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SURVEY

k-Domination and k-Independence in Graphs: A Survey

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Abstract In 1985, Fink and Jacobson gave a generalization of the concepts of domination and independence in graphs. For a positive integer k, a subset S of vertices in a graph $G = (V, E)$ is *k-dominating* if every vertex of $V - S$ is adjacent to at least k vertices in S. The subset S is *k-independent* if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k - 1$. In this paper we survey results on k-domination and k-independence.

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Contents

1 Introduction and Terminology

Let $G = (V, E) = (V(G), E(G))$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. If S is a subset of $V(G)$ then $N(S) = \cup_{x \in S} N(x)$, $N[S] = \cup_{x \in S} N[x]$ and the subgraph induced by S in G is denoted $G[S]$. By $\delta = \delta(G)$ and $\Delta = \Delta(G)$ we denote the *minimum* and *maximum degree* of the graph G, respectively. A vertex of degree zero is *isolated*, a vertex of degree one is called a *leaf* and its neighbor is called a *stem*. If u is a stem, then L_u will denote the set of leaves attached to u. If $L(G)$ is the set of leaves of a graph G, then let $\ell = \ell(G) = |L(G)|$.

The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$ and with exactly the edges that do not belong to G. The *complete graph* of order n is denoted by K_n, and K_1 is called the *trivial graph*. The *complete bipartite graph* with partition sets A, B such that $|A| = p$ and $|B| = q$ is denoted by $K_{p,q}$. If H and G are graphs, then G is called H*-free* if G does not contain any induced subgraph isomorphic to H. A *claw-free* graph is a $K_{1,3}$-free graph. The *corona* of two graphs G_1 and G_2, as defined in [71], is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. A vertex v of G is called a *cut vertex* if removing it from G increases the number of components of G. A connected graph without cut vertices is called a *block*. A *block of a graph* G is a subgraph of G which is itself a block and which is maximal with respect to that property. A graph G is a *block graph* if every block of G is a complete graph. A *block-cactus graph* is a graph for which every block is complete or a cycle. A graph is a *cactus graph* if every block is a cycle or a K_2. A cactus graph having one cycle is called a *unicyclic graph* and a connected graph with no cycles is called a *tree*. If we substitute each edge in a nontrivial tree by two parallel edges and then subdivide each edge, then we speak of a C_4-*cactus*. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with p and q leaves attached to each stem, respectively, is denoted by $S_{p,q}$. A bipartite graph is called p-*semiregular* if every vertex in one of the two partite sets has degree p. The *subdivision graph* $S(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. In the case that G is the trivial graph, we define $S(G) = G$. A *subdivided star* SS_t is the subdivision graph of the *star* $K_{1,t}$. If G is a double star $S_{p,q}$, then we write $SS_{p,q}$ instead of $S(S_{p,q})$. A *generalized star* is a tree that results from a star $K_{1,t}$ by subdividing its edges arbitrary many times.

A set $S \subseteq V$ is a *dominating set* if for each vertex $v \in V - S$, $N(v) \cap S \neq \emptyset$. The *domination number* $\gamma(G)$ and the *upper domination number* $\Gamma(G)$ are respectively the minimum cardinality of a dominating set and the maximum cardinality of a minimal dominating set of G. A subset $S \subseteq V(G)$ is said to be a *total dominating set* if every vertex in $V(G)$ has at least one neighbor in S and it is a *connected dominating set*, if it is a dominating set and the graph induced by S is connected. The *total domination number* $\gamma_t(G)$ and the *connected domination number* $\gamma_c(G)$ represent the cardinality of a *minimum total dominating set* and, respectively, of a *minimum connected dominating set* of G. A subset $S \subseteq V(G)$ is said to be *independent* if $E(G[S]) = \emptyset$. The *independent domination number* (resp. the *independence number*) of G denoted by $i(G)$ (resp. $\beta(G)$) is the size of the smallest (resp. the largest) maximal independent set in G. A *vertex cover* in G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph is called the *covering number* of G and is denoted by $\alpha(G)$. The *edge independence number* $\tau(G)$ of G is the maximum cardinality among the sets of pairwise non adjacent edges of G.

It is well known that an independent set is maximal if and only if it is also dominating. So we can say that the domination, which is defined even for non-independent sets, is the property which makes an independent set maximal. Moreover every set which is both independent and dominating is a minimal dominating set of G. This observation leads to the well known inequality chain (see [94]):

$$\gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \quad \text{for all graphs } G. \tag{1}$$

In [67,68], Fink and Jacobson generalized the concepts of independent and dominating sets. In the whole paper, k will be a positive integer. We say that a subset S of V is k-*independent* if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k - 1$. The subset S is k-*dominating* if every vertex of $V - S$

has at least k neighbors in S. The property for a subset of V to be k-independent (k-dominating) is hereditary (superhereditary). A k-independent set S of G is maximal if for every vertex $v \in V - S$, $S \cup \{v\}$ is not k-independent. A k-dominating set S is minimal if, for every vertex $v \in S$, $S - \{v\}$ is not k-dominating in G. The *k-domination number* $\gamma_k(G)$ and the *upper k-domination number* $\Gamma_k(G)$ are respectively the minimum cardinality of a k-dominating set and the maximum cardinality of a minimal k-dominating set of G. The *lower k-independence number* $i_k(G)$ is the minimum cardinality of a maximal k-independent set in G and the *k-independence number* $\beta_k(G)$ is the maximum cardinality of a k-independent set. Thus for $k = 1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets. Hence $i_1(G) = i(G)$, $\beta_1(G) = \beta(G)$, $\gamma_1(G) = \gamma(G)$ and $\Gamma_1(G) = \Gamma(G)$.

In 1989, Jacobson and Peters [103] showed that determining the numbers $\gamma_k(G)$ and $\beta_k(G)$ for an arbitrary graph is NP-Complete and gave linear algorithms to compute them in trees and in series-parallel graphs. In 1994, Bean, Henning and Swart [8] proved that for γ_k, the problem remains NP-Complete in bipartite or chordal graphs.

Note that a k-independent set is sometimes called $(k-1)$-dependent [67,68], k-dependent [104], $(k-1)$-small [101]. Also the term k-domination is sometimes used for the k-distance domination introduced by Meir and Moon [125].

Since the entire vertex set of a graph G is a k-dominating set, Fink and Jacobson [68] gave a necessary and sufficient condition for a k-dominating set to be minimal in a graph G.

Theorem 1 [68] *Let D be a k-dominating set of a graph G. Then D is minimal if and only if for every vertex $v \in D$, either:*

1. *v has less than k neighbors in D, or*
2. *there exists a vertex $u \in V - D$ such that $|N(u) \cap D| = k$ and $u \in N(v)$.*

Likewise we give below the conditions under which a k-independent set of a graph G is maximal.

Theorem 2 *Let S be a k-independent set of a graph G. Then S is maximal if and only if for every vertex $v \in V - S$ either:*

1. *v is adjacent to at least k vertices in S, or*
2. *v is adjacent to a vertex $u \in S$ that has exactly $k-1$ neighbors in S.*

From what precedes, we have the following general properties.

- Every k-dominating set of a graph G contains at least k vertices and all vertices of degree less than k; so $\gamma_k(G) \geq k$ when $n \geq k$.
- Every set with k vertices is k-independent; so $i_k(G) \geq k$ when $n \geq k$.
- From the definition we have that for every graph G and every k, $\gamma_k(G) \leq \Gamma_k(G)$ and $i_k(G) \leq \beta_k(G)$.
- Every $(k+1)$-dominating set is also a k-dominating set and so $\gamma_k(G) \leq \gamma_{k+1}(G)$. Moreover, the vertex set V is the only $(\Delta+1)$-dominating set but is not a minimal Δ-dominating set. Thus every graph G satisfies $\Gamma_\Delta(G) < n$ and

$$\gamma(G) = \gamma_1(G) \leq \gamma_2(G) \leq \cdots \leq \gamma_\Delta(G) < \gamma_{\Delta+1}(G) = n.$$

- Every k-independent set is $(k+1)$-independent; so $\beta_{k+1}(G) \geq \beta_k(G)$. Moreover, the vertex set V is the only maximal $(\Delta+1)$-independent but is not a Δ-independent set. Thus every graph G satisfies $i_{\Delta+1}(G) = n$ and

$$\beta(G) = \beta_1(G) \leq \beta_2(G) \leq \cdots \leq \beta_\Delta(G) < \beta_{\Delta+1}(G) = n.$$

- Every set S that is both k-independent and k-dominating (if any) is a maximal k-independent set.
- Every set S that is both k-independent and k-dominating (if any) is a minimal k-dominating set.
- For $k \geq 2$, a maximal k-independent set S is not necessarily a k-dominating set. It can be seen by the corona $C_n \circ K_1$ of a cycle $C_n, n \geq 3$, where $V(C_n)$ is a maximal 3-independent set but not a 3-dominating set.

We close this section by mentioning that Lu, Hou, Xu and Li [121] give equivalent conditions for trees with unique minimum k-dominating sets.

2 Inequalities Between the Four Parameters

2.1 Relationships Between Two Parameters of the Same Kind

We saw in the introduction that the sequence (γ_k) was nondecreasing. In [67,68], Fink and Jacobson raised the question of the rate at which the k-domination number increases with k. They proved that $\gamma_3(G) > \gamma(G)$ for graphs with $\Delta \geq 3$ (by Theorem 8 below) and their first conjecture in [67] was $\gamma_{2k+1}(G) > \gamma_k(G)$ if $\delta \geq k$. This strict inequality was proved for the case $k = 2$ by Chen and Jacobson.

Theorem 3 [40] *For every graph G with minimum degree $\delta \geq 2$, $\gamma_2(G) < \gamma_5(G)$.*

Theorem 3 is best possible in the sense that there exist infinitely many graphs G with minimum degree at least 2 having $\gamma_2(G) = \gamma_4(G)$.

However, Schelp (unpublished) disproved Fink and Jacobson's conjecture by constructing the following family of graphs. For $k \geq 4$, let G_k be the graph obtained from an independent set B with $|B| = k + 1$ and $k + 1$ disjoint cliques A_i, with $|A_i| = k$ for every i, by adding all possible edges between B and the vertices of A_i for every i. Then G has minimum degree $\delta = 2k$, and $\gamma_{2k}(G) = \gamma_{k(k+1)}(G) = k(k + 1)$.

The problem of the rate of growth of the sequence (γ_k) remains open under the following form.

Problem 4 [68] Find a function f such that

$$\gamma_k(G) < \gamma_{f(k)}(G)$$

for every graph with $\delta \geq k$.

By Schelp's counterexample, if f exists then $f(k) > k^2/4$.

Such a function f cannot exist in general graphs for the nondecreasing sequence (β_k) since the star $K_{1,n-1}$ satisfies

$$\beta(K_{1,n-1}) = \beta_2(K_{1,n-1}) = \cdots = \beta_{n-1}(K_{1,n-1}) = n - 1.$$

Similar problems do not exist for the sequences (i_k) and (Γ_k) since examples showing that they are not necessarily monotone are given in [57].

We give below some particular classes of graphs for which a function f has been determined for γ_k or β_k. We denote by $K_{1,3} + e$ the graph obtained from a claw $K_{1,3}$ by adding an edge between two leaves of $K_{1,3}$. Let g be the graph obtained from a cycle C_4 by adding a new vertex adjacent to all vertices of the cycle. Let h be the graph obtained from a cycle C_4 by adding two new vertices each one adjacent to all vertices of the cycle C_4.

Theorem 5 [68] *If G is a claw-free graph and k is an integer with $2 \le k \le \Delta$, then $\gamma_k(G) < \gamma_{2k}(G)$.*

Theorem 6 [57]

1. *If G is a $\{K_{1,3}, K_{1,3} + e\}$-free graph and k is a positive integer with $k \le \Delta$, then $\gamma_k(G) < \gamma_{k+2}(G)$.*
2. *If G is a $\{K_{1,3}, K_{1,3}+e, h\}$-free graph and k is a positive integer with $2 \le k \le \Delta$, then $\gamma_k(G) < \gamma_{k+1}(G)$.*

Theorem 7 [57]

1. *If G is a $K_{1,3}$-free graph and k is a positive integer with $k \le \Delta$, then $\beta_k(G) < \beta_{2k+1}(G)$.*
2. *If G is a $\{K_{1,3}, K_{1,3} + e\}$-free graph and k is a positive integer with $k \le \Delta$, then $\beta_k(G) < \beta_{k+2}(G)$.*
3. *If G is a $\{K_{1,3}, K_{1,3}+e, g\}$-free graph and k is a positive integer with $2 \le k \le \Delta$, then $\beta_k(G) < \beta_{k+1}(G)$.*

The particular question of the comparison between γ_k and γ was first raised by Fink and Jacobson.

Theorem 8 [67] *If G is a graph with $\Delta(G) \ge k \ge 2$, then $\gamma_k(G) \ge \gamma(G) + k - 2$.*

Several authors determined some classes of graphs for which the bound $\gamma(G)+k-2$ in Theorem 8 can be replaced by $\gamma(G) + k - 1$. The last results in this direction are those of Hansberg who gave some properties of the graphs satisfying the equality on the bound of Theorem 8.

Theorem 9 [76, submitted] *Let G be a connected graph and k an integer with $\Delta(G) \ge k \ge 2$. If $\gamma_k(G) = \gamma(G) + k - 2$, then every vertex of G lies on an induced cycle of length 4.*

Theorem 10 [76, submitted] *Let G be a connected graph and k an integer with $\Delta(G) \ge k \ge 2$. If $\gamma_k(G) = \gamma(G)+k-2$, then G contains at least $(\gamma(G)-1)(k-1)$ induced cycles of length 4.*

Reverting the assertions of the theorems yields a strengthening of Theorem 8.

Corollary 11 [76, submitted] *Let G be a connected graph. If there is a vertex $u \in V$ that is not contained in any induced cycle of length 4 of G, then $\gamma_k(G) \geq \gamma(G)+k-1$.*

Corollary 12 [76, submitted] *Let G be a graph with $\Delta(G) \leq n(G)-2$. If G has less than $(\gamma(G)-1)(k-1)$ induced cycles of length 4 for an integer k with $\Delta(G) \geq k \geq 2$, then $\gamma_k(G) \geq \gamma(G) + k - 1$.*

Let r and k be two positive integers, where $k \geq 2$. Let G be a graph consisting of a complete graph H on $k-1$ vertices and of vertices u_i, v_i, w_i, for $1 \leq i \leq r$, such that every u_i and w_i is adjacent to every vertex of H and to v_i. Then it is easy to see that $\gamma_k(G) = k + r - 1$, $\gamma(G) = r + 1$ and thus $\gamma_k(G) = \gamma(G) + k - 2$. Since G contains exactly $r(k-1) = (\gamma(G)-1)(k-1)$ induced cycles of length 4, it follows that the bound of Corollary 12 is sharp.

Corollaries 11 and 12 improve previous partial results by Chellali, Favaron, Hansberg and Volkmann [33].

Hansberg and Volkmann [82–84] considered the case $k = 2$. They proved that for nontrivial connected graphs, $\gamma_2(G) \geq \gamma(G) + 1$ if G is a block graph or a unicyclic graph different from C_4 and characterized the graphs of these two families satisfying $\gamma_2(G) = \gamma(G) + 1$ (comprising thus Theorem 14). In [84] they proved that if $\gamma_2(G) = \gamma(G)$ then $\delta \geq 2$ and that if G is a block-cactus graph, then $\gamma_2(G) = \gamma(G)$ if and only if G is a C_4-cactus.

Chellali, Favaron, Hansberg and Volkmann [33] proved that if G is a graph with at most $k-2$ induced cycles of length 4 for an integer k with $\Delta(G) \geq k \geq 2$, then $\gamma_k(G) \geq \gamma(G) + k - 1$. Moreover, in the particular case of trees, they obtained the following extremal result when $k \geq 3$.

Theorem 13 [33] *Let T be a tree such that $\Delta(T) \geq k \geq 3$ for an integer k. Then $\gamma_k(T) = \gamma(T) + k - 1$ if and only if T is isomorphic to a subdivided star SS_k minus p leaves for an integer $1 \leq p \leq k$.*

When $k = 2$, the extremal trees are given by

Theorem 14 [143] *Every nontrivial tree satisfies $\gamma_2(T) = \gamma(T) + 1$ if and only if T is a subdivided star SS_t or a subdivided star SS_t minus a leaf or a subdivided double star $SS_{s,t}$.*

That $\gamma_{k+1}(G)$ cannot be too large with respect to $\gamma_k(G)$ is nevertheless shown by the following two theorems.

Theorem 15 [60, 142, p. 195] *For every graph G of order n and positive integer k with $k \leq \delta - 1$, $\gamma_{k+1}(G) \leq \frac{n+\gamma_k(G)}{2}$ and this bound is sharp.*

That the bound is sharp may be seen by the graph G obtained from a clique B with $|B| = k$ and $q \geq k$ cliques A_i each of order two by adding all edges between B and all A_i's. Then $n = k+2q$, $\delta = k+1$, $\gamma_k(G) = k$ and $\gamma_{k+1}(G) = k+q = (n+\gamma_k(G))/2$. The special case $k = 2$ of Theorem 15 can be found in [19]. In [146], Volkmann characterized the connected P_4-free graphs G with the property that $\overline{G}$ is $(K_4 - e)$-free attaining equality in Theorem 15.

Theorem 16 [61] *Every graph of order n and minimum degree δ satisfies*

$$\gamma_k(G) + \frac{(k' - k + 1)}{2k' - k}\gamma_{k'}(G) \leq n$$

for all integers k and k' with $1 \leq k \leq k' \leq \delta$.

In what concerns relationships between two upper or lower independence parameters, we can cite

Theorem 17 [14] *For every graph G and integers j, k with $1 \leq j \leq k$, $\beta_{k+1}(G) \leq \beta_j(G) + \beta_{k-j+1}(G)$.*

Theorem 18 [14] *For every graph G and integers j, k with $1 \leq j \leq k$, $i_{k+1}(G) \leq (k - j + 2)i_j(G)$. Equality can occur only when $j = 1$ or $j = k$.*

As a corollary of Theorems 17 and 18, we get $\beta_{k+1}(G) \leq (k+1)\beta(G)$ and $i_{k+1}(G) \leq (k+1)i(G)$ (in particular $i_2(G) \leq 2i(G) \leq i(G) + \beta(G)$). In [14], a structural property of graphs with $\beta_{k+1}(G) = (k+1)\beta(G)$ is given and some classes of graphs satisfying $i_2(G) = i(G) + \beta(G)$ are determined.

2.2 Relationships Between Parameters of Different Kinds

According to (1), every graph G satisfies $\gamma_1(G) \leq \beta_1(G)$. In [67], Fink and Jacobson proved that $\gamma_2(G) \leq \beta_2(G)$ and conjectured that for every graph G and any positive integer k, $\gamma_k(G) \leq \beta_k(G)$. This inequality is not obvious because for $k \geq 2$, a maximal k-independent set is not necessarily k-dominating as it is for $k = 1$ (see Sect. 5.3). The conjecture has been proved by showing the following stronger result.

Theorem 19 [56] *For any graph G and positive integer k, every k-independent set D such that $\phi_k(D) = k|D| - |E(G[D])|$ is maximum is a k-dominating set of G.*

The proof of Theorem 19 allows to construct a k-independent k-dominating set from a k-independent one, and thus from any independent set. However, the algorithmic aspect is not developed in [56] (see k-insulated sets in Sect. 5.1).

By Theorem 19, any graph G admits a set S that is both k-independent and k-dominating. Since such a set is a minimal k-dominating set and a maximal k-independent set, $\gamma_k(G) \leq |S| \leq \Gamma_k(G)$ and $i_k(G) \leq |S| \leq \beta_k(G)$. Therefore

Corollary 20 [56] *For any graph G and positive integer k,*

$$\gamma_k(G) \leq \beta_k(G) \quad \text{and} \quad i_k(G) \leq \Gamma_k(G).$$

The inequality chain (1) is now partially generalized. We can wonder whether a complete generalization of (1) is possible for every positive integer k. The answer is negative as noticed in [57]. The following examples show that each of the four inequalities $\gamma_k(G) > i_k(G)$, $i_k(G) > \gamma_k(G)$, $\Gamma_k(G) > \beta_k(G)$ and $\beta_k(G) > \Gamma_k(G)$ is possible.

For a double star $S_{k-1,k-1} (k \geq 2)$ we have $\gamma_k(G) = n - 1 > i_k(G) = n-2$ while for the graph G constructed from three subdivided stars SS_p, $p \geq 2$, with centers x, y and z by adding edges xy and xz, $i_2(G) = 4p + 2$ and $\gamma_2(G) = 3p + 3$.

Now let us consider the graph G obtained from $k \geq 2$ disjoint stars $F_i \simeq K_{1,k}$ with centers c_i and leaves $u_{i,1}, u_{i,2}, \ldots, u_{i,k}$ by adding a new vertex x and the k edges xc_i, $1 \leq i \leq k$. Then $n = k^2 + k + 1$ and, since $k < \Delta$, $\Gamma_k(G) < n$ follows. Moreover, $\bigcup_{i=1}^{k} V(F_i)$ is a minimal k-dominating set of G. Therefore $\Gamma_k(G) = k^2 + k$. On the other hand, let S be a k-independent set of G. Clearly, every maximal k-independent set has at most k vertices in each star F_i and $\beta_k(G) \leq k^2 + 1$. Since $V(G) - \{c_1, c_2, \ldots, c_k\}$ is a maximal k-independent set, $\beta_k(G) = k^2 + 1$ follows. Thus $\Gamma_k(G) > \beta_k(G)$. The same graph G also provides an example for the opposite inequality since $\Gamma_{k+1}(G) = k^2 + 1 < k^2 + k = \beta_{k+1}(G)$.

However, some inequalities can be proved in particular situations as shown by Theorems 21 and 22.

Theorem 21 [104] *If $\gamma_{k+1}(G) = \gamma_k(G)$ for some positive integer k, then $\gamma_k(G) \geq i_k(G)$.*

Theorem 22 *Every graph G satisfies $\beta_\Delta(G) \geq \Gamma_\Delta(G)$.*

Proof Let S be a minimal Δ-dominating set of G with maximum order. Then the maximum degree in $G[S]$ is at most $\Delta - 1$ for otherwise a vertex $v \in S$ of degree Δ has all its neighbors in S and then $S - \{v\}$ is a Δ-dominating set, contradicting the minimality of S. Thus S is a Δ-independent set of G and so $\beta_\Delta(G) \geq |S| = \Gamma_\Delta(G)$. □

In the remaining part of the section, we give relations between two parameters of different kinds and different indices.

The authors of [104] observed that if S is a k-independent set of a graph G of minimum degree $\delta \geq k$, then $V - S$ is a $\delta - k + 1$-dominating set. Similarly, if S is a q-dominating set of a graph G of maximum degree $\Delta \geq q$, then $V - S$ is a $\Delta - q + 1$-independent set. Therefore

Theorem 23

1. (Jacobson, Peters and Rall [104]) *For every graph G and positive integer $k \leq \delta$, $\beta_k(G) + \gamma_{\delta-k+1}(G) \leq n$.*
2. (Favaron [60]) *For every graph G and positive integer $k \leq \Delta$, $\beta_k(G) + \gamma_{\Delta-k+1}(G) \geq n$. If moreover G is d-regular, then $\gamma_{d+1-k} + \beta_k = n$.*

Theorems 24 and 25 give an upper bound for the k-domination number in terms of the independence number for general graphs and r-partite graphs.

Theorem 24 [79] *Let G be a connected nontrivial graph with maximum degree Δ and let k be a positive integer such that $\delta(G) \geq k$. If G is neither isomorphic to a cycle of odd length when $k = 2$ nor to the complete graph K_{k+1}, then $\gamma_k(G) \leq (\Delta - 1)\beta(G)$. Moreover, if G is a non-regular graph, $\gamma_k(G) = (\Delta - 1)\beta(G)$ if and only if G is the corona $K_2 \circ K_k$.*

Theorem 25 [79] *If G is a connected r-partite graph and k is an integer such that $\Delta \geq k$, then $\gamma_k(G) \leq \frac{\beta(G)}{r}((r-1)r+k-1)$.*

For bipartite graphs and $k = 2$, the bound of Theorem 25 was already given by Fujisawa, Hansberg, Kubo, Saito, Sugita and Volkmann [72], who also characterized in this case the equality. The inequality $\gamma_2(G) \leq \frac{3}{2}\beta(G)$ had been previously shown for trees by Blidia, Chellali and Favaron [11].

Theorem 26 [72] *If G is a connected bipartite graph of order at least 3, then $\gamma_2(G) \leq \frac{3}{2}\beta(G)$ and equality holds if and only if G is the corona of the corona of a connected bipartite graph or G is the corona of the cycle C_4.*

The following results give a lower bound on the k-domination number in terms of the $(k-1)$-independent number for special classes of graphs. The case $k = 2$ of Theorem 27 was known by [11].

Theorem 27 [18] *If T is a tree then $\gamma_k(T) \geq \beta_{k-1}(T)$ for every integer $k \geq 2$.*

Equivalent conditions for trees to satisfy $\gamma_k(T) = \beta_{k-1}(T)$ are also given in [18]. A *weak* (resp., *weakly exact*) $\mathcal{N}_k$-tree T of special vertex w is a tree with $d_T(w) \geq k-1$ (resp., $d_T(w) = k-1$) and $d_T(x) \leq k-2$ for every vertex $x \in V(T) - N[w]$. Let $\mathcal{A}_k$ be the family of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_p (p \geq 1)$ of trees, where T_1 is a weak $\mathcal{N}_k$-tree of special vertex w of degree at least k, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by the two operations defined below. Let $A(T_1) = V(T_1) - \{w\}$.

- *Operation $\mathcal{T}_1$*: Attach a weak $\mathcal{N}_k$-tree T_0 of special vertex w of degree at least k by adding an edge from w to any vertex of T_i. Let $A(T_{i+1}) = A(T_i) \cup (V(T_0) - \{w\})$.
- *Operation $\mathcal{T}_2$*: Attach a weakly exact $\mathcal{N}_k$-tree T_0 of special vertex w by adding an edge from w to any vertex of $A(T_i)$. Let $A(T_{i+1}) = A(T_i) \cup (V(T_0) - \{w\})$.

Theorem 28 [18] *Let T be a tree and $k \geq 2$ an integer. Then the following statements are equivalent:*

1. $\gamma_k(T) = \beta_{k-1}(T)$,
2. $\Delta(T) \leq k-2$ *or* $T \in \mathcal{A}_k$,
3. *T has a unique $\gamma_k(T)$-set that is also a unique $\beta_{k-1}(T)$-set.*

The previous method recursively applying operations to construct the extremal class for some inequality between parameters of trees is commonly used. From now on we call it a *recursive construction.*

For $k = 2$, the authors of Theorem 27 extended their result to block graphs and established a similar bound for cactus graphs.

Theorem 29 [19] *If G is a block graph, then $\gamma_2(G) \geq \beta(G)$.*

Theorem 30 [19] *If G is a connected cactus graph with $c(G) \geq 1$ cycles, then*

$$\gamma_2(G) \geq \beta(G) - c(G) + 1.$$

We improve below Theorem 30 by replacing $\beta(G)$ by $\Gamma(G)$. Recall that $\beta(G) \leq \Gamma(G)$ for every graph G and that $\beta(G) = \Gamma(G)$ if G is bipartite [44] or unicyclic [139].

Theorem 31 *If G is a connected cactus graph with $c(G) \geq 1$ cycles, then $\gamma_2(G) \geq \Gamma(G) - c(G) + 1$.*

Proof We proceed by induction on the number of cycles $c(G)$. If $c(G) = 1$, then the theorem is valid since $\beta(G) = \Gamma(G)$ by [139]. Thus we assume that the result holds for all connected cactus graphs having p cycles where $1 \leq p < c(G)$. Let G be a connected cactus graph with $c(G)$ cycles and S any minimal dominating set of G with maximum order.

Assume that there are two adjacent vertices u, v on some cycle such that u, v are both in S or both not in S. Consider the spanning graph G' obtained by removing the edge uv. Then $c(G') = c(G) - 1$. Clearly S is a minimal dominating set of G' and so $\Gamma(G') \geq \Gamma(G)$. On the other hand it is easy to show that $\gamma_2(G) + 1 \geq \gamma_2(G')$. Using the induction hypothesis for G', we deduce that

$$\gamma_2(G) + 1 \geq \gamma_2(G') \geq \Gamma(G') - c(G') + 1 \geq \Gamma(G) - (c(G) - 1) + 1$$

and therefore $\gamma_2(G) \geq \Gamma(G) - c(G) + 1$.

Thus we assume that all vertices on the cycles are contained alternately in S. It follows that G contains no odd cycles and so G is bipartite and satisfies $\beta(G) = \Gamma(G)$ by [44]. By Theorem 30 we obtain $\gamma_2(G) \geq \Gamma(G) - c(G) + 1$. □

The last results of Sect. 2.2 involve the 2-domination number and the independent domination number for some classes of graphs.

Theorem 32 [87] *If G is a connected nontrivial block-cactus graph, then $\gamma_2(G) \geq i(G)$. Moreover $\gamma_2(G) = i(G)$ if and only if G is a C_4-cactus.*

Using the characterization of trees T such that $\gamma_2(T) = \gamma(T) + 1$ given in Theorem 14, Hansberg and Volkmann characterized all trees T with $\gamma_2(T) = i(T) + 1$.

Theorem 33 [87] *Let T be a nontrivial tree. Then $\gamma_2(T) = i(T) + 1$ if and only if T is a subdivided star SS_t or a subdivided star SS_t minus a leaf or a subdivided double star $SS_{s,t}$ or T is isomorphic to the tree of order 6 with two adjacent vertices of order 3 and 4 leaves.*

3 Bounds on the Four Parameters

We present in this section different bounds on $\gamma_k(G)$, $\beta_k(G)$ and $i_k(G)$ in terms of other parameters of the graph.

3.1 Bounds on γ_k

We begin by the lower bounds on the k-domination number given by Fink and Jacobson in their paper introducing k-domination and k-independence [67].

Theorem 34 [67] *For every graph G with n vertices and m edges and every positive integer k,*

1. $\gamma_k(G) \geq \dfrac{kn}{k+\Delta}$.
2. $\gamma_k(G) \geq n - \frac{m}{k}$. *Furthermore, if $m \neq 0$, then $\gamma_k(G) = n - \frac{m}{k}$ if and only if G is a bipartite k-semiregular graph.*

Since $m = n-1$ for trees, it follows from Theorem 34(2) that for every tree T of order n and every positive integer k,

$$\gamma_k(G) \geq \frac{(k-1)n+1}{k}.$$

In [143], Volkmann provides a characterization of extremal trees T achieving $\gamma_k(T) = \lceil\dfrac{(k-1)n+1}{k}\rceil$.

Theorem 35 [143] *If T is a tree of order n, then*

$$\gamma_k(T) = \left\lceil \frac{(k-1)n+1}{k} \right\rceil$$

if and only if

1. $n = kt+1$ *for an integer $t \geq 0$ and T is a k-semiregular tree or $n = 1$, or*
2. $n = kt + r$ *for integers $t \geq 0$ and $2 \leq r \leq k$ and T consists of r trees $T_1, T_2, \ldots, T_r$ which satisfy conditions in (1) and $r-1$ further edges such that the trees $T_1, T_2, \ldots, T_r$ together with these $r-1$ edges result in a tree.*

Since $m \leq n$ for graphs with at most one cycle, it follows from Theorem 34(2) that for these graphs of order n and every positive integer k, $\gamma_k(G) \geq (k-1)n/k$. Theorems 36 and 37 improve this bound. For a graph G, let $L_k(G)$ denote the set of vertices of G of degree at most $k-1$, and $S_k(G)$ the set of all vertices not in $L_k(G)$ which are adjacent to $L_k(G)$.

Theorem 36 [119] *Let T be a tree of order $n \geq 2$ and $k \geq 2$ an integer. Then*

$$\gamma_k(T) \geq \frac{n + |L_k(T)| - |S_k(T)|}{2}.$$

Theorem 37 [30] *If G is a graph of order n with at most one cycle, ℓ leaves and s stems, then $\gamma_2(G) \geq (n+\ell-s)/2$.*

The extremal trees of Theorem 36 are given in [119] by a recursive construction. The graphs G obtained by attaching at least two leaves at each vertex of a cycle show that the bound of Theorem 37 is sharp. For trees, the bound of Theorem 37 is equal to that given in Theorem 36 in the case $k = 2$.

Let $\mu_0 = \mu_0(G)$ be the minimum number of edges that can be removed from a graph G such that the remaining graph is bipartite. Theorem 38 improves Theorem 34(2) for all graphs.

Theorem 38 [85] *If G is a graph of order n and size m, then*

$$\gamma_k(G) \geq n - \frac{m - \mu_0}{k}.$$

Additionally, if $m \neq 0$, then $\gamma_k(G) = \lceil n - \frac{m-\mu_0}{k} \rceil$ if and only if G contains a bipartite k-semiregular factor H with $m(H) = m - \mu_0 - r$, where r is an integer such that $0 \leq r \leq k - 1$ and $m - \mu_0 - r \equiv 0 \pmod{k}$.

For cactus graphs G, the term $\mu_0(G)$ equals the number of odd cycles in G. Hence, with the well-known identity of cactus graphs $m = n - 1 + \nu$, where ν denotes the number of cycles in G, the following result can be derived from Theorem 38.

Theorem 39 [85] *If G is a connected cactus graph of order n with ν_e cycles of even length and matching number $\tau(G)$, then*

$$\gamma_2(G) \geq \tau(G) + 1 - \left\lceil \frac{\nu_e}{2} \right\rceil,$$

and if n and ν_e are both odd, then

$$\gamma_2(G) \geq \tau(G) + 2 - \left\lceil \frac{\nu_e}{2} \right\rceil.$$

In [85], Hansberg and Volkmann presented also different families of examples which show that the bounds in Theorem 39 are best possible. In addition, they characterized the cactus graphs G with at most one even cycle such that $\gamma_2(G) = \tau(G)$ or $\gamma_2(G) = \tau(G) + 1$.

Since any nontrivial connected dominating set is also a total dominating set, $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$ for any connected graph G with $\Delta(G) \leq n(G) - 2$. Therefore Theorems 41 and 42 improve the inequality $\gamma_k(G) \geq \gamma(G) + k - 1$ valid in trees by Corollary 11.

Theorem 40 [33] *Let G be a nontrivial block graph. If $\Delta(G) \geq k$ for an integer $k \geq 2$, then $\gamma_k(G) \geq \gamma_t(G) + k - 2$.*

In trees and for $k \geq 3$, the bound of Theorem 40 can be increased by one. Let $\mathcal{S}_t$ be the family of trees that are obtained from a star $K_{1,t}$ for $t \geq 3$ by subdividing one edge twice and the remaining edges at most twice but not all edges are subdivided twice. Let T_t be the tree that is obtained from the star $K_{1,t}$ by subdividing one edge exactly three times and let $\mathcal{T}$ be the family of graphs that are obtained from every tree $T \in \mathcal{S}_3 \cup \{T_3\}$ by attaching a leaf to one or to both stems which have distance at least 2 to the unique vertex of degree 3 in T.

Theorem 41 [33] *Let T be a tree different from a star such that $\Delta(T) \geq k \geq 3$ for an integer k. Then $\gamma_k(T) \geq \gamma_t(T) + k - 1$ with equality if and only if $T \in \mathcal{S}_k \cup \{T_k\}$ or T is isomorphic to a subdivided star SS_k minus p leaves for an integer $0 \leq p \leq k - 1$ or $T \in \mathcal{T}$ in the case $k = 3$.*

Theorem 42 [33] *Let T be a tree with $\Delta(T) \geq k \geq 3$ for an integer k. Then $\gamma_k(T) \geq \gamma_c(T) + k - 1$ with equality if and only if T is a generalized star with k leaves or, when $k = 3$, T has maximum degree 3 and no two vertices of degree 3 are adjacent to each other.*

Clearly, since $\gamma_t(G) \leq \gamma_c(G)$ for any connected graph G with $\Delta(G) < n - 1$, extremal trees in Theorem 41 with this restriction on the maximum degree belong to the family of extremal trees achieving equality in Theorem 42. Also, since $\gamma(G) \leq \gamma_t(G)$ for any nontrivial graph G, the family of extremal trees of Theorem 41 comprises the extremal trees of Theorems 13 and 14.

We consider now upper bounds on $\gamma_k(G)$. In 1962, Ore [126] proved that the domination number of a graph without isolated vertices is at most half its order. Extremal graphs attaining this upper bound have been determined by Payan and Xuong [128], and independently by Fink, Jacobson, Kinch and Roberts [69].

The first upper bound on the k-domination number generalizing Ore's upper bound was given by Cockayne, Gamble and Shepherd in 1985. This upper bound has been improved later by different authors, among them Caro and Roditty [22], Stracke and Volkmann [138], Chen and Zhou [42] and Favaron, Hansberg and Volkmann [61].

Theorem 43 [45] *If G is a graph of order n and minimum degree δ, then $\gamma_k(G) \leq \frac{k}{k+1}n$ for every integer $k \leq \delta$.*

With $k = k'$ in Theorem 16, we immediately obtain Theorem 43. Using Theorem 16, Favaron, Hansberg and Volkmann [61] characterized the extremal graphs attaining equality in Theorem 43. This characterization generalizes that of graphs G without isolated vertices realizing $\gamma(G) = n/2$.

Theorem 44 [61] *Let G be a connected graph of order n and minimum degree δ. Then G satisfies $\gamma_k(G) = \frac{k}{k+1}n$ for some integer k with $1 \leq k \leq \delta$ if and only if G is the corona $J \circ K_k$, when $k \geq 2$, and $J \circ K_1$ or $G \cong C_4$, when $k = 1$, where J is any connected graph.*

Theorem 45 [22] *Let r, k be positive integers and G a graph of order n and minimum degree $\delta \geq \frac{r+1}{r}k - 1$. Then $\gamma_k(G) \leq \frac{r}{r+1}n$.*

The bound of Theorem 45 can be stated in the following equivalent but more explicit form.

Corollary 46 [61] *If G is a graph of order n and minimum degree δ, then, for every positive integer $k \leq \delta$,*

$$\gamma_k(G) \leq \frac{\lceil k/(\delta + 1 - k)\rceil}{\lceil k/(\delta + 1 - k)\rceil + 1}n.$$

In [138], Stracke and Volkmann defined a generalization of the concept of domination as follows: For an integer-valued function f defined on $V(G)$, a set $D \subseteq V(G)$ is called an *f-dominating set* of G if each $x \in V(G) - D$ is adjacent to at least $f(x)$ vertices in D. The *f-domination number* $\gamma_f(G)$ of G was defined in [155,156] as the minimum cardinality of an f-dominating set of G. Clearly, if $f(x) = k$ for every vertex x of G, then $\gamma_f(G)$ is the k-domination number. The concept of f-domination already appeared in a slightly different way in [98]. In [138], it was used to prove Theorem 47 that also follows easily from Theorem 45.

Theorem 47 [22,138] *For every graph G and every integer $k \geq 1$,*

$$\gamma_k(G) \leq \begin{cases} \dfrac{(2k-\delta)n}{2k-\delta+1} & \text{if } \dfrac{\delta+1}{2} \leq k \leq \delta - 1 \\ \dfrac{n}{2} & \text{if } k \leq \dfrac{\delta+1}{2} \end{cases}$$

Following the ideas in [138], Chen and Zhou [42] slightly improved the bounds in Theorem 47.

Theorem 48 [42] *For every graph G and every integer $k \geq 1$,*

$$\gamma_k(G) \leq \begin{cases} \dfrac{kn}{k+1} & \text{if } k = \delta \\ \dfrac{(2k-\delta-1)n}{2k-\delta} & \text{if } \dfrac{\delta+3}{2} \leq k \leq \delta - 1 \\ \dfrac{2n}{3} & \text{if } k = \dfrac{\delta+2}{2} \\ \dfrac{n}{2} & \text{if } k \leq \dfrac{\delta+1}{2} \end{cases}$$

Finally, the following theorem improves Chen and Zhou's upper bound for $\dfrac{\delta+3}{2} \leq k \leq \delta - 1$.

Theorem 49 [61] *If G is a graph of order n and minimum degree δ and $k \leq \delta$ is a positive integer, then*

$$\gamma_k(G) \leq \frac{\delta}{2\delta+1-k}n.$$

A well-known result on the domination number, which was proved independently by Arnautov [5] in 1974 and, in 1975, by Lovász [118] and by Payan [127], states that $\gamma(G) \leq \frac{1+\ln(\delta+1)}{\delta+1}n$ for every n-vertex graph G with minimum degree $\delta \geq 1$. In 1990, Alon [2] showed that this result is asymptotical optimal for general graphs G. In [21], Caro considered the k-domination number and derived an analog result to the one obtained by Lovász for large minimum degree. He showed that $\gamma_k(G) \leq (1 + o_\delta(1))\frac{n \ln \delta}{\delta}$. Considering connected k-domination (see Sect. 5.4), Caro, West and Yuster [24] showed that the bound obtained by Lovász also holds in this much restricted case. Using the concept of (F, k)-*cores*, Caro and Yuster [25] generalized all these results. If $F = \{G_1, G_2, \ldots, G_t\}$ is a family of graphs on the same vertex set V, a subset $D \subseteq V$ is called an (F, k)-*core* if D is a *total k-dominating set* of each graph in F (a k-total k-dominating set with the notation of Sect. 4), i.e. if $|N_{G_i}(x) \cap D| \geq k$

for every vertex in V, $1 \leq i \leq t$. We denote with $c(k, F)$ the minimum cardinality of an (F, k)-core. Evidently, if $F = \{G\}$, an (F, k)-core is precisely a total k-dominating set in G and viceversa.

Theorem 50 [25] *Let k, t and δ be positive integers satisfying $k < \sqrt{\ln \delta}$ and $t < \ln(\ln \delta)$. Let F be a family of graphs on the same n-vertex set. Assume that every graph in F has minimum degree at least δ. Then:*

$$c(k, F) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)).$$

Since every total k-dominating set is also a k-dominating set, Caro's [21] result follows.

Corollary 51 [21] *Let k and δ be positive integers satisfying $k < \sqrt{\ln \delta}$ and let G be a graph on n vertices with minimum degree at least δ. Then:*

$$\gamma_k(G) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)).$$

Note that from the definition of an (F, k)-core, Theorem 50 implies a stronger result than Corollay 51, namely

$$\gamma_{k,k}(G) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)),$$

where $\gamma_{k,k}$ is the k-total k-dominating number which will be defined in Sect. 4.

Caro and Yuster [25] obtained exactly the same result for connected (F, k)-cores, where, given $F = \{G_1, G_2, \ldots, G_t\}$, the underlying induced graphs $G_i[D]$ are connected for $1 \leq i \leq t$. This is an even more stronger result and will be presented in Sect. 5.4.

Weakening considerably the condition on the minimum degree, Rautenbach and Volkmann [132] presented the following upper bound on the k-domination number γ_k. Due to the weaker conditions, this bound is as expected not as strong as Caro's bound.

Theorem 52 [132] *Let G be a graph on n vertices with minimum degree $\delta \geq 1$ and let $k \in \mathbb{N}$. If $\frac{\delta+1}{\ln(\delta+1)} \geq 2k$, then*

$$\gamma_k(G) \leq \frac{n}{\delta + 1}\left(k \ln(\delta + 1) + \sum_{i=0}^{k-1} \frac{1}{i!\,(\delta + 1)^{k-1-i}}\right).$$

Following the same probabilistic method as in [132], Hansberg and Volkmann [86] presented two new bounds for the k-domination number γ_k. The first one is better than the bound of Rautenbach and Volkmann, even though it preserves the same assumptions.

Theorem 53 [86] *Let G be a graph on n vertices with minimum degree* $\delta \geq 1$ *and let* $k \in \mathbb{N}$*. If* $\frac{\delta+1}{\ln(\delta+1)} \geq 2k$*, then*

$$\gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) + \sum_{i=0}^{k-1} \frac{\delta^i}{i!\,(\delta+1)^{k-1}}\right).$$

An easy consequence of this theorem is the following corollary.

Corollary 54 [86] *Let G be a graph on n vertices with minimum degree* $\delta \geq 1$ *and let* $k \in \mathbb{N}$*. If* $\frac{\delta+1}{\ln(\delta+1)} \geq 2k$*, then*

$$\gamma_k(G) \leq \frac{n}{\delta+1}(k\ln(\delta+1)+1).$$

For the case $k = 1$, this corollary and also Theorem 52 yield directly the well-known bound for the usual domination number γ.

Corollary 55 [5,118,127] *Let G be a graph on n vertices and minimum degree* $\delta \geq 1$*. Then*

$$\gamma(G) \leq \frac{n}{\delta+1}(\ln(\delta+1)+1).$$

The second bound of Hansberg and Volkmann in [86] weakens a little more the assumption on the minimum degree δ and, for $k \geq 3$, it is even better than the first one.

Theorem 56 [86] *Let G be a graph on n vertices with minimum degree* $\delta \geq k$*, where* $k \in \mathbb{N}$*. If* $\frac{\delta+1+2\ln(2)}{\ln(\delta+1)} \geq 2k$ *then*

$$\gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) - \ln(2) + 2\sum_{i=0}^{k-1} \frac{\delta^i}{i!\,(\delta+1)^{k-1}}\right).$$

Hansberg and Volkmann [86] gave the following observation, which improves Corollary 54 for $k \geq 4$.

Observation 57 [86] *Let* $k \geq 4$ *be an integer and G a graph of minimum degree* $\delta \geq k$*.*

(i) *If* $2k \leq \frac{\delta+1}{\ln(\delta+1)}$*, then*

$$\gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) + 1 - \frac{k-1}{\delta}\right).$$

(ii) *If* $\frac{\delta+1+2\ln(2)}{\ln(\delta+1)} \geq 2k$*, then*

$$\gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) - \ln(2) + 2 - 2\frac{k-1}{\delta}\right).$$

The following two theorems give upper bounds on $\gamma_k(G)$ for multipartite graphs and cactus graphs. Recall that $L_k(G) = \{x \in V(G) : d_G(x) \leq k-1\}$. For bipartite graphs G, Blidia, Chellali and Volkmann [18] proved that $\gamma_k(G) \leq (n(G)+|L_k(G)|)/2$. This bound was generalized later to r-partite graphs.

Theorem 58 [79] *Let $k \geq 1$ be an integer. If G is an r-partite graph, then*

$$\gamma_k(G) \leq ((r-1)n(G) + |L_k(G)|)/r.$$

From their upper bound $(n(G)+|L_k(G)|)/2$ in bipartite graphs, the authors derived a sharp upper bound on $\gamma_k(G)$ for cactus graphs.

Theorem 59 [17] *Let $k \geq 2$ be an integer. If G is a connected cactus graph with $\nu_0(G)$ odd cycles, then*

$$\gamma_k(G) \leq (n(G) + |L_k(G)| + \nu_0(G))/2.$$

Extremal trees attaining the bound in Theorem 59 are described by a recursive construction in [11] for $k = 2$ and in [18] for the general case.

Using Ore's upper bound, Theorems 43, 45 and two further results in the paper by Stracke and Volkmann [138], Volkmann [144] recently derived the following Nordhaus-Gaddum bound for the 2-domination number.

Theorem 60 [144] *If G is a graph of order n, then*

$$\gamma_2(G) + \gamma_2(\overline{G}) \leq n + 2.$$

Jaeger and Payan [105] proved that $\gamma_1(G) + \gamma_1(\overline{G}) \leq n + 1$, and Theorem 60 says that $\gamma_2(G)+\gamma_2(\overline{G}) \leq n+2$ for any graph G. So one could suspect that $\gamma_k(G)+\gamma_k(\overline{G}) \leq n + k$ for $k \geq 3$. However, the following example by Volkmann [144] demonstrates that this is not valid in general. Let t be a positive integer, and let G be a $2t$-regular graph of order $n = 4t + 1$. Then $\overline{G}$ is also a $2t$-regular graph, and we observe that

$$\gamma_{2t+1}(G) + \gamma_{2t+1}(\overline{G}) = 2n = n + 4t + 1 > n + 2t + 1.$$

Using the pigeonhole principle, Prince [130] proved the next Nordhaus-Gaddum-type result.

Theorem 61 [130, in press] *For any graph G on n vertices*

$$\gamma_k(G) + \gamma_k(\overline{G}) \leq n + 2k - 1.$$

The example above shows that this bound is sharp, at least for small n and odd k.

Theorem 62 [130, in press] *Let $k \geq 2$ be an integer and G be a graph of order n. Then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq 8kn.$$

For different special cases, Prince [130] derived some better upper bounds for $\gamma_k(G)\gamma_k(\overline{G})$, as for example:

Theorem 63 [130, in press] *Let $k \geq 2$ be an integer. If G is a graph of order $n \geq 72(40k)^{80k}$, then*

$$\gamma_k(G)\gamma_k(\overline{G}) \leq (2k-1)(n-k+2).$$

3.2 Bounds on β_k and i_k

Several lower bounds are known on the independence number in terms of the maximum degree, the average degree d^* or the degree sequence of the graph, as $\beta(G) \geq \dfrac{n}{d^*+1} \geq \dfrac{n}{\Delta+1}$ [9] or $\beta(G) \geq \sum_{v\in V} \dfrac{1}{1+d(v)}$ [149]. These bounds were generalized for the k-independence number by several authors. In [101], the term k-small is used for k-independent.

Theorem 64 [101] *For every graph G of order n and every positive integer $k \leq \Delta$,*

$$\beta_k(G) \geq HS_k(G) := \frac{n}{1+\lfloor \Delta/k \rfloor}.$$

Theorem 65 [57] *For every graph G and every positive integer k,*

$$\beta_k(G) \geq \sum_{v\in V} \frac{k}{1+kd(v)}.$$

In [23], Caro and Tuza established a lower bound on the k-independence number in uniform hypergraphs in terms of the degree sequence whose restriction to graphs improved Theorem 65. The formula was printed incorrectly in [23] and has been corrected by Jelen [107].

Theorem 66 [23,107] *If G is a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ and k a positive integer, then*

$$\beta_k(G) \geq CT_k(G) := \sum_{i=1}^{n} f_k(d_i),$$

where $f_k(x) = \begin{cases} 1 - \dfrac{x}{2k} & \textit{if } \ 0 \leq x \leq k \\ \dfrac{k+1}{2x+1} & \textit{if } \ x > k. \end{cases}$

Jensen's inequality applied to $\sum_{i=1}^{n} f_k(d_i)$ gives a bound of easier use, in terms of the average degree $d^* = 2m/n$ of the graph.

Corollary 67 [23,107] *If G is a graph with average degree d^* and k a positive integer, then $\beta_k(G) \geq CT_k(G) \geq nf_k(d^*)$.*

Examples given in [107] show that none of the two bounds HS_k and CT_k is better than the other one.

Another interesting lower bound on β_k is the k-residue. The concept of residue was introduced by Fajtlowicz [55] and his program GRAFFITI conjectured that the independence number of any graph was bounded below by its residue. The conjecture was proven by Favaron, Mahéo and Saclé [66]. In [107], Jelen generalized the concept of residue by defining the k-residue. We give below the definition of the residue and of the k-residue and illustrate it by a small example. Given a sequence $\pi := (d_1 \geq d_2 \geq \cdots \geq d_n)$, the action of the Havel-Hakimi operator H consists in removing d_1, subtracting 1 from the d_1 following terms of π and sorting down the $(n-1)$-tuple $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. If $d_1 \leq n-1$, we get a new sequence $H(\pi) := (d_2^1 \geq d_3^1 \geq \cdots \geq d_n^1)$ which is graphical, i. e. which is the degree sequence of a graph, if and only if π is graphical. In this case, the operator H can be applied i times for some $i \in \{0, 1, \ldots, n-1\}$ such that $H^i(\pi) := (d_{i+1}^i \geq \cdots \geq d_n^i)$ is a sequence of $n-i$ zeroes. The length of this last sequence is the residue $R(\pi)$. If π is the degree sequence of the graph G, then $R(G) = R(\pi)$. Let $E(\pi)$ be the sequence of terms deleted from the degree sequence π by using the Havel-Hakimi process, namely the sequence $(d_1^0 := d_1, d_2^1, \ldots, d_n^{n-1})$, and let $b_j(\pi)$ be the number of terms with value j in $E(\pi)$. For each positive integer k, the k-residue of the sequence, or of the graph G with degree sequence π, is defined by

$$R_k(G) = R_k(\pi) = \frac{1}{k}\sum_{j=0}^{k-1}(k-j) \cdot b_j(\pi).$$

We note that $R_1(\pi) = b_0(\pi) = R(\pi)$.

Example The graph G obtained by deleting one edge in a triangle of the prism $K_3 \times K_2$ has degree sequence $\pi(G) = (3, 3, 3, 3, 2, 2)$. The Havel-Hakimi process can be presented as

$$\begin{array}{cccccccl}
3 & 3 & 3 & 3 & 2 & 2 & & \pi \\
 & 2 & 2 & 2 & 2 & 2 & & H(\pi) \\
 & & 2 & 2 & 1 & 1 & & H^2(\pi) \\
 & & & 1 & 1 & 0 & & H^3(\pi) \\
 & & & & 0 & 0 & & H^4(\pi) \\
 & & & & & 0 & & H^5(\pi)
\end{array}$$

For this sequence, $n = 6$, $E(\pi) = (3, 2, 2, 1, 0, 0)$, $b_0 = 2$, $b_1 = 1$, $b_2 = 2$, $b_3 = 1$ and $b_j = 0$ for $j > \Delta = 3$. Therefore $R_1 = 2$, $R_2 = 5/2$, $R_3 = 10/3$ and for $k \geq 4$, $R_k = 6 - 8/k < n$.

Theorem 68 [107] *For every positive integer k and every graph G, $\beta_k(G) \geq R_k(G)$.*

Jelen proved that the bound R_k is always better than the bound CT_k of Theorem 66 (Theorem 8 of [107]) and gave examples showing that R_k can be arbitrarily smaller or larger than the bound HS_k of Theorem 64.

For nontrivial trees T, Maddox generalized the well known bound $\beta(T) \geq n/2$ and proved $\beta_k(T) \geq kn/(k+1)$ [123]. In [13], Blidia, Chellali, Favaron and Meddah gave another proof of this result which allowed one to characterize the extremal trees with a recursive construction. Let $\mathcal{F}(k)$ be the set of trees which can be constructed from $T_1 = K_{1,k}$ by recursively joining by an edge any vertex of T_i to any vertex of a new star $K_{1,k}$.

Theorem 69 [13,123] *Let T be a tree of order n and maximum degree Δ. Then for every integer k with $2 \leq k \leq \Delta$, $\beta_k(T) \geq M_k := \dfrac{kn}{k+1}$, with equality if and only if $T \in \mathcal{F}(k)$.*

It was shown in [13] that for trees the bound M_k of Theorem 69 is always better than the bound HS_k of Theorem 64 while M_k can be smaller or arbitrarily larger than the bound R_k of Theorem 68.

In the particular case $k = 2$, it is proved in [13] that the 2-independence number $\beta_2(G)$ for nontrivial bipartite graphs with s stems is at least $(n+s)/2$. Using a constructive technique, the authors of [13] also gave a characterization of the extremal trees T with $\beta_2(T) = (n+s)/2$.

Lower and upper bounds valid for every graph have also been obtained for i_k. They respectively generalize the well known bounds $i(G) \geq n/(\Delta+1)$ and $i(G) \leq n - \Delta$.

Theorem 70 [57] *Every graph G of maximum degree $\Delta \geq 1$ satisfies $i_k(G) \geq \dfrac{n+k-1}{\Delta+1}$ for $1 \leq k \leq n-1$.*

For $k = 2$, the bound of Theorem 70 was slightly improved in [14] and the extremal graphs were characterized.

Theorem 71 [14] *Let G be a connected graph of order $n \geq 2$ and maximum degree Δ. Then $i_2(G) \geq (n+2)/(\Delta+1)$, with equality if and only if $G = P_2$ or G is obtained from a double star $S_{\Delta-1,\Delta-1}$ by adding zero or more edges between its leaves without creating a vertex of degree larger than Δ.*

Theorem 72 [13] *For $2 \leq k \leq \Delta$, every graph G of order n and maximum degree Δ satisfies $i_k(G) \leq n - \Delta + k - 1$. If moreover G is connected and $\Delta < n-1$, then $i_2(G) \leq n - \Delta$.*

In order to characterize the trees different from stars satisfying $i_2 = n - \Delta$, the authors of [13] defined generalized stars as follows: for $r+1$ integers $p_i \geq 0$ such that $\sum_{i=0}^{r} p_i \geq 1$, the generalized star $GS_{p_0,p_1,\ldots,p_r}$ is the tree of order $n = 1 + \sum_{i=0}^{r}(i+1)p_i$ and maximum degree $\Delta = \sum_{i=0}^{r} p_i$ obtained from the star $K_{1,p_0+p_1+\cdots+p_r}$ by subdividing p_1 rays once, p_2 rays twice, …, p_r rays r times. Let $S^*_{p,q}$ be the tree obtained from a double star $S_{p,q}$ by subdividing once the edge joining the two stems.

Theorem 73 [13] *A tree T of order $n \geq 4$ and maximum degree $\Delta < n-1$ satisfies $i_2(T) = n - \Delta$ if and only if T is a generalized star GS_{p_0,p_1,p_2} with $p_0 + p_1 \geq 1$ and $p_1 + p_2 \geq 1$ or $GS_{p_0,0,0,1}$ with $p_0 \geq 1$, or is equal to $S^*_{p,2}$ with $p \geq 2$.*

In the particular case $k = 2$, it was shown in [13] that $i_2(G) \leq (n + s(G))/2$ for nontrivial bipartite graphs with s stems and the extremal trees were given by a recursive construction. This result has been extended in [14] to cactus graphs.

Theorem 74 [14] *If G is a connected nontrivial cactus graph with ν_0 odd cycles and $s(G)$ stems, then*

$$i_2(G) \leq \frac{n + s(G) + \nu_0}{2}.$$

Odd cycles are examples of extremal graphs for Theorem 74.

As an extension of classical bounds by Chartrand and Schuster [28], Blidia, Bouchou and Volkmann [10] proved the following Nordhaus-Gaddum-type results.

Theorem 75 [10] *If G is a graph of order n such that $k \leq \min\{\Delta(G), \Delta(\overline{G})\}$, then*

$$\beta_k(G) + \beta_k(\overline{G}) \leq n + 2k - 1 \quad \text{and} \quad \beta_k(G)\beta_k(\overline{G}) \leq (n + 2k - 1)^2/4.$$

4 *l*-Total *k*-Domination and *k*-Tuple Domination

The concept of l-total k-domination was introduced to generalize both domination and total domination.

Definition 76 [64] For two integers $l \geq 0$ and $k > 0$, the subset S of vertices of G is l-total k-dominating if every vertex x has at least l neighbors in S if $x \in S$ and k neighbors in S if $x \in V - S$. The l-total k-dominating number $\gamma_{l,k}(G)$ is the minimum cardinality of an l-total k-dominating set of G.

Clearly, $\gamma_{0,k}(G) = \gamma_k(G)$ for any $k \geq 1$ and if $\delta \geq 1$, $\gamma_{1,1}(G) = \gamma_t(G)$. When $l = k$, k-total k-dominating sets (which are such that $|N(v) \cap S| \geq k$ for all $v \in V$) were already defined by Kulli under the term of *k-total dominating sets* and $\gamma_{k,k}(G)$ was denoted by $\gamma_{tk}(G)$ [116]. These sets were called total k-dominating sets by Caro and Yuster [25] and reintroduced under the term of *total k-tuple dominating sets* by Dorbec, Gravier, Klavžar and Špacapan [52] who denoted $\gamma_{k,k}(G)$ by $\gamma_t^{(\times k)}(G)$. When $l = k - 1$, $(k - 1)$-total k-dominating sets (which are such that $|N[v] \cap S| \geq k$ for all $v \in V$) have been introduced by Harary and Haynes under the term of *k-tuple dominating sets* and $\gamma_{k-1,k}(G)$ is denoted by $\gamma_{\times k}(G)$ [92].

The first question raised by this definition is that of the existence of l-total k-dominating sets. If such a set S exists, every vertex of degree δ belongs to S or to $V - S$, thus implying $\delta \geq \min\{l, k\}$. Conversely, if $\delta \geq l$, then V itself is a l-total k-dominating set. Therefore if $l \leq k$, then l-total k-dominating sets exist if and only if $\delta \geq l$. In particular, $\gamma_{k-1,k}(G)(= \gamma_{\times k}(G))$ is defined if and only if $k \leq \delta + 1$ and $\gamma_{k,k}(G)(= \gamma_{tk}(G))$ is defined if and only if $k \leq \delta$. But if $k \leq \delta < l$, l-total k-dominating sets may or not exist.

In the following theorem, the lower bound on $\gamma_{l,k}(G)$ is an immediate consequence of the definition.

Theorem 77 [64] *If* $\delta \geq \max\{l, k-1\}$*, then*

$$\max\{l+1, k\} \leq \gamma_{l,k}(G) \leq \max\{n-\delta+l, n-\delta+k-1\}.$$

Let S be a $\gamma_{l,k}$-set and let $0 \leq l' \leq l$ and $1 \leq k' \leq k$. If $Y \subseteq S$ with $|Y| \leq \min\{l-l', k-k'\}$, then $S-Y$ is an l'-total k'-dominating set. Therefore

Theorem 78 [64] *If* $\gamma_{l,k}(G)$ *exists, then* $\gamma_{l',k'}(G)$ *exists for every* l', k' *with* $0 \leq l' \leq l, 1 \leq k' \leq k$ *and*

$$\gamma_{l',k'}(G) \leq \gamma_{l,k}(G) - \min\{l-l', k-k'\}.$$

Theorem 78 generalizes results related to particular values of l and k as $(\gamma_{1,1}(G) =)$ $\gamma_t(G) \leq \gamma_{\delta,\delta}(G) - \delta + 1$ [116] or $(\gamma_{0,1}(G) =)\gamma(G) \leq \gamma_{\times 2}(G) - 1$ [92]. It implies also $\gamma_t(G) \leq \gamma_{\times 2}(G)$.

In the class of claw-free graphs, we have some stronger inequalities.

Theorem 79 [64] *If the graph G is claw-free, then*

$$\gamma(G) \leq \frac{k+1}{2k}\gamma_{1,k}(G) \quad \text{if } \delta \geq 1 \quad \text{and} \quad \gamma(G) \leq \frac{3k+5}{8k}\gamma_{3,k}(G) \quad \text{if } \delta \geq 3.$$

The parameter $\gamma_{l,k}$ has essentially been studied when $l = k-1$, i.e., in the context of the k-tuple domination, and more precisely in the particular case $k = 2$. The following two subsections are devoted to this study. We adopt the usual notation $\gamma_{\times k}$ for $\gamma_{k-1,k}$.

4.1 k-Tuple Domination

Recall that $\gamma_{\times k}(G)$ exists if and only if $\delta \geq k-1$. When $k = 1$, $\gamma_{\times 1}(G) = \gamma(G)$, so we can suppose $k \geq 2$.

It is easy to see that for the complete bipartite graph $G = K_{k-1,n-k+1}$ with $n \geq 2k-2$, $\gamma_{\times k}(G) = n$ and that for the clique $G = K_n$ with $n \geq k$, $\gamma_{\times k}(G) = \gamma_k(G) = k = n - \delta + k - 1$. The graphs such that $\gamma_{\times k}(G) = k$ are easy to characterize.

Observation 80 [92] for $k = 2$ $\gamma_{\times k}(G) = k$ *if and only if G contains a clique* K_k*, each vertex of which has degree* $n-1$.

In [117] Liao and Chang proved that the k-tuple domination problem is NP-Complete even for bipartite and split graphs. They also gave a linear-time algorithm to compute $\gamma_{\times k}$, and with some adaptation $\gamma_{k,k}$, in strongly chordal graphs. In [114], Klasing and Laforest gave an $(\ln n + 1)$-approximation algorithm for computing $\gamma_{\times k}(G)$ in general graphs.

The first immediate bounds are obtained by putting $l = k-1$ in Theorems 77 and 78.

Corollary 81 *If G is a graph with $\delta \geq k-1$, then*

1. $\gamma(G) \leq \gamma_{\times k}(G) - k + 1$,
2. $\gamma_t(G) \leq \gamma_{\times k}(G) - k + 2$,
3. $k \leq \gamma_k(G) \leq \gamma_{\times k}(G) \leq n - \delta + k - 1 \leq n$.

In their paper introducing the k-tuple domination, Harary and Haynes [92] gave two lower bounds on $\gamma_{\times k}(G)$ in terms of n, m and Δ.

Theorem 82 [92] *If $\delta \geq k-1$, then*

$$\gamma_{\times k}(G) \geq \frac{kn}{\Delta+1} \quad \text{and} \quad \gamma_{\times k}(G) \geq \frac{2kn-2m}{k+1}$$

and the two bounds are sharp.

Other lower bounds on $\gamma_{\times k}$ were found by Harant and Henning in terms of the independence number.

Theorem 83 [89] *Let G be a graph of order n, size m, independence number β and maximum degree Δ, and let $k \leq \delta + 1$.*

1. $\gamma_{\times k} \geq \frac{\beta}{2}(\Delta + k + 1) - \sqrt{(\frac{\beta}{2}(\Delta + k + 1))^2 - \beta kn}$.
2. $\gamma_{\times k} \geq n + \beta(k+1) - \sqrt{(\beta(k+1))^2 + x}$, *where* $x = 2\beta(2m - (k-1)n)$.

Klasing and Laforest also gave a lower bound on $\gamma_{\times k}$ in terms of the independence number in graphs with no induced star $K_{1,p}$, i.e. in p-claw-free graphs.

Theorem 84 [114] *If G is a p-claw-free graph with $\delta \geq k-1$, then*

$$\gamma_{\times k}(G) \geq \frac{k\beta(G)}{p-1}.$$

Many people have been looking for upper bounds on $\gamma_{\times k}(G)$ in terms of the order and the degrees of G. Some bounds have been obtained by probabilistic methods already mentioned in Sect. 3.1 for similar bounds on γ_k. First recall that the bound on Corollary 51 is valid for $\gamma_{k,k}$. Hence

$$\gamma_{\times k}(G) \leq \gamma_{k,k}(G) \leq n\frac{\ln\delta}{\delta}(1 + o_\delta(1)) \quad \text{for } k < \sqrt{\ln\delta}.$$

For $k = 2$, Harant and Henning [88] proved in 2005 that

$$\text{if } \delta \geq 1, \quad \text{then } \gamma_{\times 2}(G) \leq \frac{\ln(1+d^*) + \ln\delta + 1}{\delta} n$$

where $d^* = 2m/n$ is the average degree of G.

In [132], Rautenbach and Volkmann extended this formula to $k = 3$ by proving that if $\delta \geq 2$, then

$$\gamma_{\times 3}(G) \leq \frac{n}{\delta-1}\left(\ln(\delta-1)+\ln\left(\sum_{v\in V}\binom{d(v)+1}{2}\right)-\ln(n)+1\right).$$

They also proposed a conjecture generalizing the two formulas to any value of k and which has been proved independently by Xu, Kang, Shan and Yan and by Zverovich.

Theorem 85 [151,157] *For any graph G with $\delta \geq k-1$,*

$$\gamma_{\times k}(G) \leq \frac{n}{\delta-k+2}\left(\ln(\delta-k+2)+\ln\left(\sum_{v\in V}\binom{d(v)+1}{k-1}\right)-\ln(n)+1\right).$$

Actually, Zverovich proved Theorem 85 after a weaker result was previously obtained by Gagarin and Zverovich [73] under the equivalent form

$$\gamma_{\times k}(G) \leq \frac{n}{\delta-k+2}\left(\ln(\delta-k+2)+\ln(\hat{d}_{k-1}+\hat{d}_{k-2})+1\right)$$

where $\hat{d_p} = \frac{1}{n}\sum_{v\in V}\binom{d(v)}{p}$.

Theorem 86 [132] *If* $(\delta+1)/\ln(\delta+1) \geq 2k$ *then*

$$\gamma_{\times k}(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1)+\sum_{i=0}^{k-1}\frac{k-i}{i!(\delta+1)^{k-1-i}}\right).$$

Theorem 87 [48] *If k is fixed and δ is large,*

$$\gamma_{\times k}(G) \leq \frac{n}{\delta}\left(\ln\delta+(k-1+o(1))\ln\ln\delta\right).$$

The last upper bound on $\gamma_{\times k}$ involves the number of edges of G.

Theorem 88 [89] *Let G be a graph of order n and size m, and let $k \leq \delta+1$. Then*

$$\gamma_{\times k} < \frac{(2k+1)n-2m}{k+2}.$$

A relationship between $\gamma_{\times k}(G)$ and the connected k-domination number will be given in Sect. 5.4.

When $k=3$, $\gamma_{\times 3}(G)$ is defined in graphs with $\delta \geq 2$. The following result concerns 3-tuple domination in grids $P_h \times P_q$ of small height h.

Theorem 89 [124]

1. $\gamma_{\times 3}(P_2 \times P_q) = \lceil\frac{3(q+1)}{2}\rceil$ *if* $q \geq 3$.
2. $\gamma_{\times 3}(P_3 \times P_q) = \lceil\frac{9q-\lceil\frac{q-2}{5}\rceil+4}{4}\rceil$ *if* $q \geq 2$.
3. $\gamma_{\times 3}(P_4 \times P_q) = \lceil\frac{11q+\lfloor\frac{q-4}{5}\rfloor+9}{4}\rceil$ *if* $q \geq 2$.

To finish this paragraph, we can note that Araki considered *k-tuple domination in digraphs* and determined the value of $\gamma_{\times k}$ for de Bruijn and Kautz digraphs [4].

4.2 Double Domination

Many results on the k-tuple domination concern the particular case $k = 2$. The 2-tuple domination number $\gamma_{\times 2}(G)$ is often called the *double domination number* of G and sometimes denoted $\gamma_d(G)$ or $dd(G)$. We observe that every double dominating set of a graph G contains all the leaves and stems of G. For paths of order $n \geq 2$ and cycles, it has been observed that $\gamma_{\times 2}(C_n) = \lceil \frac{2n}{3} \rceil$ [92] and $\gamma_{\times 2}(P_n) = \lceil \frac{2n+2}{3} \rceil$ [16,112]. We first consider the results related to general graphs with $\delta \geq 1$ or $\delta \geq 2$ and then those valid in particular classes of connected graphs, claw-free graphs or trees.

Obviously, all the bounds given on $\gamma_{\times k}$ in the previous subsection give bounds on $\gamma_{\times 2}$ by putting $k = 2$. In 2008, Cockayne and Thomason gave a new upper bound on $\gamma_{\times 2}$ in terms of n and δ which improved the bound

$$\gamma_{\times 2}(G) \leq \frac{\ln(1 + \bar{d}) + \ln\delta + 1}{\delta} n$$

of Harant and Henning in [88] by replacing $\bar{d}$ by δ.

Theorem 90 [48] *If G is a graph with $\delta \geq 1$, then*

$$\gamma_{\times 2}(G) \leq \frac{\ln(1 + \delta) + \ln\delta + 1}{\delta} n.$$

When $\delta \geq 2$, Henning gave an upper bound on $\gamma_{\times 2}$ which is better than the previous one for $2 \leq \delta \leq 6$. Let $\mathcal{H}$ be the family of graphs obtained from any connected graph H and $|V(H)|$ copies of C_4 by identifying each vertex of H with one vertex of one of the copies of C_4. Let F_1 be obtained from the prism $P_3 \times P_2$, where the two paths P_3 are $x_1x_2x_3$ and $y_1y_2y_3$, by subdividing once the edges x_1y_1 and x_3y_3.

Theorem 91 [99] *If $G \neq C_5$ is a graph of order n with $\delta \geq 2$, then $\gamma_{\times 2}(G) \leq 3n/4$ with equality if and only if each component of G belongs to $\{F_1, C_8\} \cup \mathcal{H}$.*

We gather in the following theorems various results obtained by different authors relating $\gamma_{\times 2}$ to other graph parameters. The *paired domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a dominating set whose induced subgraph admits a perfect matching. The maximum cardinality of a minimal total dominating set, the maximum cardinality of a 2-packing and the minimum cardinality of a set of edges incident to all the edges of G are respectively denoted $\Gamma_t(G)$, $\rho(G)$ and $\gamma'(G)$.

Theorem 92 *Let G be a graph with $\delta \geq 1$.*

1. (Harant and Henning [89])
   $\gamma_{\times 2}(G) \leq \left(\frac{1+\ln 2\delta}{\delta}\right) n + \left(\frac{\delta-1-\ln 2\delta}{\delta}\right) \gamma.$
   $\gamma_{\times 2}(G) \leq \left(\frac{1+\ln \delta}{\delta}\right) n + \left(\frac{\delta-1-\ln \delta}{\delta}\right) \gamma_t.$
2. (Blidia, Chellali and Favaron [12])
   $\gamma_{\times 2}(G) \leq i(G) + \beta(G) \leq 2\beta(G)$ *and the bound* $2\beta(G)$ *is sharp.*
   If $\gamma_{\times 2}(G) \neq 2$*, then* $\gamma_{\times 2}(G) \geq \dfrac{\gamma_{pr}(G)}{2} + 2$ *and the bound is sharp.*

Theorem 93 *Let G be a graph with $\delta \geq 2$.*

1. (Harary and Haynes [92])
   $\gamma_{\times 2}(G) \leq 2\tau(G)$ *and* $\gamma_{\times 2}(G) \leq 2\gamma'(G)$.
   $\gamma_{\times 2}(G) \leq \lfloor \frac{n}{2} \rfloor + \gamma(G) - 1$ *if* $n \neq 3, 5$, $\gamma_{\times 2}(G) \leq \lfloor \frac{n}{2} \rfloor + \gamma(G)$ *if* $n \in \{3, 5\}$, *and the bounds are sharp.*
2. (Chellali and Haynes [35])
   $\gamma_{\times 2}(G) \leq n - \rho(G)$ *and* $\gamma_{\times 2}(G) \leq \frac{3n}{2} - \delta\rho(G)$.
   $\gamma_{\times 2}(G) \leq \frac{n + \gamma_t(G)}{2}$.

When the graph is connected, $\gamma_{\times 2}(G)$ can be compared to more specific parameters as the connected domination number $\gamma_c(G)$, the diameter $\mathrm{diam}(G)$ and the girth $g(G)$.

Theorem 94 [41,78] *If G is connected, then*

$$\gamma_{\times 2}(G) \geq \frac{\gamma_c(G)}{2} + 1.$$

Theorem 95 [112] *Let G be a connected graph.*

1. *If $diam(G) = 2$, then $\gamma_{\times 2}(G) \leq \delta + \Delta$ and the bound is sharp.*
2. $\gamma_{\times 2}(G) \geq \lceil \frac{2\mathrm{diam}(G)+4}{3} \rceil$ *if* $diam(G) \equiv 0, 1 \bmod 3$ *and*
   $\gamma_{\times 2}(G) \geq \lceil \frac{2\mathrm{diam}(G)+4}{3} \rceil - 1$ *if* $diam(G) \equiv 2 \bmod 3$.
   These bounds are sharp.
3. *If $\delta \geq 2$ then $\gamma_{\times 2}(G) \leq n + \lceil \frac{1-\mathrm{diam}(G)}{3} \rceil$.*

Since $\gamma(G) \leq \lceil \frac{n - \lfloor \frac{g}{3} \rfloor}{2} \rceil$ by [20], the upper bound $\gamma_{\times 2}(G) \leq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n - \lfloor \frac{g}{3} \rfloor}{2} \rceil - 1$ valid if $g \geq 5$, $n \geq 6$ and $\delta \geq 2$ is an immediate consequence of Theorem 93. Theorem 96 gives lower bounds on $\gamma_{\times 2}(G)$ in terms of the girth of G.

Theorem 96 [112] *Let G be a connected graph with finite girth g.*

1. $\gamma_{\times 2}(G) \geq \lceil \frac{2g}{3} \rceil$. *When $g = 4$, $\gamma_{\times 2}(G) = 3$ if and only if $G = K_{2,n-2}$.*
2. *If $g \geq 5$, then $\gamma_{\times 2}(G) \geq 2\delta$. When $g \geq 6$, $\gamma_{\times 2}(G) = 2\delta$ if and only if $G = C_6$.*
3. *If $g \geq 5$, then $\gamma_{\times 2}(G) \geq \Delta + \lceil \frac{2g-7}{3} \rceil$.*
   (i) *When $g \geq 6$, then $\gamma_{\times 2}(G) = \Delta + 2$ if and only if*
   $V = \{v, w, v_i, u_j \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ *and*
   $E = \{vv_i, v_j u_j, u_j w \mid 1 \leq i \leq p, 1 \leq j \leq q\}$
   for some p, q with $2 \leq q \leq p$.
   (ii) *When $g \geq 8$, then $\gamma_{\times 2}(G) = \Delta + 3$ if and only if*
   $V = \{v, w, v_i, u_j, w_j \mid 1 \leq i \leq p, j = 1, 2\}$ *and*
   $E = \{vv_i, v_j u_j, u_j w_j, w_j w \mid 1 \leq i \leq p, j = 1, 2\}$
   for some $p \geq 3$.
4. *If $g \geq 7$ and $\delta \geq 2$, then $\gamma_{\times 2}(G) \geq 2\Delta + 1$ and the bound is sharp for $g = 7$.*

For claw-free graphs, we have the following results. Note that it was remarked in [35] that for general graphs G, $\gamma_{pr}(G)$ may be larger or smaller than $\gamma_{\times 2}(G)$.

Theorem 97 *Let G be a claw-free graph with $\delta \geq 1$. Then*

1. (Chellali and Haynes [35]) $\gamma_{\times 2}(G) \geq \gamma_{pr}(G)$.
2. (Blidia, Chellali and Favaron [12])
   $\gamma_{\times 2}(G) \leq 3\gamma(G)$ *and this bound is sharp.*
   $\gamma_{\times 2}(G) \leq 2\gamma_t(G) \leq 2\gamma_{pr}(G)$ *and the two bounds are sharp.*
   $\beta(G) \leq \gamma_2(G) \leq \gamma_{\times 2}(G)$ *and the two bounds are sharp.*

When $k = 2$, the first inequality $\gamma(G) \leq \dfrac{k+1}{2k}\gamma_{1,k}(G)$ of Theorem 79 is valid with the weaker hypothesis that G belongs to the class of claw-free block graphs, i.e. all the blocks (maximal 2-connected subgraphs) of G are claw-free. For instance, claw-free graphs and trees are claw-free block graphs.

Theorem 98 [64] *If all the blocks of the graph G are claw-free and $\delta \geq 1$,*

$$\gamma(G) \leq \frac{3}{4}\gamma_{\times 2}(G)$$

and the bound is sharp.

Since for graphs with minimum degree 1, the k-tuple domination is only defined for $k \leq \delta + 1 = 2$, it is quite natural to particularly study double domination in trees. Several results on $\gamma_{\times 2}$ involve the numbers ℓ and s of leaves and stems of the tree.

Theorem 99 [16] *Let T be a nontrivial tree with s stems and ℓ leaves. Then*

1. $\gamma_{\times 2}(T) \leq \frac{2n+\ell+s}{3}$.
2. $\gamma_{\times 2}(T) \geq \gamma(T) + \ell \geq \gamma(T) + \Delta$ *and* $\gamma_{\times 2}(T) = \gamma(T) + \Delta$ *if and only if T is obtained from a star $K_{1,t}$ with $t \geq 3$ by either subdividing each edge exactly once, or by subdividing at most $t-1$ edges ot most twice.*
3. $\gamma_{\times 2}(T) \leq 2\gamma(T) + \ell - 1$ *and the extremal family is described.*
4. $\gamma_{\times 2}(T) \leq \frac{n+\gamma_t(T)+\ell}{2}$.

Theorem 100 [29] *Let T be a nontrivial tree with s stems and ℓ leaves. Then*

$$\gamma_{\times 2}(T) \geq \frac{2n+\ell-s+2}{3}.$$

A recursive construction of extremal trees is given.

Another proof of Theorem 100 can be found in [78].

Other results relating in trees $\gamma_{\times 2}$ to various parameters are given in the following three theorems.

Theorem 101 [16,111] *Let T be a nontrivial tree. Then*

$$\gamma_{\times 2}(T) \geq 2i(T) \geq 2\gamma(T).$$

Equality $\gamma_{\times 2}(T) = 2\gamma(T)$ (respectively $\gamma_{\times 2}(T) = 2i(T)$) occurs if and only if T has two disjoint γ-sets (respectively i-sets).

Note that in [7] Bange, Barkauskas and Slater gave a constructive characterization of trees with two disjoint γ-sets and that Haynes and Henning [96] gave a recursive construction to obtain trees with two disjoint i-sets.

Blidia, Chellali and Haynes [15] showed that, as for claw-free graphs (Theorem 97), the double domination number of nontrivial trees is bounded from below by the paired domination number.

Theorem 102 [15] *Every nontrivial tree T satisfies*

$$\gamma_{pr}(T) \leq \gamma_{\times 2}(T).$$

For $n \geq 3$, equality occurs if and only if no two stems are adjacent, every stem has exactly one pendant leaf, and T has a unique $\gamma_{\times 2}(T)$-set consisting of the set of stems and leaves of T.

Theorem 103 [34] *Every nontrivial tree T satisfies*

1. $\gamma_{\times 2}(T) \geq \beta_2(T) \geq \max\{\Gamma_t(T), \gamma_{pr}(T)\}$.
2. $\gamma_{\times 2}(T) \geq 2i_2(T) + 1$ *if* $n \geq 3$.
3. $\gamma_{\times 2}(T) \leq 2\gamma_2(T) - 2$ *if* $n \geq 4$.
4. $\gamma_{\times 2}(T) \leq \frac{3\beta_2(T)}{2}$.

All these bounds are sharp.

Chellali and Haynes also gave a recursive construction of trees T with a unique $\gamma_{\times 2}(T)$-set [36].

In [97], Haynes and Thacker studied the *double domination edge critical* graphs ($\gamma_{\times 2}$*-critical* for short), i.e. the graphs G such that $\gamma_{\times 2}(G + uv) < \gamma_{\times 2}(G)$ for any edge $uv \in E(\overline{G})$, and proved the following proposition.

Proposition 104 [97] *For any graph G without isolated vertices and edge $uv \in E(\overline{G})$,*

$$\gamma_{\times 2}(G) - 2 \leq \gamma_{\times 2}(G + uv) \leq \gamma_{\times 2}(G).$$

Moreover, they characterized the $\gamma_{\times 2}$-critical trees and the 3-critical graphs, that is, the double domination edge critical graphs G with $\gamma_{\times 2}(G) = 3$.

Proposition 105 [97] *A tree T is double domination edge critical if and only if T is a nontrivial star or a double star.*

Theorem 106 [97] *A graph G is 3-critical if and only if its complement $\overline{G}$ is either the union of m copies of a K_2 for $m \geq 2$ or a disjoint union of stars with exactly one isolated vertex.*

They also showed that the $\gamma_{\times 2}$-critical cycles are those of length 3, 4 and 5 and gave subfamilies of the 4-critical graphs, namely those with maximum diameter and those with a leaf. In [148], Wang and Xiang studied the double domination number in random graphs $G(n, p)$ (see [3] for more information on random graphs). Their main result states, as it was proved by Wieland and Godbole [150] for the usual domination number, that the double domination number of $G(n, p)$ with fixed $p \in (0, 1)$ has asymptotically almost sure (a.a.s.) a two-point concentration.

Theorem 107 [148] *Let $p \in (0, 1)$ be a constant and $b = 1/(1 - p)$. In $G(n, p)$, a.a.s.,*

$$g(n, p) + 1 \leq \gamma_{\times 2}(G(n, p)) \leq g(n, p) + 2,$$

where $g(n, p) = \lfloor \log_b n - \log_b \log n + \log_b \frac{p}{1-p} \rfloor$.

4.2.1 Variants of the k-Tuple Domination

We mention some variants of the concept of k-tuple domination and some results related to these variants.

A subset $S \subseteq V$ is an *efficient* [92], *exact* [38] or *perfect* [124] *k-tuple dominating set* if $|N[v] \cap S| = k$ for each vertex v of G. Perfect k-tuple dominating sets do not necessarily exist in graphs with $\delta \geq k - 1$. Some results on the existence and the size of such sets in particular classes of graphs and for $k = 2$ or 3 are given below. Note that in [38] Chellali, Khelladi and Maffray proved that the existence of a perfect double dominating set is an NP-Complete problem.

Theorem 108 [38]

1. *If a graph has a perfect double dominating set, then all such sets have the same size lying between $\frac{2n}{\Delta+1}$ and $\frac{2n}{\delta+1}$.*
2. *A path P_n (resp. cycle C_n) has perfect double dominating sets if and only if $n \equiv 2$ (mod 3) (resp. $n \equiv 0$ (mod 3)) and their size is $\frac{2(n+1)}{3}$ (resp. $\frac{2n}{3}$).*

Characterizations of connected cubic graphs or of trees with perfect double dominating sets are also given in [38].

The grids $P_p \times P_q$, cylinders $C_p \times P_q$ and tori $C_p \times C_q$ admiting perfect double dominating sets or perfect triple dominating sets have been respectively determined by Khodkar and Sheikholeslami in [113] and by Mehri, Mirnia and Sheikholeslami in [124].

A subset $S \subseteq V$ is a *weak double dominating set* [16] if $|N[v] \cap S| \geq 2$ for each vertex v of degree at least 2 of G. The minimum cardinality of a weak double dominating set of G is denoted $\gamma^w_{\times 2}(G)$. Let $\mathcal{T}$ be the family of trees that can be obtained from the disjoint union of $p \geq 1$ copies of P_3 by adding $p - 1$ edges so that each added edge joins two leaves from different P_3's.

Theorem 109 [16] *If T is a tree of order $n \geq 2$, then $\gamma^w_{\times 2}(T) \leq \frac{2n}{3}$ with equality if and only if $T \in \mathcal{T}$.*

As mentioned in the introduction of this section, $\gamma_{k,k}(G)$-sets were called $\gamma_t^{(\times 2)}(G)$-sets by Dorbec, Gravier, Klavžar and Špacapan in the context of the study of the total domination number of a direct product. They proved

Theorem 110 [52] *Let G and H be graphs with $2 \le \delta(G) \le \Delta(G) < \gamma_t(H)$ and let $p \ge \gamma_{2,2}(G)$. Then*

$$\gamma_t(G \times H) \ge \gamma_{2,2}(G) \quad \text{and} \quad \gamma_t(G \times K_p) \le \gamma_{2,2}(G).$$

Liao and Chang observed in [117] that for strongly chordal graphs, their linear algorithm constructing a $\gamma_{\times k}(G)$-set can be adapted to construct a $\gamma_{k,k}(G)$-set in linear time. Similarly, one can slightly modify their transformations proving the NP-completeness of the determination of $\gamma_{\times k}(G)$ when G is split or bipartite to show that the determination of $\gamma_{k,k}(G)$ is also NP-Complete in these two classes of graphs.

In [108], Kala and Nirmala Vasantha initialized the study of the *restrained double domination number* of a graph, which is the minimum cardinality of a double dominating set S such that $G[V - S]$ has no isolated vertices.

4.2.2 Nordhaus-Gaddum Bounds

To study Nordhaus-Gaddum-type bounds on $\gamma_{\times k}$, we suppose $\delta(G) \ge k - 1$ and $\delta(\overline{G}) \ge k - 1$. Therefore, $\Delta(G), \Delta(\overline{G}) \le n - k \le n - 2$ and by Observation 80, $\gamma_{\times k}(G) \ge k + 1$. Hence

$$2k + 2 \le \gamma_{\times k}(G) + \gamma_{\times k}(\overline{G}) \le 2n \quad \text{and} \quad 4(k+1)^2 \le \gamma_{\times k}(G)\gamma_{\times k}(\overline{G}) \le n^2.$$

The previous bounds were given in [90] for $k = 2$, with the determination of P_4 as the unique extremal graph for the upper bounds.

Other Nordhaus-Gaddum bounds can of course be obtained by using known bounds on $\gamma_{\times k}(G)$. For instance, the inequality $\gamma_{\times k}(G) \le n - \delta + k - 1$ of Corollary 81 gives

$$\gamma_{\times k}(G) + \gamma_{\times k}(\overline{G}) \le n + \Delta - \delta + 2k - 1$$

(observed in [35] for $k = 2$). When k is odd, let G be a $(k - 1)$-regular graph of order $2k - 1$. When k is even, let G be obtained from a $(k - 1)$-regular graph of order $2k$ by adding a matching of $n/2$ edges. These graphs are extremal for the two upper bounds $2n$ and $n + \Delta - \delta + 2k - 1$ on $\gamma_{\times k}(G) + \gamma_{\times k}(\overline{G})$.

The result of Theorem 111 was conjectured in [90] for $k = 2$ (note that in [54], the k-tuple domination number $\gamma_{\times k}$ is denoted γ_k).

Theorem 111 [54] *For any integer $k \ge 1$, if a graph G has $\gamma(G), \gamma(\overline{G}) \ge k + 2$, then $\gamma_{\times k}(G) + \gamma_{\times k}(\overline{G}) \le n - \Delta + \delta - 1$.*

For the particular case $k = 2$, Chen and Sun proved

Theorem 112 [41] *If G is a graph with $\delta(G) \ge 1, \delta(\bar{G}) \ge 1$ and $diam(G) > 2$ or $diam(\overline{G}) > 2$, then $\gamma_{\times 2}(G) + \gamma_{\times 2}(\overline{G}) \le n + 4$.*

It is also known [112] that if $\text{diam}(G) \ge 4$ then $\gamma_{\times 2}(\overline{G}) \le 4$, and if moreover G is triangle-free or $\text{diam}(G) \ge 6$, then $\gamma_{\times 2}(\overline{G}) = 3$.

5 Varieties of k-Domination and k-Independence

5.1 k-Independent j-Dominating Sets

Subsets S of vertices of G which are both k-independent and j-dominating were considered in 1985 by Fink and Jacobson [68] and later on by Favaron [57]. By Theorem 19, such sets exist when $j = k$ with $1 \leq k \leq \Delta$, and consequently for each $j \leq k$ since for $j' < j$, every j-dominating set is j'-dominating. For $j > k$, k-independent j-dominating sets may exist or not. The minimum and maximum cardinalities of a k-independent j-dominating set, if any, are respectively denoted by $i_j^k(G)$ and $I_j^k(G)$ (shortly i_j^k and I_j^k) in [57] while Fink and Jacobson used the notation $i(k-1, j, G)$ for $i_j^k(G)$. Clearly for any graph, $i_j^k \leq I_j^k$, $i_1^1 = i$, $I_1^1 = \beta$, $I_1^k = \beta_k$ and $i_k^\Delta \geq i_k^{\Delta+1} = \gamma_k$. Moreover, a γ_k-set S cannot contain a vertex x with Δ neighbors in S for otherwise $S - \{x\}$ is still k-dominating, and thus $i_k^\Delta = \gamma_k$. Therefore, the first part of the following theorem is obvious.

Theorem 113 [57] *Let G be a graph of maximum degree Δ and $1 \leq k \leq \Delta$.*

1. $\gamma_k = i_k^\Delta \leq i_k^{\Delta-1} \leq \cdots \leq i_k^{k+1} \leq i_k^k \leq I_k^k \leq I_{k-1}^k \leq \cdots \leq I_2^k \leq I_1^k = \beta_k$.
2. *If $k \leq \Delta - 1$ then $\gamma_k = i_k^{\Delta-1}$ and if $k \geq 2$ then $\beta_k = I_2^k$. However, $k \leq \Delta - 2$ does not imply $\gamma_k = i_k^{\Delta-2}$ and $k \geq 3$ does not imply $\beta_k = I_3^k$.*

Theorem 114 concerns particular classes of graphs. The graphs B and $B + e$ are defined by $V(B) = V(B + e) = \{c, x_1, x_2, x_3, x_4\}$, $E(B) = \{x_1x_2, x_3x_4, cx_i$ for $1 \leq i \leq 4\}$ and $E(B + e) = E(B) + \{x_2x_3\}$. Since $i_1^1(G) = i(G)$, the first result of Theorem 114 generalizes Allan and Laskar's [1] result saying that $\gamma(G) = i(G)$ in claw-free graphs.

Theorem 114 *Let G be a graph of maximum degree Δ and $1 \leq k \leq \Delta$.*

1. (Fink and Jacobson [68])
   If G is $K_{1,3}$-free then $\gamma_k(G) = i_k^{2k-1}(G)$.
   If G is $\{K_{1,3}, K_{1,3} + e\}$-free or $\{K_{1,3}, B, B + e\}$-free, then $\gamma_k(G) = i_k^k(G)$.
2. (Favaron [57])
   If G is $\{K_{1,3}, K_{1,3} + e\}$-free then $\beta_k = I_k^k$.

Different authors studied the previous parameters for particular values of j and k. The notation differs from one article to another and we continue to use the previous notation to cite the main definitions and results.

Perfect independent sets

In 1983, before Fink and Jacobson's papers, Croitoru and Suditu [49] defined a *perfect stable set* or *perfect independent set* of G as a subset of vertices which is both 1-independent and 2-dominating. There are graphs without perfect independent sets, for example odd cycles and even paths. The authors of [49] proved that the problem of deciding if a given graph G has a perfect independent set is NP-Complete in general, but polynomial for claw-free graphs. Clearly, any perfect independent set is a maximal independent set, but not necessarily a maximum independent set.

For example, the complete bipartite graph $K_{2,p}$ with the partite sets $A = \{a, b\}$ and $B = \{x_1, x_2, \ldots, x_p\}$ with $p \geq 3$, has the perfect independent set A and the maximum independent set B. However, the next observation shows that this is valid for claw-free graphs.

Observation 115 [140] *If G is a claw-free graph, then every perfect independent set is also a maximum independent set.*

For more information on perfect independent sets, we refer the reader to the article by Volkmann [141], where the author proved the following result.

Theorem 116 [141] *Let G be a graph with the property that every even cycle contains a chord, and let I be an independent set. Then I is a perfect independent set if and only if I is a unique maximum independent set.*

k-Independent dominating sets

[63] mainly deals with subsets of vertices which are k-independent and 1-dominating. The main results concern i_1^k (denoted γ^{k-1} in [63]). Recall that $i = i_1^1$ and that if $\Delta \geq 2$, then $i_1^{\Delta-1} = \gamma$ by Theorem 113(2).

Theorem 117 [63] *Let G be a graph of maximum degree Δ and $1 \leq k \leq \Delta - 2$. Then*

1. $i_1^k \leq \max\{k-1, (\Delta-k)\gamma - k(\Delta-k-1)\}$.
2. $i_1^k \leq \frac{\Delta-k+1}{2} i_1^{k+1}$ *(hence $i \leq \frac{3}{2}\gamma$ if $\Delta = 3$).*
3. *If $4 \leq \Delta \leq 6$ then $i \leq \frac{\Delta}{2} i_1^3$ (hence $i \leq 2\gamma$ if $\Delta = 4$).*
   If $\Delta \geq 7$ then $i \leq \frac{2\Delta-3}{3} i_1^3$.

k-Insulated sets

In 2001, Jagota, Narasimhan and Šoltés [106], apparently independently from the existing literature on k-independence, called *k-insulated* a subset of vertices of G which is both $(k+1)$-independent and $(k+1)$-dominating and i_k the maximum cardinality I_{k+1}^{k+1} of these sets. They gave algorithms for finding general k-insulated sets and k-insulated sets of order at least $\frac{c\log^2 n}{kn} I_k^k(G)$ for a constant $c > 0$. The first algorithm, based on the proof that every set D fulfilling that $\psi_k(D) = m(G[D]) - (k - \frac{1}{2})|D|$ is minimum turns to be k-independent and k-dominating, constructs a k-independent k-dominating set from any set D (compare to Theorem 19).

In [74], Grigorescu studied the *insulation sequence* $I_k^k(G)$ for $k \geq 1$. She obtained results on I_k^k for paths, cycles, fans, wheels and for some products, compositions or induced subgraphs of graphs. She showed that for all graphs, $I_1^1 \leq I_2^2 \leq I_k^k$ for $k \geq 2$ but that except these inequalities and $I_\Delta^\Delta < I_{\Delta+1}^{\Delta+1}$, no other constraint between I_k^k and I_{k+1}^{k+1} exists for general graphs. However, the sequence is non-decreasing for trees.

Theorem 118 [74] *Let T be a tree and S_k any k-independent k-dominating set of T with $1 \leq k \leq \Delta$. Then*

1. $|S_k| \leq |S_{k+1}|$. *Therefore $I_k^k(T) \leq I_{k+1}^{k+1}(T)$.*
2. $|S_k| \geq \frac{(k-1)n+1}{k}$. *Therefore $I_k^k(T) \geq \frac{(k-1)n+1}{k}$.*

The result of item 1 remains true for unicyclic graphs. Note that the consequences of the two properties of Theorem 118 are stronger than indicated in the theorem. They imply $I_k^k(T) \le i_{k+1}^{k+1}(T)$ and $i_k^k(T) \ge \frac{(k-1)n+1}{k}$.

5.2 (p, k)-Domination and (p, k)-Independence

In [8], Bean, Henning and Swart defined (p, k)-dominating and (p, k)-independent sets as follows: let p and k be positive integers. A set D of vertices of a graph G is a (p, k)-*dominating set* if every vertex not in D is within distance p from at least k vertices of D. The subset D is a (p, k)-*independent set* if every vertex of D is within distance p from at most $k - 1$ other vertices of D. The (p, k)-*domination number*, $\gamma_{(p,k)}(G)$, is the minimum cardinality among all (p, k)-dominating sets of G, and the (p, k)-*independence number*, $\beta_{(p,k)}(G)$, is the maximum cardinality among all (p, k)-independent sets of G. The concept of (p, k)-domination is a generalization of the two concepts of distance domination and k-domination, and the concept of (p, k)-independence is a generalization of the two concepts of distance independence and k-independence. In particular, for $p = 1$, a (p, k)-dominating set of G is a k-dominating set and a (p, k)-independent set of G is a k-independent set.

For any positive integer r and any graph G, the *rth power* G^r of G is that graph with the same vertex set and an edge uv belongs to $E(G^r)$ if and only if the distance between u and v is at most r in G. Let $\delta_r(G) = \delta(G^r)$ and $\Delta_r(G) = \Delta(G^r)$. It can be easily seen that

$$\gamma_{(p,k)}(G) = \gamma_k(G^p) \quad \text{and} \quad \beta_{(p,k)}(G) = \beta_k(G^p)$$

for any graph G and arbitrary positive integers k and p. Consequently, by using the above relations, several results obtained for $\gamma_k(G)$ and $\beta_k(G)$ can be generalized to $\gamma_{(p,k)}(G)$ and $\beta_{(p,k)}(G)$. In particular, since $\gamma_k(G) \le \beta_k(G)$ for every integer $k \ge 1$ and every graph G [56], it follows that $\gamma_{(p,k)}(G) \le \beta_{(p,k)}(G)$ for all positive integers k and p. Similarly, $\gamma_{(p,k)}(G) \ge \gamma_{p,1}(G) + k - 2$ for $2 \le k \le \Delta_p$ and $\gamma_{(p,k)}(G) \ge \frac{kn}{\Delta_p + k}$ for $k \ge 1$ are immediate generalizations of Theorems 8 and 34(1).

To generalize Theorem 43 of Cockayne, Gamble, and Shepherd [45], Bean, Henning, and Swart posed the following conjecture.

Conjecture 119 [8] *Let k and p be arbitrary positive integers and let G be a graph of minimum p-degree $\delta_p(G) \ge k + p - 1$. Then*

$$\gamma_{(p,k)}(G) \le \frac{kn}{p + k}.$$

The conjecture is true for $p = 1$ by Theorem 43 and is also true for $k = 1$ and all integers $p \ge 1$ as proven by Henning, Oellermann, and Swart [100] in 1991.

The conjecture remains open and has been essentially investigated by Fischermann and Volkmann [70]. They presented upper bounds on $\gamma_{(p,k)}(G)$ depending on $\frac{k}{p}$ and $\delta_p(G)$. Their results prove the conjecture when k is a multiple of p.

Theorem 120 [70] *Let k and p be two arbitrary positive integers, and let G be a graph. Then*

1. *If $\frac{k}{p} \in \mathbb{N}$ and $\delta_p(G) \geq k + p - 1$, then $\gamma_{(p,k)}(G) \leq \frac{k}{p+k} n(G)$.*
2. *If $\frac{k}{p} \notin \mathbb{N}$ and $\delta_p(G) \geq k + p - 1$, then $\gamma_{(p,k)}(G) \leq \frac{p+k-1}{k+2p-1} n(G)$.*
3. *If $\frac{k}{p} \notin \mathbb{N}$, $k > p$ and $\delta_p(G) \geq p\frac{k}{k+1-p} + k - 1$, then $\gamma_{(p,k)}(G) < \frac{k}{p+k} n(G)$.*

In the particular case that $k = p = 2$, Korneffel, Meierling and Volkmann gave an upper bound for $\gamma_{(2,2)}(G)$ that confirms the conjecture without the hypothesis $\delta_2(G) \geq 3$ for all connected graphs with the exception of a certain family.

Theorem 121 [115] *If G is a connected graph of order $n \geq 3$, then $\gamma_{(2,2)}(G) \leq \frac{n+1}{2}$ with equality if and only if G is a subdivided star.*

The main results given by Bean, Henning and Swart in [8] concern the complexity of the determination of $\gamma_{(p,k)}(G)$ and $\beta_{(p,k)}(G)$. They showed that finding the (p, k)-domination number with p and $k \geq 2$ is NP-Complete even when restricted to bipartite graphs and chordal graphs and that the finding the (p, k)-independence number with p even ≥ 2 and $k \geq 2$ is NP-Complete even when restricted to bipartite graphs.

In a recent paper, Lu, Hou and Xu establish a lower and an upper bound on the $(2, 2)$-domination number for trees in term of the domination number. Moreover, they characterize all trees attaining each bound.

Theorem 122 [120] *For every nontrivial tree T,*

$$\frac{2(\gamma(T) + 1)}{3} \leq \gamma_{(2,2)}(T) \leq 2\gamma(T).$$

5.3 k-Star-Forming Sets

As seen in Sect. 2, the relationships between the four parameters $\gamma_k, i_k, \beta_k, \Gamma_k$ do not completely generalize the inequality chain $\gamma \leq i \leq \beta \leq \Gamma$. Indeed for all graphs and all k, $i_k \leq \beta_k$ and $\gamma_k \leq \Gamma_k$ by definition, $\gamma_k \leq \beta_k$ and $i_k \leq \Gamma_k$ by Corollary 20, but for every value of k there exist graphs satisfying $i_k(G) < \gamma_k(G)$ or $\Gamma_k(G) < \beta_k(G)$. The reason of this difference is that every maximal independent set of G is dominating while for $k \geq 2$, a maximal k-independent set is not necessarily k-dominating. More precisely, a k-independent set S is maximal if and only if for every vertex $v \in V - S$, $d_S(v) \geq k$ or v has a neighbor u in S such that $d_S(u) = k - 1$. In other words, for every vertex $v \in V - S$, there exist k vertices $u_1, u_2, \ldots, u_k$ in S such that $G[\{v, u_1, u_2, \ldots, u_k\}]$ contains a star $K_{1,k}$ as a (not necessarily induced) subgraph. Following the terminology of [93], Chellali and Favaron called such sets *$K_{1,k}$-forming* or *k-star-forming* [32] and denoted by $\mathrm{sf}_k(G)$ ($\mathrm{SF}_k(G)$) the minimum (maximum) cardinality of a minimal $K_{1,k}$-forming set. Clearly, $\mathrm{sf}_1(G) = \gamma(G)$ and $\mathrm{SF}_1(G) = \Gamma(G)$. Consider for instance the star $K_{1,p}$ with $p \geq 5$, center c and set of leaves X, and an integer k with $2 \leq k < p$. Let $Y \subseteq X$ with $|Y| = k - 1$. Then X is the unique k-dominating set while $Y \cup \{c\}$ is a maximal k-independent set, i.e., a k-star-forming set, which is not k-dominating.

Replacing the concept of k-dominating set by that of k-star forming set allows to completely generalize (1).

Theorem 123 [32] *Let G be a graph of maximum degree Δ and $1 \leq k \leq \Delta$.*

1. *A k-independent set is maximal if and only if it is k-star-forming.*
2. *A k-independent and k-star-forming set is a minimal k-star-forming set.*
3. *$sf_k(G) \leq i_k(G) \leq \beta_k(G) \leq \mathrm{SF}_k(G)$.*
4. *$\gamma_k(G^2) \leq \mathrm{sf}_k(G) \leq \gamma_k(G)$ where G^2 is the square of G but $SF_k(G)$ may be larger or smaller than $\Gamma_k(G)$.*

Every $(k+1)$-star-forming set is k-star-forming and, as the sequence $\gamma_k(G)$, the sequence $\mathrm{sf}_k(G)$ is non-decreasing:

$$\gamma(G) = \mathrm{sf}_1(G) \leq \mathrm{sf}_2(G) \leq \cdots \leq \mathrm{sf}_\Delta(G) < \mathrm{sf}_{\Delta+1}(G) = n.$$

The following results concern particular classes of graphs. The first one generalizes the fact that a well-covered tree satisfies $\gamma(T) = \Gamma(T)$. The description of well-k-covered trees, i.e., trees T such that $i_k(T) = \beta_k(G)$ was given in [62].

Theorem 124 [32] *If T is a well-k-covered tree then $sf_k(T) = \mathrm{SF}_k(T)$.*

In general graphs, $\gamma_t(G)$ is at least $\mathrm{sf}_2(G)$ but can be smaller or larger than $\mathrm{sf}_k(G)$ for any $k > 2$. More precise relations between $\gamma_t(G)$ and $\mathrm{sf}_k(G)$ are known in trees and in $K_{1,k}$-free graphs.

Theorem 125 [32]

1. *If G is chordal then $\gamma_t(G) = \mathrm{sf}_2(G)$ (proved for trees in [93]).*
2. *If G is $K_{1,k}$-free with $k \geq 2$ then $\gamma_t(G) \leq \mathrm{sf}_k(G)$.*

5.4 Connected k-Domination

A subset $D \subseteq V(G)$ is a *connected k-dominating set* of a connected graph G, if D is a k-dominating set of G and the subgraph induced by the vertex set D is connected. The *connected k-domination number* $\gamma_k^c(G)$ is the minimum cardinality among the connected k-dominating sets of G.

The following characterization of all connected graphs G with the property that $\gamma_k^c(G) = n(G)$ is easy to verify.

Proposition 126 [145] *Let $k \geq 2$ be an integer, and let G be a connected graph of order $n \geq 2$. Then $\gamma_k^c(G) = n$ if and only if all vertices of G are either cut vertices or vertices of degree less than k.*

Corollary 127 [145] *Let $k \geq 2$ be an integer. If T is a tree, then $\gamma_k^c(T) = n(T)$.*

Corollary 128 [145] *If $k \geq 2$ is an integer, and G a connected graph with $\delta(G) \geq k$, then $\gamma_k^c(G) \leq n(G) - 1$.*

Next theorem gives the characterization of all connected graphs G with $\gamma_k^c(G) = n(G) - 1$ when $\delta(G) \geq k \geq 2$.

Theorem 129 [145] *Let $k \geq 2$ be an integer, and let G be a connected graph of order n and minimum degree δ.*

1. *If $\delta \geq 2$, then $\gamma_2^c(G) = n - 1$ if and only if G is a cycle.*
2. *If $\delta \geq k \geq 3$, then $\gamma_k^c(G) = n - 1$ if and only if G is isomorphic to the complete graph K_{k+1}.*

In [77], Hansberg showed a similar result to Theorem 8 of Fink and Jacobson [67].

Theorem 130 [77] *Let G be a connected graph and k an integer with $1 \leq k \leq \delta(G)$. Then*

$$\gamma_k^c(G) \geq \gamma_c(G) + k - 2.$$

For $k = 2$ and $k = 3$ the following slightly better bounds are valid.

Theorem 131 [145] *If G is a connected graph of order $n \geq 3$, then $\gamma_2^c(G) \geq \gamma_c(G)+1$ and $\gamma_3^c(G) \geq \gamma_c(G) + 2$.*

Volkmann [145] has given families of graphs with $\gamma_k^c(G) = \gamma_c(G) + k - 2$ for $k \geq 4$. Consequently, Theorem 130 is sharp for $k \geq 4$. If $\gamma(G) \geq k - 1$, then the next result implies that $\gamma_k^c(G) \geq \gamma(G) + k - 1$.

Theorem 132 [145] *If G is a connected graph of order $n \geq 2$, then*

$$\gamma_k^c(G) \geq \min\{2\gamma(G), \gamma(G) + k - 1\}.$$

By defining the parameter $\kappa_{\max}(G)$ of a graph G, which denotes the maximum number of components of $G - u$ among all vertices $u \in V(G)$, Hansberg [77] gave a similar result to Theorem 131 for $\gamma_2^c(G)$.

Theorem 133 [77] *Let G be a connected graph on $n \geq 2$ vertices. Then*

$$\gamma_2^c(G) \geq \gamma_c(G) + \kappa_{\max}(G).$$

This result implies that, if $\gamma_2^c(G) = \gamma_c(G) + q$, then $\kappa_{\max}(G) \leq q$. Consequently, every graph G with $\gamma_2^c(G) = \gamma_c(G) + 1$ does not have cut vertices. Easily it follows that, if G is a block-cactus graph of order $n \geq 2$, then $\gamma_2^c(G) = \gamma_c(G) + 1$ if and only if $G \cong K_n$ or $n \geq 3$ and $G \cong C_n$.

More generally, Hansberg presented the following related theorem.

Theorem 134 [77] *Let G be a connected graph and k an integer with $2 \leq k \leq \delta(G)$. Then*

$$\gamma_k^c(G) \geq \gamma_c(G) + (k - 2)\kappa_{\max}(G).$$

Following results concerning the connected k-domination number are closely related to Theorem 34 of Fink and Jacobson [67].

Theorem 135 [145] *Let $k \geq 2$ be an integer, and let G be a connected graph of order n and maximum degree $\Delta \geq 1$. If $\gamma_k^c(G) \leq n-1$, then*

$$\gamma_k^c(G) \geq \left\lceil \frac{kn-2}{\Delta+k-2} \right\rceil. \tag{2}$$

Theorem 136 [145] *If $k \geq 2$ is an integer, and G is a connected graph of order n and size m, then*

$$\gamma_k^c(G) \geq n + \frac{n-m-1}{k-1} \tag{3}$$

Next theorems give upper bounds for the connected k-domination number and were obtained by applying the general method on domination proposed in [78].

Theorem 137 [78] *Let G be a connected graph and $k \geq 1$ an integer. Then*

$$\gamma_k^c(G) \leq \frac{k+2}{k}\gamma_{\times k}(G) - 2.$$

Theorem 138 [77] *Let G be a connected graph and k an integer with $2 \leq k \leq \Delta(G)$. Then*

$$\gamma_k^c(G) \leq 2\gamma_k(G) - k + 1.$$

Analogous to the bound of Arnautov [5], Lovász [118] and Payan [127] for the domination number (see Corollary 55), Caro, West and Yuster [24] presented the following theorem for the connected k-domination number.

Theorem 139 [24] *Let k and δ be positive integers satisfying $k < \sqrt{\ln \delta}$ and let G be a graph on n vertices with minimum degree at least δ. Then:*

$$\gamma_k^c(G) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)).$$

This result was later generalized by Caro and Yuster [25], using the concept of connected (F, k)-cores. Hereby, given a family of graphs $F = \{G_1, G_2, \ldots, G_t\}$ sharing the same vertex set V, a *connected (F, k)-core* is a subset $D \subseteq V$ of vertices such that D is a connected and total k-dominating set for each graph in F. $c_c(k, F)$ denotes the minimum cardinality of a connected (F, k)-core.

Theorem 140 [25] *Let k, t and δ be positive integers satisfying $k < \sqrt{\ln \delta}$ and $t < \ln(\ln \delta)$. Let F be a family of connected graphs on the same n-vertex set. Assume that every graph in F has minimum degree at least δ. Then:*

$$c_c(k, F) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)).$$

Moreover, since in a connected graph G every connected total k-dominating set is a total k-dominating set (or, as in Sect. 4, a k-total k-dominating set) and the latter is always a k-dominating set, it follows that

$$\gamma_k(G) \leq \gamma_{k,k}(G) = \gamma_k^t(G) \leq \gamma_k^{c,t}(G) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)).$$

Hence, this result comprises all of this form mentioned before (see Corollary 51 and Sect. 4.1). For the proof of Theorem 140, Caro and Yuster gave the following lemma involving the total and the connected total k-domination numbers and generalizing the result a of Duchet and Meyniel [53] for the case $k = 1$.

Lemma 141 [25] *If G is a connected graph, then*

$$\gamma_k^t(G) \leq \gamma_k^{c,t}(G) \leq 3\gamma_k^t(G) - 2.$$

Next theorem improves this bound considerably.

Theorem 142 [77] *Let G be a connected graph and $k \geq 1$ an integer. Then*

$$\max\{\gamma_k^c(G), \gamma_k^t(G)\} \leq \gamma_k^{c,t}(G) \leq \frac{k+3}{k+1}\gamma_k^t(G) - 2.$$

In particular, it follows that $\gamma_c(G) \leq 2(\gamma_t(G) - 1)$ for any connected nontrivial graph G, which was shown by Favaron and Kratsch in [65]. Hansberg also gives examples that prove the sharpness of Theorem 142.

We close this section on the connected k-domination number with the following open problems by Volkmann [145].

Problem 143 [145] Let $k \geq 2$ be an integer. Characterize the connected graphs G with one of the following properties.

(1) $\gamma_2^c(G) = \gamma(G) + 1$,
(2) $\gamma_2^c(G) = \gamma_c(G) + 1$,
(3) $\gamma_3^c(G) = \gamma(G) + 2$,
(4) $\gamma_3^c(G) = \gamma_c(G) + 2$,
(5) $\gamma_k^c(G) = \gamma(G) + k - 2$, for $k \geq 4$,
(6) $\gamma_k^c(G) = \gamma_c(G) + k - 2$, for $k \geq 4$,
(7) $\gamma_k^c(G) = \gamma_k(G)$,
(8) $\gamma_k^c(G) = n(G) - 2$ for $\delta(G) \geq k \geq 2$,
(9) $\gamma_k^c(G) = \lceil n(G) + \frac{n(G)-m-1}{k-1} \rceil$,
(10) $\gamma_k^c(G) = \lceil \frac{kn(G)-2}{\Delta+k-2} \rceil$.

5.5 The k-Domatic and k-Tuple Domatic Numbers

A *k-domatic partition* (*k-tuple domatic partition*) of a graph G is a partition of $V(G)$ into k-dominating (k-tuple dominating) sets. The *k-domatic number* $d_k(G)$ (*k-tuple* *domatic number* $d_{\times k}(G)$) is the maximum number of sets in a partition of $V(G)$ into k-dominating (k-tuple dominating) sets. For $k = 1$, $d_1(G) = d_{\times 1}(G)$ is simple the usual *domatic number* $d(G)$ studied first by Cockayne and Hedetniemi [47]. Zelinka [153] defined the k-domatic number $d_k(G)$ and called it k-ply domatic number, and Harary and Haynes [91] introduced the k-tuple domatic number. Remember that not every graph has a k-tuple domination number for $k \geq 2$. However, any graph G with $\delta(G) \geq k-1$ has a k-tuple dominating set and thus a k-tuple domatic number. For more information on the domatic number and their variants, we refer the reader to the survey article by Zelinka [154].

Proposition 144 [153] *If K_n is the complete graph, then $d_k(K_n) = \lfloor n/k \rfloor$. If C_n is the cycle of length n, then $d_2(C_n) = 2$ when n is even and $d_2(C_n) = 1$ when n is odd.*

Theorem 145 [153] *If $K_{p,q}$ is the complete bipartite graph, then $d_k(K_{p,q}) = 1$ for $\min\{p, q\} < k$, $d_k(K_{p,q}) = 2$ for $k \leq \min\{p, q\} < 2k$ and $d_k(K_{p,q}) = \lfloor \min\{p, q\}/k \rfloor$ for $\min\{p, q\} \geq 2k$.*

Proposition 146 [109] *If G is a graph of order n, then*

$$d_k(G) + \gamma_k(G) \leq d_k(G) + \frac{n}{d_k(G)} \leq n + 1. \tag{4}$$

The graphs with equality in the inequality chain (4) are characterized in the next result.

Theorem 147 [109] *Let G be a graph of order n. Then $d_k(G) + \gamma_k(G) = n + 1$ if and only if $\Delta(G) < k$ or $G = K_n$ when $k = 1$.*

Corollary 148 [47] *For any graph G with n vertices, $d(G) + \gamma(G) \leq n + 1$, with equality if and only if $G = K_n$ or $\overline{K_n}$.*

Theorem 149 [91] *If $k \geq 2$ is an integer, and G is a graph of order n with $\delta(G) \geq k-1$ and $d_{\times k}(G) \geq 2$, then*

$$\gamma_{\times k}(G) + d_{\times k}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2. \tag{5}$$

Harary and Haynes [91] also characterized the graphs with equality in the inequality (5). The following result is an extension of a lower bound on the classical domatic number given by Zelinka [152].

Theorem 150 [109] *For any graph G of order n and minimum degree δ,*

$$d_k(G) \geq \left\lfloor \frac{n}{k(n-\delta)} \right\rfloor.$$

Next we present interesting upper bounds in terms of minimum degree and order.

Theorem 151 [153] *For any graph G,*

$$d_k(G) \le \frac{\delta(G)}{k} + 1.$$

Theorem 152 [91] *If G is a graph with $\delta(G) \ge k-1$, then*

$$d_{\times k}(G) \le \frac{\delta(G)+1}{k}.$$

The special case $k = 1$ in Theorems 151 and 152 can be found in the article by Cockayne and Hedetniemi [47]. Simple applications of Theorem 151 and 152 lead to the following Nordhaus-Gaddum-type results.

Theorem 153 [109] *For every graph G of order n,*

$$d_k(G) + d_k(\overline{G}) \le \frac{n-1}{k} + 2, \tag{6}$$

and equality in (6) implies that G is a regular graph.

Theorem 154 [91] *If G is a graph of order n with $\delta(G), \delta(\overline{G}) \ge k-1$, then*

$$d_{\times k}(G) + d_{\times k}(\overline{G}) \le \frac{n - \Delta(G) + \delta(G) + 1}{k} \le \frac{n+1}{k}. \tag{7}$$

Theorems 153 and 154 yield immediately a classical bound by Cockayne and Hedetniemi [47].

Corollary 155 [47] *For every graph G having n vertices, $d(G) + d(\overline{G}) \le n + 1$.*

Using Theorems 151 and 153, Kämmerling and Volkmann derived the following main result of their work [109].

Theorem 156 [109] *Let G be a graph of order $n \ge 2$ such that*

$$d_k(G) + d_k(\overline{G}) = \frac{n-1}{k} + 2.$$

If we assume, without loss of generality, that $d_k(G) \ge d_k(\overline{G})$, then

$$d_k(G) = \frac{n}{r}$$

for an integer $r \in \{k, k+1, \ldots, 2k-1\}$.

If $k = 1$, then G is isomorphic to the complete graph K_n.

If $k \ge 2$, then $k + 1 \le r \le 2k - 1$ and $n < kr^2/(r-k)$.

Since $d(K_n) + d(\overline{K_n}) = n + 1$, the next well-known result is an immediate consequence of Theorem 156.

Corollary 157 [47] *If G is a graph of order n, then $d(G) + d(\overline{G}) = n + 1$ if and only if $G = K_n$ or $G = \overline{K_n}$.*

Corollary 158 [109] *Let $k \geq 2$ be an integer. Then there are only a finite number of graphs G such that*

$$d_k(G) + d_k(\overline{G}) = \frac{n(G) - 1}{k} + 2.$$

For $k = 2$ and $k = 3$ there exist more precisely results.

Theorem 159 [109] *If G is a graph of order $n \geq 3$ such that*

$$d_2(G) + d_2(\overline{G}) = \frac{n - 1}{2} + 2, \tag{8}$$

then $n = 9$ and G is 4-regular. In addition, there exists a 4-regular graph of order 9 such that the identity (8) holds.

Theorem 160 [109] *If G is a graph of order $n \geq 4$ such that*

$$d_3(G) + d_3(\overline{G}) = \frac{n - 1}{3} + 2, \tag{9}$$

then $n = 25$ and G is 12-regular or $n = 28$ and G or $\overline{G}$ is 9-regular. In addition, there exists a 12-regular graph of order 25 such that the identity (9) holds.

Recently, Volkmann [147] discussed the graphs fulfilling equality in inequality (7). The next result shows that equality is not possible when k is even.

Theorem 161 [147] *If $k \geq 2$ is an even integer, and G is a graph of order n with $\delta(G), \delta(\overline{G}) \geq k - 1$, then*

$$d_{\times k}(G) + d_{\times k}(\overline{G}) \leq \frac{n}{k}.$$

In the case that k is odd, equality in (7) is possible, but then the following is valid.

Theorem 162 [147] *If $k \geq 3$ is an odd integer, then there is only a finite number of graphs G with $\delta(G)$, $\delta(\overline{G}) \geq k - 1$ such that*

$$d_{\times k}(G) + d_{\times k}(\overline{G}) = \frac{n(G) + 1}{k}.$$

5.6 k-Irredundance

An irredundant set has been defined in 1977 by Cockayne and Hedetniemi [47] as a subset S of V such that

(P_1) $\forall x \in S$, either $d_S(x) < 1$ or x admits a neighbor x' in $V - S$ such that $d_S(x') = 1$.

This property can also be expressed by $|N[S]| > |N[S - \{x\}]|$ for every vertex x of S. The lower and upper irredundant numbers $ir(G)$ and $IR(G)$ are the minimum and maximum cardinalities of a maximal (by inclusion) irredundant set. Irredundance is usually presented as the property of making a dominating set minimal. Indeed, a dominating set S of G is minimal if and only if it is irredundant. This allows us to extend (1) to the inequality chain

$$ir(G) \le \gamma(G) \le i(G) \le \beta(G) \le \Gamma(G) \le IR(G) \tag{10}$$

valid for every graph G. Jacobson, Peters and Rall [104] extended the concept of irredundance to k-irredundance. Let the closed k-neighborhood of a subset $S \subseteq V$ be $N_k[S] = S \cup \{v \in V - S \mid d_S(v) \ge k\}$. A set S is *k-irredundant* if it satisfies $|N_k[S]| > |N_k[S - \{x\}]|$ for every vertex x of S or, equivalently, if

(P_k) $\forall x \in S$, either $d_S(x) < k$ or x admits a neighbor x' in $V - S$ such that $d_S(x') = k$.

A k-irredundant set S is maximal if no proper superset of S is k-irredundant. Note that a single vertex extension is not sufficient to define maximal k-irredundant sets because for $k \ge 2$ a subset of a k-irredundant set is not necessarily k-irredundant as shown by an example given in [104]. The minimum and maximum cardinalities of maximal k-irredundant sets of G are denoted by $ir_k(G)$ and $IR_k(G)$. The first question raised by this definition is the determination of the inequalities of (10) which can be generalized for all graphs or in particular families.

Theorem 163 [104] *For every graph*

1. *A minimal k-dominating set is a maximal k-irredundant set. Hence*

$$ir_k(G) \le \gamma_k(G) \le \Gamma_k(G) \le IR_k(G).$$

2. *Every k-independent set is k-irredundant. Hence*

$$\beta_k(G) \le IR_k(G).$$

However $ir_k(G)$ can be larger or smaller than $i_k(G)$.
Stronger properties hold in claw-free graphs.

Theorem 164 [104]

1. *If G is $\{K_{1,3}, K_{1,3} + e\}$-free then $\Gamma_k(G) = IR_k(G)$.*
2. *If G is $\{K_{1,3}, K_{1,3} + e, W_4\}$-free, where W_4 is a wheel obtained by joining a vertex to the four vertices of a cycle C_4, then $\beta_k(G) = \Gamma_k(G) = IR_k(G)$ and $ir_k(G) = i_k(G)$.*

Motivated by the two inconvenients of this definition of the k-iredundance, namely the facts that the property is not hereditary and that the sequence $IR_k(G)$ is not necessarily monotone (because a k-irredundant set may be not $(k+1)$-irredundant), Favaron proposed a second notion of k-irredundance by considering another generalization of Property (P_1). A set S is *k-irredundant*' if it satisfies

(P'_k) $\forall x \in S$, either $d_S(x) < k$ or x admits a neighbor x' in $V - S$ such that $d_S(x') \leq k$.

Property (P'_k) is clearly hereditary and thus a k-irredundant' set S is maximal if $S \cup \{x\}$ is not k-irredundant' for all $x \in V - S$. The minimum and maximum cardinalities of a maximal k-irredundant' set of G are denoted $ir'_k(G)$ and $IR'_k(G)$. For $k = 1$, $ir_1(G) = ir'_1(G) = ir(G)$ and $IR_1(G) = IR'_1(G) = IR(G)$. Since Property ($P_k$) implies ($P'_k$), every k-irredundant set is k-irredundant' and $IR_k(G) \leq IR'_k(G)$ for every graph. More precisely, it is proved in [58] that for every graph G and every value of k,

$$IR'_k(G) \geq \max\{IR_j(G) \mid 1 \leq j \leq k\}.$$

Every k-irredundant' set is $(k+1)$-irredundant', implying that as (γ_k) and (β_k), the sequence (IR'_k) is non-decreasing.

The properties of Theorem 163 are strengthened with the second definition of the k-irredundance.

Theorem 165 [58]

1. *A k-dominating set is minimal if and only if it is k-irredundant' and a minimal k-dominating set is a maximal k -irredundant' set. Hence for every graph,*

$$ir'_k(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq IR'_k(G).$$

2. *For every graph, $\beta_k(G) \leq IR_k(G) \leq IR'_k(G)$ and $\beta_\Delta(G) = IR'_\Delta(G)$.*

But as $ir_k(G)$, $ir'_k(G)$ can be larger or smaller than $i_k(G)$.

Since $IR(G) = IR'_1(G) \leq IR'_\delta(G)$, the following result improves the inequality $\gamma(G) + IR(G) \leq n$ valid if $\delta \neq 0$ [44].

Theorem 166 [58] *In every graph G without isolated vertices,*

$$\gamma(G) + IR'_\delta(G) \leq n \quad \text{and thus} \quad IR'_\delta(G) \leq n\Delta/(\Delta + 1).$$

Concerning 1-irredundance, let us mention that in [63] the sets which are both irredundant and $(k+1)$-independent were considered. The minimum and maximum cardinality of maximal such sets are denoted in this paper ir^k and IR^k. It is proved that the sequence (IR^k) is non-decreasing, that $ir^{k+1}(G) \leq 2ir^k(G)$ for every graph G and integer $k \geq 1$ and that the determination of these two parameters is NP-Complete.

5.7 Roman k-Domination

In this section, we focus on results about an extension of the *Roman dominating function* which was suggested by ReVelle and Rosing [133] and Stewart [137]. According to [43], Constantine the Great (Emperor of Rome) issued a decree in the fourth century AD for the defense of his cities. He decreed that any city without a legion stationed to secure it must neighbor another city having two stationed legions. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself. The objective, of course, is to minimize the total number of legions needed. However, the Roman Empire had a lot of enemies, and if a number of k enemies attack k cities without a legion, then the cities are secured in the above sense if they are neighbored to at least k cities having two stationed legions. This leads in a natural way to the following generalization of the Roman dominating function.

A *Roman k-dominating function* on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least k vertices $v_1, v_2, \ldots, v_k$ with $f(v_i) = 2$ for $i = 1, 2, \ldots, k$. The *weight* of a Roman k-dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman k-dominating function on a graph G is called the *Roman k-domination number* $\gamma_{kR}(G)$ of G. The Roman k-domination number was introduced by Kämmerling and Volkmann [110] in 2009. Note that the Roman 1-domination number $\gamma_{1R}(G)$ is the usual *Roman domination number* $\gamma_R(G)$. A Roman k-dominating function of minimum weight is called a γ_{kR}*-function*. If $f : V(G) \to \{0, 1, 2\}$ is a Roman k-dominating function, then let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) | f(v) = i\}$ for $i = 0, 1, 2$. Note that there is a 1–1 correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. Thus we will write $f = (V_0, V_1, V_2)$.

The first observation is an extension of a corresponding inequality chain in [43] for $k = 1$.

Proposition 167 [110] *For any graph G*

$$\gamma_k(G) \leq \gamma_{kR}(G) \leq 2\gamma_k(G).$$

Proposition 168 [110] *If G is a graph of order n, then the following conditions are equivalent:*

1. $\gamma_k(G) = \gamma_{kR}(G)$,
2. $\gamma_k(G) = n$,
3. $\Delta(G) \leq k - 1$.

Corollary 169 [43] *Let G be a graph of order n. Then $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.*

Proposition 170 [110] *Let G be a graph of order n.*

1. *If $n \leq 2k$, then $\gamma_{kR}(G) = n$.*
2. *If $n \geq 2k + 1$, then $\gamma_{kR}(G) \geq 2k$.*
3. *If $n \geq 2k + 1$ and $\gamma_k(G) = k$, then $\gamma_{kR}(G) = \gamma_k(G) + k = 2k$.*

Using Proposition 170, Kämmerling and Volkmann [110] derived the following Nordhaus-Gaddum-type result for the Roman k-domination number.

Theorem 171 [110] *If G is a graph of order n, then*

$$\gamma_{kR}(G) + \gamma_{kR}(\overline{G}) \geq \min\{2n, 4k + 1\}. \tag{11}$$

Furthermore, equality holds in (11) if and only if $n \leq 2k$ or $k \geq 2$ and $n = 2k + 1$ or $k = 1$ and G or $\overline{G}$ has a vertex of degree $n - 1$ and its complement has a vertex of degree $n - 2$.

Corollary 172 [27] *If G is a graph of order $n \geq 3$, then $\gamma_R(G) + \gamma_R(\overline{G}) \geq 5$ with equality if and only if G or $\overline{G}$ has a vertex of degree $n - 1$ and its complement has a vertex of degree $n - 2$.*

The special case $k = 1$ of the following lower bound on the Roman k-domination number can be found in [46].

Theorem 173 [110] *If G is a graph of order n and maximum degree $\Delta \geq k$, then*

$$\gamma_{kR}(G) \geq \frac{2n}{\frac{\Delta}{k} + 1}.$$

Now we present a characterization of the graphs G with $\gamma_{kR}(G) < n(G)$.

Theorem 174 [110] *Let G be a graph of order n. Then $\gamma_{kR}(G) < n$ if and only if G contains a bipartite subgraph H with bipartition X, Y such that $|X| > |Y| \geq k$ and $d_H(v) \geq k$ for each $v \in X$.*

As an application of Theorem 174, one can prove the next result by induction on the *cyclomatic number* $c(G) = m(G) - n(G) + \omega(G)$, where $\omega(G)$ is the number of components of G.

Theorem 175 [110] *Let G be a graph of order n. If $k \geq 2$, then*

$$\gamma_{kR}(G) \geq \min\{n, n + 1 - c(G)\}.$$

Corollary 176 [110] *If G is a graph of order n with at most one cycle, then $\gamma_{kR}(G) = n$ when $k \geq 2$.*

The graph G of order 7 consisting of two cycles $x_1x_2x_3x_4x_1$ and $y_1y_2y_3y_4y_1$ with $x_1 = y_1$ and the Roman 2-dominating function f such that $f(x_1) = f(x_3) = f(y_3) = 2$ and $f(x_2) = f(x_4) = f(y_2) = f(y_4) = 0$ shows that Corollary 176 is no longer true if the graph contains more than one cycle.

Applying this example, it is easy to see that the Roman 2-domination number $\gamma_{2R}(G_{i,j}) < ij$ for each $i \times j$ grid $G_{i,j}$ when $i, j \geq 3$. In addition, it is a simple matter to prove that $\gamma_{3R}(G_{i,j}) < ij$ when $i \geq 5$ and $j \geq 9$ and $\gamma_{kR}(G_{i,j}) = ij$ when $k \geq 4$.

As a further application of Theorem 174, one can show that $\gamma_{kR}(G) = n$ for each cactus graph of order n when $k \geq 3$.

Using the probabilistic method, Hansberg and Volkmann [86] presented the following upper bounds for the Roman k-domination number.

Theorem 177 [86] *Let G be a graph of order n and minimum degree δ. If $k \leq \delta$ and $2k \leq \frac{\delta+1+2\ln(2)}{\ln(\delta+1)}$, then*

$$\gamma_{kR}(G) \leq \frac{2n}{\delta + 1}(k \ln(\delta + 1) - \ln(2) + 1).$$

Corollary 178 [43] *If G is a graph of order n and minimum degree $\delta \geq 1$, then*

$$\gamma_R(G) \leq \frac{2n}{\delta+1}(\ln(\delta+1) - \ln(2) + 1).$$

Theorem 179 [86] *Let $k \geq 4$ be an integer, and let G be a graph of order n and minimum degree δ. If $k \leq \delta$ and $2k \leq \frac{\delta+1+2\ln(2)}{\ln(\delta+1)}$, then*

$$\gamma_{kR}(G) \leq \frac{2n}{\delta+1}\left(k\ln(\delta+1) - \ln(2) + 1 - \frac{k-1}{\delta}\right).$$

5.8 The 2-Domination Subdivision Number

The 2-*domination subdivision number* $\mathrm{sd}_{\gamma_2}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the 2-domination number. For example, $\mathrm{sd}_{\gamma_2}(K_n) = 2$ when $n \geq 3$ and $\mathrm{sd}_{\gamma_2}(K_{p,q}) = 3$ when $p, q \geq 4$. In 2008, Atapour, Sheikholeslami, Hansberg, Volkmann and Khodkar [6] initialized the study of the 2-domination subdivision number. The next result is one of the main tools for obtaining upper bounds on the 2-domination subdivision number.

Theorem 180 [6] *Let G be a connected graph. If $v \in V(G)$ has degree at least two, then $\mathrm{sd}_{\gamma_2}(G) \leq d_G(v)$.*

Using the simple observation that $\gamma_2(H) \leq \alpha(H)$ when $\delta(H) \geq 2$ and Theorem 180, following upper bound can be obtained for arbitrary graphs.

Theorem 181 [6] *If G is a connected graph of order $n \geq 3$, then $\mathrm{sd}_{\gamma_2}(G) \leq \gamma_2(G)$.*

Corollary 182 [6] *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\mathrm{sd}_{\gamma_2}(G) \leq \lfloor n/2 \rfloor$.*

Next we present better bounds for some special graph classes and two open problems.

Theorem 183 [6] *If G is a connected claw-free graph of order $n \geq 3$, then $\mathrm{sd}_{\gamma_2}(G) \leq 4$.*

Theorem 184 [6] *If G is a connected block-cactus graph of order $n \geq 3$, then $\mathrm{sd}_{\gamma_2}(G) \leq 2$.*

The complete graph K_n is an example of a block-cactus graph with $\mathrm{sd}_{\gamma_2}(K_n) = 2$ and thus this bound is sharp.

Corollary 185 [6] *If T is a tree of order $n \geq 3$, then $\mathrm{sd}_{\gamma_2}(T) \leq 2$.*

The path P_{2n} of even order $2n$ is an example where the upper bound of this corollary is attained. Therefore, the following problem has been stated.

Problem 186 [6] Characterize the family of trees achieving the bound in Corollary 185.

Problem 187 [6] Prove or disprove: If G is a graph of order $n \geq 3$, then $\mathrm{sd}_{\gamma_2}(T) \leq 3$.

6 Last Results

During the process of submission of this survey, other results were found. We give here the essential ones. In [37], Chellali, Haynes and Volkmann were interested in graphs whose k-independence number is unaffected by the deletion of any edge. A graph G is called β_k^-*-stable* if $\beta_k(G-e)=\beta_k(G)$ for every edge e of $E(G)$. A vertex in a k-independent set S is said to be *full* if it has exactly $k-1$ neighbors in S. They gave a necessary and sufficient condition for β_k^--stable graphs.

Theorem 188 [37] *A graph G is β_k^--stable if and only if for every $\beta_k(G)$-set S, each vertex $x \in V-S$ is $(k+1)$-dominated by S or there are at least two full vertices in $N(x)\cap S$.*

For the class of trees, the same authors established four equivalent conditions for β_k^--stable trees. In particular, they showed that a tree T is a β_k^--stable if and only if it has a unique $\beta_k(T)$-set. We note that the result in [75] concerning β_1^--stable trees is a special case of the results presented in [37].

Similarly to β_k^--stable graphs, Chellali in [31] studied γ_k^--stable graphs and gave a necessary and sufficient condition for such graphs. Moreover, a recursive construction for $k\geq 2$ of γ_k^--stable trees has been also provided.

Theorem 189 [31] *Let k be a positive integer. A graph G is γ_k^--stable if and only if for each pair of adjacent vertices $u, v \in V(G)$, there exists a $\gamma_k(G)$-set D such that one of the following conditions holds:*

(i) *u, v are both in D or both in $V(G)-D$,*
(ii) *if $u \in D$ and $v \notin D$, then v is $(k+1)$-dominated by D.*

Recently, DeLaViña, Goddard, Henning, Pepper and Vaughan presented two bounds on the k-domination number of a graph. Their results are inspired by two conjectures of the computer program Graffiti.pc.

Theorem 190 [50] *Let k be a positive integer, and G a graph of order n. Let $H \subseteq V(G)$ be the set of vertices of degree less than $2k-1$. Then $\gamma_k(G)\leq \tau(G-H)+|H|$.*

Theorem 191 [50] *Let k be a positive integer, and G a graph of order n. Suppose that in G no two vertices of degree less than $2k-2$ are adjacent. Let $H\subseteq V(G)$ be the set of vertices of degree less than $2k-1$. Then $\gamma_k(G)\leq (n+\beta(G[H]))/2$.*

Note that if $k=2$ and G is a connected bipartite graph of order at least three, then $n\leq 2\beta(G)$ and $\beta(G[H])\leq \beta(G)$ and so the bound of Theorem 26 immediately follows from Theorem 191.

Concerning Fink and Jacobson's Theorem 8 when $k=2$, Hansberg, Randerath and Volkmann [81] (submitted) were able to characterize all claw-free graphs with equal 2-domination and domination numbers. From this characterization follows also the one for line-graphs. Moreover, we should mention that a recursive construction of trees with equal 2-domination and total domination numbers was given by Lu, Hou, Xu and Li. [122]. Also using the characterization of trees T such that $\gamma_2(T)=\gamma(T)+1$

given in Theorem 14, Chellali and Volkmann [39] characterized all trees with equal 2-domination number and domination number plus two.

In [129], Pepper and later Hansberg and Pepper in [80] studied more relations between the k-domination number and the j-independence number and proved the following theorem.

Theorem 192 *Let G be a graph of order n, let k, j, m be positive integers such that $m = k + j - 1$, and let H_m and G_m denote, respectively, the subgraphs induced by the vertices of degree at least m and by the vertices of degree at most m. Then,*

1. (Pepper [129, in press]) $\gamma_k(G) + \beta_j(H_m) \leq n$, *and*
2. (Hansberg and Pepper [80, submitted]) $\gamma_k(G_m) + \beta_j(G) \geq n(G_m)$.

Note that this Theorem generalizes Theorem 23. As it is given in Theorem 23 (ii), regular graphs fulfill equality in that bound (and hence also in item (i)). Hansberg and Pepper analyzed also when the converse is true and obtained the following theorems.

Theorem 193 [80, submitted] *Let G be a connected graph on n vertices with maximum degree Δ and minimum degree $\delta \geq 1$. Then*

$$\gamma_k(G) + \beta_j(G) = n \quad \textit{and} \quad \gamma_{k'}(G) + \beta_{j'}(G) = n$$

for every pair of integers k, j and k', j' such that $k + j - 1 = \delta$ and $k' + j' - 1 = \Delta$ if and only if G is regular.

Theorem 194 [80, submitted] *For any graph G the following statements are equivalent:*

(i) $\gamma(G) + \beta_\delta(G) = n$ *and* $\gamma_{\Delta-\delta+1}(G) + \beta_\delta(G) = n$.
(ii) *G is either regular or $\gamma(G) = \gamma_2(G)$ and G is semiregular.*

Moreover, Pepper gave also the following bound involving the k-domination and the $(k-1)$-domination numbers.

Theorem 195 [129, in press] *Let k be a positive integer and let G be a graph of order n and let n_{k-1} denote the number of vertices of degree $k - 1$. Then*

$$\gamma_k(G) \leq \frac{n + \gamma_{k-1}(G) + n_{k-1}}{2}$$

For $k = 1$, we shall set $\gamma_0(G) = 0$. This result is similar to the one given by Favaron and Volkmann in Theorem 15 with the difference that here is no restriction on the minimum degree and that is valid for any $k \geq 1$.

Concerning the k-star-forming numbers, we saw that the parameters Γ_k and SF_k were not comparable. Theorem 196 gives upper bounds on them in terms of n and δ. The bounds are the same but the classes of extremal graphs characterized in [59] are different.

Theorem 196 [59] *Let G be a graph of order n and k a positive integer.*

1. *If* $1 \leq k \leq \delta$ *then* $max\{\Gamma_k(G), \mathrm{SF}_k(G)\} \leq n - \delta + k - 1$.
2. *If* $\delta \leq k \leq \Delta$ *then* $max\{\Gamma_k(G), \mathrm{SF}_k(G)\} \leq n - 1$.

In [26], Chaluvaraju, Chellali and Vidya initiated the study of *perfect k-dominating graphs* defined as graphs G that admit a k-dominating set D such that every vertex not in D is adjacent to exactly k vertices in D. They showed that the problem of deciding whether a graph admits a perfect k-dominating set is NP-Complete.

Finally, Sheikholeslami and Volkmann [136] introduced the Roman k-domatic number. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Roman k-dominating functions on a graph G with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(G)$ is called a *Roman k-dominating family* (of functions) on G. The maximum number of functions in a Roman k-dominating family on G is the *Roman k-domatic number* of G, denoted by $d_{kR}(G)$. The Roman k-domatic number is well-defined and $d_{kR}(G) \geq 1$.

The special case $k = 1$ of the following three results can be found in [135].

Theorem 197 [136] *If G is a graph of order n, then $\gamma_{kR}(G) \cdot d_{kR}(G) \leq 2n$.*

Theorem 198 [136] *If G is a graph with minimum degree δ, then $d_{kR}(G) \leq \lfloor \frac{\delta}{k} \rfloor + 2$.*

As an application of Theorems 173 and 197 one can prove the following improvement of Theorem 198 for regular graphs.

Theorem 199 [136] *If G is a δ-regular graph, then $d_{kR}(G) \leq \lfloor \frac{\delta}{k} \rfloor + 1$.*

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