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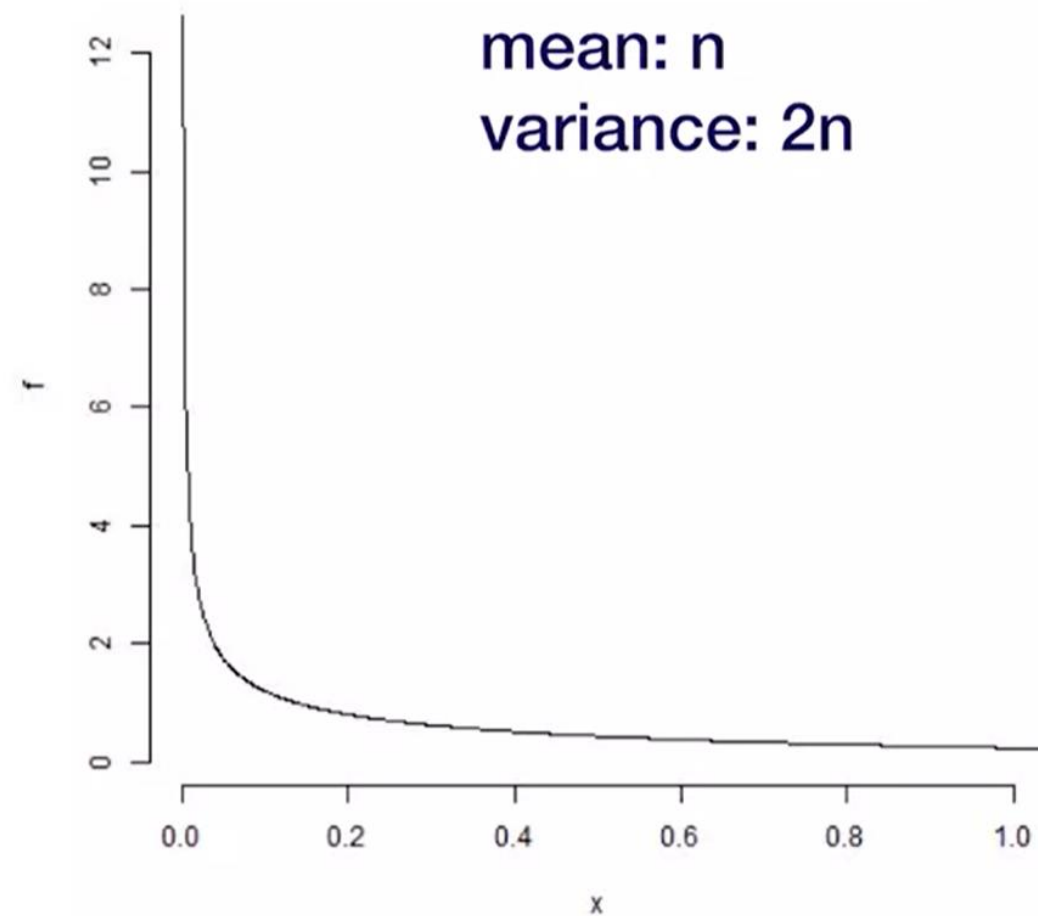
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“A chi-squared distribution with  $n$  **degrees of freedom**.”

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## Things About $\chi^2(n)$

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So,

$$X \sim \chi^2(n) = \Gamma(n/2, 1/2)$$

$$\Rightarrow M_X(t) = \left( \frac{1/2}{1/2 - t} \right)^{n/2}$$

## Things About $\chi^2(n)$

Suppose that  $X_1, X_2, \dots, X_k$  are independent random variables with  $X_i \sim \chi^2(n_i)$ .

$$\text{Let } Y = \sum_{i=1}^k X_i$$

$$M_Y(t) = E[e^{tY}] = E \left[ e^{t \sum_{i=1}^k X_i} \right]$$

$$= E \left[ \prod_{i=1}^k e^{tX_i} \right] \stackrel{\text{indep}}{=} \prod_{i=1}^k E \left[ e^{tX_i} \right]$$

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$$M_Y(t) = \prod_{i=1}^k E[e^{tX_i}] = \prod_{i=1}^k M_{X_i}(t)$$

$$= \prod_{i=1}^k \left( \frac{1/2}{1/2 - t} \right)^{n_i/2} = \left( \frac{1/2}{1/2 - t} \right)^{\sum_{i=1}^k n_i/2}$$

## A Chi-Squared Property

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$$M_Y(t) = \prod_{i=1}^k \left( \frac{1/2}{1/2 - t} \right)^{\sum_{i=1}^n n_i/2}$$

$$\Rightarrow Y \sim \chi^2(n_1 + n_2 + \dots + n_k)$$

**The sum of chi-squareds is chi-squared!**

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Let  $Y = X^2$ .

Then  $Y \sim \chi^2(1)$ .

We could show this using moment generating functions, as described in Module 1, Lesson 8.

Recall from Module 1, Lesson 4, the pdf of the transformation  $Y = g(x)$  is

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

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There is a bivariate version of this to go from  $X_1$  and  $X_2$  to  $Y_1 = g_1(X_1, X_2)$  and  $Y_2 = g_2(X_1, X_2)$ .

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)) \cdot |J|$$

where

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$



## The t-Distribution

Let  $Z \sim N(0, 1)$  and  $W \sim \chi^2(n)$  be independent random variables..

Define

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Do the “Jacobian transformation” with

$$\begin{aligned} X_1 &= Z & Y_1 &= \frac{X_1}{\sqrt{X_2/n}} \\ X_2 &= W \end{aligned}$$

$$Y_2 = g_2(X_1, X_2)$$

## The t-Distribution

Can show that  $T$  has pdf

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\sqrt{n\pi}} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

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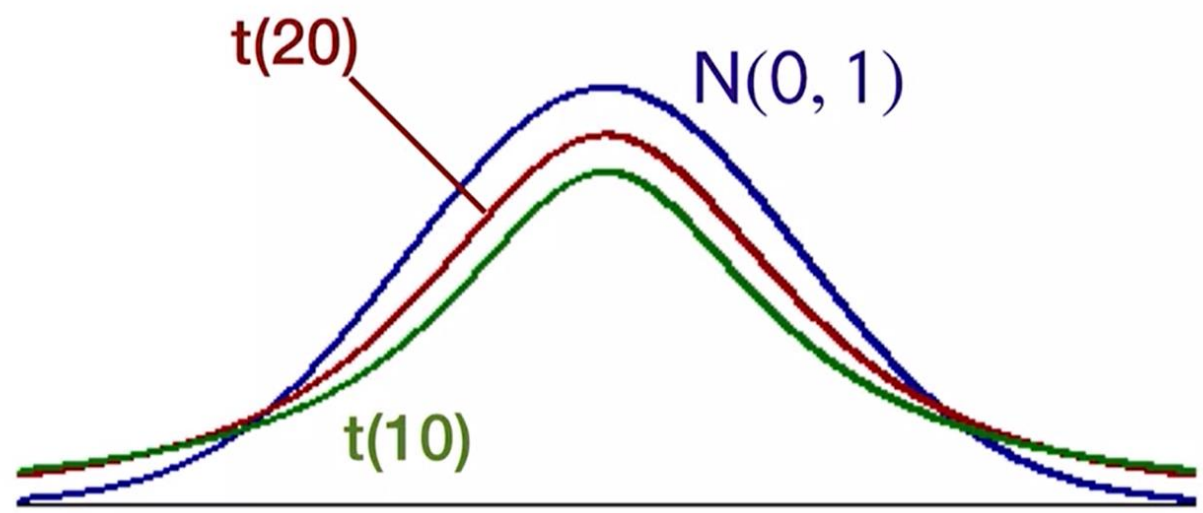
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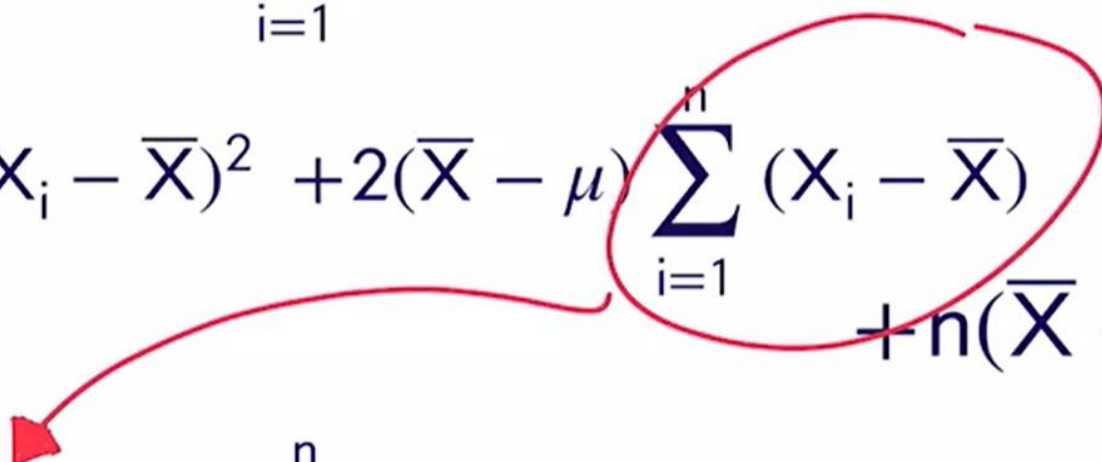
$$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$$

$$\begin{aligned} \sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n (X_i - \bar{X} + \bar{X} - \mu)^2 \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + 2(\bar{X} - \mu) \sum_{i=1}^n (X_i - \bar{X}) \\ &\quad + n(\bar{X} - \mu)^2 \end{aligned}$$



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$$\begin{aligned} \sum_{i=1}^n (X_i - \bar{X}) &= \sum_{i=1}^n X_i - n\bar{X} \\ &= \sum_{i=1}^n X_i - n \frac{1}{n} \sum_{i=1}^n X_i = 0 \end{aligned}$$

Why now?

$$\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

Divide through by  $\sigma^2$ .

$$\sum_{i=1}^n \left( \frac{X_i - \mu}{\sigma} \right)^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2$$

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$$X_i \sim N(\mu, \sigma^2)$$

$$\sum_{i=1}^n \left( \frac{X_i - \mu}{\sigma} \right)^2$$



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$$Y_1 = Y_2 + Y_3$$

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- $Y_1 = \sum_{i=1}^n \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$

- $Y_2 = \frac{(n-1)S^2}{\sigma^2} \sim ?$

- $Y_3 = \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \sim \chi^2(1)$

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independent

???

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$$M_{Y_1}(t) = M_{Y_2+Y_3}(t)$$

$$\stackrel{\text{indep}}{=} M_{Y_2}(t) \cdot M_{Y_3}(t)$$

$$\Rightarrow M_{Y_2}(t) = \frac{M_{Y_1}(t)}{M_{Y_3}(t)}$$

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$$\Rightarrow Y_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

for  $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$ .

