PhysHW9

November 8, 2024

[1]:
```
#----------------------------1.1--------------------------------------------
```

Solve for steady state solution I(t) for:

$$\frac{dI}{dt} + \frac{I}{\tau} = \frac{V_o}{L}\cos(\omega t + \phi)$$

Steady state solution: $I(t) = Re[\bar{I}e^{j\omega t}]$ Input voltage in phasor form: $\frac{V_o}{L}\cos(\omega t + \phi) = Re[\bar{V}e^{j\omega t}]$ where $\bar{V} = \frac{V_o}{L}e^{j\phi}$

Substituting in original equation:

$$j\omega\bar{I} + \frac{1}{\tau}\bar{I} = \bar{V}$$

$$\bar{I}(j\omega + \frac{1}{\tau}) = \frac{V_o}{L}e^{j\phi}$$

Multiply by conjugate to solve:

$$\bar{I} = \frac{V_o/L}{j\omega + 1/\tau}e^{j\phi} \cdot \frac{-j\omega + 1/\tau}{-j\omega + 1/\tau} = \frac{V_o/L \cdot e^{j\phi}(-j\omega + 1/\tau)}{(\omega^2 + 1/\tau^2)}$$

For numerator $(-j\omega + 1/\tau)$ in form $a + bj$: * $a = 1/\tau$ * $b = -\omega$ Therefore: * Magnitude = $\sqrt{\omega^2 + 1/\tau^2}$ * Phase = $-\arctan(\omega\tau)$

Final steady state solution:

$$I(t) = \frac{V_o/L}{\sqrt{\omega^2 + 1/\tau^2}}\cos(\omega t + \phi - \arctan(\omega\tau))$$

[2]:
```
#-----------------------------------1.2---------------------------------------
```

Solve for steady state solution I(t) for:

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = V_o\cos(\omega t)$$

Consider $I(t) = Re[\bar{I}e^{j\omega t}]$ and Input voltage in phasor form: $V_o\cos(\omega t) = Re[\bar{V}e^{j\omega t}]$ where $\bar{V} = V_o$

Note that $Q = \int I dt$ so in phasor form: $\bar{Q} = \frac{\bar{I}}{j\omega}$

Substituting in original equation:

$$\bar{I}(R + j\omega L + \frac{1}{j\omega C}) = \bar{V}$$

$$\bar{I} = \frac{\bar{V}}{R + j(\omega L - \frac{1}{\omega C})}$$

Multiply by conjugate to solve:

$$\bar{I} = \frac{\bar{V}[R - j(\omega L - \frac{1}{\omega C})]}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

For term $[R - j(\omega L - \frac{1}{\omega C})]$ in form $a + bj$: * $a = R$ * $b = -(\omega L - \frac{1}{\omega C})$ Therefore: * Magnitude = $\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$ * Phase = $-\arctan(\frac{\omega L - \frac{1}{\omega C}}{R})$

Final steady state solution:

$$I(t) = \frac{V_o}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \cos(\omega t - \arctan(\frac{\omega L - \frac{1}{\omega C}}{R}))$$

Which can be written as:

$$I(t) = \frac{V_o}{Z} \cos(\omega t - \phi)$$

where:

$$Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$\phi = \arctan(\frac{\omega L - \frac{1}{\omega C}}{R})$$

[3]:
```
#------------------------------2-----------------------------------------
```

For R-C circuit and using similar analogy in 1.2 for Q 1):

$$V(t) = IR + \frac{Q}{C}$$

where $I = \frac{dQ}{dt}$

2) Phasors also:

$$V(t) = Re[\bar{V}_o e^{j\omega t}]$$
$$I(t) = Re[\bar{I}_o e^{j\omega t}]$$

Therefore:

$$Q(t) = \int I(t)dt = Re[\frac{\bar{I}_o}{j\omega} e^{j\omega t}]$$

3) Substituting in original equation:

$$\bar{V}_o e^{j\omega t} = \bar{I}_o R e^{j\omega t} + \frac{\bar{I}_o}{j\omega C} e^{j\omega t}$$

4) Since $e^{j\omega t}$ terms cancel:

$$\bar{V}_o = \bar{I}_o R + \bar{I}_o \frac{1}{j\omega C}$$

5) Factor out $\bar{I}_o$:

$$\bar{V}_o = \bar{I}_o (R + \frac{1}{j\omega C})$$

Therefore:

$$\bar{V}_o - \bar{I}_o R - \bar{I}_o X_C = 0$$

where $X_C = \frac{1}{j\omega C}$ is the capacitive impedance

This shows capacitor acts like a resistor with impedance X_C, similar to inductor with $X_L = j\omega L$

[4]:
```
##----------------------------------3.1---------------------------------------
#    0    L     1     L    2
# o--o---oooo---o---oooo---o
# |  |          |          |
# Vo C          C          ZL
# |  |          |          |
# o--o----------o----------o
# |
# ground
```

KCL definitions:

- $\bar{I_{0c}} = \bar{I}_0 - \bar{I}_1$
- $\bar{I_{1c}} = \bar{I}_1 - \bar{I}_2$

Given: $V_0(t) = \cos(\omega t)$ and circuit parameters: L, C

KVL definitions:

1) $-\bar{V}_0 + \frac{\bar{I_{0c}}}{j\omega C} = 0$
2) $-\bar{V}_0 + \bar{I}_1 j\omega L + \frac{\bar{I_{1c}}}{j\omega C} = 0$
3) $-\bar{V}_0 + \bar{I}_1 j\omega L + \bar{I}_2 j\omega L + \bar{I}_2 Z_L = 0$
4) $-\frac{\bar{I_{0c}}}{j\omega C} + \bar{I}_1 j\omega L + \frac{\bar{I_{1c}}}{j\omega C} = 0$
5) $-\frac{\bar{I_{1c}}}{j\omega C} + \bar{I}_2 j\omega L + \bar{I}_2 Z_L = 0$

Solving backwards:

1) From equation 5:

$$\frac{\bar{I_{1c}}}{j\omega C} = \bar{I}_2 (j\omega L + Z_L)$$

$$\bar{I_{1c}} = \bar{I}_2 (j\omega L + Z_L) j\omega C$$

2) Using $\bar{I_{1c}} = \bar{I_1} - \bar{I_2}$:

$$\bar{I_1} - \bar{I_2} = \bar{I_2}(j\omega L + Z_L)j\omega C$$
$$\bar{I_1} = \bar{I_2}[1 + (j\omega L + Z_L)j\omega C]$$

3) From equation 1:

$$\bar{V_0} = \frac{\bar{I_{0c}}}{j\omega C}$$
$$\bar{I_{0c}} = \bar{V_0} j\omega C$$

4) Using $\bar{I_{0c}} = \bar{I_0} - \bar{I_1}$:

$$\bar{V_0} j\omega C = \bar{I_0} - \bar{I_2}[1 + (j\omega L + Z_L)j\omega C]$$

5) Solving for I2 using equation 3:

$$\bar{I_2} = \frac{\bar{V_0}}{(1 + (j\omega L + Z_L)j\omega C)j\omega L + j\omega L + Z_L}$$

Final expressions:

For currents through elements:

$$\bar{I_1} = \bar{I_2}[1 + (j\omega L + Z_L)j\omega C]$$
$$\bar{I_0} = \bar{V_0} j\omega C + \bar{I_2}[1 + (j\omega L + Z_L)j\omega C]$$

For currents through capacitors:

$$\bar{I_{0c}} = \bar{V_0} j\omega C$$
$$\bar{I_{1c}} = \bar{I_2}(j\omega L + Z_L)j\omega C$$

[5]:
```
#----------------------------3.3-------------------------------------------
```

Across C1:

$$V_{C1} = \frac{\bar{I_{0c}}}{j\omega C} = \frac{\bar{V_0} j\omega C}{j\omega C} = \bar{V_0} = \cos(\omega t)$$

Across C2:

$$V_{C2} = \frac{\bar{I_{1c}}}{j\omega C} = \frac{\bar{I_2}(j\omega L + Z_L)j\omega C}{j\omega C} = \bar{I_2}(j\omega L + Z_L)$$

Across L1:

$$V_{L1} = \bar{I_1} j\omega L = \bar{I_2}[1 + (j\omega L + Z_L)j\omega C]j\omega L$$

Across L2:

$$V_{L2} = \bar{I_2} j\omega L$$

Across ZL:

$$V_{ZL} = \bar{I_2} Z_L$$

where:

$$\bar{I_2} = \frac{1}{(1 + (j\omega L + Z_L)j\omega C)j\omega L + j\omega L + Z_L}$$

[6]:
```
#--------------------------3.3------------------------------------------------
#Check of impedance seen by source
```

Starting with:

$$\bar{I_0} = \bar{V_0} j\omega C + \bar{I_2}[1 + (j\omega L + Z_L) j\omega C]$$

$$\bar{I_2} = \frac{\bar{V_0}}{(1 + (j\omega L + Z_L) j\omega C) j\omega L + j\omega L + Z_L}$$

Substitute $\bar{I_2}$:

$$\bar{I_0} = \bar{V_0}\left(j\omega C + \frac{1 + (j\omega L + Z_L) j\omega C}{(1 + (j\omega L + Z_L) j\omega C) j\omega L + j\omega L + Z_L}\right)$$

Therefore:

$$Z_{in} = \frac{\bar{V_0}}{\bar{I_0}} = \frac{1}{j\omega C + \frac{1 + (j\omega L + Z_L) j\omega C}{(1 + (j\omega L + Z_L) j\omega C) j\omega L + j\omega L + Z_L}}$$

Proof: For $\omega = 1$ and $Z_L = 1$:

$$Z_{in} = \frac{1}{j + \frac{1 + (j+1)j}{(1 + (j+1)j)j + j + 1}} = \frac{1}{1 + j}$$

Multiplying numerator and denominator by conjugate $(1 - j)$:

$$Z_{in} = \frac{1}{1 + j} \cdot \frac{1 - j}{1 - j} = \frac{1 - j}{(1 + j)(1 - j)} = \frac{1 - j}{1 + 1} = \frac{1 - j}{2}$$

[7]:
```
#----------------------------------3.4.1--------------------------------------
```

Given $V_0(t) = \cos(\omega t)$, in phasor form:

$$\bar{V_0} = e^{j \cdot 0} = 1$$

Then for currents:

$$\bar{I_2} = \frac{\bar{V_0}}{(1 + (j\omega L + Z_L) j\omega C) j\omega L + j\omega L + Z_L}$$

$$\bar{I_2} = \frac{\bar{V_0}}{(1 + (j\omega L + Z_L) j\omega C) j\omega L + j\omega L + Z_L}$$

Let $Z_L = R$:

$$\begin{aligned}\bar{I_2} &= \frac{1}{(1 + (j\omega L + R) j\omega C) j\omega L + j\omega L + R} \\ &= \frac{1}{(1 + j^2\omega^2 LC + j\omega RC) j\omega L + j\omega L + R} \\ &= \frac{1}{(1 - \omega^2 LC + j\omega RC) j\omega L + j\omega L + R}\end{aligned}$$

$$= \frac{1}{j\omega L - j\omega^3 L^2 C + j^2\omega^2 RLC + j\omega L + R}$$
$$= \frac{1}{2j\omega L - j\omega^3 L^2 C - \omega^2 RLC + R}$$

Let: $a = R - \omega^2 RLC$ (real part) $b = 2\omega L - \omega^3 L^2 C$ (coefficient of j in denominator)

Then:

$$= \frac{1}{jb + a} \cdot \frac{-jb + a}{-jb + a}$$
$$= \frac{a - jb}{a^2 + b^2}$$

Therefore: $I_2(t) = A\cos(\omega t + \phi)$ where:

$$A = \frac{1}{\sqrt{a^2 + b^2}}$$

$$\phi = -\tan^{-1}\left(\frac{b}{a}\right)$$

For $\bar{I}_1$

$$\bar{I}_1 = \bar{I}_2[1 + (j\omega L + Z_L)j\omega C]$$
$$\bar{I}_1 = \frac{a - jb}{a^2 + b^2}[1 + (j\omega L + R)j\omega C]$$

$$= \frac{a - jb}{a^2 + b^2}[1 + j^2\omega^2 LC + j\omega RC]$$

$$= \frac{a - jb}{a^2 + b^2}[1 - \omega^2 LC + j\omega RC]$$

Let: $c = 1 - \omega^2 LC$ (real part of bracket) $d = \omega RC$ (coefficient of j in bracket)

$$= \frac{(a - jb)(c + jd)}{a^2 + b^2}$$

$$= \frac{(ac + bd) + j(ad - bc)}{a^2 + b^2}$$

Therefore: $I_1(t) = A_1\cos(\omega t + \phi_1)$ where:

$$A_1 = \frac{\sqrt{(ac + bd)^2 + (ad - bc)^2}}{a^2 + b^2}$$

$$\phi_1 = \tan^{-1}\left(\frac{ad - bc}{ac + bd}\right)$$

Finally,

$$\bar{I_0} = \bar{V_0} j\omega C + \bar{I_2}[1 + (j\omega L + Z_L) j\omega C]$$

$$\bar{I_0} = j\omega C + \frac{a - jb}{a^2 + b^2}[1 - \omega^2 LC + j\omega RC]$$

$$= j\omega C + \frac{a - jb}{a^2 + b^2}[c + jd]$$

$$= j\omega C + \frac{(ac + bd) + j(ad - bc)}{a^2 + b^2}$$

$$= \frac{(ac + bd)}{a^2 + b^2} + j[\omega C + \frac{ad - bc}{a^2 + b^2}]$$

Therefore: $I_0(t) = A_0 \cos(\omega t + \phi_0)$ where:

$$A_0 = \sqrt{\frac{(ac + bd)^2}{(a^2 + b^2)^2} + [\omega C + \frac{ad - bc}{a^2 + b^2}]^2}$$

$$\phi_0 = \tan^{-1}\left(\frac{\omega C(a^2 + b^2) + ad - bc}{ac + bd}\right)$$

[8]:
```
#----------------------------------3.4.2-------------------------------------
```

Using previous expressions where:

For V_{C2} from $\bar{I_2}(j\omega L + R)$:

$$\frac{a - jb}{a^2 + b^2}(j\omega L + R)$$

$$= \frac{(a - jb)(j\omega L + R)}{a^2 + b^2}$$

$$= \frac{(aR + b\omega L) + j(a\omega L - bR)}{a^2 + b^2}$$

$$V_{C2}(t) = \frac{\sqrt{(aR + b\omega L)^2 + (a\omega L - bR)^2}}{a^2 + b^2} \cos(\omega t + \tan^{-1}(\frac{a\omega L - bR}{aR + b\omega L}))$$

For V_{L2} from $\bar{I_2} j\omega L$:

$$\frac{a - jb}{a^2 + b^2} j\omega L$$

$$= \frac{\omega L(b + ja)}{a^2 + b^2}$$

$$V_{L2}(t) = \frac{\omega L}{\sqrt{a^2 + b^2}} \cos(\omega t + \tan^{-1}(\frac{a}{b}))$$

For V_{ZL} from $\bar{I}_2 R$:

$$\frac{a - jb}{a^2 + b^2} R$$
$$= \frac{R(a - jb)}{a^2 + b^2}$$
$$V_{ZL}(t) = \frac{R}{\sqrt{a^2 + b^2}} \cos(\omega t - \tan^{-1}(\frac{b}{a}))$$

Finally, for V_{L1}:

$$V_{L1} = \frac{a - jb}{a^2 + b^2}[1 + (j\omega L + R)j\omega C]j\omega L$$
$$= \frac{a - jb}{a^2 + b^2}[1 - \omega^2 LC + j\omega RC]j\omega L$$
$$= \frac{a - jb}{a^2 + b^2}[c + jd]j\omega L$$
$$= \frac{(ac + bd) + j(ad - bc)}{a^2 + b^2} j\omega L$$
$$= \frac{\omega L[-(ad - bc) + j(ac + bd)]}{a^2 + b^2}$$

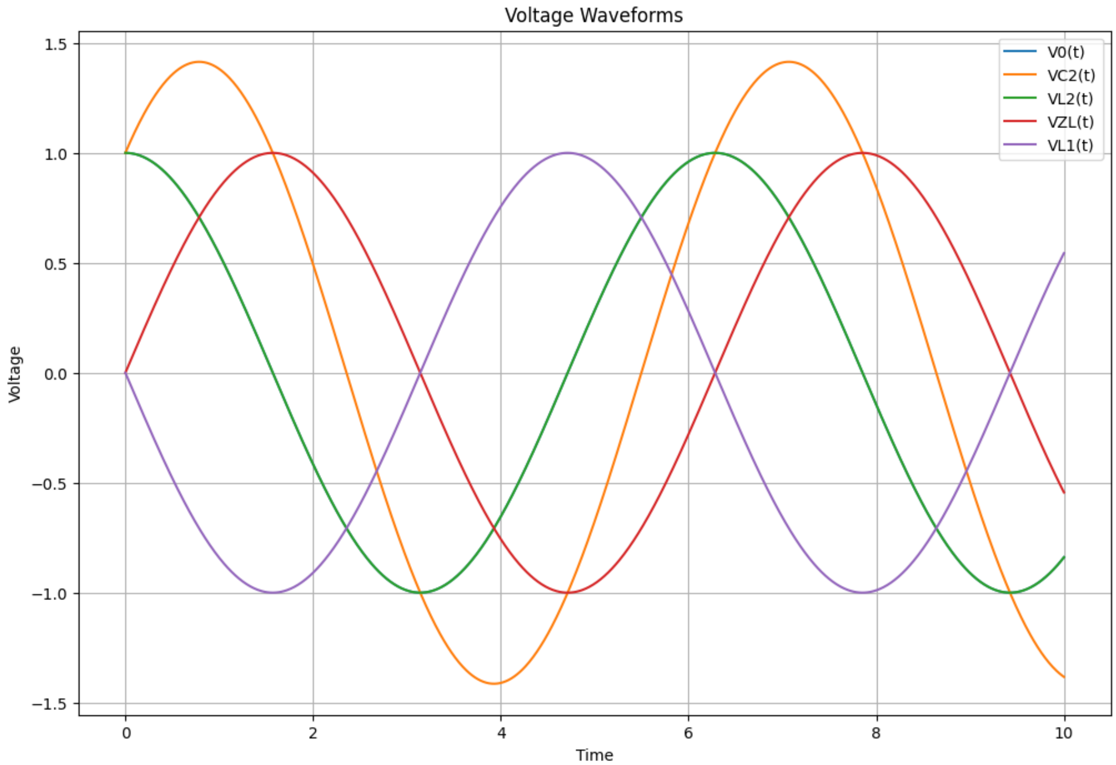
Therefore:

$$V_{L1}(t) = \frac{\omega L \sqrt{(ad - bc)^2 + (ac + bd)^2}}{a^2 + b^2} \cos(\omega t + \tan^{-1}(\frac{ac + bd}{-(ad - bc)}))$$

where: $a = R - \omega^2 RLC$ $b = 2\omega L - \omega^3 L^2 C$ $c = 1 - \omega^2 LC$ $d = \omega RC$

[10]:
```
#-----------------------------------------4--------------------------------
# plotting
import numpy as np
import matplotlib.pyplot as plt

# Params
omega = 1
C = w = L = R = 1
t = np.linspace(0, 10, 1000)  # Time range for plotting

# Parameters for a, b, c, d
a = R - omega**2 * R * L * C
b = 2 * omega * L - omega**3 * L**2 * C
c = 1 - omega**2 * L * C
d = omega * R * C

# Input voltage
v0 = np.cos(w*t)

# VC2 calculation
vc2_amp = np.sqrt((a*R + b*w*L)**2 + (a*w*L - b*R)**2)/(a**2 + b**2)
```

```python
vc2_phase = np.arctan2(a*w*L - b*R, a*R + b*w*L)
vc2 = vc2_amp * np.cos(w*t + vc2_phase)

# VL2 calculation
vl2_amp = w*L/np.sqrt(a**2 + b**2)
vl2_phase = np.arctan2(a, b)
vl2 = vl2_amp * np.cos(w*t + vl2_phase)

# VZL calculation
vzl_amp = R/np.sqrt(a**2 + b**2)
vzl_phase = -np.arctan2(b, a)
vzl = vzl_amp * np.cos(w*t + vzl_phase)

# VL1 calculation
vl1_amp = w*L*np.sqrt((a*d - b*c)**2 + (a*c + b*d)**2)/(a**2 + b**2)
vl1_phase = np.arctan2(a*c + b*d, -(a*d - b*c))
vl1 = vl1_amp * np.cos(w*t + vl1_phase)

# Plotting
plt.figure(figsize=(12, 8))
plt.plot(t, v0, label='V0(t)')
plt.plot(t, vc2, label='VC2(t)')
plt.plot(t, vl2, label='VL2(t)')
plt.plot(t, vzl, label='VZL(t)')
plt.plot(t, vl1, label='VL1(t)')
plt.grid(True)
plt.xlabel('Time')
plt.ylabel('Voltage')
plt.title('Voltage Waveforms')
plt.legend()
plt.show()

# Amps and phases
print(f"VC2 amplitude: {vc2_amp:.3f}, phase: {vc2_phase*180/np.pi:.1f}°")
print(f"VL2 amplitude: {vl2_amp:.3f}, phase: {vl2_phase*180/np.pi:.1f}°")
print(f"VZL amplitude: {vzl_amp:.3f}, phase: {vzl_phase*180/np.pi:.1f}°")
print(f"VL1 amplitude: {vl1_amp:.3f}, phase: {vl1_phase*180/np.pi:.1f}°")
```




```
VC2 amplitude: 1.414, phase: -45.0°
VL2 amplitude: 1.000, phase: 0.0°
VZL amplitude: 1.000, phase: -90.0°
VL1 amplitude: 1.000, phase: 90.0°
```

[ ]: