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8.

a. Consider the case where we only spin once, t = 1, and we will generalize the probability of $Pr[M \geq j]$. If j = 1, clearly $Pr[M \geq j] = 1$. If j = 2, $Pr[M \geq j] = 9/10$. We can see that this probability can be generalized to (n - j + 1) / n, for any j when t = 1. When t = 2, we consider the case when *both* spins have M as greater or equal to than j → $Pr[M \geq j] \cap Pr[M \geq j] = Pr[M \geq j] * Pr[M \geq j] \rightarrow \frac{(n-j+1)}{n} * \frac{(n-j+1)}{n}$, given that each spin is independent. More generally, for any t and j = 1, …, n, $Pr[M \geq j] = \frac{(n-j+1)^t}{n^t}$ .

b. Given the tail-sum formula, $E[M] = \sum_{j=1}^{n} Pr[M \geq j] \rightarrow \sum_{j=1}^{n} \frac{(n-j+1)^t}{n^t} \rightarrow \frac{1}{n^t} \sum_{j=1}^{n} (n-j+1)^t$ → To simplify the sum, we can compare the values of j and n - j + 1. When j = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, n - j + 1 = 10, 9, 8, 7, 6, 5, 4, 3, 2, 1. Therefore, we can rearrange the sum to be $\sum_{i=1}^{n} i^{\,t}$, as the set of numbers in j perfectly spans/covers the set of numbers represented by n - j + 1. Therefore, $E[M] = \frac{1}{n^t} \sum_{i=1}^{n} i^{\,t}$.

c. $E[M] = \frac{1}{n^t} \sum_{i=1}^{n} i^{\,t} \rightarrow \frac{1}{n^t} \sum_{i=1}^{n} i^{\,t} \rightarrow$ Approximating as an integral $\rightarrow \frac{1}{n^t} \int_{1}^{n} x^{\,t}\, dx + M \rightarrow$

$\frac{1}{n^t} \left( \left[ \frac{x^{t+1}}{t+1} \right]_1^n + max(1^t, n^t) \right) \rightarrow$

$\frac{1}{n^t} \left( \frac{n^{t+1}}{t+1} - \frac{1}{t+1} + \frac{n^t}{n^t} \right) \rightarrow \frac{n^{t+1}-1}{n^t(t+1)} + \frac{1}{1} \rightarrow \frac{n * n^t}{n^t(t+1)} - \frac{1}{n^t(t+1)} + \frac{1}{1} \rightarrow$

$\frac{n}{(t+1)} - \frac{1}{n^t(t+1)}$ , we can see that for all sufficiently large n, the term c, where c = $\frac{1}{n^t(t+1)}$ + 1, bounds the sum such that $\left| E[M] - \frac{n}{(t+1)} \right| \leq c$. Additionally, we can say that the constant c is significantly less than $\frac{n}{(t+1)}$ (especially if n will never be smaller than 1), so we can generalize the formula to be $E[M] = \frac{n}{(t+1)} + O(1)$.