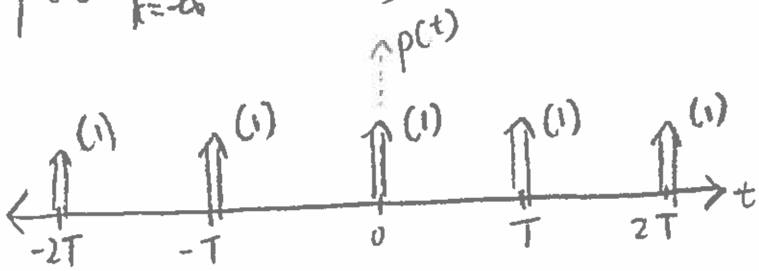
Dennis Chen

PS07

1a. Sketch $p(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$



b. Find the fourier series representation

$$C_k=\frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=-\infty}^{\infty}\delta(t-kT)\,e^{-j\frac{2\pi}{T}kt}\,dt$$

$$=\frac{1}{T}\left[\ldots\underbrace{\cancel{\int_{-T/2}^{T/2}\delta(t-T)\,e^{-j\frac{2\pi}{T}kt}\,dt}}_{\text{0, unit impulse at } T \text{ is zero over range of } (-\frac{T}{2},\frac{T}{2})}+\underbrace{\int_{-T/2}^{T/2}\delta(t)\,e^{-j\frac{2\pi}{T}kt}\,dt}_{\text{"samples" } e^{-j\frac{2\pi}{T}kt}dt \text{ at } t=0\text{, so integral is } 1}+\underbrace{\cancel{\ldots}}_{0}\right]$$

$$\boxed{C_k=\frac{1}{T}}\quad\text{so}\quad\boxed{p(t)=\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{j\frac{2\pi}{T}kt}}$$

c. Given $x(t)=\sum_{k=-\infty}^{\infty}C_k e^{j\frac{2\pi}{T}kt}$, find $X(\omega)$ $\quad\cancel{\omega=\frac{2\pi}{T}}$

$$\cancel{x(t)=\sum_{k=-\infty}^{\infty}C_k e^{j\omega kt}}\qquad X(\omega)=\int_{-\infty}^{\infty}x(t)\,e^{-j\omega t}\,dt$$

$$X(\omega)=\int_{-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}C_k e^{j\frac{2\pi}{T}kt}\right]e^{-j\omega t}\,dt$$

$$\ldots+\int_{-\infty}^{\infty}C_{-1}e^{-j\frac{2\pi}{T}t}\cdot e^{-j\omega t}\,dt+\int_{-\infty}^{\infty}C_0\cdot e^{0}\cdot e^{-j\omega t}\,dt+\int_{-\infty}^{\infty}C_1\cdot e^{j\frac{2\pi}{T}t}\cdot e^{-j\omega t}\,dt+\ldots$$

$$\cancel{\ldots+C_{-1}\int_{-\infty}^{\infty}e^{jt(-\frac{2\pi}{T}-\omega)}\,dt+C_0\int_{-\infty}^{\infty}e^{jt(0-\omega)}\,dt+C_1\int_{-\infty}^{\infty}e^{jt(\frac{2\pi}{T}-\omega)}\,dt+\ldots}$$

1d. Given b & c, find $P(\omega)$

for $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$, $C_k = \frac{1}{T} = \frac{\omega}{2\pi}$

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty} \frac{\omega}{2\pi} e^{j\omega kt}$$

$$P(\omega) = \sum_{k=-\infty}^{\infty} \frac{\omega}{2\pi} \cdot \frac{e^{(k-1)j\omega t}}{(k-1)j\omega} = \boxed{\sum_{k=-\infty}^{\infty} \frac{e^{(k-1)j\omega t}}{2\pi(k-1)j}}$$

1c. continued

$$X(\omega) = \cdots + C_{-1}\int_{-\infty}^{\infty} e^{-j\frac{2\pi}{T}t} \cdot e^{-j\omega t} dt + C_0 \int_{-\infty}^{\infty} e^{0} \cdot e^{-j\omega t} dt + \cdots$$

Table of fourier transform pairs tells us that

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega t} dt = 2\pi\delta(\omega - \omega_0), \text{ so}$$

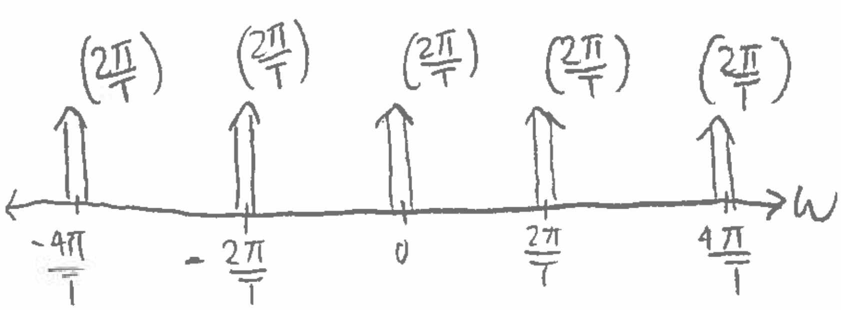
$$X(\omega) = \cdots + C_{-1} 2\pi\delta\left(\omega - \left(-\tfrac{2\pi}{T}\right)\right) + C_0 2\pi\delta(\omega - 0) + C_1 2\pi\delta\left(\omega - \left(\tfrac{2\pi}{T}\right)\right)$$

$$\boxed{X(\omega) = \sum_{k=-\infty}^{\infty} C_k 2\pi\delta\left(\omega - k\left(\tfrac{2\pi}{T}\right)\right)}$$

1d. find $P(\omega)$ ← Substitute C_k we found into above expression $[C_k = \frac{1}{T}]$

$$\boxed{P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \left(\tfrac{2\pi}{T}\right)k\right)}$$

1e. Sketch $P(\omega)$. How does changing T affect $p(t)$ & $P(\omega)$?



↑T: $p(t)$ has more time inbetween impulses

$P(\omega)$ has impulses w/ smaller areas, & less frequencies b/w impulses

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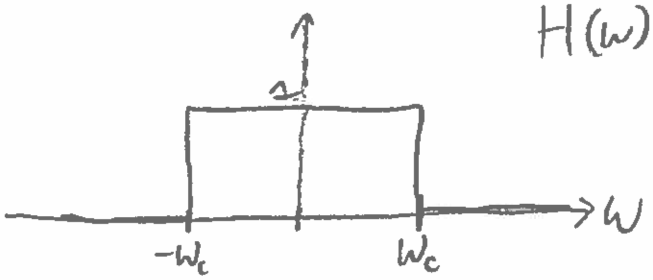
2a. $x(t) \xrightarrow{\text{system}} y(t)$

impulse response $h(t)$

$H(\omega)$



$$H(\omega) = \begin{cases} 1 & \text{if } -W_c \le \omega \le W_c \\ 0 & \text{elsewhere} \end{cases}$$

Use ICFT to find $h(t)$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-W_c}^{W_c} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W_c}^{W_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{jW_c t}}{jt} - \frac{e^{-jW_c t}}{jt}\right]$$

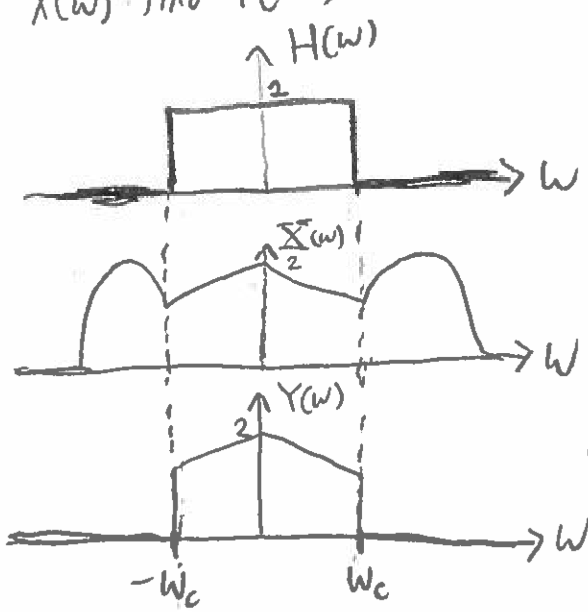
Note: $\sin\theta = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$

$$= \frac{1}{\pi t}\left[\frac{e^{jW_c t}}{2j} - \frac{e^{j(-W_c t)}}{2j}\right]$$

$$= \frac{1}{\pi t}\left[\sin(W_c t)\right]$$

So $\boxed{h(t) = \frac{\sin(W_c t)}{\pi t}}$

b. Given $X(\omega)$ find $Y(\omega)$



$Y(\omega)$ looks like this b/c the output of an LTI system is the input convoluted with the impulse response. Convolution is just multiplication in the frequency domain, so $Y(\omega)$ is just the product of $H(\omega)$ & $X(\omega)$

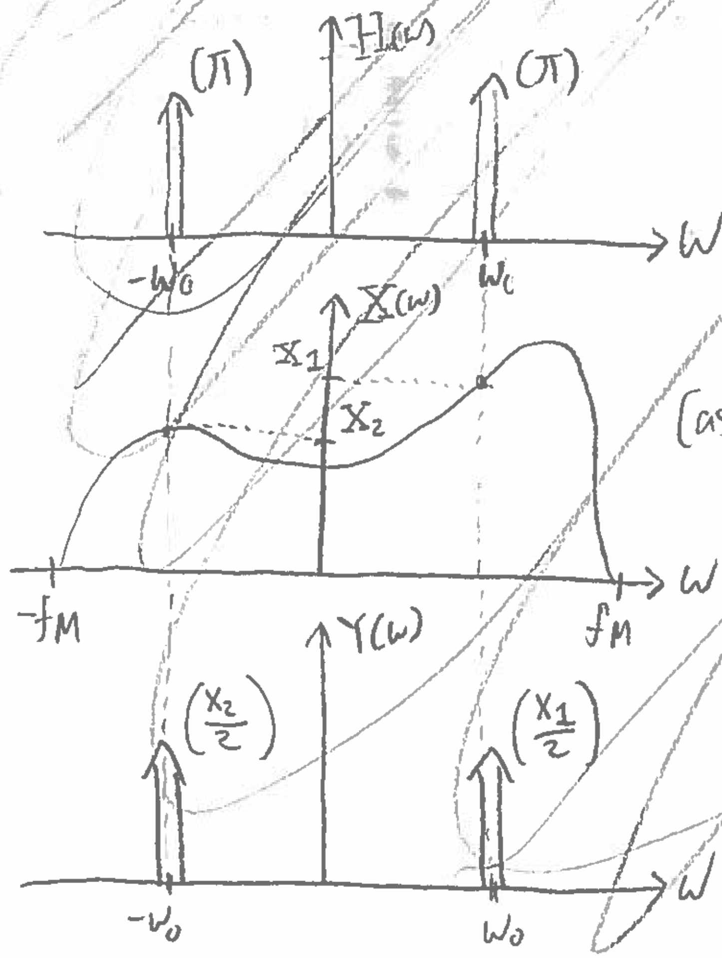
?C. This LTI system is known as an ideal low-pass filter w/ cutoff W_c b/c it leaves frequency components between $-W_c$ & W_c alone, & completely kills off higher frequency components. Since inputs are convolved (multiplication in the frequency domain) with the impulse response, you see that the impulse response is zero & kills off $w > W_c$, & is 1 for $-W_c < w < W_c$ & preserves those frequencies

3. $y(t) = x(t)\cos(W_0 t)$. Given a sketch of $X(w)$, sketch $Y(w)$

fourier transform property: if $y(t) = x(t)h(t)$ $\quad Y(w) = \frac{1}{2\pi} X * H(w)$

if $h(t) = \cos(W_0 t)$, what is $H(w)$?

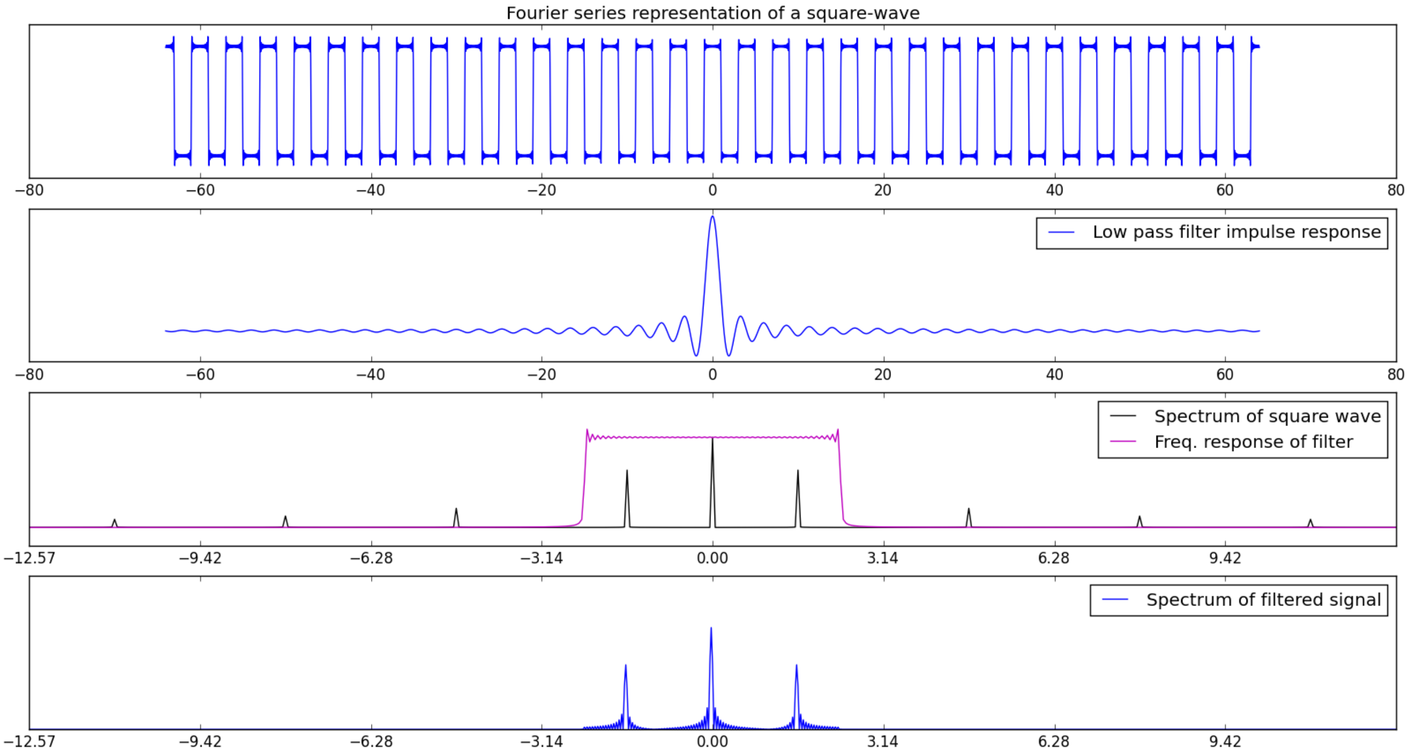
$$H(w) = \pi\delta(w - W_c) + \pi\delta(w + W_c)$$



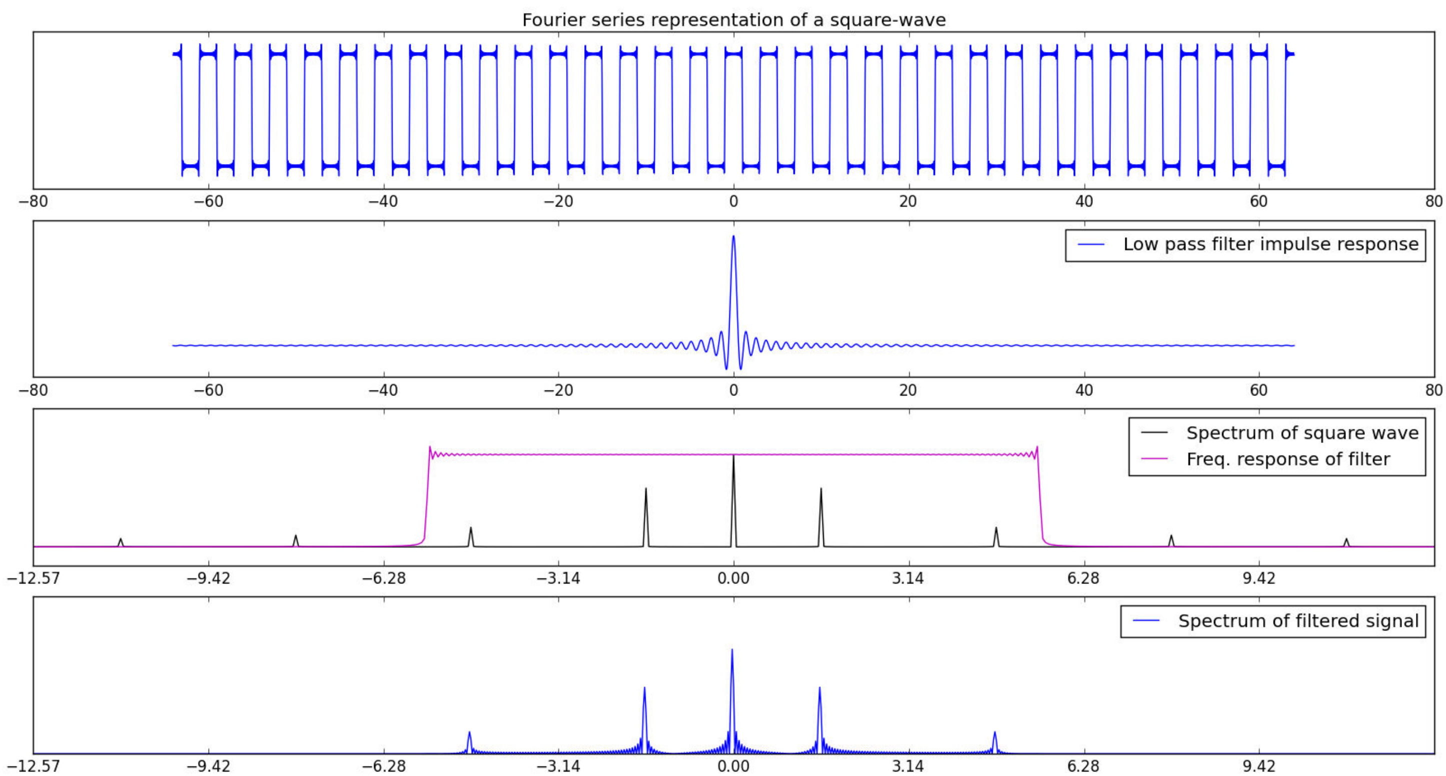
(assume $f_M > W_0$)

... otherwise, if $f_M < W_0$

$Y(w) = 0$



Wc = 0.75*pi

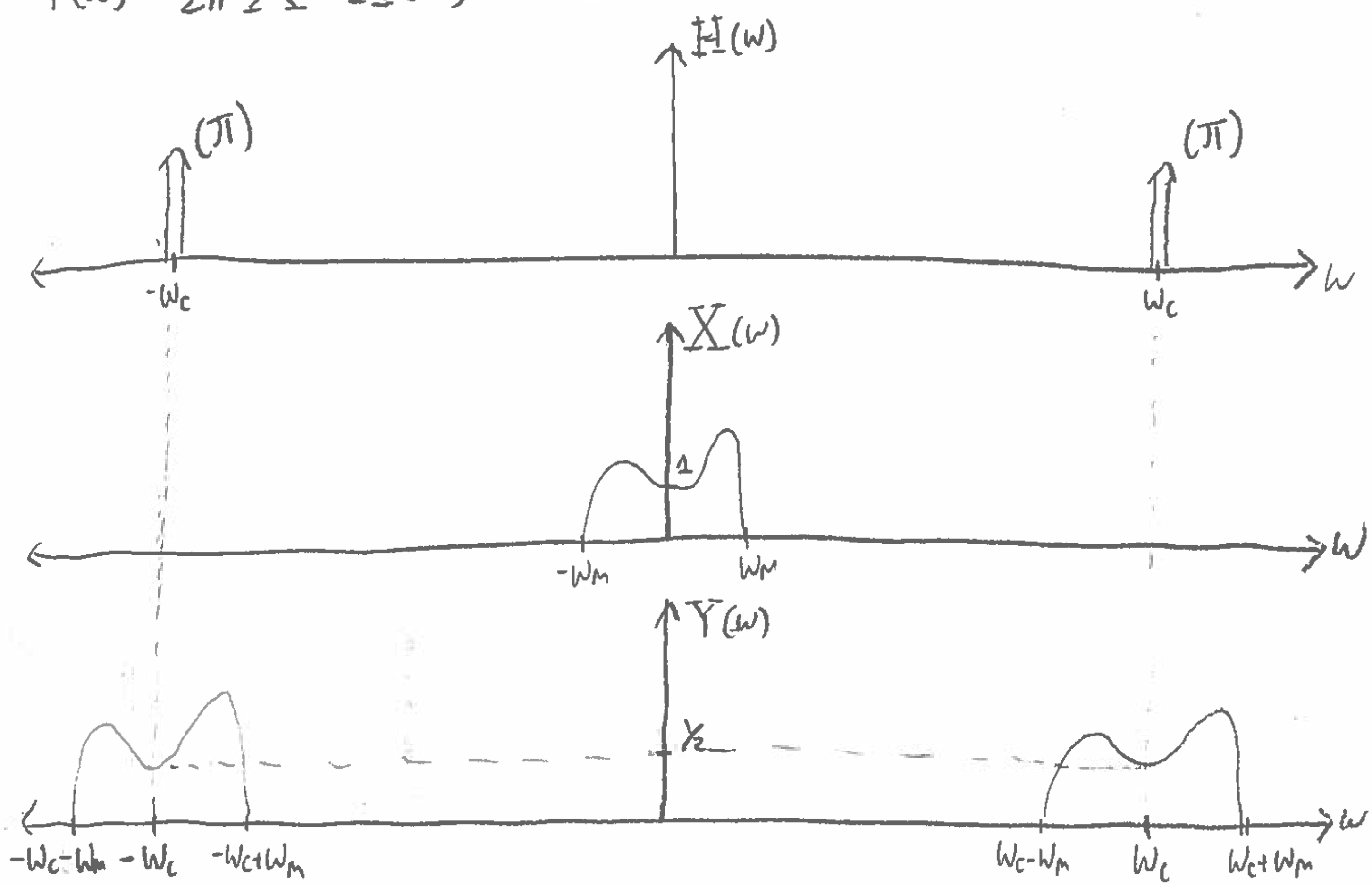


Wc = 1.75*pi

3. $x(t)$ w/ band-limited range $[-W_m, W_m]$

$y(t) = x(t)\cos(W_c t)$, $W_c \gg W_m$. Sketch $Y(w)$

$Y(w) = \frac{1}{2\pi} X * H(w)$ if $h(t) = \cos(W_c t)$, $H(w) = \pi\delta(w - W_c) + \pi\delta(w + W_c)$



$Y(w)$ works this way b/c convolution w/ an impulse shifts & scales a signal so that it is centered at the impulse.