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A linear-time algorithm for paired-domination problem in strongly chordal graphs ☆

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ABSTRACT

Let $G = (V, E)$ be a simple graph without isolated vertices. A vertex set $S \subseteq V$ is a paired-dominating set if every vertex in $V - S$ has at least one neighbor in S and the induced subgraph $G[S]$ has a perfect matching. In this paper, we present a linear-time algorithm to find a minimum paired-dominating set in strongly chordal graphs if the strong (elimination) ordering of the graph is given in advance.

1. Introduction

Let $G = (V, E)$ be a simple graph without isolated vertices. For a vertex $v \in V$, the *open neighborhood* of v in G is defined as $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v. The *degree* of v in G, denoted by $d_G(v)$, is defined as $|N_G(v)|$. We use $N(v)$ for $N_G(v)$, $N[v]$ for $N_G[v]$ and $d(v)$ for $d_G(v)$ if there is no ambiguity. For a subset S of V, the subgraph of G induced by the vertices in S is denoted by $G[S]$. A *matching* in a graph G is a set of pairwise nonadjacent edges in G. A *perfect matching* M in G is a matching such that every vertex of G is incident to an edge of M. Some other notation and terminology not introduced in here can be found in [20].

Domination and its variations in graphs have been extensively studied [2,10,11]. A set $S \subseteq V$ is a *paired-dominating set*, denoted by PDS, of G if every vertex in $V - S$ has at least one neighbor in S and the induced subgraph $G[S]$ has a perfect matching M. Two vertices u, v joined by an edge of M are said to be paired with regard to M and we call u (v, respectively) is the paired vertex of v (u, respectively) with regard to M. For a subset U of S, U is perfect with regard to M if $M \cap E(G[U])$ is a perfect matching in $G[U]$. The *paired-domination number*, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a PDS. A paired-dominating set S with cardinality $\gamma_{pr}(G)$ is called a MPDS of G. The *paired-domination problem* is to

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determine the paired-domination number of a graph without isolated vertices. The paired-domination problem was proposed by Haynes and Slater in 1998 [12].

It was proved that the paired-domination problem is *NP*-complete, even restricted to split graphs and bipartite graphs [5]. Hence, researchers tried to find efficient algorithms for paired-domination problem on special class of graphs due to the complexity of paired-domination problem in graphs. Among these special graphs, chordal graph is the most important one. A graph is *chordal* if every cycle of length at least four has a chord. Chordal graphs are raised in the theory of perfect graphs, see [9]. It contains trees, split graphs, interval graphs, block graphs, directed path graphs, undirected path graphs, strongly chordal graphs ... as subclasses. The subclasses of chordal graphs are of most interesting in the study of many graphs optimization problem [2]. Up to now, for paired-domination problem, linear-time algorithms have been found in trees [18], interval graphs [5], and block graphs [5]. Polynomial time algorithms for paired-domination problem in circular-arc graphs and permutation graphs are given in [6] and [7], respectively.

Strongly chordal graph is a subclass of chordal graphs and includes directed path graphs, interval graphs, block graphs and trees as subclasses. In this paper, we will study the paired-domination problem in strongly chordal graphs. The remainder of the paper is organized as follows. In Section 2, we give a simple introduction on strongly chordal graphs. In Section 3, a linear-time algorithm will be given for paired-domination problem in strongly chordal graphs, which generalizes the algorithms in [5,18].

2. Strongly chordal graphs

Strongly chordal graphs are introduced by several researchers [4,8,13] in the study of domination problem. It is a very important subclass of chordal graphs as most of the variations of the domination problem are solvable in strongly chordal graphs [3,4,14,15,21]. On the other hand, it includes directed path graphs, interval graphs, block graphs, trees as subclasses. There are many equivalent ways to define strongly chordal graphs. In this paper, we adopt the notation from Farber's paper [8]. A vertex u is *simple* if $N_G[u] = \{u_1, u_2, \ldots, u_k\}$, where $u = u_1$, satisfies $N_G[u_i] \subseteq N_G[u_j]$ for $1 \leqslant i \leqslant j \leqslant k$. A graph $G = (V, E)$ is *strongly chordal* if every induced subgraph has a simple vertex. It is also the case that $G = (V, E)$ is strongly chordal if and only if it admits a vertex ordering $v_1, v_2, \ldots, v_n$ of V such that the following condition holds.

SEO:

$$\text{If } i \leqslant j \leqslant k \text{ and } v_j, v_k \in N_i[v_i], \text{ then } N_i[v_j] \subseteq N_i[v_k]$$

where $N_i[v_j] = \{v_k \mid v_k \in N_G[v_j] \text{ and } k \geqslant i\}$. Any vertex ordering satisfying SEO is called a *strong* (*elimination*) *ordering*. Note that v_i is a simple vertex of the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_n\}$.

There are many results to recognize whether the given graph is a strongly chordal graph. In [1,13], authors gave $O(|V|^3)$-time algorithms for testing whether a graph $G = (V, E)$ is a strongly chordal graph. Later, Improvements to an $O(L(\log L)^2)$-time algorithm was given in [16], where $L = |V| + |E|$, to an $O(L(\log L))$-time algorithm in [17], and to an $O(|V|^2)$-time algorithm in [19]. These algorithms also give a vertex ordering satisfying SEO in case the answer is positive.

3. Algorithm

Let $G = (V, E)$ be a strongly chordal graph and $v_1, v_2, \ldots, v_n$ be the strong (elimination) ordering of G. For each i, define $M_i = \{k \mid v_i v_k \in E \textit{ and } k > i\} \cup \{i\}$ and $F(v_i) = v_j$ such that j is the maximum integer in M_i. In order to determine which vertex may be selected, we need define two labels on each vertex as follows:

$$D(v_i) = \begin{cases} 0 & \text{if } v_i \text{ is not dominated;} \\ 1 & \text{if } v_i \text{ is dominated.} \end{cases}$$

$$L(v_i) = \begin{cases} 0 & \text{if } v_i \text{ is not selected;} \\ 1 & \text{if } v_i \text{ is selected, but its paired vertex} \\ & \quad \text{has not been determined;} \\ 2 & \text{if } v_i \text{ is selected, and its paired vertex} \\ & \quad \text{has been determined.} \end{cases}$$

Next, we present an algorithm to determine a minimum paired-dominating set in a strongly chordal graph without isolated vertices.

Algorithm MPDS. Find a MPDS in a strongly chordal graph without isolated vertices.
Input. A strongly chordal graph $G = (V, E)$ with a strong (elimination) ordering $v_1, v_2, \ldots, v_n$ $(n \geqslant 2)$. Each vertex v_i has labels $(D(v_i), L(v_i)) = (0, 0)$, and $F(v_i) = v_j$ with $j = \max\{k \mid k \in M_i\}$.
Output. A minimum paired-dominating set *PD* of G.
Method.

```
For i = 1 to n do
   If (D(v_i) = 0 and F(v_i) ≠ v_i) then
      L(F(v_i)) = 1;
      D(u) = 1 for u ∈ N[F(v_i)];
   else if (D(v_i) = 0 and F(v_i) = v_i) then
      L(v_i) = 2;
      L(u) = 2 for some u ∈ N(v_i) such that L(u) = 0;
      D(v_i) = 1;
   else if (L(v_i) = 1) then
      Let C = N(v_i) ∩ {v_j | L(v_j) = 1 and j > i};
      If (C = ∅ and L(F(v_i)) = 0) then
         L(v_i) = 2;
         L(F(v_i)) = 2;
         D(u) = 1 for all u ∈ N[F(v_i)];
      else if (C = ∅) then
         L(v_i) = 2;
         L(u) = 2 for some u ∈ N(v_i) such that L(u) = 0;
      else
         Take v_j such that j = min{k | v_k ∈ C};
         L(v_i) = 2;
         L(v_j) = 2;
      endif
   endif
endfor
Output PD = {v | L(v) = 2}
end
```

Next, we will prove that algorithm MPDS can find a minimum paired-dominating set of G. By the observation of algorithm MPDS, we easily know

Lemma 1. *When v_i is the considering vertex for $1 \leqslant i \leqslant n$, we have*

(i) $L(v_j) = 0$ *or* 2 *for all* $1 \leqslant j \leqslant i-1$;
(ii) $D(v_j) = 1$ *for all* $1 \leqslant j \leqslant i-1$.

For each $1 \leqslant i \leqslant n$, let $S_i = \{v \mid L(v) > 0\}$ and $S'_i = \{v \mid L(v) = 2\}$ when v_i has just been considered. In special, $S_0 = S'_0 = \emptyset$. It is obvious that $S_n = S'_n = PD$ is a paired-dominating set of G. Therefore, to prove the correctness of the algorithm MPDS, it is sufficient to prove that there is a minimum paired-dominating set S of G such that $PD \subseteq S$. We will prove the following more powerful lemma.

Lemma 2. *For each $0 \leqslant i \leqslant n$, there is a MPDS S of G such that $S_i \subseteq S$ and S'_i is perfect with regard to a perfect matching M in $G[S]$.*

Proof. We use induction on i. Clearly, when $i = 0$, there is a MPDS S of G such that $S_0 \subseteq S$ and S'_0 is perfect with regard to any perfect matching in $G[S]$. Suppose it also holds for any integer less than i, i.e., there is a MPDS S^* of G such that $S_{i-1} \subseteq S^*$ and S'_{i-1} is perfect with regard to a perfect matching M^* in $G[S^*]$. If $S_i = S_{i-1}$ and $S'_i = S'_{i-1}$, then let $S = S^*$ and $M = M^*$, the lemma follows. So, according to algorithm MPDS, we discuss the following cases.

Case 1. $D(v_i) = 0$ and $F(v_i) \neq v_i$.

It is obvious that $S_i = S_{i-1} \cup \{F(v_i)\}$ and $S'_i = S'_{i-1}$. If $F(v_i) \in S^*$, then let $S = S^*$ and $M = M^*$, the lemma follows. Thus we assume that $F(v_i) \notin S^*$. Let $v_k \in S^*$ be the vertex dominating v_i with the smallest index. Let v_l be the paired vertex of v_k with regard to M^*. As $D(v_i) = 0$, we know that $v_k \notin S_{i-1}$. If $k = i$, the choice of v_k implies that $l > i$. Clearly, $N_i[v_i] \subseteq N_i[F(v_i)]$ and $v_l F(v_i) \in E$. Let $S = S^* - \{v_i\} \cup \{F(v_i)\}$ and $M = M^* - \{v_i v_l\} \cup \{v_l F(v_i)\}$, then S and M are the desired sets by Lemma 1(ii). So in the following discussion we assume $k \neq i$.

Suppose $k < i$. If $l = i$, then let $S = S^* - \{v_k\} \cup \{F(v_i)\}$ and $M = M^* - \{v_k v_i\} \cup \{v_i F(v_i)\}$, by $N_i[v_k] \subseteq N_i[v_i]$ and Lemma 1(ii), we have S and M are the desired sets. If $l < i$, then $v_l \notin S_{i-1}$ by Lemma 1(i) and $(N_i[v_k] \cup N_i[v_l]) \subseteq N_i[v_i]$ wherever $l < k$ or $k < l < i$. On the other hand, $v_i \notin S^*$, for otherwise $S^* - \{v_k, v_l\}$ is a smaller PDS of G. Let $S = S^* - \{v_l, v_k\} \cup \{v_i, F(v_i)\}$ and $M = M^* - \{v_l v_k\} \cup \{v_i F(v_i)\}$, then S and M are the desired sets by Lemma 1(ii). If $l > i$, then $v_l v_i \in E$ and $v_l F(v_i) \in E$. Let $S = S^* - \{v_k\} \cup \{F(v_i)\}$ and $M = M^* - \{v_k v_l\} \cup \{v_l F(v_i)\}$. By $N_i[v_k] \subseteq N_i[v_l]$ and Lemma 1(ii), S and M are the desired sets.

Suppose $k > i$. In this case, $v_k F(v_i) \in E$. If $l < i$, then $v_l \notin S_{i-1}$ by Lemma 1(i). Let $S = S^* - \{v_l\} \cup \{F(v_i)\}$ and $M = M^* - \{v_l v_k\} \cup \{v_k F(v_i)\}$. By the fact $N_i[v_l] \subseteq N_i[v_k]$ and Lemma 1(ii), S and M are the desired sets. If $l > i$, then $v_l F(v_i) \in E$. Let $S = S^* - \{v_k\} \cup \{F(v_i)\}$ and $M = M^* - \{v_k v_l\} \cup \{v_l F(v_i)\}$, by the fact $N_i[v_k] \subseteq N_i[F(v_i)]$ and Lemma 1(ii), S and M are the desired sets.

Case 2. $D(v_i) = 0$ and $F(v_i) = v_i$.

In this case, $S_i = S_{i-1} \cup \{v_i, u\}$ and $S'_i = S'_{i-1} \cup \{v_i, u\}$, where $u \in N(v_i)$ with $L(u) = 0$. Note that u always can be found as $D(v_i) = 0$. Since $D(v_i) = 0$, thus $v_i \notin S_{i-1}$. Clearly $N_i[v_i] = \{v_i\}$ due to $F(v_i) = v_i$. In addition, for $v_j \in N[v_i]$, $N_i[v_j] = \{v_i\}$. Let $v_k \in S^*$ be a vertex adjacent to v_i and v_l be its paired vertex with regard to M^*. Then $v_k \in N[v_i]$ and $k \leqslant i$ and $l \leqslant i$. Thus $v_k, v_l \notin S_{i-1}$ by Lemma 1(i) and the fact $N_i[v_l] \subseteq \{v_i\}$. Let $S = S^* - \{v_k, v_l\} \cup \{v_i, u\}$ and $M = M^* - \{v_k v_l\} \cup \{v_i u\}$, thus S and M are the desired sets by Lemma 1(ii).

Case 3. $L(v_i) = 1$.

Let $C = N(v_i) \cap \{v_j \mid L(v_j) = 1 \text{ and } j > i\}$. We divide this case according to whether $C = \emptyset$ or not.

Subcase 3.1. $C = \emptyset$ and $L(F(v_i)) = 0$.

It is obvious that $S_i = S_{i-1} \cup \{F(v_i)\}$ and $S'_i = S'_{i-1} \cup \{F(v_i), v_i\}$. $v_i \in S^*$ due to $L(v_i) = 1$. Let v_l be the paired vertex of v_i with regard to M^*. Then $v_l \notin S_{i-1}$ due to $C = \emptyset$ and Lemma 1(i). If $v_l = F(v_i)$, let $S = S^*$ and $M = M^*$, then S and M are the desired sets. So we assume that $v_l \neq F(v_i)$.

Suppose $F(v_i) \notin S^*$. If $l < i$, then $N_i[v_l] \subseteq N_i[v_i]$. If $l > i$, then $v_l F(v_i) \in E$ and $N_i[v_l] \subseteq N_i[F(v_i)]$. In any case, let $S = S^* - \{v_l\} \cup \{F(v_i)\}$ and $M = M^* - \{v_l v_i\} \cup \{v_i F(v_i)\}$. It is easy to check that S and M are the desired sets.

Suppose $F(v_i) \in S^*$. Let v_k be the paired vertex of $F(v_i)$ with regard to M^*. Similar to the former case, either $N_i[v_l] \subseteq N_i[v_i]$ or $N_i[v_l] \subseteq N_i[F(v_i)]$ holds. If $N[v_k] \subseteq S^*$, then $S^* - \{v_l, v_k\}$ is a smaller PDS of G. Thus there is a neighbor w of v_k such that $w \notin S^*$. Let $S = S^* - \{v_l\} \cup \{w\}$ and $M = M^* - \{v_i v_l, F(v_i) v_k\} \cup \{v_i F(v_i), v_k w\}$. It is easy to check that S and M are the desired sets.

Subcase 3.2. $C = \emptyset$ and $L(F(v_i)) \neq 0$.

It is obvious that $S_i = S_{i-1} \cup \{u\}$ and $S'_i = S'_{i-1} \cup \{u, v_i\}$, where $u \in N(v_i)$ such that $L(u) = 0$. Note that u can always be found as $L(v_i) = 1$. Clearly, $v_i \in S^*$ since $L(v_i) = 1$.

Suppose $L(F(v_i)) = 1$. $C = \emptyset$ and $L(F(v_i)) = 1$ implies that $F(v_i) = v_i$. Similar to Case 2, $N_i[v_i] = \{v_i\}$ and $N_i[v_j] = \{v_i\}$ for any $v_j \in N[v_i]$. Let v_k be the paired vertex of v_i with regard to M^*, then $k < i$ and $v_k \notin S_{i-1}$. Let $S = S^* - \{v_k\} \cup \{u\}$ and $M = M^* - \{v_k v_i\} \cup \{v_i u\}$. It is easy to check that S and M are the desired sets.

Suppose $L(F(v_i)) = 2$. Then $F(v_i) \in S'_{i-1}$ and let v_l be the paired vertex of $F(v_i)$ with regard to M^*. Clearly, $l < i$. Let v_k be the paired vertex of v_i with regard to M^*. Due to $C = \emptyset$ and Lemma 1(i), $v_k \notin S_{i-1}$. If $k < i$, then $N_i[v_k] \subseteq N_i[v_i]$. If $k > i$, then $N_i[v_k] \subseteq N_i[F(v_i)]$. Let $S = S^* - \{v_k\} \cup \{u\}$ and $M = M^* - \{v_k v_i\} \cup \{u v_i\}$. It is easy to check that S and M are the desired sets.

Subcase 3.3. $C \neq \emptyset$.

Take v_j such that $j = \min\{k \mid v_k \in C\}$. It is obvious that $S_i = S_{i-1}$ and $S'_i = S'_{i-1} \cup \{v_i, v_j\}$. $v_i, v_j \in S^*$ as $L(v_i) = L(v_j) = 1$. Let v_k (v_l, respectively) be the paired vertex of v_i (v_j, respectively) with regard to M^*. If $k = j$ or $l = i$, then let $S = S^*$ and $M = M^*$, and hence the lemma follows. Thus we assume that $k \neq j$ and $l \neq i$.

Suppose $k < i$ and $l < i$. Then $v_k, v_l \notin S_{i-1}$ by Lemma 1(i). Since $N_i[v_k] \subseteq N_i[v_i]$ and $N_i[v_l] \subseteq N_i[v_j]$, thus $S^* - \{v_k, v_l\}$ is a smaller PDS of G, a contradiction.

Suppose $k < i$ and $l > i$. In this case, $v_k \notin S_{i-1}$ by Lemma 1(i) and there is a neighbor w of v_l such that $w \notin S^*$. For otherwise, $S^* - \{v_k, v_l\}$ is a smaller PDS of G as $N_i[v_k] \subseteq N_i[v_i]$. Let $S = S^* - \{v_k\} \cup \{w\}$ and $M = M^* - \{v_k v_i, v_l v_j\} \cup \{v_i v_j, v_l w\}$. It is easy to check that S and M are the desired sets. Suppose $k > i$ and $l < i$. The argument is similar. Thus it is omitted.

Suppose $k > i$ and $l > i$. If $i < k < j$, then $v_k \notin S_{i-1}$ by the choice of v_j. In addition, $N_i[v_k] \subseteq N_i[v_j]$. It is clear that there is a neighbor w of v_l such that $w \notin S^*$. Let $S = S^* - \{v_k\} \cup \{w\}$ and $M = M^* - \{v_i v_k, v_j v_l\} \cup \{v_i v_j, v_l w\}$. It is easy to check that S and M are the desired sets. If $k > j$, then $v_l v_k \in E$. Let $S = S^*$ and $M = M^* - \{v_i v_k, v_j v_l\} \cup \{v_i v_j, v_k v_l\}$. Then S and M are the desired sets. $\square$

Theorem 1. *Algorithm MPDS can produce a minimum paired-dominating set of a strongly chordal graph G without isolated vertices in $O(m+n)$ time, where $m = |E(G)|$ and $n = |V(G)|$, if the strong (elimination) ordering of G is given in advance.*

Proof. By Lemma 2, we obtain that there is a MPDS S of G such that $S_n \subseteq S$ and S'_n is perfect with regard to a perfect matching M in $G[S]$. On the other hand, $PD = S_n = S'_n$ and PD is a PDS of G. Thus PD is a MPDS of G. In addition, every vertex and edge in G are used in a constant number. Thus algorithm MPDS can finish in $O(m+n)$ time, where $m = |E(G)|$ and $n = |V(G)|$. $\square$

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