

# PhsHW11

November 22, 2024

[ ]: # Question 1.1

```
#/-----
#/
#/
#(~)
#/
#/      ---> (V0+)      ---> (V1+)
#/-----
#      <--(V0-)
```

For  $V_0(x, t)$ :

$$V_0(x, t) = \text{Re}(e^{j\omega t}[V_f^+ e^{-j\beta x} + V_f^- e^{+j\beta x}])$$

where  $-L \leq x \leq 0$

$$V_1(x, t) = \text{Re}(e^{j\omega t}[V_t^+ e^{-j\beta x}])$$

where  $x \geq 0$

$$I_0(x, t) = \text{Re}(e^{j\omega t}[\frac{V_f^+}{Z_0} e^{-j\beta x} - \frac{V_f^-}{Z_0} e^{+j\beta x}])$$

$$I_1(x, t) = \text{Re}(e^{j\omega t}[\frac{V_t^+}{Z_1} e^{-j\beta x}])$$

At  $x = -L$ :

$$V_0(-L, t) = V_{in}(t) = \text{Re}(V_s e^{j\omega t})$$

$$\text{Re}(e^{j\omega t}[V_f^+ e^{j\beta L} + V_f^- e^{-j\beta L}]) = \text{Re}(V_s e^{j\omega t})$$

Boundary conditions:

At  $x = -L$ :

$$V_f^+ e^{j\beta L} + V_f^- e^{-j\beta L} = V_s$$

At  $x = 0$  (voltage continuity):

$$V_f^+ + V_f^- = V_t^+$$

At  $x = 0$  (current continuity):

$$\frac{V_f^+}{Z_0} - \frac{V_f^-}{Z_0} = \frac{V_t^+}{Z_1}$$

With reflection coefficient:

$$\rho = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

$$V_f^+(e^{j\beta L} + \rho e^{-j\beta L}) = V_s$$

Therefore:

$$V_f^+ = \frac{V_s}{e^{j\beta L} + \rho e^{-j\beta L}}$$

Since  $V_f^- = \rho V_f^+$ :

$$V_f^- = \frac{\rho V_s}{e^{j\beta L} + \rho e^{-j\beta L}}$$

And from voltage continuity at  $x = 0$ :

$$V_t^+ = V_f^+ + V_f^- = V_f^+(1 + \rho)$$

Therefore:

$$V_t^+ = \frac{V_s(1 + \rho)}{e^{j\beta L} + \rho e^{-j\beta L}}$$

[ ]: # Question 1.2

We know:

$$V_f^+ = \frac{V_s}{e^{j\beta_0 L} + \rho e^{-j\beta_0 L}}$$

$$V_f^- = \frac{\rho V_s}{e^{j\beta_0 L} + \rho e^{-j\beta_0 L}}$$

where  $\rho = 1/2$

Substituting these in:

$$V_0(x, 0) = Re[V_s(\frac{e^{-j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}} + \frac{\frac{1}{2}e^{+j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}})]$$

$$= Re[V_s(\frac{e^{-j\beta_0 x} + \frac{1}{2}e^{+j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}})]$$

Similarly for second cable ( $x = 0$ ):

$$V_1(x, 0) = Re[V_t^+ e^{-j\beta_0 x}]$$

$$= Re[V_s \frac{\frac{3}{2}e^{-j\beta_0 x}}{e^{j\beta_0 L} + \frac{1}{2}e^{-j\beta_0 L}}]$$

Since  $e^{-j\beta_0 L} = e^{-j\beta_0 L} = 1$  Therefore:

$$V_0(x, 0) = \text{Re}[V_s(\frac{2}{3})(e^{-j\beta_0 x} + \frac{1}{2}e^{+j\beta_0 x})]$$

$$V_1(x, 0) = \text{Re}[V_s e^{-j\beta_0 x}]$$

For  $-L \leq x \leq 0$ :

$$V_0(x, 0) = V_s \cos(\beta_0 x)$$

For  $x \geq 0$ :

$$V_1(x, 0) = V_s \cos(\beta_0 x)$$

```
[ ]: import numpy as np
import matplotlib.pyplot as plt

# Parameters
Vs = 1.0
beta0_l0 = 4*np.pi # l = 4

# For first cable (-L ≤ x ≤ 0)
x1_l0 = np.linspace(-1, 0, 500) # x/l from -1 to 0
V1 = Vs * np.cos(beta0_l0 * x1_l0) # cos(4 x/l)

# For second cable (0 ≤ x ≤ 2)
x2_l0 = np.linspace(0, 2, 500) # x/l from 0 to 2
V2 = Vs * np.cos(beta0_l0 * x2_l0) # cos(4 x/l)

# Plotting
plt.figure(figsize=(12, 6))
plt.plot(x1_l0, V1, 'blue', label='First Cable (Z)', linewidth=2)
plt.plot(x2_l0, V2, 'red', label='Second Cable (3Z)', linewidth=2)
plt.axvline(x=0, color='black', linestyle='--', alpha=0.5) # marking x=0

plt.grid(True)
plt.xlabel('x/l ')
plt.ylabel('V(x,0)/Vs')
plt.title('Voltage Distribution Along the Transmission Lines ( l = 4 )')
plt.legend()

plt.ylim(-1.5, 1.5)
plt.xlim(-1, 2)
plt.show()
```

```
[ ]: # Question 1.3
```

Recall,

$$V_0(x) = V_f^+ e^{-j\beta_0 x} + V_f^- e^{+j\beta_0 x}$$

and  $V_f^+ = \frac{2}{3}V_s$   $V_f^- = \frac{1}{3}V_s$

So:

$$\begin{aligned} V_0(x) &= \frac{2}{3}V_s e^{-j\beta_0 x} + \frac{1}{3}V_s e^{+j\beta_0 x} \\ &= V_s \left[ \frac{2}{3}(\cos(\beta_0 x) - j \sin(\beta_0 x)) + \frac{1}{3}(\cos(\beta_0 x) + j \sin(\beta_0 x)) \right] \\ &= V_s \left[ \cos(\beta_0 x) - j \frac{1}{3} \sin(\beta_0 x) \right] \end{aligned}$$

For polar form  $|V_0(x)|e^{j\phi}$ :

$$|V_0(x)| = V_s \sqrt{\cos^2(\beta_0 x) + \frac{1}{9} \sin^2(\beta_0 x)}$$

$$\phi = -\tan^{-1}\left(\frac{\frac{1}{3}\sin(\beta_0 x)}{\cos(\beta_0 x)}\right)$$

[ ]: **## Question 2.1**

1) KCL at V node (first capacitor):

- Current entering: I
- Current leaving: I
- Current through capacitor: C dV /dt

$$C \frac{dV_1}{dt} = I_1 - I_2$$

In state variables (C=1):

$$\dot{x}_3 = x_1 - x_2$$

2) KVL around first L loop:

$$L \frac{dI_1}{dt} = V_0 - V_1$$

In state variables (L=1):

$$\dot{x}_1 = \cos(t) - x_3$$

3) KVL around second L loop: Since V = I R and R=1:

$$L \frac{dI_2}{dt} = V_1 - I_2$$

In state variables (L=1):

$$\dot{x}_2 = x_3 - x_2$$

This gives us:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos(t)$$

[ ]: **## Question 2.2**

Given:

$$\dot{X} = AX + BU$$

where:

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad U = \begin{bmatrix} \cos(t) \\ 0 \\ 0 \end{bmatrix}$$

### 0.0.1 (1) Steady-State Assumptions

Since  $U(t) = \begin{bmatrix} \cos(t) \\ 0 \\ 0 \end{bmatrix}$ , we write it in complex exponential form using Euler's formula:

$$U(t) = \text{Re} \left( \begin{bmatrix} e^{j\omega t} \\ 0 \\ 0 \end{bmatrix} \right)$$

where  $\omega = 1$ .

Assume the steady-state solution is:

$$X(t) = \text{Re} \left( \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} e^{j\omega t} \right)$$

where  $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$  are the phasors for the steady-state solution.

### 0.0.2 (2) Frequency-Domain Representation

From the derivation, we have:

$$(j\omega I - A) \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = B \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Substituting  $\omega = 1$ , we get:

$$(jI - A) \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

### 0.0.3 (3) Matrix Substitution

$$jI - A = \begin{bmatrix} j & 0 & 1 \\ 0 & j+1 & -1 \\ -1 & 1 & j \end{bmatrix}$$

Solving the linear system of equations:

$$\begin{bmatrix} j & 0 & 1 \\ 0 & j+1 & -1 \\ -1 & 1 & j \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

#### 0.0.4 (4) Solve the Linear System

1. From the first equation:  $jX_1 + X_3 = 1$
2. From the second equation:  $(j+1)X_2 - X_3 = 0$
3. From the third equation:  $-X_1 + X_2 + jX_3 = 0$

Solving this system, we get:

$$\begin{aligned}X_1 &= 1 \\X_2 &= \frac{1+j}{j+1} \\X_3 &= 1-j\end{aligned}$$

#### 0.0.5 (5) Convert to Time Domain

The steady-state solution is the real part of  $\mathbf{X}e^{j\omega t}$ , where:

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+j}{j+1} \\ 1-j \end{bmatrix}$$

$\mathbf{X}_1(t)$ :

$$X_1(t) = \text{Re}(X_1 e^{j\omega t}) = \text{Re}(1 \cdot e^{j\omega t}) = \cos(\omega t) = \cos(t)$$

$\mathbf{X}_2(t)$ :

$$X_2(t) = \text{Re}\left(\frac{1+j}{j+1} e^{j\omega t}\right)$$

Simplifying the fraction:

$$\frac{1+j}{j+1} = \frac{(1+j)(j-1)}{(j+1)(j-1)} = \frac{2j}{-2} = -j$$

Therefore:

$$X_2(t) = \text{Re}(-je^{j\omega t}) = \text{Re}(-j(\cos(\omega t) + j\sin(\omega t))) = \text{Re}(-\sin(\omega t) - j\cos(\omega t)) = \sin(t)$$

$\mathbf{X}_3(t)$ :

$$X_3(t) = \text{Re}((1-j)e^{j\omega t}) = \text{Re}((1-j)(\cos(\omega t) + j\sin(\omega t)))$$

Expanding:

$$X_3(t) = \text{Re}(\cos(\omega t) - j\cos(\omega t) + j\sin(\omega t) + \sin(\omega t)) = \cos(t) + \sin(t)$$

Thus, the steady-state solutions are:

$$I_1(t) = \mathbf{X}_1(t) = \cos(t)$$

$$I_2(t) = \mathbf{X}_2(t) = \sin(t)$$

$$V_1(t) = \mathbf{X}_3(t) = \cos(t) + \sin(t)$$

```
[ ]: import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# Define the system of differential equations
def system(t, x):
    # State variables: x1, x2, x3
    x1, x2, x3 = x

    # System matrix A
    A = np.array([[0, 0, -1],
                  [0, -1, 1],
                  [1, -1, 0]])

    # Input vector
    u = np.array([np.cos(t), 0, 0])

    # Compute the derivatives
    dxdt = A @ np.array([x1, x2, x3]) + u
    return dxdt

# Initial conditions: x1(0), x2(0), x3(0)
initial_conditions = [0, 0, 0]

# Time span for the solution (e.g., from t=0 to t=10)
t_span = (0, 20)

# Solve the system using solve_ivp
sol = solve_ivp(system, t_span, initial_conditions, t_eval=np.linspace(0, 20,
↪1000))

# Steady-state solutions
t = np.linspace(0, 20, 1000)
I1_steady = np.cos(t) # I1(t) = cos(t)
I2_steady = np.sin(t) # I2(t) = sin(t)
V1_steady = np.cos(t) + np.sin(t) # V1(t) = cos(t) + sin(t)

# Plot the solutions
plt.figure(figsize=(10, 6))

# Numerical solutions
plt.plot(sol.t, sol.y[0], label=r'$I_1(t)$ (Numerical)', color='r',
↪linestyle='--', linewidth=2)
plt.plot(sol.t, sol.y[1], label=r'$I_2(t)$ (Numerical)', color='b',
↪linestyle='--', linewidth=2)
plt.plot(sol.t, sol.y[2], label=r'$V_1(t)$ (Numerical)', color='g',
↪linestyle='--', linewidth=2)
```

```

# Steady-state solutions
plt.plot(t, I1_steady, label=r'$I_1(t) = \cos(t)$ (Steady-state)', color='r',
↪linewidth=2)
plt.plot(t, I2_steady, label=r'$I_2(t) = \sin(t)$ (Steady-state)', color='b',
↪linewidth=2)
plt.plot(t, V1_steady, label=r'$V_1(t) = \cos(t) + \sin(t)$ (Steady-state)',
↪color='g', linewidth=2)

plt.title('Comparison of Numerical and Steady-State Solutions')
plt.xlabel('Time $t$')
plt.ylabel('Amplitude')
plt.legend(loc="center left", bbox_to_anchor=(1, 0.5), fontsize=12)
plt.grid(True)
plt.show()

```

## 0.1 Comment

The transient state of the system converges to the steady-state solution after some time.