2 Limits and Their Properties

2.1 A Preview of Calculus

Objectives

- Understand what calculus is and how it compares with precalculus.
- Understand that the tangent line problem is basic to calculus.
- Understand that the area problem is also basic to calculus.

What Is Calculus?

What Is Calculus?

Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus.

Precalculus mathematics is more static, whereas calculus is more dynamic.

What Is Calculus?

Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.

- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.

What Is Calculus?

- The curvature of a circle is constant and can be analyzed with precalculus mathematics. To analyze the variable curvature of a general curve, you need calculus.

- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

What Is Calculus?

Each of these situations involves the same general strategy—the reformulation of precalculus mathematics through the use of a limit process.

So, one way to answer the question “What is calculus?” is to say that calculus is a “limit machine” that involves three stages.

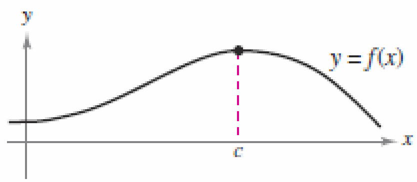
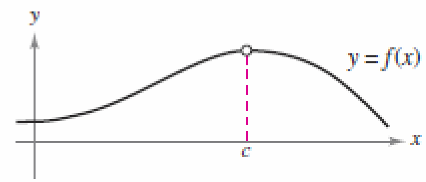
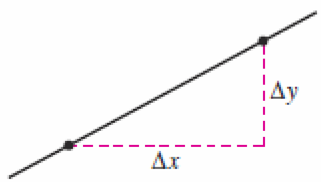
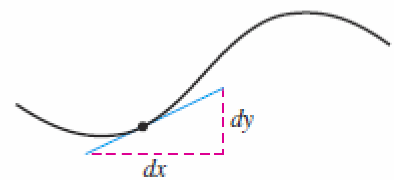
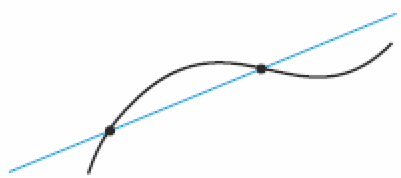
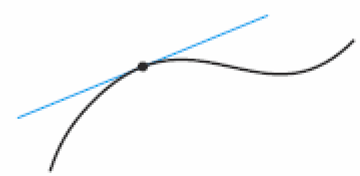
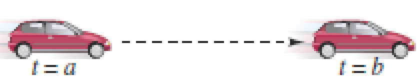
The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle.

The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.

Precalculus mathematics $\Rightarrow$ Limit process $\Rightarrow$ Calculus

What Is Calculus?

Here are some familiar precalculus concepts coupled with their calculus counterparts:

Without Calculus	With Differential Calculus
Value of $f(x)$ when $x = c$ 	Limit of $f(x)$ as x approaches c 
Slope of a line 	Slope of a curve 
Secant line to a curve 	Tangent line to a curve 
Average rate of change between $t = a$ and $t = b$ 	Instantaneous rate of change at $t = c$ 

What Is Calculus?

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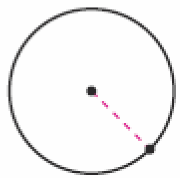
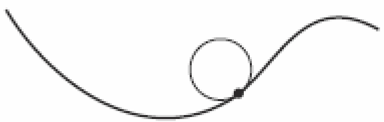
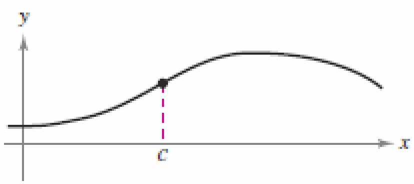
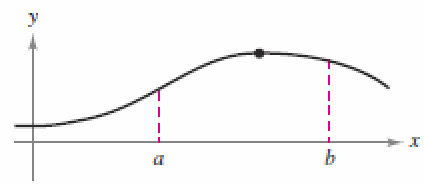
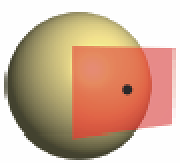
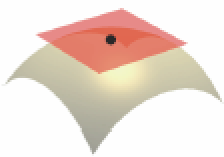
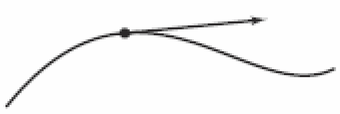
Without Calculus		With Differential Calculus	
Curvature of a circle		Curvature of a curve	
Height of a curve when $x = c$		Maximum height of a curve on an interval	
Tangent plane to a sphere		Tangent plane to a surface	
Direction of motion along a line		Direction of motion along a curve	

What Is Calculus?

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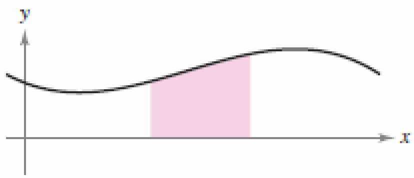
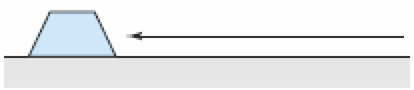
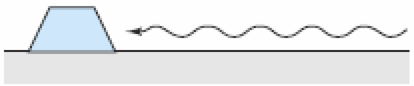
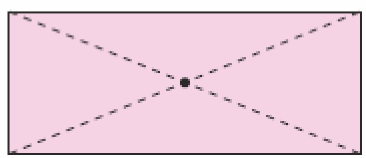
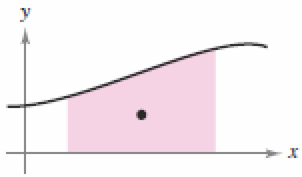
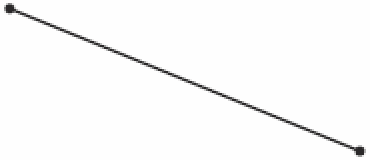
Without Calculus		With Integral Calculus	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	

What Is Calculus?

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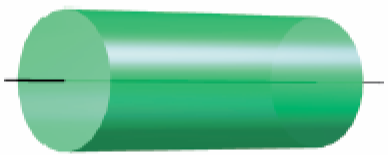
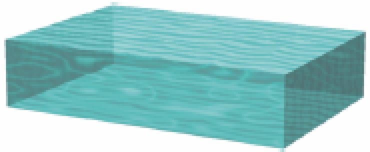
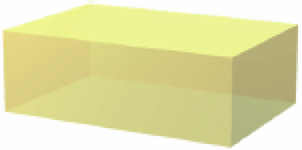
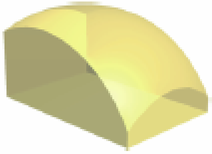
Without Calculus		With Integral Calculus	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdots + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdots = S$

The Tangent Line Problem

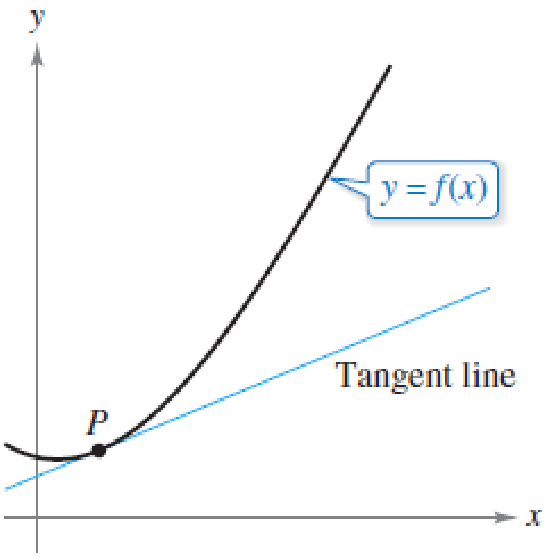
The Tangent Line Problem

The notion of a limit is fundamental to the study of calculus.

The following brief descriptions of two classic problems in calculus—*the tangent line problem* and *the area problem*—should give you some idea of the way limits are used in calculus.

The Tangent Line Problem

In the tangent line problem, you are given a function *f* and a point *P* on its graph and are asked to find an equation of the tangent line to the graph at point *P*, as shown in Figure 2.1.



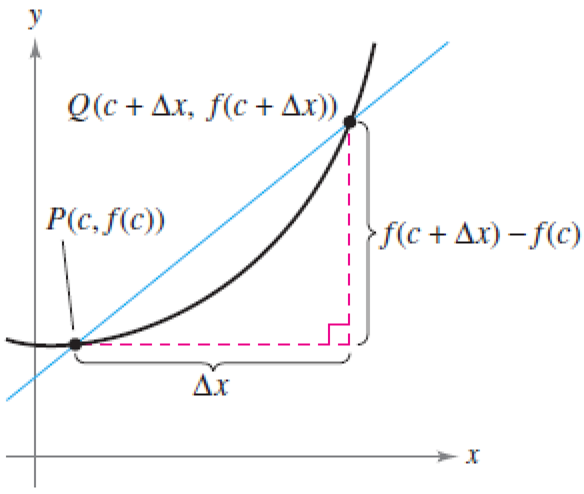
The tangent line to the graph of *f* at *P*

Figure 2.1

The Tangent Line Problem

Except for cases involving a vertical tangent line, the problem of finding the **tangent line** at a point *P* is equivalent to finding the *slope* of the tangent line at *P*.

You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 2.2(a). Such a line is called a **secant line**.



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

Figure 2.2(a)

The Tangent Line Problem

If $P(c, f(c))$ is the point of tangency and

$$Q(c + \Delta x, f(c + \Delta x))$$

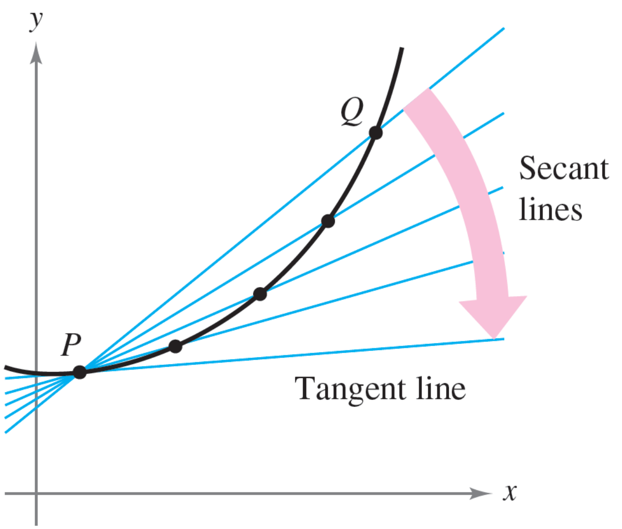
is a second point on the graph of *f*, then the slope of the secant line through these two points can be found using precalculus and is

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

The Tangent Line Problem

As point Q approaches point P, the slopes of the secant lines approach the slope of the tangent line, as shown in Figure 2.2(b).

When such a "limiting position" exists the slope of the tangent line is said to be the **limit** of the slopes of the secant lines.



(b) As Q approaches P, the secant lines approach the tangent line.

Figure 2.2(b)

The Area Problem

The Area Problem

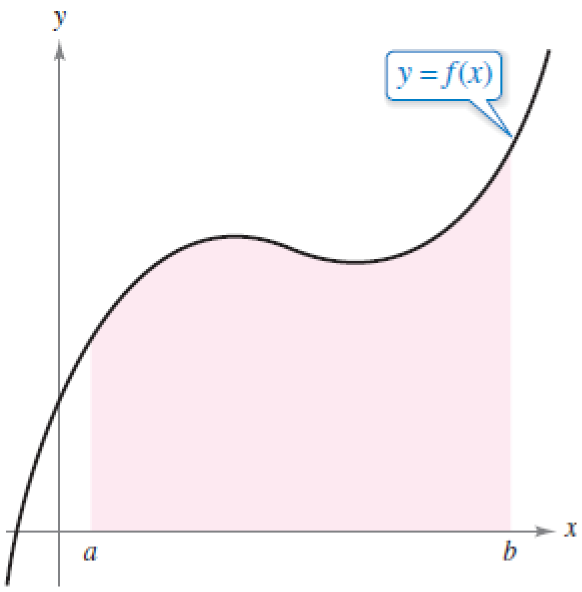
A second classic problem in calculus is finding the area of a plane region that is bounded by the graphs of functions.

This problem can also be solved with a limit process.

In this case, the limit process is applied to the area of a rectangle to find the area of a general region.

The Area Problem

As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 2.3.

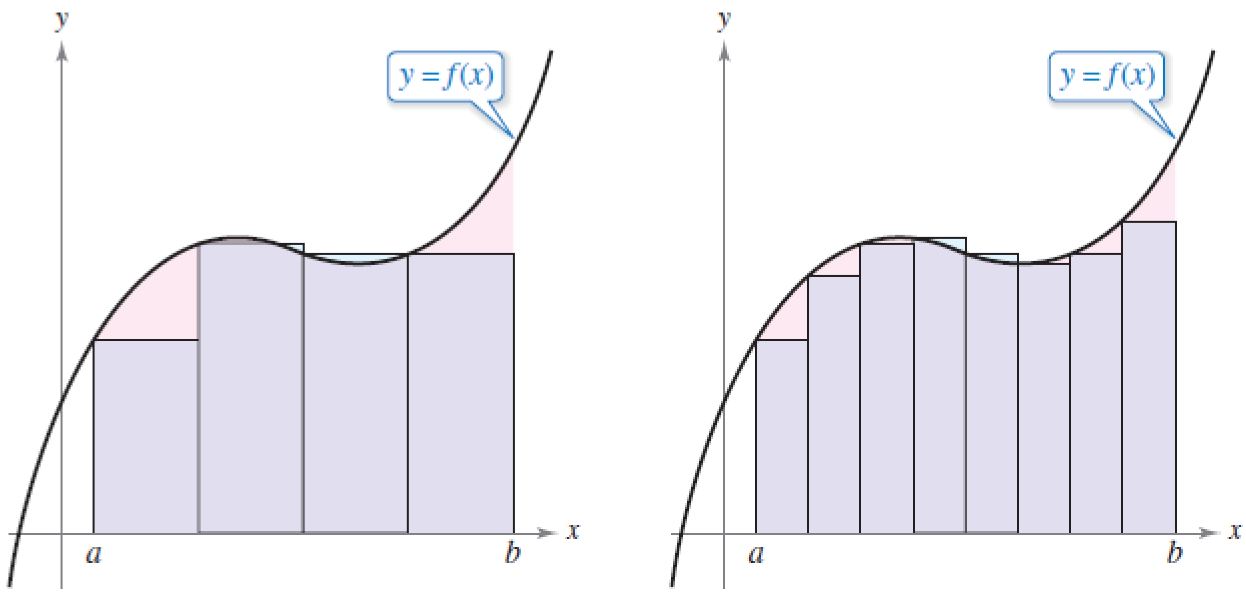


Area under a curve

Figure 2.3

The Area Problem

You can approximate the area of the region with several rectangular regions, as shown in Figure 2.4.



Approximation using four rectangles

Approximation using eight rectangles

Figure 2.4

The Area Problem

As you increase the number of rectangles, the approximation tends to become better and better because the amount of area missed by the rectangles decreases.

Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound.