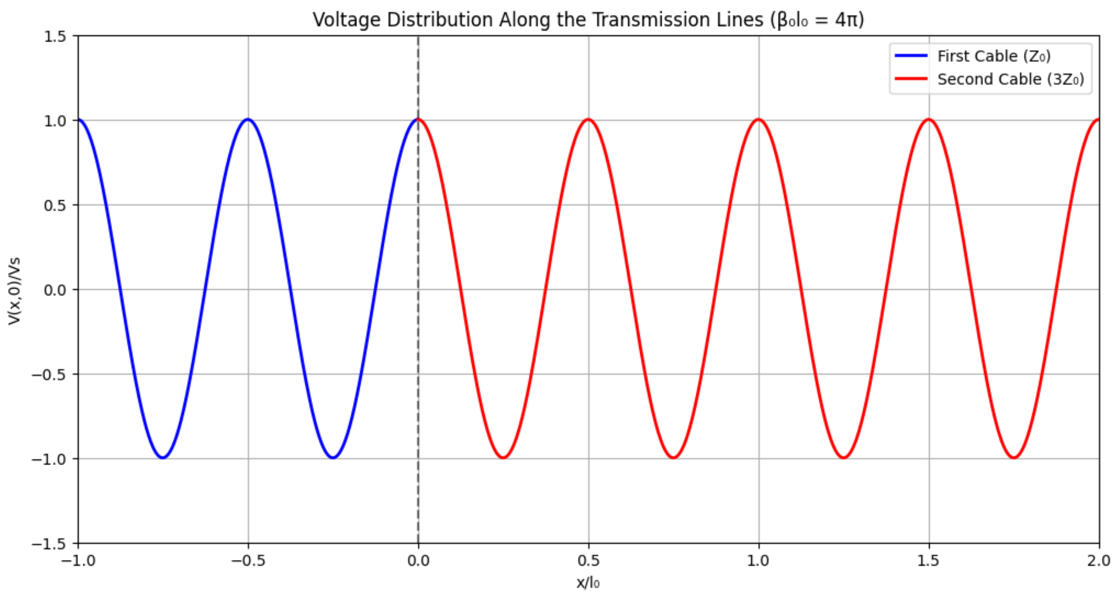
# For first cable (-L  x  0)
x1_l0 = np.linspace(-1, 0, 500)  # x/l  from -1 to 0
V1 = Vs * np.cos(beta0_l0 * x1_l0)  # cos(4 x/l )

# For second cable (0  x)
x2_l0 = np.linspace(0, 2, 500)  # x/l  from 0 to 2
V2 = Vs * np.cos(beta0_l0 * x2_l0)  # cos(4 x/l )

# Plotting
plt.figure(figsize=(12, 6))
plt.plot(x1_l0, V1, 'blue', label='First Cable (Z )', linewidth=2)
plt.plot(x2_l0, V2, 'red', label='Second Cable (3Z )', linewidth=2)
plt.axvline(x=0, color='black', linestyle='--', alpha=0.5)  # marking x=0

plt.grid(True)
plt.xlabel('x/l ')
plt.ylabel('V(x,0)/Vs')
plt.title('Voltage Distribution Along the Transmission Lines ( l  = 4 )')
plt.legend()

plt.ylim(-1.5, 1.5)
plt.xlim(-1, 2)
plt.show()
```



[4]:
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# Question 1.3
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Recall,

$$V_0(x) = V_f^+ e^{-j\beta_0 x} + V_f^- e^{+j\beta_0 x}$$

and $V_f^+ = \frac{2}{3}V_s$ $V_f^- = \frac{1}{3}V_s$

So:

$$V_0(x) = \frac{2}{3}V_s e^{-j\beta_0 x} + \frac{1}{3}V_s e^{+j\beta_0 x}$$

$$= V_s[\frac{2}{3}(\cos(\beta_0 x) - j\sin(\beta_0 x)) + \frac{1}{3}(\cos(\beta_0 x) + j\sin(\beta_0 x))]$$

$$= V_s[\cos(\beta_0 x) - j\frac{1}{3}\sin(\beta_0 x)]$$

For polar form $|V_0(x)|e^{j\phi}$:

$|V_0(x)| = V_s\sqrt{\cos^2(\beta_0 x) + \frac{1}{9}\sin^2(\beta_0 x)}$

$\phi = -\tan^{-1}(\frac{\frac{1}{3}\sin(\beta_0 x)}{\cos(\beta_0 x)})$

[5]:
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## Question 2.1
```

1) KCL at V node (first capacitor):

- Current entering: I
- Current leaving: I

- Current through capacitor: C dV /dt

$$C\frac{dV_1}{dt} = I_1 - I_2$$

In state variables (C=1):

$$\dot{x_3} = x_1 - x_2$$

2) KVL around first L loop:

$$L\frac{dI_1}{dt} = V_0 - V_1$$

In state variables (L=1):

$$\dot{x_1} = cos(t) - x_3$$

3) KVL around second L loop: Since V = I R and R=1:

$$L\frac{dI_2}{dt} = V_1 - I_2$$

In state variables (L=1):

$$\dot{x_2} = x_3 - x_2$$

This gives us:

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{x_3}\end{bmatrix} = \begin{bmatrix}0 & 0 & -1\\ 0 & -1 & 1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} cos(t)$$

[6]:
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## Question 2.2
```

Given:

$$\dot{X} = AX + BU$$

where:

$$A = \begin{bmatrix}0 & 0 & -1\\ 0 & -1 & 1\\ 1 & -1 & 0\end{bmatrix}, \quad B = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}, \quad U = \begin{bmatrix}\cos(t)\\ 0\\ 0\end{bmatrix}$$

0.0.1 (1) Steady-State Assumptions

Since $U(t) = \begin{bmatrix}\cos(t)\\ 0\\ 0\end{bmatrix}$, we write it in complex exponential form using Euler's formula:

$$U(t) = \text{Re}\left(\begin{bmatrix}e^{j\omega t}\\ 0\\ 0\end{bmatrix}\right)$$

where $\omega = 1$.

Assume the steady-state solution is:

$$X(t) = \mathrm{Re}\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} e^{j\omega t}\right)$$

where $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ are the phasors for the steady-state solution.

0.0.2 (2) Frequency-Domain Representation

From the derivation, we have:

$$(j\omega I - A)\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = B\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Substituting $\omega = 1$, we get:

$$(jI - A)\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

0.0.3 (3) Matrix Substitution

$$jI - A = \begin{bmatrix} j & 0 & 1 \\ 0 & j+1 & -1 \\ -1 & 1 & j \end{bmatrix}$$

Solving the linear system of equations:

$$\begin{bmatrix} j & 0 & 1 \\ 0 & j+1 & -1 \\ -1 & 1 & j \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

0.0.4 (4) Solve the Linear System

1. From the first equation: $jX_1 + X_3 = 1$
2. From the second equation: $(j+1)X_2 - X_3 = 0$
3. From the third equation: $-X_1 + X_2 + jX_3 = 0$

Solving this system, we get:

$$X_1 = 1$$

$$X_2 = \frac{1+j}{j+1}$$

$$X_3 = 1 - j$$

0.0.5 (5) Convert to Time Domain

The steady-state solution is the real part of $\mathbf{X}e^{j\omega t}$, where:

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+j}{j+1} \\ 1-j \end{bmatrix}$$

$\mathbf{X}_1(t)$:

$$X_1(t) = \text{Re}(X_1 e^{j\omega t}) = \text{Re}(1 \cdot e^{j\omega t}) = \cos(\omega t) = \cos(t)$$

$\mathbf{X}_2(t)$:

$$X_2(t) = \text{Re}\left(\frac{1+j}{j+1} e^{j\omega t}\right)$$

Simplifying the fraction:

$$\frac{1+j}{j+1} = \frac{(1+j)(j-1)}{(j+1)(j-1)} = \frac{2j}{-2} = -j$$

Therefore:

$$X_2(t) = \text{Re}(-je^{j\omega t}) = \text{Re}(-j(\cos(\omega t) + j\sin(\omega t))) = \text{Re}(-\sin(\omega t) - j\cos(\omega t)) = \sin(t)$$

$\mathbf{X}_3(t)$:

$$X_3(t) = \text{Re}((1-j)e^{j\omega t}) = \text{Re}((1-j)(\cos(\omega t) + j\sin(\omega t)))$$

Expanding:

$$X_3(t) = \text{Re}(\cos(\omega t) - j\cos(\omega t) + j\sin(\omega t) + \sin(\omega t)) = \cos(t) + \sin(t)$$

Thus, the steady-state solutions are:

$$I_1(t) = \mathbf{X}_1(t) = \cos(t)$$

$$I_2(t) = \mathbf{X}_2(t) = \sin(t)$$

$$V_1(t) = \mathbf{X}_3(t) = \cos(t) + \sin(t)$$

[7]:

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# Define the system of differential equations
def system(t, x):
    # State variables: x1, x2, x3
    x1, x2, x3 = x

    # System matrix A
    A = np.array([[0, 0, -1],
                  [0, -1, 1],
                  [1, -1, 0]])
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```
    # Input vector
    u = np.array([np.cos(t), 0, 0])

    # Compute the derivatives
    dxdt = A @ np.array([x1, x2, x3]) + u
    return dxdt

# Initial conditions: x1(0), x2(0), x3(0)
initial_conditions = [0, 0, 0]

# Time span for the solution (e.g., from t=0 to t=10)
t_span = (0, 20)

# Solve the system using solve_ivp
sol = solve_ivp(system, t_span, initial_conditions, t_eval=np.linspace(0, 20,␣
 ↪1000))

# Steady-state solutions
t = np.linspace(0, 20, 1000)
I1_steady = np.cos(t)  # I1(t) = cos(t)
I2_steady = np.sin(t)  # I2(t) = sin(t)
V1_steady = np.cos(t) + np.sin(t)  # V1(t) = cos(t) + sin(t)

# Plot the solutions
plt.figure(figsize=(10, 6))

# Numerical solutions
plt.plot(sol.t, sol.y[0], label=r'$I_1(t)$ (Numerical)', color='r',␣
 ↪linestyle='--', linewidth=2)
plt.plot(sol.t, sol.y[1], label=r'$I_2(t)$ (Numerical)', color='b',␣
 ↪linestyle='--', linewidth=2)
plt.plot(sol.t, sol.y[2], label=r'$V_1(t)$ (Numerical)', color='g',␣
 ↪linestyle='--', linewidth=2)

# Steady-state solutions
plt.plot(t, I1_steady, label=r'$I_1(t) = \cos(t)$ (Steady-state)', color='r',␣
 ↪linewidth=2)
plt.plot(t, I2_steady, label=r'$I_2(t) = \sin(t)$ (Steady-state)', color='b',␣
 ↪linewidth=2)
plt.plot(t, V1_steady, label=r'$V_1(t) = \cos(t) + \sin(t)$ (Steady-state)',␣
 ↪color='g', linewidth=2)

plt.title('Comparison of Numerical and Steady-State Solutions')
plt.xlabel('Time $t$')
plt.ylabel('Amplitude')
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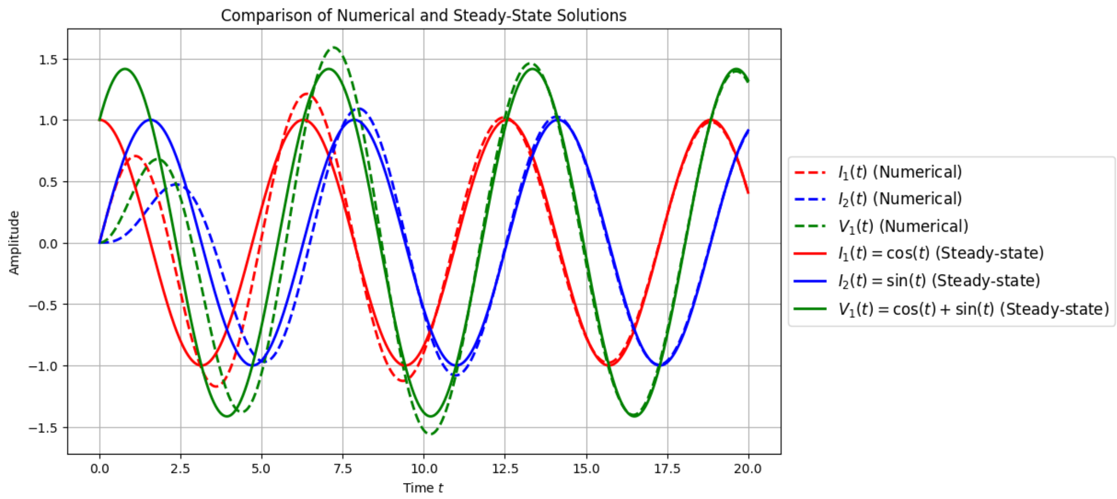
```
plt.legend(loc="center left", bbox_to_anchor=(1, 0.5), fontsize=12)
plt.grid(True)
plt.show()
```



0.1 Comment

The transient state of the system converges to the steady-state solution after some time.