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CSCI-UA 310-001 PS6

4. Bob should choose the guessed number (or a number out of a set of numbers) by enumerating all the possibilities of 2 dice summing to ℓ and picking the number appearing in the most possibilities. For example, ℓ= 4, possibilities: { {1, 3} {2, 2} {3, 1} }. The optimal choice would be 1 or 3 because they appear in 2 of three possibilities while 2 only appears in 1. Pr[Z = ℓ] = number of possibilities where sum = ℓ / total possibilities = 36
   Case where ℓ = 2: Bob should choose 1, and the probability that he win, p = 1. E[W | Z = 2] = 2 because he should double down, as p > q, where q = the probability he will lose. We find E[W | Z = l] by computing (1)p + (-1)q to find expected earnings. If this result is positive, this will indicated that Bob is expected to earn money, so he should double down to double his expected earnings. Pr[Z = 2] = 1/36
   Case ℓ= 3: Bob should choose 1 or 2, and p = 1. E[W | Z = 3] = 2 because he should double down, as p > q. Pr[Z = 3] = 2/36
   Case ℓ= 4: Bob should choose 1 or 3, and p = 2/3. E[W | Z = 4] = 1/3 → 2/3 because he should double down, as p > q. Pr[Z = 4] = 3/36
   Case ℓ= 5: Bob should choose 1, 2, 3, or 4, and p = 2/4 = 1/2. E[W | Z = 5] = 0. It does not matter if Bob doubles down because p = q. Pr[Z = 5] = 4/36
   Case ℓ= 6: Bob should choose 1, 2, 4, or 5, and p = 2/5. E[W | Z = 6] = -1/5. Bob should not double down, as he is expected to lose because p < q. Pr[Z = 6] = 5/36
   Case ℓ= 7: Bob should choose 1, 2, 3, 4, 5, or 6, and p = 2/6 = 1/3. E[W | Z = 7] = -1/3. Bob should not double down, as he is expected to lose because p < q. Pr[Z = 7] = 6/36
   Case ℓ= 8: Bob should choose 2, 3, 5, or 6, and p = 2/5. E[W | Z = 8] = -1/5. Bob should not double down, as he is expected to lose because p < q. Pr[Z = 8] = 5/36
   Case ℓ= 9: Bob should choose 3, 4, 5, or 6, and p = 2/4 = 1/2. E[W | Z = 9] = 0. It does not matter if Bob doubles down because p = q. Pr[Z = 9] = 4/36
   Case ℓ= 10: Bob should choose 4 or 6, and p = 2/3. E[W | Z = 10] = 1/3 → 2/3 because he should double down, as p > q. Pr[Z = 10] = 3/36
   Case ℓ= 11: Bob should choose 5 or 6, and p = 1. E[W | Z = 11] = 2 because he should double down, as p > q. Pr[Z = 11] = 2/36
   Case ℓ= 12: Bob should choose 6, and p = 1. E[W | Z = 12] = 2 because he should double down, as p > q. Pr[Z = 12] = 1/36
   By the law of total expectation

   $$\mathrm{E}[W] = \sum_{\ell=2}^{12} \mathrm{E}[W \mid Z = \ell] \Pr[Z = \ell].$$

   Summing, we get
   2(1/36)+2(2/36)+(2/3)(3/36)+0(4/36)+(-1/5)(5/36)+(-1/3)(6/36)+(-1/5)(5/36)+0(4/36)+(2/3)(3/36)+2(2/36)+2(1/36) = 1/3