Computational Simplifications

$$I_n(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)^2\right]$$

$\boxed{1}$

$$E\left[\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right] = 0$$

Proof:

$$E\left[\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right] = \int\left(\frac{\partial}{\partial\theta}\ln f(\vec{x};\theta)\right) f(\vec{x};\theta)\, dx$$

$$E\left[\frac{\partial}{\partial \theta}\ln f(\vec{X};\theta)\right] = \int \left(\frac{\partial}{\partial \theta}\ln f(\vec{x};\theta)\right) f(\vec{x};\theta)\, dx$$

$$= \int \left(\frac{\frac{\partial}{\partial \theta} f(\vec{x};\theta)}{f(\vec{x};\theta)}\right) f(\vec{x};\theta)\, d\vec{x}$$

$$= \int \frac{\partial}{\partial \theta} f(\vec{x};\theta)\, d\vec{x} \quad = \frac{\partial}{\partial \theta}\int f(\vec{x};\theta)\, d\vec{x}$$

$$= \frac{\partial}{\partial \theta} 1 \quad = 0$$

Computational Simplifications

$$I_n(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)^2\right]$$

$\boxed{2}$

$$I_n(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(\vec{X};\theta)\right]$$

Proof: From 1, we have that

$$\int\left(\frac{\partial}{\partial\theta}\ln f(\vec{x};\theta)\right) f(\vec{x};\theta)\,d\vec{x} = 0$$

Take the derivative on both sides WRT θ.

$$\int \left(\frac{\partial}{\partial \theta} \ln f(\vec{x};\theta) \right) f(\vec{x};\theta)\, d\vec{x} = 0$$

$$\int \left(\frac{\partial}{\partial \theta} \ln f(\vec{x};\theta) \right) \frac{\partial}{\partial \theta} f(\vec{x};\theta)\, d\vec{x}$$

$\frac{\partial}{\partial \theta} \ln f(\vec{x};\theta)$

$\cdot f(\vec{x};\theta)$

$$+ \int \left(\frac{\partial^2}{\partial \theta} \ln f(\vec{x};\theta) \right) f(\vec{x};\theta)\, d\vec{x} = 0$$

$$\underbrace{E\left[\left(\frac{\partial}{\partial \theta} \ln f(\vec{X};\theta) \right)^2 \right]}_{I_n(\theta)} + E\left[\frac{\partial^2}{\partial \theta^2} \ln f(\vec{X};\theta) \right] = 0$$

Computational Simplifications

$$I_n(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)^2\right]$$

$\boxed{3}$ **If $X_1, X_2, \ldots, X_n$ are iid, then:**

$$I_n(\theta) = n\,I_1(\theta)$$

Proof:

$$I_n(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right)^2\right]$$

$$= E\left[\left(\frac{\partial}{\partial\theta}\ln \prod_{i=1}^{n} f(X_i;\theta)\right)^2\right]$$

$$= E\left[\left(\frac{\partial}{\partial \theta}\sum_{i=1}^{n} \ln f(X_i;\theta)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} \frac{\partial}{\partial \theta} \ln f(X_i;\theta)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} \frac{\partial}{\partial \theta} \ln f(X_i;\theta)\right)\left(\sum_{j=1}^{n} \frac{\partial}{\partial \theta} \ln f(X_j;\theta)\right)\right]$$

Now pull the “j sum” into the “i sum”.

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial}{\partial\boldsymbol{\theta}}\ln f(X_i;\boldsymbol{\theta})\right)\left(\frac{\partial}{\partial\boldsymbol{\theta}}f(X_j;\boldsymbol{\theta})\right)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\underbrace{\left(\frac{\partial}{\partial\boldsymbol{\theta}}\ln f(X_i;\boldsymbol{\theta})\right)}\underbrace{\left(\frac{\partial}{\partial\boldsymbol{\theta}}f(X_j;\boldsymbol{\theta})\right)}\right]$$

independent if $j \neq i$

In this case, the expectation factors and both are zero by

$$\boxed{1} \qquad E\left[\frac{\partial}{\partial\theta}\ln f(\vec{X};\theta)\right] = 0$$

The surviving terms are

$$= \sum_{i=1}^{n} E\left[\left(\frac{\partial}{\partial \theta} \ln f(X_i;\theta)\right)\left(\frac{\partial}{\partial \theta} \ln f(X_i;\theta)\right)\right]$$

$$= \sum_{i=1}^{n} E\left[\left(\frac{\partial}{\partial \theta} \ln f(X_i;\theta)\right)^2\right]$$

Because the X_i are iid, these expectations are all the same!

$$= n \cdot E\left[\left(\frac{\partial}{\partial \theta} \ln f(X_i;\theta)\right)^2\right] = n\, I_1(\theta)$$

Example:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \exp(\text{rate} = \lambda)$$

Find the Cramér-Rao lower bound of the variance of all unbiased estimators of λ.

$$\tau(\lambda) = \lambda \qquad \mathrm{Var}[T] \geq \frac{[\tau'(\lambda)]^2}{I_n(\lambda)}$$

The Fisher Information:

$$I_n(\lambda) := E\left[\left(\frac{\partial}{\partial p}\ln f(\vec{X};\lambda)\right)^2\right]$$

pdf: $f(x;\lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$

joint pdf:

$$f(\vec{x};\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x} \prod_{i=1}^{n} I_{(0,\infty)}(x_i)$$

Take the log:

$$\ln f(\vec{x};\lambda) = n\ln\lambda - \lambda\sum_{i=1}^{n} x_i$$

Take the derivative:

$$\frac{\partial}{\partial\lambda}\ln f(\vec{x};\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

Put the random variables in, square, and take the expectation.

$$I_n(\lambda) := E\left[\left(\frac{\partial}{\partial\lambda}\ln f(\vec{X};\lambda)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} X_i - \frac{n}{\lambda}\right)^2\right]$$

Let $Y = \sum_{i=1}^{n} X_i$. We know that $Y \sim \Gamma(n,\lambda)$.

We know that $E[Y]=n/\lambda$.

$$I_n(\lambda) := E\left[\left(Y - \frac{n}{\lambda}\right)^2\right] = Var[Y] = \frac{n}{\lambda^2}$$

$$\mathrm{CRLB}_{\lambda} \geq \frac{[\;1\;]^2}{I_n(\lambda)} = \frac{1}{n/\lambda^2} = \frac{\lambda^2}{n}$$

Alternatively:

$$\frac{\partial}{\partial \lambda} \ln f(\vec{x};\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

$$\Downarrow$$

$$\frac{\partial^2}{\partial \lambda^2} \ln f(\vec{x};\lambda) = -\frac{n}{\lambda^2}$$

$$\frac{\partial^2}{\partial \lambda^2} \ln f(\vec{x};\lambda) = -\frac{n}{\lambda^2}$$

$$I_n(\lambda) = -E\left[\frac{\partial^2}{\partial \lambda^2} \ln f(\vec{X};\lambda)\right]$$

$$= -E\left[-\frac{n}{\lambda^2}\right] = \frac{n}{\lambda^2}$$

which is the same thing we got with the single derivative expression.

Either of these methods could have been done with a single X_1 as opposed to the vector $\vec{X}$.

Example:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \exp(\text{rate} = \boldsymbol{\lambda})$$

Find the Cramér-Rao lower bound of the variance of all unbiased estimators of $\exp[-\lambda]$.

We already did all of the work!

$$\tau(\lambda)=e^{-\lambda} \qquad \mathrm{Var}[T] \geq \frac{[\tau'(\lambda)]^2}{I_n(\lambda)}$$

The CRLB on the variance of all unbiased estimators of $e^{-\lambda}$ is

$$\frac{[-\lambda e^{-\lambda}]^2}{n/\lambda^2} = \boxed{\frac{\lambda^4 e^{-2\lambda}}{n}}$$