Dual Channel Structure Optimization

from a managerial perspective of view

Xiang Liu

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Privious studies substantiate the value of market competition in supply chain management. Openning a direct channel has the advantage of reduced double marginalization even if no sales occur at this channel, but it has introduced fairness concern which can be hurtful to manufacturer in the long run. This paper extends previous direct channel design and examines how indirect dual channel structure affects a manufacturer's wholesale pricing problem in an environment where consumers can choose to buy from two competing retailers. The model suggests that an indirect dual channel design can achieve no worse than direct channel design. Furthermore, the model suggests that when the manufacturer chooses to sell through both channels while offering them different wholesale price can be both profitable and sustainable. This is achieved through increased price competition between the two channels such that both channels are able to make sales. To avoid a dual channel evolving to a single channel, we suggest manufacturer to make products of equal quality but label them differently and sell low to the "less-competitive" retailer so as to prevent inter-channel transactions.

1. Introduction

Channel design is one of the most important aspects of marketing decisions. There are several different kinds of channel structures in current marketing systems, such as the traditional retail channel, the direct channel through the Internet, and the dual-channel. In the traditional retail channel, manufacturers sell the products to retailers who then sell the products to end consumers. This had been a common channel design until the commencement of the Internet (Chiang et al., 2003). More companies decide to sell directly to their consumers directly through online store. However, products are usually made in assembly line and it's uneconomical to have them in stock. Some manufacturers choose to combine direct channel with indirect channel (e.g., Dell's online store together with retail store). This can help the manufacturer improve overall profitability by reducing the degree of inefficient price double marginalization. However, manufacturer's acting both as a supplier and a competitor may trigger the fairness concern of the retailers (Li et al., 2016; Zhu et al., 2017). For example, Compaq, a personal computer maker, used to own 25% market share in early 2000, blindly followed Dell's direct channel strategy. It was counterattacked by its retailers and finally acquired by Hewlett-Packard. Aiura et al. (2007) analyzes wholesale pricing and retail pricing when a monopolistic manufacturer sells its product to a high street retailer and an online electronic retailer. He suggests future studies on the impact of wholesale price discrimination on social welfares. In our study, we focus on how to prevent retailers from pushing up prices without direct competition. That is, we are exploring the effect of horizontally aligned indirect channel structure on solving this problem.

We have some interesting observations. First, indirect channel design can achieve the same effect as direct channel design if manufacturer is offering both retailers with the same wholesale price. When the added retailer is not too worse off than the existing one, dual channel structure can bring additional profit to the manufacturer even when the added channel doesn't make sales. If the added channel is as good as the existing one within a certain range, both the manufacturer and the exisiting retailer are better off with the establishment of the added less-famous retailer. Second, two horizontally aligned retailers with different wholesale price can further profit manufacturer and both retailers are making profits if consumer's preference on the added retailer reaches a certain threshold.

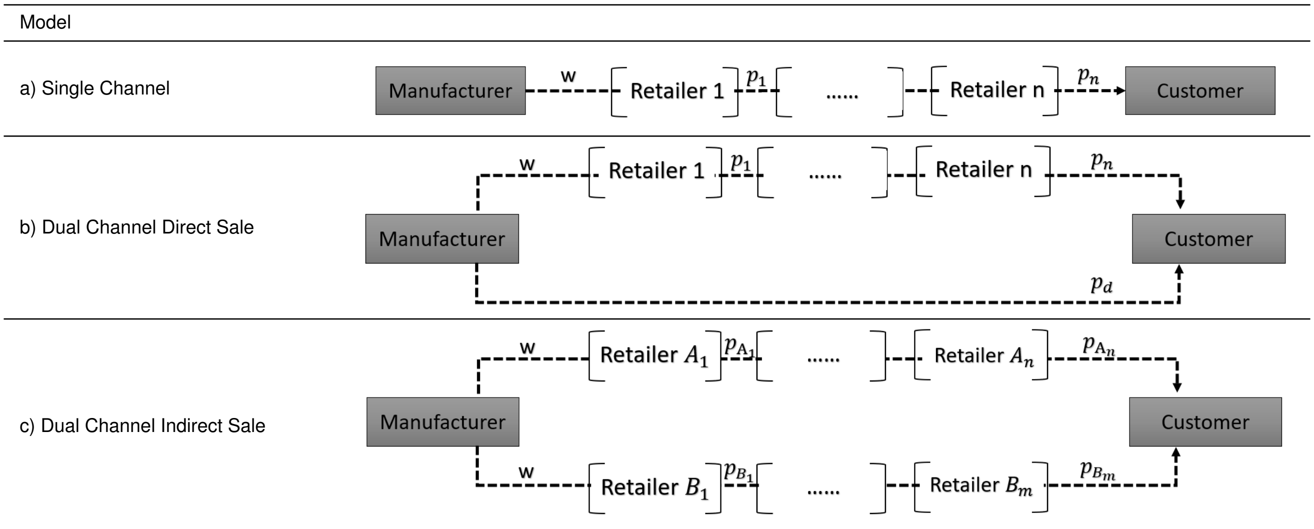
2. Consumer's Response

For analytic simplicity, it is assumed that consumer's valuation of the products is ranged between 0 and 1 and uniformly distributed. If retailers are all vertically aligned (i.e., single channel setting), no pricing competition occurs between retailers. Since each retailer wants to extract their profits and in doing so pushing up the retail price that is significantly higher than its wholesale price. consumer's final decision depends only on the last retailer on the chain. If it is finally priced at p, according to our uniform valuation assumption, the demand is $D(p) = 1 - p$. Now suppose we add an inferior channel to the single channel structure, it can be either a direct channel by selling directly to consumers or an indirect channel by introducing horizontally aligned retailers and thereby gaining from their pricing competitions. We assume consumer's acceptance of this added channel (either direct or indirect) is $\theta, 0 \leq \theta \leq 1$, which can be seen as a discounted factor. If a product is valued as v by a consumer, a depreciated value of θv is perceived by this consumer if this product is sold on the inferior channel.

Now suppose it's priced as p_1 on the stronger channel and p_2 on the weaker/inferior channel. Consumer decide if this product is worth its price first: the marginal consumer whose valuations $v = \frac{p_2}{\theta}$ is indifferent to buying the products from the weaker channel and likewise whose valuation $v = p_1$ is indifferent to buying the products from the stronger channel. If both are worth its price, consumer needs to determine which price is more competitive by comparing their net valuation surplus (i.e., $\theta v - p_2$ with $v - p_1$). If $v - p_1 \geq \theta v - p_2 \Rightarrow v \geq \frac{p_1 - p_2}{1-\theta}$, then the stronger channel is preferred to the weaker one and customer with valuation expectation higher than $\frac{p_1-p_2}{1-\theta}$ would choose to buy from the stronger channel.

The demand function is deduced from Chiang et al's dual channel customer's demand function. He investigated customer's choice between a retail channel and an inferior direct channel under the uniform valuation assumption. In this paper, we extend the model to a situation where there is two horizontally aligned retailers with one being weaker to the other. Essentially, our paper share with Chiang et al. (2003) the same

Figure 1. Different Channel Structure Discussed in This Paper



For simplicity reason, we assume there is only one retailer at each channel and no intermediate transactions between retailers

Table 1. Notation Used in This Paper

Notation	Definition
θ	Channel acceptance or substitution, to what degree do we discount value of product sold on the weaker channel
c	Manufacturering cost of a product
w_1,w_2	Manufacturer's wholesale price offered to retailers. $w_1 = w_2$ if both stronger and weaker retailers receive the same wholesale price
p_1,p_2	Retailer's retail price observed by consumers
π_1, π_2, π_m	Profit earned by stronger retailer, weaker retailer and manufacturer
$D_{R_1}(p_1, p_2)$	Stronger retailer's sales as a function of retailing prices, see Eq.1
$D_{R_2}(p_1, p_2)$	Weaker retailer's sales as a function of retailing prices, see Eq.2

here we assume manufacturing cost is the same for both channels

idea of channel substitution or channel acceptance. Therefore, it's the same choice model faced by customer: should I buy from the stronger channel or the weaker one.

$$D_{R_1}(p_1, p_2) = \begin{cases} 1 - p_1 & \text{if } \frac{p_2}{\theta} \leq p_1 \Rightarrow 0 \leq \theta \leq \frac{p_2}{p_1} \\ 1 - \frac{p_1 - p_2}{1-\theta} & \text{if } \frac{p_2}{p_1} \leq \theta \leq 1 - (p_1 - p_2) \\ 0 & \text{if } 1 - (p_1 - p_2) \leq \theta \leq 1 \end{cases} \quad [1]$$

$$D_{R_2}(p_1, p_2) = \begin{cases} 0 & \text{if } 0 \leq \theta \leq \frac{p_2}{p_1} \\ \frac{p_1 - p_2}{1-\theta} - \frac{p_2}{\theta} & \text{if } \frac{p_2}{p_1} \leq \theta \leq 1 - (p_1 - p_2) \\ 1 - \frac{p_2}{\theta} & \text{if } 1 - (p_1 - p_2) \leq \theta \leq 1 \end{cases} \quad [2]$$

3. Impact of Indirect Dual Channel Design

In the decentralized dual-channel supply chain where the manufacture and the retailer are independent decision-makers who seek to maximize their individual profit, we assume that the manufacture acts as the Stackelberg leader who determines the wholesale price first, and the retailer acts as the Stackelberg follower who determines the retail price in response to the manufacturer's pricing decision. We first analyze the second-stage retailer's decisions followed by the first-stage manufacturer's decisions. Since manufacturer has control over only wholesale price to maximize its profit, we explore the impact of different wholesale pricing strategy made by manufacturer on its profit and the overall supply chain's long run sustainability.

Same Wholesale Price . If both retailers receive the same wholesale price, manufacturer cares only about total sales since the marginal profit is indifferent to products sold in either channels. The manufacturer's optimization problem is outlined as

$$\pi_m = \{(D_{R_1} + D_{R_2})(p_1, p_2)\} \cdot (w - c) \quad [3]$$

The main result of this section is that implementation of indirect dual channel given that the added channel is competitive enough achieves the same effect as what direct channel does with two additional benefits: 1) avoid retailer's fairness concern; 2) save manufacturers operational fees to maintain the direct channel. Manufacturers favor all sales occuring on the stronger channel. The weaker, added retailer won't be able to make sales unless it's identically popular to the stronger, existing retailer. Therefore, this dual channel structure may not work in the long run unless manufacturers acuiquire and merge the weaker retailers, which turns this structure back to the dual channel direct design.

Stage Two Retailer's Response. Instead of there being a chain of retailers at each channel, we focus on the case where there is only one retailer and competition occurs between two channels (i.e., two retailers are horizontally aligned). Suppose *Retailer 1* receives better reputation than *Retailer 2* does. Given the wholesale price w, the marginal profit of *Retailer 1* is $p_1 - w$ and its profit is determined by

$$\pi_{R_1} = (p_1 - w) \cdot D_{R_1}(p_1, p_2) \quad [4]$$

Likewise, the marginal profit of *Retailer 2* is $p_2 - w$ and its profit is

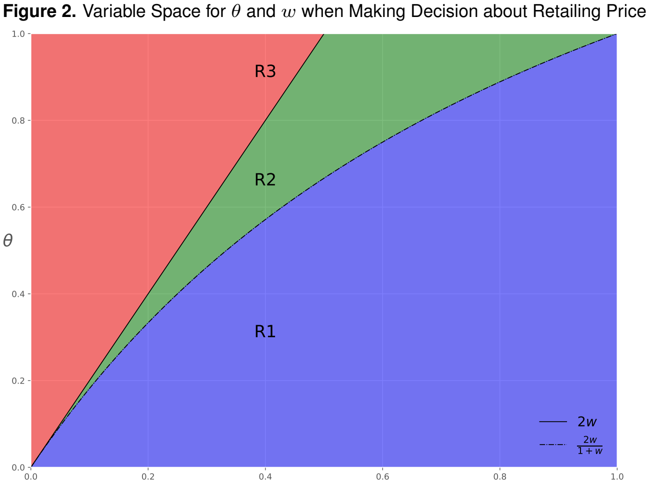
$$\pi_{R_2} = (p_2 - w) \cdot D_{R_2}(p_1, p_2) \qquad [5]$$

The retailers have control over their retail price and they must take into account their respective piecewise-linear demand function (see Eq.1 and Eq.2).

Proposition 1. *Retailer 1 can fully monopolize the market when the added channel is relatively weak (i.e., $\theta \leq 2w$). Specifically, its decision model is to set $p_1 = \frac{w}{\theta}$ when $\theta \geq \frac{2w}{1+w}$ and $p_1 = \frac{1+w}{2}$ otherwise.*

When channel acceptance is relatively weak, we have zero demand for the weaker channel. It follows that introducing *Retailer 2* won't have any influence on *Retailer 1*'s decision making about retail price. *Retailer 1* can still monopolize the market while pursueing highest profit with or without the presence of *Retailer 2*. As the added channel gets more fame, *Retailer 2* would lower its margin up to zero (i.e., $p_2 = w$) to get order and market share. To maintain monopoly, *Retailer 1* has to beat the lowest retail price *Retailer 2* can set. To maximize its profit, *Retailer 1* has to weigh the gains from monopoly agaisnt the loss from reduced retail price. It turns out for *Retailer 1* it's always more benefitial to monopolize the market if its competitor is not strong enough.

Figure 2. Variable Space for θ and w when Making Decision about Retailing Price



Region R_1 : $\frac{2w}{1+w} \geq \theta$ No effect of adding Retailer 2
Region R_2 : $2w \geq \theta \geq \frac{2w}{1+w}$ Suffered monopoly of Retailer 1
Region R_3 : $\theta \geq 2w$ Both channel are making sales

Proposition 2. *Retailer 1 would rather give away some market shares to Retailer 2 than cut off its retail price to maintain monopoly when channel acceptance is relatively strong (i.e., $\theta \geq 2w$).*

When its competitor gets relatively strong, it's no longer *Retailer 1*'s interest to maintain monopoly. Instead, *Retailer 1* and *Retailer 2* would both make sales. If *Retailer 1* raises its retail price, more consumers would switch to *Retailer 2* which partially reduces double marginalization. As can be seen in Fig.2, given a wholesale price w, as θ increases, *Retailer 1* starts to give away parts of market share to *Retailer 2* in order to maximize its profits. When the alternative channel is very weak (i.e., $\theta \to 0$), changing the wholesale price w does little effect on Retailer's pricing strategy. However, if the alternative channel is very strong (i.e., $\theta \to 1$), varying the wholesale price can fundamentally influence how two retailers compete.

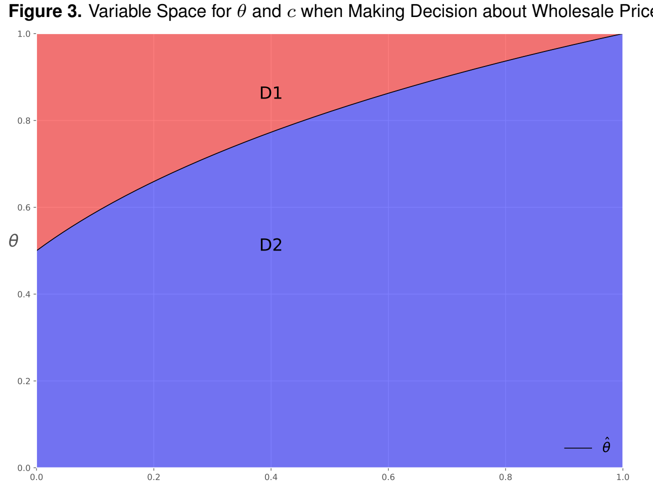
Theorem 1. *(Best Response of Retail Price). Given Manufacturer's decision of wholesale price w and channel acceptance θ, the optimal pair of prices for the Retailer 1 and 2 are:*

$$\begin{cases} p_1^* = \frac{1+w}{2}, p_2^* = w & \text{if } \{w, \theta\} \in R_1 \\ p_1^* = \frac{w}{\theta}, p_2^* = w & \text{if } \{w, \theta\} \in R_2 \\ p_1^* = \frac{-2\theta+3w+2}{4-\theta}, p_2^* = \frac{\theta\left(1-\theta+w+\frac{2w}{\theta}\right)}{4-\theta} & \text{if } \{w, \theta\} \in R_3 \end{cases} \qquad [6]$$

Stage One Manufacturer's Response. Anticipating the retailer's choices in different $\{\theta, w\}$ space, the manufacturer's problem is to maximize its total profits by choosing the wholesale price w (see Eq.3). There are three regions to be examined, as seen in Fig.2. Manufacturer has to measure the gain of increased sales with the loss of marginal profits if it wants to set wholesale price low. It turns out manufacturer will set its wholesale price w such that no sales occured in *Retailer 2*. Compared to Chiang's (2003) paper in which direct channel is built merely to control the independent retailer's pricing, an independent weaker retailing channel does the same work but need not incur additional maintenance cost.

As can be seen in Fig.3, manufacturer considers openning up additional horizontally aligned retail channel only if this added channel is competitive enough. We also observe from Fig.4 that manufacturer drops its wholesale price exactly when it decides to open up another retail channel. To what degree the wholesale price gets cut off depends on how threatening the added channel is. Reduction of wholesale price seems to compensate for the competitiveness of the added weaker channel.

Figure 3. Variable Space for θ and c when Making Decision about Wholesale Price



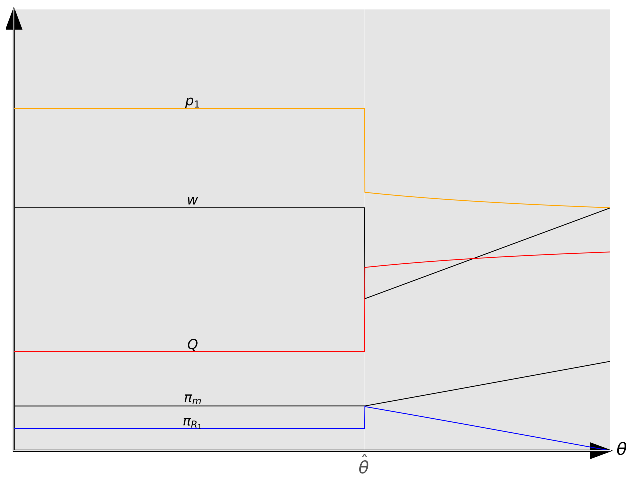
Region D_1 : $\theta \geq \hat{\theta}$ Switch to dual channel
Region D_2 : $\theta \geq \hat{\theta}$ Maintain a single channel

If channel acceptance reaches beyond the cannibalistic threshold within a certain range, manufacturer's adopting dual channel strategy actually benefits the monopolist retailer as well. Both manufacturer and *Retailer 1*'s utility are better off than what they are under single channel structure, i.e., when $\theta \leq \hat{\theta}$ (see Fig.4). As mentioned, wholesale reduction compensates for channel weakness and if channel is strong enough, the degree to which wholesale price drops may not offset the pressure of retail price reduction pushed by this newly added channel.

Theorem 2. *(Best Response of Wholesale Price). Given manufacturing cost c and channel acceptance θ, the optimal wholesale price for Manufacturer is:*

$$\begin{cases} w^* = \frac{c+\theta}{2} & \text{if } \{c,\theta\} \in D_1 \\ w^* = \frac{c+1}{2} & \text{if } \{c,\theta\} \in D_2 \end{cases} \qquad [7]$$

Figure 4. Retail and Wholesale Prices with Change of Channel Acceptance θ



Different Wholesale Price . This dual channel indirect seems to work well except we forget the added weak channel does not work for manufacturer. It may start trying to gain market share and earn low margins. However in the long run, it can withdraw from market competition if manufacturer's pricing strategy is unfavorable to it. We think manufacturer always benefits from retailers' competition and double marginalization is completely reduced if we have two equally competitive horizontally aligned retailers (Iyer, 1998). Our hypothesis is that most sales still occur at *Retailer 1* (stronger channel) however *Retailer 1*'s retail price is further reduced if manufacturer provides *Retailer 2* with lower wholesale price. The manufacturer's optimization problem is outlined as

$$\pi_m = D_{R_1}(p_1, p_2) \cdot (w_1 - c) + D_{R_2}(p_1, p_2) \cdot (w_2 - c) \qquad [8]$$

The main result of this section is that with implementation of wholesale price differentiation, we can switch to dual channel design with less acceptable retailers (i.e., cannibalistic threshold becomes smaller). Also, similar to cannibalistic threshold, we bring up the term "sustainable threshold" beyond which the added weaker channel cannot only benefit manufacturer but make profits on its own. To avoid retailers' buy and sell from its horizontally aligned competitor, which makes this dual channel evolves into a vertically aligned single channel, we offer some managerial advices in the end.

Stage Two Retailer's Response. To be consistent, we still assume *Retailer 1* is the monopolist retailer in single channel design and manufacturer has to make a decision about whether it's beneficial to add *Retailer 2,* another retailer with lower reputation (i.e., $0 \le \theta \le 1$). The only difference here is $w_1 \neq w_2$ to the previous Section. If manufacturer prices w_2 to be higher than w_1, this won't change anything we concluded previously other than increased space of D_2 in Fig.3 which implies weakened threat from the added channel.

Given the wholesale price w_1, w_2, the marginal profit of *Retailer 1* is $p_1 - w_1$ and its profit is determined by

$$\pi_{R_1} = (p_1 - w_1) \cdot D_{R_1}(p_1, p_2) \qquad [9]$$

Likewise, the marginal profit of *Retailer 2* is $p_2 - w$ and its profit is

$$\pi_{R_2} = (p_2 - w_2) \cdot D_{R_2}(p_1, p_2) \qquad [10]$$

The retailers have control over their retail price and they must take into account their respective piecewise-linear demand function (see Eq.1 and Eq.2).

As can be seen in Fig. 5(a), given the wholesale price of weaker channel is $w_2 = 0.3$, higher channel acceptance (i.e., θ) and retailing cost (i.e., w_1) make it less profitable for *Retailer 1* to adopt monopoly. Likewise, for *Retailer 2*, increasing its channel reputation and decreasing its retailing cost (i.e., θ and w_2) are conducive to its monopoly, see Fig. 5(b).

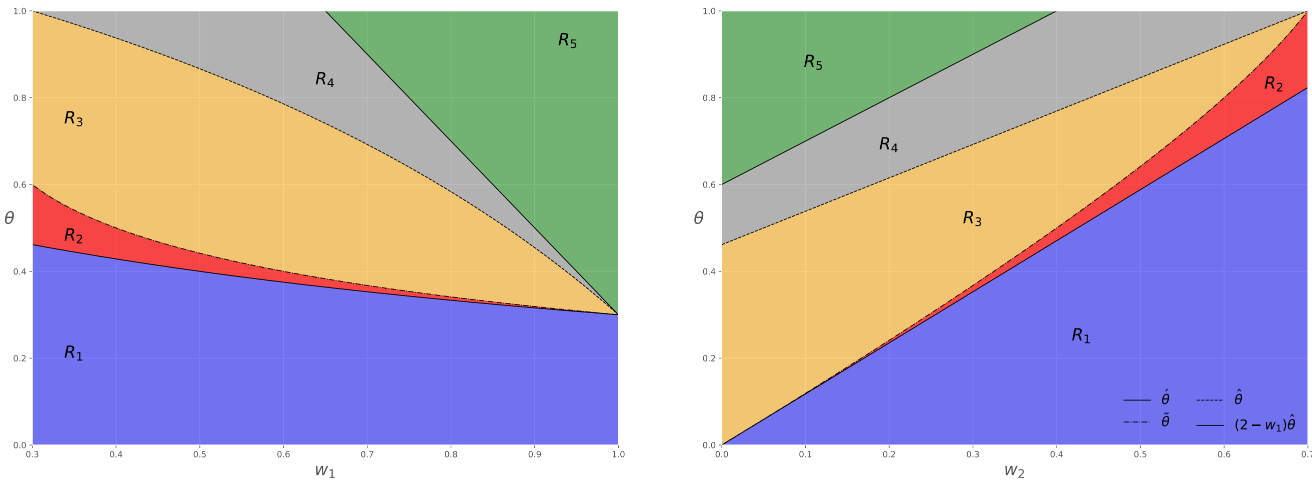
In the same-wholesale-price scenario, increasing wholesale price w incurs less marginal profit for both retailers but it does more harm to *Retailer 2* as it requires higher channel acceptance to force *Retailer 1* to change its pricing strategy. In the different-wholesale-price situation, raising alternative channel's cost (i.e., w_2) indirectly helps to promote *Retailer 1*'s monopoly and hence achieving its highest possible utility. In addition, manufacturer's maneuver to adjust wholesale price have opposite effect on the area of R_2 which is a buffered space where *Retailer 1* can still enjoy monopoly with reduced profit. In the same-wholesale-price scenario, a reduction of wholesale price leaves *Retailer 1* less room to maintain monopoly (i.e., $R_1$and R_2) but it squeezes the area of R_2. But in the different-wholesale-price situation, a reduction in w_1 or a rise in w_2 does the opposite by enlarging the area of R_2.

Theorem 3. *(Best Response of Retail Price). Given Manufacturer's decision of wholesale price $\{w_1, w_2\}$ and channel acceptance θ, the optimal pair of prices for the Retailer 1 and 2 are:*

$$\begin{cases} p_1^* = \frac{1+w_1}{2}, p_2^* = w_2 & \text{if } \{w_1, w_2, \theta\} \in R_1 \\ p_1^* = \frac{w_2}{\theta}, p_2^* = w_2 & \text{if } \{w_1, w_2, \theta\} \in R_2 \\ p_1^* = \frac{-2\theta+2w_1+w_2+2}{4-\theta}, & \text{if } \{w_1, w_2, \theta\} \in R_3 \\ p_2^* = \frac{-\theta^2+\theta w_1+\theta+2w_2}{4-\theta} & \\ p_1^* = w_1, p_2^* = w_1 + \theta - 1 & \text{if } \{w_1, w_2, \theta\} \in R_4 \\ p_1^* = w_1, p_2^* = \frac{\theta+w_2}{2} & \text{if } \{w_1, w_2, \theta\} \in R_5 \end{cases} \qquad [11]$$

Stage One Manufacturer's Response. Now it's manufacturer's play to price w_1, w_2 in order to maximize its profit. Earlier we have shown introducing a weak channel can successfully dampen the "double marginalization" and raise profit if channel acceptance is higher enough. Here, we are trying to achieve higher utility by offering different wholesale price. There are five regions to be examined, as seen in Figure. 6.

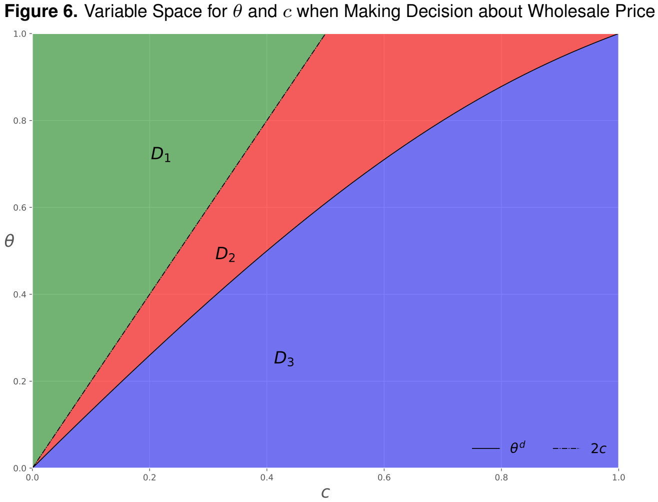
Figure 5. Variable Space for θ and $\{w_1, w_2\}$ when Making Decision about Retailing Price



(a) Decision Space by $\{w_1, \theta \mid 0.3 = w_2 \leq w_1\}$

(b) Decision Space by $\{w_2, \theta \mid w_2 \leq w_1 = 0.7\}$

Figure 6. Variable Space for θ and c when Making Decision about Wholesale Price



Region D_1 : $1 \geq \theta \geq 2c$ Sustainable region where both retailers make profits
Region D_2 : $2c \geq \theta \geq \theta^d$ Switch to dual channel but may not sustain
Region D_3 : $\theta^d \geq \theta \geq 0$ Maintain a single channel

Theorem 4. *(Manufacturer's Pricing Strategy). Cannibalistic threshold is updated to be θ^d defined as*

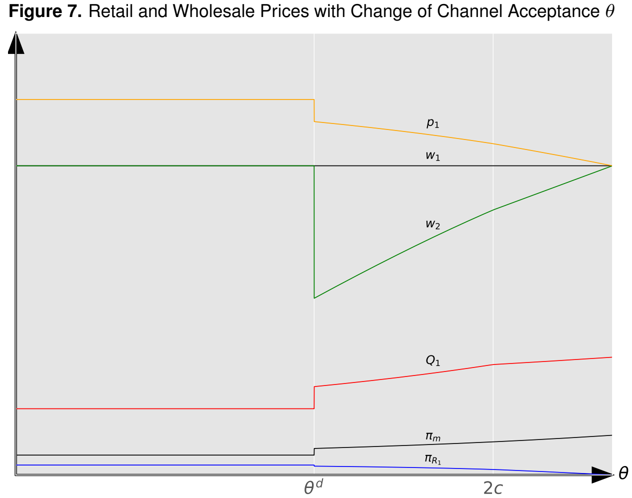
$$\theta^d = \frac{3(c+1) - \sqrt{9(c+1)^2 - 32c}}{4} \qquad [12]$$

such that when θ exceeds the threshold, dual channel is adopted by the manufacturer and a retailer with channel acceptance of θ is added to the supply chain

Theorem 5. *(Manufacturer's Pricing Strategy). There exists a consumer's acceptance of retail channel which is double the normalized score of manufactuering cost such that when θ exceeds the threshold, no monopolist retailer exists.*

Manufacturer avoids vertical competition but brings more horizontal competition between retailers through wholesale price discrimination. Moreover, weaker channel is making profits while helping manufactuer to profit more. However, observed from Fig. 7, wholesale price manufacturer provides to *Retailer 1* is even higher than *Retailer 2*'s retail price. This would certainly trigger buy and sell in between two retailers which would turn this dual channel structure into vertically aligned single channel which manufacturer wants to avoid. One way to avoid this happening is to sell low-quality product to the weaker retailer, but this can hurt customers' perception of brand image (Geller and Wahba, 2010). Another way is to make the same product but label them differently (e.g., "outlets factory"). By doing so, manufacturer can 1). maintain good brand image; 2). enhance channel competitions and make profits; 3). attract more customers with various purchasing power.

Figure 7. Retail and Wholesale Prices with Change of Channel Acceptance θ



Theorem 6. *(Best Response of Wholesale Price). Given manufacturing cost c and channel acceptance θ, the optimal wholesale price for Manufacturer is:*

$$\begin{cases} w_1^* = \frac{c+1}{2}, w_2^* = \frac{\theta+c}{2} & \text{if } \{c, \theta\} \in D_1 \\ w_1^* = \frac{c+1}{2}, w_2^* = \frac{\theta(c-2\theta+3)}{2(2-\theta)} & \text{if } \{c, \theta\} \in D_1 \\ w_1^* = \frac{c+1}{2}, w_2^* = \frac{1+c}{2} & \text{if } \{c, \theta\} \in D_3 \end{cases} \qquad [13]$$

Appendix

Proof of Proposition 1.

Proof. a) $\theta \geq \frac{2w}{1+w} \Rightarrow w \leq \frac{\theta}{2-\theta}$, given a wholesale price w setting in this range, if *Retailer 1* set $p_1 = \frac{w}{\theta}$, how will *Retailer 2* respond to maximize its profit*?

If *Retailer 2* wants to fully occupy the market, $1-(p_1-p_2)$ has to be less or equal than θ and this will lead to $p_2 \leq \frac{w}{\theta} - (1-\theta)$ and $p_2 - w \leq (1-\theta)\left(\frac{w}{\theta}-1\right)$. Since $w \leq \frac{\theta}{2-\theta}$, we have $p_2 - w \leq (1-\theta)\left(\frac{\theta-1}{2-\theta}\right) < 0\ \forall\theta \in [0,1)$. Because a rational retailer won't make $p_2 < w$, *Retailer 2* cannot fully control the market without "self-sacrificing".

Instead if *Retailer 2* just wants to have nonzero sales, p_2 has to be set such that $\frac{p_2}{p_1} = \frac{\theta p_2}{w} \leq \theta \Rightarrow p_2 \leq w$ which is only reasonable when $p_2 = w$ and this is exactly how p_2 is priced under monopoly analysis of *Retailer 1*

b) $\theta < \frac{2w}{1+w} \Rightarrow w > \frac{\theta}{2-\theta}$, given a *wholesale* price w setting in this range, if *Retailer 1* set $p_1 = \frac{w+1}{2}$, how will *Retailer 2* respond to maximize its profit?

If *Retailer 2* wants to fully occupy the market, $1-(p_1-p_2)$ has to be less or equal than θ and this will lead to $p_2 \leq \frac{1+w}{2} - (1-\theta)$ and $p_2 - w \leq \frac{1-w}{2} - (1-\theta)$. Since $w > \frac{\theta}{2-\theta} \Rightarrow \frac{1-w}{2} < \frac{1-\theta}{2-\theta}$, we have $p_2 - w \leq (1-\theta)\left(\frac{1}{2-\theta}-1\right) = \frac{-(1-\theta)^2}{2-\theta} < 0\ \forall\theta \in [0,1)$. Because a rational retailer won't make $p_2 < w$, *Retailer 2* cannot fully control the market without "self-sacrificing".

Instead if *Retailer 2* just wants to have nonzero sales, p_2 has to be set such that $\frac{p_2}{p_1} = \frac{2p_2}{w+1} \leq \theta \Rightarrow p_2 \leq \frac{\theta(1+w)}{2}$, since $\theta < \frac{2w}{1+w}$, we have $p_2 \leq \frac{\theta(1+w)}{2} < w$ which will incur negative marginal profit for *Retailer 2*.

Therefore, by a) and b), we prove that when p_1 is set to be $\frac{w}{\theta}$ when $\theta \geq \frac{2w}{1+w}$ and $\frac{1+w}{2}$ otherwise, *Retailer 1* fully occupies the market. □

Proof of Proposition 2.

Proof. When $\frac{p_2}{p_1} \leq \theta \leq 1-(p_1-p_2)$, both channel make sales.

For *Retailer 1*, the optimization problem is

$$\begin{aligned} \max \quad & \pi_{R_1} = \left(1 - \tfrac{p_1-p_2}{1-\theta}\right)(p_1 - w) \\ \text{s.t.} \quad & \max\left(\tfrac{p_2}{\theta}, w\right) \leq p_1 \leq \min(1+p_2-\theta, 1) \end{aligned}$$

For *Retailer* 2, the optimization problem is

$$\begin{aligned} \max \quad & \pi_{R_2} = \left(\tfrac{p_1-p_2}{1-\theta} - \tfrac{p_2}{\theta}\right)(p_2 - w) \\ \text{s.t.} \quad & \max(w, \theta + p_1 - 1) \leq p_2 \leq \theta p_1 \end{aligned}$$

Since θ is continuous and optimal solution is obtained in either $\frac{p_2}{p_1} < \theta < 1-(p_1-p_2)$, $\theta = \frac{p_2}{p_1}$ or $\theta = 1-(p_1-p_2)$

Case 1: Suppose no constraints are binding, i.e., $\frac{p_2}{p_1} < \theta < 1-(p_1-p_2)$

$$\begin{aligned} p_1^* &= \tfrac{-2\theta+3w+2}{4-\theta} \\ p_2^* &= \tfrac{-\theta^2+\theta w+\theta+2w}{4-\theta} \end{aligned}$$

We can easily verify that $p_1^* + \theta - 1 \leq w$ and $\frac{p_2^*}{\theta} \geq w$. Therefore, we only need to check four constraints, namely, $\frac{p_2^*}{\theta} \leq p_1^* \leq \min(1+p_2^*-\theta, 1)$, $w \leq p_2^* \leq \theta p_1^*$.

a) $0 \leq \theta \leq \frac{2w}{3-w} \Rightarrow 1 + p_2^* - \theta \geq 1$, we examine four constraints specified above and get the following

$$\begin{aligned} p_1^* - \tfrac{p_2^*}{\theta} \geq 0 \quad &\Rightarrow \quad \theta \geq 2w \\ 1 - p_1^* \geq 0 \quad &\Rightarrow \quad \theta \geq 3w - 2 \\ p_2^* - w \geq 0 \quad &\Rightarrow \quad \theta \geq 2w \\ \theta p_1^* - p_2^* \geq 0 \quad &\Rightarrow \quad \theta \geq 2w \end{aligned}$$

Since $2w > \frac{2w}{3-w}$, it is not feasible to satisfy all constraints regardless of choice of θ and w

b) $1 \geq \theta \geq \frac{2w}{3-w} \Rightarrow 1 + p_2^* - \theta \leq 1$, all constraints stay the same except now we have $1 + p_2^* - \theta - p_1^* \geq 0$ instead of $1 - p_1^* \geq 0$

Since $1 + p_2^* - \theta - p_1^* \geq 0$ is always true and $2w > \frac{2w}{3-w}$, all constraints are satisfied as long as $\theta \geq 2w$.

To sum up, when $\theta \geq 2w$, *Retailer 1*'s response is $p_1^* = \frac{-2\theta+3w+2}{4-\theta}$ and utility is $\pi_{R_1}^* = \frac{(1-\theta)(w-2)^2}{(\theta-4)^2}$. *Retailer 2*'s response is $p_2^* = \frac{\theta\left(1-\theta+w+\frac{2w}{\theta}\right)}{4-\theta}$ and the utility is $\pi_{R_2}^* = \frac{(2w-\theta)^2(1-\theta)}{\theta(4-\theta)^2}$.

Case 2: Suppose optimal equilibrium solution is obtained at $\theta = \frac{p_2}{p_1}$

The dual problem of *Retailer i* is the following

$$\begin{aligned} &\min_{\lambda_i \leq 0} \quad L(p_i, \lambda_i) \\ &L(p_i, \lambda_i) \quad = \sup_{p_i} \left\{\pi_{R_i} + \lambda_i (p_2 - \theta p_1)\right\} \end{aligned}$$

We differentiate $L(p_1, \lambda_1)$ and $L(p_2, \lambda_2)$ with respect to p_1 and p_2 and solve system of equations. Substituting expression of p_1 and p_2 in terms of λ_1 and λ_2 and then set the first-order condition with respect to λ_1, λ_2 to zero and solve system of equations. As a result,

$$\lambda_1^* = \frac{1-w}{2(\theta-1)}$$
$$\lambda_2^* = \frac{\theta w + \theta - 2w}{2\theta(1-\theta)}$$

$\lambda_1^* \leq 0\ \forall\theta \in [0,1)$, $\lambda_2^* \leq 0 \Rightarrow \theta \leq \frac{2w}{1+w}$. The corresponding p_1^* and p_2^* are $p_1^* = \frac{1+w}{2}, p_2^* = \frac{\theta(1+w)}{2}$, and rational assumption requires $p_2^* \geq w \Rightarrow \theta \geq \frac{2w}{1+w}$. But if $\theta > \frac{2w}{1+w}$, the dual problem of *Retailer 2* is not feasible and p_2^* would fall outside of $\frac{p_2}{p_1} \leq \theta$ which is not under the scope of current case analysis†. Therefore we have $\theta = \frac{2w}{1+w} \Rightarrow \lambda_2^* = 0$ and $p_1^* = \frac{1+w}{2}, p_2^* = w$ which is already included in Case 1 monopoly analysis.

Case 3: Suppose optimal equilibrium solution is obtained at $\theta = 1-(p_1-p_2)$

The dual problem of *Retailer i* is the following

$$\begin{aligned} &\min_{\lambda_i \leq 0} \quad L(p_i, \lambda_i) \\ &L(p_i, \lambda_i) \quad = \sup_{p_i} \left\{\pi_{R_i} + \lambda_i (\theta - 1 + p_1 - p_2)\right\} \end{aligned}$$

We have to make sure both λ_1 and λ_2 are non-positive so that both *Retailer 1* and *Retailer 2* are happy with the Nash-Equilibrium solutions. Following the same trick, we obtain the optimal solution of λ_1, λ_2:

$$\lambda_1^* = \frac{2-\theta-w}{2(1-\theta)}$$
$$\lambda_2^* = \frac{w-\theta}{2(1-\theta)}$$

Since $\theta + w \leq 2 \Rightarrow \lambda_1^* \geq 0$ is always true, there is no solution to the above optimization problem.

*Although it's a Nash game where Retailer 1 and Retailer 2 make decisions about retailing price simultaneously but here we are claiming that when p_1 is set this way, Retailer 2 has no share of market even he/she is notified of *Retailer 1*'s pricing.

†Since if we require both retailers have nonzero sales, we need to make sure optimal solution of both retailers (i.e., p_1^*, p_2^*) satisfy $\frac{p_2}{p_1} \leq \theta$

There is no need to solve constraint optimization with adding both $\frac{p_2}{p_1} \leq \theta$ and $\theta \leq 1-(p_1-p_2)$ because the unconstrained optimal solution falls outside the constraints and the only possible sub-optimal solutions are obtained at the boundary (i.e., $\{p_1^*, p_2^*\} \mid \frac{p_2}{p_1} = \theta$ or $\theta = 1-(p_1-p_2)$) which are already included in the analysis of above two sub-cases.

Therefore, we have the following pairs of pricing strategies such that both retailers have none-zero sales: $p_1^* = \frac{-2\theta+3w+2}{4-\theta}, p_2 = \frac{\theta-\theta^2+w\theta+2w}{4-\theta}$ when $\theta \geq 2w$. $\square$

Proof of Theorem 2.

Proof. Case 1: $0 \leq \theta \leq \frac{2w}{1+w}$, decision space in R_1 and no threat brought by the weaker channel

In region R_1, $p_1^* = \frac{1+w}{2}$ and $p_2^* = w$. *Retailer 1* has the market all to itself and the total demand is $1-p_1 = \frac{1-w}{2}$. The *Manufacturer*'s optimization problem is to maximize its profit by charging the right wholesale price w. The dual problem is the following:

$$\begin{array}{ll} \min_{\lambda \leq 0} & L(w,\lambda) \\ L(w,\lambda) & = \sup_w \{\pi_m + \lambda(\theta(1+w) - 2w)\} \end{array}$$

L is concave because $\frac{\partial^2 L}{\partial w^2} = -1 < 0$. Hence, the unique maximizer w^* of $L(w,\lambda)$ is determined by equating the gradient to be zero (i.e., $1 + c + \lambda(2\theta - 4) - 2w = 0$). Therefore, $w^* = \frac{1}{2}(c + \lambda(2\theta-4) + 1)$. Substituting in $L(w,\lambda)$, it follows that $\frac{\partial^2 L \mid w=w^*}{\partial \lambda^2} = (2-\theta)^2 > 0$ which suggests a convex function and existence of minimizer $\lambda^* = \{\lambda \mid \frac{\partial L \mid w=w^*}{\partial \lambda} = 0\} = \frac{(2c-\theta(c+3)+2)}{2(\theta-2)^2}$. The condition that dual solution is feasible is $\lambda^* \leq 0 \Rightarrow \theta \geq \theta^1 = \frac{2(c+1)}{c+3}$ and optimal wholesale price is set as $w^* = \frac{1}{2}(c + \lambda(2\theta-4)+1) \mid_{\lambda=\lambda^*} = \frac{\theta}{2-\theta}$ ‡.The highest possible profit *Manufacturer* can obtain is $\pi_m^* = \frac{(1-\theta)\left(\frac{\theta}{(2-\theta)} - c\right)}{(2-\theta)}$. And if $\theta < \theta^1$, which suggests the optimal solution can be obtained within the constraint (i.e., no need to restrict the decision space). Therefore, $w^* = \arg\max_w \pi_m = \frac{1+c}{2}$. Thus we have in decision space R_1:

$$\begin{cases} w^* = \frac{1+c}{2}, \pi_m^* = \frac{(1-c)^2}{8} & \text{if } \theta < \theta^{(1)} = \frac{2(1+c)}{c+3} \\ w^* = \frac{\theta}{2-\theta}, \pi_m^* = \frac{(1-\theta)\left(\frac{\theta}{(2-\theta)}-c\right)}{(2-\theta)} & \text{otherwise} \end{cases}$$

Case 2: $\frac{2w}{1+w} < \theta < 2w$, decision space in R_2 and little threat of the alternative channel

In region R_2, $p_1^* = \frac{w}{\theta}$ and $p_2^* = w$. *Retailer 1* has the market all to itself and the total demand is $1 - p_1 = \frac{\theta - w}{\theta}$ and $\pi_m = (w-c)(1-p_1) = \frac{1}{\theta}(\theta - w)(w-c)$. Since $\frac{\partial^2 \pi_m}{\partial w^2} < 0$, we have best wholesale price $w^* = \{w \mid \frac{\partial \pi_m}{\partial w} = 0\} = \frac{c+\theta}{2}$. To make sure $w^*$is feasible in region R_2, we substitute w^*in the constraint

$$\begin{aligned} \theta > \frac{2w^*}{1+w^*} &\Rightarrow \theta > \theta^{(2)} = \frac{1}{2}\left(\sqrt{c^2+8c} - c\right) \\ \theta < 2w^* &\Rightarrow all\ \theta \end{aligned}$$

If $\theta > \theta^{(2)}$, then $w^* = \frac{c+\theta}{2}$ can be obtained in R_2 and $\pi_m^* = \pi_m \mid w = w^* = \frac{(\theta-c)^2}{4\theta}$. If $\theta \leq \theta^{(2)}$, $\frac{2w}{1+w} < \theta$ is violated and we do constraint optimization by incorporating this restriction to the decision space of w, the dual problem of *Manufacturer* is

$$\begin{array}{ll} \min_{\lambda \leq 0} & L(w,\lambda) \\ L(w,\lambda) & = \sup_w \{\pi_m + \lambda(2w - \theta(1+w))\} \end{array}$$

Because $\theta \leq \theta^{(2)} \Rightarrow \lambda^* = \frac{c\theta - 2c + \theta^2}{\theta(\theta-2)^2} \leq 0$ and optimal solution is achieved in the boundary of R_1 and R_2 (i.e., $w^* = \frac{\theta}{2-\theta}$). Therefore we have in decision space R_2:

$$\begin{cases} w^* = \frac{\theta}{2-\theta}, \pi_m^* = \frac{(1-\theta)\left(\frac{\theta}{(2-\theta)} - c\right)}{(2-\theta)} & \text{if } c \leq \theta < \theta^{(2)} \\ w^* = \frac{c+\theta}{2}, \pi_m^* = \frac{(\theta-c)^2}{4\theta} & \text{if } \theta^{(2)} \leq \theta \end{cases}$$

Case 3: $2w \leq \theta \leq 1$, decision space in R_3 and full competition of channels

In region R_3, $p_1^* = \frac{-2\theta+3w+2}{4-\theta}$ and $p_2^* = \frac{\theta\left(1-\theta+w+\frac{2w}{\theta}\right)}{4-\theta}$. Both retailers have their own market shares. The total demand is $\left(1-v^{12}\right) + \left(v^{12} - v^2\right) = \left(1 - v^2\right) = 1 - \frac{p_2}{\theta} = \frac{\theta(3-w)-2w}{\theta(4-\theta)}$. We start with unconstrained optimization and get $w^* = \arg\max_w \pi_m = \frac{c(\theta+2)+3\theta}{2(\theta+2)}$ and we require that

$$w^* \geq c \Rightarrow \theta \geq \frac{2c}{3-c}$$
$$2w^* - \theta \leq 0 \Rightarrow \theta \in \emptyset$$

Since $2w^* - \theta \leq 0$ cannot be obtained, we have to restrict decision space. Therefore, we turn to a constraint optimization problem. The dual problem of *Manufacturer* is the following:

$$\begin{array}{ll} \min_{\lambda \leq 0} & L(w,\lambda) \\ L(w,\lambda) & = \sup_w \{\pi_m + \lambda(2w - \theta)\} \end{array}$$

Since $\frac{\partial^2 L}{\partial w^2} = \frac{2(\theta+2)}{\theta(\theta-4)} < 0$, $L(w,\lambda)$ is concave of w therefore $w^* = \arg\max_w L(w,\lambda) = \{w \mid \frac{\partial L}{\partial w} = 0\} = \frac{c(\theta+2) - 2\lambda\theta^2 + 8\lambda\theta + 3\theta}{2(\theta+2)}$. Substituting w^* and $\frac{\partial^2 L}{\partial \lambda^2} = \frac{2\theta(4-\theta)}{\theta+2} > 0$ which suggests a convex function of L with respect to λ. Therefore the global minimizer $\lambda^* = \arg\min_\lambda L(\lambda) = \{\lambda \mid \frac{\partial L}{\partial \lambda} = 0\} = \frac{\theta(1-\theta)+c(\theta+2)}{2\theta(\theta-4)} < 0\ \forall \theta \in (0,1]$ since numerator is always positive and denominator is negative. Thus dual optimal can be obtained and $w^* = \frac{\theta}{2}$. The $w^* \geq c \Rightarrow \theta \geq \theta^{(3)} = 2c$ gives the range of θ where this w^*can be obtained. If $\theta < 2c$, w^*cannot be achieved. *Manufacturer* will price w to be c which leads to zero marginal profit for *Manufacturer*. This is definitely not the strategy *Manufacturer* wants to adopt. Therefore we have the optimal response of Manufacturer in decision space $R_3$where both *Retailer 1* and *Retailer 2* have non-negative sales to be:

$$\begin{cases} w^* = c, \pi_m^* = 0 & \text{if } \theta < \theta^{(3)} \\ w^* = \frac{\theta}{2}, \pi_m^* = \frac{\theta - 2c}{4} & \text{otherwise} \end{cases}$$

$\square$

Proof of Theorem 3.

Case 1: $0 \leq \theta \leq \frac{p_2}{p_1}$***, zero demand for the weaker channel.*** For *Retailer 1*, the profit is $\pi_{R_1} = (1-p_1)(p_1 - w_1)$. To guarantee $p_1 \leq \frac{p_2}{\theta}$ is always true, p_1 has to be less than the lower bound of $\frac{p_2}{\theta}$, which is $\frac{w_2}{\theta}$. The dual

$$\begin{array}{ll} \max & \pi_{R_1} = (1-p_1)(p_1 - w_1) \\ \text{s.t.} & w_1 \leq p_1 \leq \min\left(\frac{w_2}{\theta}, 1\right) \end{array}$$

‡ under the condition that $\theta \geq \frac{2(c+1)}{(c+3)}$, $w^* - c \geq \frac{1-c}{2} > 0$. Therefore a positive marginal profit for *Manufacturer* is always guaranteed.

Case 1.1: If $\theta \geq \frac{w_2}{w_1}$, we would have $1 \geq w_1 \geq \frac{w_2}{\theta}$***.*** The only feasible p_1 that falls into the range $\left[w_1, \frac{w_2}{\theta}\right]$ is $p_1^* = w_1 = \frac{w_2}{\theta}$. But this will incur no marginal profits for both *Retailer 1* and *Retailer 2* which are surely not desired for both retailers

Case 1.2: If $\theta < \frac{w_2}{w_1}$ and $w_2 \geq \theta \Rightarrow \frac{w_2}{\theta} \geq 1$. The goal is to maximize π_{R_1} subject to $p_1 \in [w_1, 1]$. It is not difficult to see $w_1 \leq p_1^* = \frac{1+w_1}{2} \leq 1$ is true. Thus $p_1^* = \frac{1+w_1}{2}$, $p_2 = w_2$ can be achieved. Since $w_1 \leq 1 \Rightarrow \frac{w_2}{w_1} \geq w_2 \geq \theta$, the condition for this to hold true is $0 \leq \theta \leq w_2$

Case 1.3: If $\theta < \frac{w_2}{w_1}$ and $w_2 < \theta \Rightarrow \frac{w_2}{\theta} < 1$. The goal is to maximize π_{R_1} subject to $p_1 \in \left[w_1, \frac{w_2}{\theta}\right]$. The optimal solution of unconstrained problem is $p_1^* = \frac{1+w_1}{2} \geq w_1$. Define $\acute{\theta} = \frac{2w_2}{1+w_1}$ and if $p_1^* \leq \frac{w_2}{\theta} \Rightarrow \theta \leq \acute{\theta}$, $p_1^* = \frac{1+w_1}{2}$ can be obtained. Otherwise $p_1^* = \frac{w_2}{\theta}$ since the objective function is a quadratic function of p_1 and $\frac{w_2}{\theta}$ is its maximum feasible value. Since $w_1 \leq \frac{1+w_1}{2} \Rightarrow \frac{w_2}{w_1} \geq \acute{\theta} \geq w_2$, we have $p_1^* = \frac{1+w_1}{2}$ when $w_2 < \theta \leq \acute{\theta}$ and $p_1^* = \frac{w_2}{\theta}$ when $\acute{\theta} \leq \theta < \frac{w_2}{w_1}$.

To combine Case 1.2 and Case 1.3, we have the following

$$\begin{cases} p_1^* = \frac{w_2}{\theta}, p_2^* = w_2 & \text{if } \frac{w_2}{w_1} > \theta \geq \acute{\theta} \\ p_1^* = \frac{1+w_1}{2}, p_2^* = w_2 & \text{if } \acute{\theta} > \theta \geq 0 \end{cases}$$

When $\frac{w_2}{w_1} \leq \theta < 1$, there is simply no way for *Retailer 1* to fully occupy the market under the rational assumption.

Case 2: $\frac{p_2}{p_1} \leq \theta \leq 1 - (p_1 - p_2)$, non-negative demand for both channels. For *Retailer 1*, the optimization problem is

$$\begin{array}{ll} \max & \pi_{R_1} = \left(1 - \frac{p_1 - p_2}{1-\theta}\right)(p_1 - w_1) \\ \text{s.t.} & \max\left(\frac{p_2}{\theta}, w_1\right) \leq p_1 \leq \min(1 + p_2 - \theta, 1) \end{array}$$

For *Retailer* 2, the optimization problem is

$$\begin{array}{ll} \max & \pi_{R_2} = \left(\frac{p_1 - p_2}{1-\theta} - \frac{p_2}{\theta}\right)(p_2 - w_2) \\ \text{s.t.} & \max(w_2, \theta + p_1 - 1) \leq p_2 \leq \theta p_1 \end{array}$$

Case 2.1: The Nash-Equilibrium solution to the two optimization problem without taking any constraints into consideration is.

$$\begin{array}{ll} p_1^* & = \frac{-2\theta + 2w_1 + w_2 + 2}{4-\theta} \\ p_2^* & = \frac{-\theta^2 + \theta w_1 + \theta + 2w_2}{4-\theta} \end{array}$$

It can be verified that $p_1^* + \theta - 1 \leq w_2$. Therefore, we start verifying constraints of *Retailer 2* dual problem.

$$\begin{array}{lll} p_2^* \geq w_2 & \Rightarrow & \theta \geq \tilde{\theta} \\ \theta p_1^* \geq p_2^* & \Rightarrow & \text{same range of } \theta \text{ as above} \end{array}$$

[§]

where $\tilde{\theta} = \frac{1}{2}\left(1 + w_1 + w_2 - \sqrt{(w_1 + w_2 - 1)^2 + 4(w_1 - w_2)}\right)$ and assume $\theta \geq \tilde{\theta}$. Verify constraints of *Retailer 1*'s optimization problem: $\theta p_1^* \leq \theta$ and $p_2^* \leq \theta p_1^* \Rightarrow 1 + p_2^* - \theta \leq 1$, but we cannot infer the relationship between $\frac{p_2}{\theta}$ and w_1. Therefore we have two cases to discuss

a) $0 \leq \theta \leq \bar{\theta} \Rightarrow \frac{p_2}{\theta} \geq w_1$ ***where =1*** $\frac{}{2(1-w_1)\left(1+\sqrt{(1-3w_1)^2+8w_2(1-w_1)}-3w_1\right)}$***.***

Claim. $\bar{\theta} \geq \tilde{\theta}\ \forall \{w_1, w_2 \mid 0 \leq w_2 \leq w_1 \leq 1\}$

Proof. OK, the technique is to take the first derivative of $\bar{\theta} - \tilde{\theta}$ with respect to w_1, w_2 and solve system of equations to get all critical points of $\{w_1, w_2\}$. Then examine each critical points to infer their local maximum/minimum properties by taking the second derivative tests of these points. By simulation, I can prove it . □

Claim. Define $\hat{\theta} = \frac{2-2w_1+w_2}{2-w_1}$, $\hat{\theta} \geq \bar{\theta}\ \forall \{w_1, w_2 \mid 0 \leq w_2 \leq w_1 \leq 1\}$

Proof. Still proved by simulation only. Rigorous mathematical proof needs some tricks. □

We need to verify two constraints of *Retailer 1*'s problem

$$\begin{array}{rll} p_1^* \geq \frac{p_2^*}{\theta} & \Rightarrow & \theta \geq \tilde{\theta} \\ 1 + p_2^* - \theta \geq p_1^* & \Rightarrow & \theta \leq \hat{\theta} \end{array}$$

Therefore, since $\theta \leq \bar{\theta} \leq \hat{\theta}$, the condition for *Retailer 1* and *Retailer 2* to price this way i.e., $\left(p_1^* = \frac{-2\theta+2w_1+w_2+2}{4-\theta}, p_2^* = \frac{-\theta^2+\theta w_1+\theta+2w_2}{4-\theta}\right)$ is $\bar{\theta} \geq \theta \geq \tilde{\theta}$.

b) $\bar{\theta} \leq \theta \leq 1 \Rightarrow \frac{p_2}{\theta} \leq w_1$***.*** We need to verify two constraints of *Retailer 1*'s problem

$$\begin{array}{rll} p_1^* \geq w_1 & \Rightarrow & \theta \leq \hat{\theta} \\ 1 + p_2^* - \theta \geq p_1^* & \Rightarrow & \text{same range as above} \end{array}$$

Therefore we can extend previous satisfactory range of θ to be $\left[\tilde{\theta}, \hat{\theta}\right]$. *Retailer 1*'s response is $p_1^* = \frac{-2\theta+2w_1+w_2+2}{4-\theta}$ and utility is $\pi_{R_1}^* = \frac{(\theta w_1 - 2\theta - 2w_1 + w_2 + 2)^2}{(\theta-4)^2(1-\theta)}$. *Retailer 2*'s response is $p_2^* = \frac{-\theta^2+\theta w_1+\theta+2w_2}{4-\theta}$ and the utility is $\pi_{R_2}^* = \frac{\left(-\theta^2+\theta w_1+\theta w_2+\theta-2w_2\right)^2}{\theta(\theta-4)^2(1-\theta)}$.

Case 2.2: After unconstrained optimization, let's do constraint optimization and see if* Retailer 1 *and* Retailer 2 *can achieve Nash-Equilibrium outside of range $\left[\tilde{\theta}, \hat{\theta}\right]$. First we add $\frac{p_2}{p_1} \leq \theta$. The dual problem of *Retailer i* is the following

$$\begin{array}{ll} \min_{\lambda_i \leq 0} & L(p_i, \lambda_i) \\ L(p_i, \lambda_i) & = \sup_{p_i} \{\pi_{R_i} + \lambda_i (p_2 - \theta p_1)\} \end{array}$$

First we verify L is a concave function of p and then take first order differential equation with respect to p. Solve system of equations and substitute p in terms of λ. Verify L is a convex function of λ and set the first-order condition with respect to λ to zero[¶]. As defined in Case 1 that $\acute{\theta} = \frac{2w_2}{1+w_1}$[‖], to make both retailers happy with the equilibrium price, we have:

[§] We also have $\theta \leq \frac{1}{2}\left(1 + w_1 + w_2 + \sqrt{(w_1 + w_2 - 1)^2 + 4(w_1 - w_2)}\right)$ but since we have $1 + w_1 + w_2 + \sqrt{(w_1 + w_2 - 1)^2 + 4(w_1 - w_2)} \geq 1 + w_1 + w_2 + |w_1 + w_2 - 1| = f(w_1, w_2)$. If $w_1 + w_2 \geq 1, f(w_1, w_2) = 2(w_1 + w_2) \geq 2$ otherwise $f(w_1, w_2) \geq 2$. In either case $\frac{1}{2}\left(1 + w_1 + w_2 + \sqrt{(w_1 + w_2 - 1)^2 + 4(w_1 - w_2)}\right) \geq 1$. Therefore we do not need to add this constraint since $\theta \leq 1$

[¶] $\frac{\partial^2 L_1}{\partial p_1^2} = \frac{2}{\theta-1} < 0, \frac{\partial^2 L_2}{\partial p_2^2} = \frac{2}{\theta(\theta-1)} < 0; \frac{\partial^2 L_1 \mid_{p_1=p_1^*, p_2=p_2^*}}{\partial \lambda_1^2} = \frac{2\theta^3(1-\theta)}{(\theta-4)^2} > 0, \frac{\partial^2 L_2 \mid_{p_1=p_1^*, p_2=p_2^*}}{\partial \lambda_2^2} = \frac{2\theta(1-\theta)(\theta-2)^2}{(\theta-4)^2} > 0$

[‖] It can be shown $\tilde{\theta} \geq \acute{\theta}\ \forall \{w_1, w_2 \mid w_1 \geq w_2\}$

$$\lambda_1^* = \frac{1-w_1}{2(\theta-1)} \le 0 \Rightarrow \text{all range of } \theta$$

$$\lambda_2^* = \frac{\theta w_1 + \theta - 2w_2}{2\theta(1-\theta)} \le 0 \Rightarrow \theta \le \acute{\theta}$$

The corresponding p_1^* and p_2^* are $p_1^* = \frac{1+w_1}{2}, p_2^* = \frac{\theta(1+w_1)}{2}$. Rational assumption would require $p_2^* \ge w_2 \Rightarrow \theta \ge \acute{\theta}$. Thus we can obtain $\{p_1^*, p_2^*\}$ only when $\theta = \acute{\theta}$, and $p_1^* = \frac{1+w_1}{2}, p_2^* = \theta p_1^* = w_2$ which is already included in Case 1 analysis.

Case 2.3: Now we try adding the other constraint $\theta \le 1-(p_1-p_2)$ ***to the unconstrained problem .*** The dual problem of *Retailer i* is the following

$$\begin{array}{ll} \min_{\lambda_i \le 0} & L(p_i,\lambda_i) \\ L(p_i,\lambda_i) & = \sup_{p_i} \{\pi_{R_i} + \lambda_i(\theta - 1 + p_1 - p_2)\} \end{array}$$

We have to make sure both λ_1 and λ_2 are non-positive so that both *Retailer 1* and *Retailer 2* are happy with the Nash-Equilibrium solutions. Recall $\hat{\theta} = \frac{2-2w_1+w_2}{2-w_1}$, to make both retailers happy with the equilibrium solutions, we have to make sure the achievement of dual feasibility:

$$\lambda_1^* = \frac{2+w_2-2w_1-\theta}{2(1-\theta)} \le 0 \Rightarrow \theta \ge (2-w_1)\hat{\theta}$$

$$\lambda_2^* = \frac{w_2-\theta}{2(1-\theta)} \le 0 \Rightarrow \theta \ge w_2$$

Since $(2-w_1)\hat{\theta} = 2(1-w_1)+w_2 \ge w_2$, we have if $\ddot{\theta} \le \theta \le 1$, both λ_1^*, λ_2^* are non-positive. The corresponding solutions are $p_1^* = 1 - \frac{\theta - w_2}{2}, p_2^* = \frac{w_2+\theta}{2}$. since $p_1^* - w_1 = \frac{(2-w_1)\hat{\theta}-\theta}{2} \le 0$, the rational assumption requires $\theta = (2-w_1)\hat{\theta}$. Since $(2-w_1)\hat{\theta} > \hat{\theta}$, we incorporate this special case to Case 3 analysis.

Therefore, in order to make both channel have non-zero sales, channel acceptance has to be in between $\tilde{\theta} = \frac{1+w_1+w_2-\sqrt{(w_1+w_2-1)^2+4(w_1-w_2)}}{2}$ and $\hat{\theta} = \frac{2-2w_1+w_2}{2-w_1}$. Two retailers' equilibrium price would be $p_1^* = \frac{-2\theta+2w_1+w_2+2}{4-\theta}$ and $p_2 = \frac{-\theta^2+\theta w_1+\theta+2w_2}{4-\theta}$.

Case 3: $1 \ge \theta \ge 1-(p_1-p_2)$***, zero demand for the stronger channel .*** When $\hat{\theta} \le \theta < 1$, under the assumption of $w_1 \ge w_2$, *Retailer 2* with moderately strong acceptance and lower cost has no reason to share the market with *Retailer 1*. For *Retailer* 2, its profit is $\pi_{R_2} = \left(1-\frac{p_2}{\theta}\right)(p_2-w_2)$. To guarantee $\theta \ge 1-(p_1-p_2) \Rightarrow p_2 \le \theta + p_1 - 1$ is always true, p_2 has to be less than the lower bound of $\theta + p_1 - 1$, which is $\theta + w_1 - 1$. The dual problem of *Retailer i* is the following

$$\begin{array}{ll} \max & \pi_{R_2} = \left(1-\frac{p_2}{\theta}\right)(p_2-w_2) \\ \text{s.t.} & w_2 \le p_2 \le \theta + w_1 - 1 \end{array}$$

To guarantee *Retailer 2* has room to gain monopoly over *Retailer 1*, we have to assume $\theta + w_1 - 1 \ge w_2$. Differentiate with respect to p_2 gives $p_2^* = \frac{1}{2}(\lambda\theta + w_2 + 1)$. Substituting p_2^* in $L(p_2,\lambda)$ and differentiate with respect to λ gives us $\lambda^* = \frac{1}{\theta}(\theta + 2w_1 - w_2 - 2) = \frac{\theta-(2-w_1)\hat{\theta}}{\theta}$.

Case 3.1: If $\lambda^* \le 0 \Rightarrow \theta \le (2-w_1)\hat{\theta}$***, dual optimal solution can be achieved .*** The corresponding solutions are $p_2^* = w_1 + \theta - 1$ and $p_1^* = w_1$. The required interval of θ for this to be hold is $w_2 + 1 - w_1 \le \theta \le (2-w_1)\hat{\theta} = w_2 + 2(1-w_1)$

Case 3.2: If $\theta > (2-w_1)\hat{\theta} \Rightarrow \lambda^* > 0$***, dual is not feasible and the constraint*** $p_2 \le \theta + w_1 - 1$ ***can be dropped.*** We are simply solving an unconstrained optimization of *Retailer 2*'s problem.

$p_2^* = \arg\max_{p_2}\left(1-\frac{p_2}{\theta}\right)(p_2-w_2) = \frac{\theta+w_2}{2}$ and the intersection of $\left\{\theta \mid \lambda^* > 0 \bigcap p_2^* \ge w_2\right\}$ gives $\left\{\theta \mid \theta > (2-w_1)\hat{\theta}\right\}$[**]. Also we can extend it with the special case we left in Case 2.3 and hence we have $p_2^* = \frac{\theta+w_2}{2}, p_1^* = w_1$ when $\theta \ge (2-w_1)\hat{\theta}$.

Define $\ddot{\theta} = w_2 + 1 - w_1$, and combine Case 3.1 and 3.2, we have

$$\begin{cases} p_1^* = w_1, p_2^* = w_1 + \theta - 1 & \text{if } (2-w_1)\hat{\theta} > \theta \ge \ddot{\theta} \\ p_1^* = w_1, p_2^* = \frac{\theta+w_2}{2} & \text{if } 1 > \theta \ge (2-w_1)\hat{\theta} \end{cases}$$

Claim. *Retailer 1* would rather compete than reduce its retail price to gain monopoly over *Retailer 2* when $\tilde{\theta} \le \theta \le \frac{w_2}{w_1}$.

Proof. First it can be shown that $\acute{\theta} \le \tilde{\theta} \le \frac{w_2}{w_1} \le \hat{\theta}$. When $\tilde{\theta} \le \theta \le \frac{w_2}{w_1}$, the monopoly strategy provides *Retailer 1* with $\pi_{R_1}^{(1)} \mid_{p_1 = \frac{w_2}{\theta}} = \frac{w_2(1-\theta)(\theta-w_2)}{\theta^2}$ and the competing strategy yields $\pi_{R_1}^{(2)} \mid_{p_1 = \frac{-2\theta+2w_1+w_2+2}{4-\theta}, p_2 = \frac{-\theta^2+\theta w_1+\theta+2w_2}{4-\theta}} = \frac{(\theta w_1 - 2\theta - 2w_1 + w_2 + 2)^2}{(\theta-4)^2(1-\theta)}$.

Recall we obtained $\tilde{\theta}$ via the equation of $\theta^2 + \theta(-w_1 - w_2 - 1) + 2w_2 = 0$. Suppose the other root which is much larger than $\tilde{\theta}$ to be $\tilde{\theta}^\dagger > 1$ so that we can rewrite $\theta^2 + \theta(-w_1-w_2-1) + 2w_2$ as $\left(\theta - \tilde{\theta}\right)\left(\theta - \tilde{\theta}^\dagger\right)$

$$\pi_{R_1}^{(2)} - \pi_{R_1}^{(1)} = \frac{\left(\theta-\tilde{\theta}\right)\left(\theta-\tilde{\theta}^\dagger\right)f(\theta)}{\theta^2(\theta-4)^2(\theta-1)}$$

where $f(\theta) = \theta(\theta-4)(\theta w_1 - w_2) - 4\left(\theta-\tilde{\theta}\right)\left(\theta+\tilde{\theta}\right)$, $\frac{\partial^2 f}{\partial\theta^2} = 2(3\theta w_1 - 4w_1 - w_2 - 4) < 0$ and

$$g(\theta) = \frac{\partial f}{\partial \theta} = 3\theta^2 w_1 + 2\theta(-4w_1 - w_2 - 4) + 4w_1 + 8w_2 + 4$$

Since $g(0) = 4w_1 + 8w_2 + 4 > 0$ and $g'(\theta) < 0$, we examine the point $g\left(\frac{w_2}{w_1}\right) = \frac{4w_1^2 + 4w_1 + w_2^2 - 8w_2}{w_1}$

a) If $g\left(\frac{w_2}{w_1}\right) \ge 0$, $f(\theta)$ is a non-decreasing function over $\tilde{\theta} \le \theta \le \frac{w_2}{w_1}$. Its minimum value is obtained at $\theta = \tilde{\theta}$. And $f(\tilde{\theta}) = \tilde{\theta}\left(\tilde{\theta}-4\right)\left(\tilde{\theta}w_1 - w_2\right)$. Since $\tilde{\theta} \le \theta \le \frac{w_2}{w_1} \Rightarrow \tilde{\theta}w_1 - w_2 \le 0$ and $\tilde{\theta} - 4 < 0$, we have $f(\theta) \ge f(\tilde{\theta}) \ge 0\ \forall\theta \in \left[\tilde{\theta}, \frac{w_2}{w_1}\right]$.

b) If $g\left(\frac{w_2}{w_1}\right) < 0$, $f(\theta)$ obtains maximum in between 0 and $\frac{w_2}{w_1}$. The minimum functional value over $\left[\tilde{\theta}, \frac{w_2}{w_1}\right]$ can be obtained at either $\tilde{\theta}$ or $\frac{w_2}{w_1}$. We have verified $f(\tilde{\theta}) \ge 0$ above and $f\left(\frac{w_2}{w_1}\right) = \frac{4w_2(1-w_1)(w_1-w_2)}{w_1^2} \ge 0$. Therefore we are guaranteed that $f(\theta) \ge 0\ \forall\theta \in \left[\tilde{\theta}, \frac{w_2}{w_1}\right]$.

Since $\left(\theta-\tilde{\theta}\right)\left(\theta-\tilde{\theta}^\dagger\right) \le 0$ and $f(\theta) \ge 0\ \forall\theta \in \left[\tilde{\theta}, \frac{w_2}{w_1}\right]$,

$$\pi_{R_1}^{(2)} - \pi_{R_1}^{(1)} = \frac{\left(\theta-\tilde{\theta}\right)\left(\theta-\tilde{\theta}^\dagger\right)f(\theta)}{\theta^2(\theta-4)^2(\theta-1)} \ge 0$$

□

Claim. For *Retailer* 2, define monopoly to be the strategy which if adopted, brings no sale to *Retailer 1* no matter how lower p_1 is. And define competition to be the strategy which if adopted, make both retailers have non-zero sale. *Retailer 2* would rather compete than reduce its retail price to gain monopoly over *Retailer 1* when $\ddot{\theta} \le \theta \le \hat{\theta}$.

[**] $p_2^* \ge w_2 \Rightarrow \theta \ge w_2$ and $w_2 \le 2(1-w_1) + w_2$

Proof. First it can be shown that $\tilde{\theta} \le \ddot{\theta} \le \hat{\theta} \le (2-w_1)\,\hat{\theta}$. When $\ddot{\theta} \le \theta \le \hat{\theta}$., the monopoly strategy provides *Retailer 2* with $\pi_{R_2}^{(1)}|_{p_2=w_1+\theta-1} = \frac{(1-w_1)(\theta+w_1-w_2-1)}{\theta}$ and the competing strategy yields $\pi_{R_1}^{(2)} \mid_{p_1=\frac{-2\theta+2w_1+w_2+2}{4-\theta}, p_2=\frac{-\theta^2+\theta w_1+\theta+2w_2}{4-\theta}} = \frac{\left(\theta^2+\theta(-w_1-w_2-1)+2w_2\right)^2}{\theta(\theta-4)^2(1-\theta)}$.

Recall we obtained $\hat{\theta}$ via the equation of $\theta\,(w_1-2)-2w_1+w_2+2=0$.

$$\pi_{R_1}^{(2)} - \pi_{R_1}^{(1)} = \frac{(w_1-2)\left(\theta-\hat{\theta}\right) f(\theta)}{\theta\,(\theta-4)^2\,(\theta-1)}$$

where $f(\theta) = \theta\,(2-\theta)^2 - \ddot{\theta}\theta^2 + 4\hat{\theta}\,(1-\theta)$, $\frac{\partial^2 f}{\partial \theta^2} = 2\,(3\theta+w_1-w_2-5) < 0$ and

$$g(\theta) = \frac{\partial f}{\partial \theta} = 3\theta^2 + \theta\,(2w_1-2w_2-10) - 8w_1 + 4w_2 + 12$$

$g(\ddot{\theta}) = (w_1-w_2)^2 + 5 - 2w_1 - 2w_2 > 0$ and $g(\hat{\theta}) = \frac{(5-4w_1)(w_1-2)^2+(w_1-w_2)(3w_1+w_2-2w_1w_2)}{(w_1-2)^2} > 0$. Since the first order derivative at both end points are positive and f is concave, we can obtain the maximum functional value of $f(\theta)$ at $\theta = \ddot{\theta}$: $f(\ddot{\theta}) = 4\,(w_1-w_2)\,(w_1-1) \le 0$. Because the maximum functional value is non-positive, we have $f(\theta) \le f(\ddot{\theta}) \le 0\ \forall \theta \in \left[\ddot{\theta}, \hat{\theta}\right]$.

And $\pi_{R_1}^{(2)} - \pi_{R_1}^{(1)} = \frac{(2-w_1)\left(\theta-\hat{\theta}\right)f(\theta)}{\theta(\theta-4)^2(1-\theta)} \ge 0$ since $\theta - \hat{\theta} \le 0$ and $f(\theta) \le 0$. ☐

Proof of Theorem 4.

Proof. In region R_1, $p_1^* = \frac{1+w_1}{2}$ and $p_2^* = w_2$. *Retailer 1* has the market all to itself and the total demand is $1-p_1 = \frac{1-w_1}{2}$. The *Manufacturer*'s optimization problem is to maximize its profit $\pi_m = \frac{(w_1-c)(1-w_1)}{2}$ by charging the right wholesale price w_1 subject to $\theta\,(1+w_1) \le 2w_2$. We obtain its optimal solution $w_1^* = \arg\max \pi_m = \frac{1+c}{2}$. But can we achieve w_1^*? By examining all constraints involving w_1, we have:$w_2 \le w_1 \le \frac{2w_2-\theta}{\theta}$. Since the objective is not a function of w_2, Manufacturer can freely adjust w_2 as long as it is within the range of $w_2 \le \frac{1+c}{2} \le \frac{2w_2-\theta}{\theta} \Leftrightarrow \frac{\theta(3+c)}{4} \le w_2 \le \frac{1+c}{2} \Rightarrow \theta \le \theta^a = \frac{2(1+c)}{3+c}$.

If $0 \le \theta \le \theta^a$, *Manufacturer* can achieve $w_1^* = \frac{1+c}{2}$ and furthermore, it can freely adjust w_2 to be in between $\frac{\theta(3+c)}{4}$ and w_1^*. Optimal utility is $\pi_m^* = \frac{(c-1)^2}{8}$

If $\theta > \theta^a$, *Manufacturer* cannot be able to achieve $w_1^* = \frac{1+c}{2}$ because there is no room for w_2 to grow higher than w_1. Instead, *Manufacturer* would price $w_2^* = w_1^* = \frac{\theta}{2-\theta}$ and sub-optimal utility $\pi_m^* = \frac{(1-\theta)(c\theta-2c+\theta)}{(\theta-2)^2}$

All in all we have in decision space R_1:

$$\begin{cases} w_1^* = \frac{1+c}{2}, \pi_m^* = \frac{(1-c)^2}{8} & \text{if } 0 \le \theta \le \theta^a \\ w_2^* = w_1^* = \frac{\theta}{2-\theta} & \text{otherwise} \end{cases}$$

In region R_2, $p_1^* = \frac{w_2}{\theta}$ and $p_2^* = w_2$. *Retailer 1* has the market all to itself and the total demand is $1-p_1 = \frac{\theta-w_2}{\theta}$ and $\pi_m = (w_1-c)\,(1-p_1) = \frac{(\theta-w_2)(w_1-c)}{\theta}$. It is a decreasing function of w_2 and an increasing function of w_1.

Let's first solve the unconstrained problem (i.e., do not restrict $\{w_1, w_2\}$ to satisfy $\acute{\theta} \le \theta \le \tilde{\theta}$ and verify optimal solutions are indeed located in R_2)

Since the maximum w_1 we can obtain is $p_1^* = \frac{w_2}{\theta}$. Substitute and the objective function becomes $\pi_m\,(w_2) = \frac{(\theta-w_2)(w_2-c\theta)}{\theta^2}$. It is a concave function of w_2 and the optimal solution can be achieved if we set the first derivative equal to zero: $w_2^* = \arg\max \pi_m\,(w_2) = \frac{\theta(c+1)}{2}$. But can this w_2^* be obtained? Since $p_1^* = \frac{w_2}{\theta} \le 1$, $c \le w_2 \le \theta$

If $\theta > \theta^b = \frac{2c}{1+c} \Rightarrow c < \frac{\theta(1+c)}{2}$, $w_2^* = \frac{\theta(c+1)}{2}$ can be achieved and verify$\left\{w_1^* = \frac{1+c}{2}, w_2^* = \frac{\theta(c+1)}{2}\right\}$ is locate in region R_2

$$\acute{\theta} \le \theta \Rightarrow \text{all range of } \theta$$
$$\theta \le \tilde{\theta} \Rightarrow \theta \in \emptyset$$

Therefore, we cannot obtain $\{w_1^*, w_2^*\}$in R_2.

If $\theta \le \theta^b \Rightarrow c \ge \frac{\theta(1+c)}{2}$, $w_2^* = c$ and the corresponding $w_1^* = \frac{c}{\theta}$. Verify$\left\{w_1^* = \frac{c}{\theta}, w_2^* = c\right\}$ is locate in region R_2

$$\acute{\theta} \le \theta \Rightarrow \theta \le c$$
$$\theta \le \tilde{\theta} \Rightarrow \theta \le c$$

But θ cannot be less than c, therefore no optimal solutions to unconstrained problem have been found. Instead, we try to examine sub-optimal solutions at the boundary[††].

After unconstrained optimization, let's try examining optimal solution obtained in the boundary of R_2

Assume optimal solution is obtained at $\theta = \acute{\theta} \Rightarrow w_2^* = \frac{\theta(1+w_1)}{2}$. Substitute and the objective function becomes $\pi_m = \frac{(w_1-c)(1-w_1)}{2}$ and $w_1^* = \arg\max \pi_m = \frac{1+c}{2}$. But can we obtain this w_1^*? Examine constraints involving w_1: Since we require $c \le w_2^* \le w_1 \le \frac{w_2^*}{\theta} \le 1$,

$$c \le w_2^* \le w_1 \Rightarrow w_1 \ge \max\left\{\frac{2c}{\theta}-1, \frac{\theta}{2-\theta}\right\}$$
$$w_1 \le \frac{w_2^*}{\theta} \le 1 \Rightarrow w_1 \le 1$$

$\theta \le \theta^b = \frac{2c}{1+c} \Rightarrow \frac{2c}{\theta}-1 \ge \frac{\theta}{2-\theta}$, the range of w_1 is $\frac{2c}{\theta}-1 \le w_1 \le 1$.

If $\theta \ge \theta^c = \frac{4c}{3+c} \Rightarrow \frac{2c}{\theta}-1 \le w_1^*$, $w_1^* = \frac{1+c}{2}$ is achievable and the corresponding $w_2^* = \frac{\theta(1+w_1)}{2} = \frac{\theta(3+c)}{4} \ge c$ if $\theta \ge \theta^c$. The optimal utility is $\pi_m^* = \frac{(c-1)^2}{8}$.

If $\theta < \theta^c \Rightarrow \frac{2c}{\theta}-1 > w_1^*$, sub-optimal w_1 is obtained at $\frac{2c}{\theta}-1$[‡‡]. The corresponding $w_2^* = c$ and the utility is $\pi_m^* = \frac{(1+c)(\theta-c)\left(\theta^b-\theta\right)}{\theta^2}$.

It can be shown $\theta^b \ge \theta^c$. Thus we have $w_1^* = \frac{1+c}{2}$ if $\theta^c \le \theta \le \theta^b$ and $w_1^* = \frac{2c-\theta}{\theta}$ if $c \le \theta \le \theta^c$

$\theta > \theta^b \Rightarrow \frac{2c}{\theta}-1 < \frac{\theta}{2-\theta}$, the range of w_1 is $\frac{\theta}{2-\theta} \le w_1 \le 1$.

If $\theta {\le} \theta^a = \frac{2(1+c)}{3+c} \Rightarrow \frac{\theta}{2-\theta} \le w_1^*$, $w_1^* = \frac{1+c}{2}$ is achievable and following what we have in Case 1.ii, we have its optimal utility $\pi_m^* = \frac{(c-1)^2}{8}$.

If $\theta > \theta^a \Rightarrow \frac{\theta}{2-\theta} > w_1^*$, sub-optimal w_1 is obtained at $\frac{\theta}{2-\theta}$. The corresponding $w_2^* = w_1^* = \frac{\theta}{2-\theta}$ and the utility is $\pi_m^* = \frac{(1+c)(1-\theta)\left(\theta-\theta^b\right)}{(\theta-2)^2}$.

All in all, since $c \le \theta^c \le \theta^b \le \theta^a \le 1$, we have optimal solutions obtained in the boundary of R_1 and R_2 to be:

[††] This implies for *Manufacturer*, the global optimal pricing of $\{w_1, w_2\}$would never fall in R_2 which is quite opposite to what we have found in same-wholesale price situation.

[‡‡] If $\theta < \theta^c < \theta^b = \frac{2c}{1+c}$, $\frac{2c}{\theta}-1 > c$

$$\begin{cases} w_1^* = \frac{2c-\theta}{\theta}, w_2^* = c, \pi_m^* = \frac{(1+c)(\theta-c)\left(\theta^b-\theta\right)}{\theta^2} & \text{if } c \le \theta \le \theta^c \\ w_1^* = \frac{1+c}{2}, w_2^* = \frac{\theta(3+c)}{4}, \pi_m^* = \frac{(c-1)^2}{8} & \text{if } \theta^c \le \theta \le \theta^a \\ w_1^* = \frac{\theta}{2-\theta}, w_2^* = \frac{\theta}{2-\theta}, \pi_m^* = \frac{(1+c)(1-\theta)\left(\theta-\theta^b\right)}{(\theta-2)^2} & \text{if } \theta^a \le \theta \le 1 \end{cases}$$

let's assuming optimal solutions are obtained at the boundary of R_2 and R_3, i.e., $\theta = \tilde{\theta}$

Since we can equivalently write $\theta = \tilde{\theta}$ as $\theta^2 - \theta\left(1 + w_1 + w_2\right) + 2w_2 = 0$. Substitute $w_2 = \frac{\theta(1-\theta+w_1)}{2-\theta}$ and the objective function becomes $\pi_m = \frac{(w_1-c)(1-w_1)}{2-\theta}$ and $w_1^* = \arg\max \pi_m = \frac{1+c}{2}$. But can we obtain this w_1^*? Examine constraints involving w_1: Since we require $c \le w_2^* \le w_1 \le \frac{w_2^*}{\theta} \le 1$,

$$c \le w_2^* \le w_1 \Rightarrow w_1 \ge \max\left\{\frac{(2-\theta)\,c + \theta^2 - \theta}{\theta}, \frac{\theta}{2}\right\}$$

$$w_1 \le \frac{w_2^*}{\theta} \le 1 \Rightarrow w_1 \le 1$$

$\theta \ge 2c \Rightarrow \frac{\theta}{2} \ge \frac{(2-\theta)c+\theta^2-\theta}{\theta}$, the range of w_1 is $\frac{\theta}{2} \le w_1 \le 1$.

It can be shown $\frac{\theta}{2} \le \frac{1+c}{2}$ always. Thus $w_1^* = \frac{1+c}{2}$ is achievable and the corresponding $w_2^* = \frac{\theta(c-2\theta+3)}{2(2-\theta)}$. Let $\theta^d = \frac{3(c+1)-\sqrt{9(c+1)^2-32c}}{4}$ and it can be shown $\theta^d \le 2c$. If $\theta \ge 2c \ge \theta^d$, we can verify $w_2^* \ge c$. The optimal utility is $\pi_m = \frac{(c-1)^2}{4(2-\theta)}$

$\theta < 2c \Rightarrow \frac{\theta}{2} < \frac{(2-\theta)c+\theta^2-\theta}{\theta}$, the range of w_1 is $\frac{(2-\theta)c+\theta^2-\theta}{\theta} \le w_1 \le 1$.

If $\theta \ge \theta^d = \frac{3(c+1)-\sqrt{9(c+1)^2-32c}}{4} \Rightarrow \frac{(2-\theta)c+\theta^2-\theta}{\theta} \le w_1^*$, $w_1^* = \frac{1+c}{2}$ is achievable and following what we have in Case 1, we have its optimal utility $\pi_m = \frac{(c-1)^2}{4(2-\theta)}$

If $\theta < \theta^d \Rightarrow \frac{(2-\theta)c+\theta^2-\theta}{\theta} > w_1^*$, sub-optimal w_1 is obtained at $\frac{(2-\theta)c+\theta^2-\theta}{\theta}$. The corresponding $w_2^* = c$ and the utility is $\pi_m^* = \frac{(\theta-c)(1-\theta)(2c-\theta)}{\theta^2}$.

All in all, we have optimal solutions obtained in the boundary of R_2 and R_3 to be:

$$\begin{cases} w_1^* = \frac{(2-\theta)c+\theta^2-\theta}{\theta}, w_2^* = c & \text{if } c \le \theta \le \theta^d \\ w_1^* = \frac{1+c}{2}, w_2^* = \frac{\theta(c-2\theta+3)}{2(2-\theta)} & \text{if } \theta^d \le \theta \le 1 \end{cases}$$

□

Proof of Theorem 5.

Proof. In region R_3, $p_1^* = \frac{-2\theta+2w_1+w_2+2}{4-\theta}$ and $p_2^* = \frac{-\theta^2+\theta w_1+\theta+2w_2}{4-\theta}$. Both retailers have their own market shares. □

Case 1:We start solving an unconstrained problem and later verify if optimal solution can be located in R_3

$$\begin{aligned} \pi_m &= (w_1 - c)\left(1 - \frac{p_1 - p_2}{1-\theta}\right) + (w_2 - c)\left(\frac{p_1 - p_2}{1-\theta} - \frac{p_2}{\theta}\right) \\ &= \frac{\left((2-\theta)(1-w_1) - (\theta - w_2)\right)(w_1 - c)}{(\theta-4)(\theta-1)} + \frac{\left((2-\theta)(\theta-w_2) - \theta(1-w_1)\right)(w_2 - c)}{\theta(\theta-4)(\theta-1)} \end{aligned}$$

Proof. Since $\frac{\partial^2 \pi_m}{\partial w_1^2} = \frac{2(\theta-2)}{(\theta-4)(\theta-1)} < 0$, $\frac{\partial^2 \pi_m}{\partial w_2^2} = \frac{2(\theta-2)}{\theta(\theta-4)(\theta-1)} < 0$ and $\det\left(\mathbf{Hess}\pi_m\right) = \frac{\partial^2 \pi_m}{\partial w_1^2}\frac{\partial^2 \pi_m}{\partial w_2^2} - \left(\frac{\partial \pi_m}{\partial w_1 \partial w_2}\right)^2 = \frac{4}{\theta(\theta-4)(\theta-1)} > 0$, π_m is jointly concave in w_1 and w_2. The first order condition yields optimal wholesale prices of $w_1^* = \frac{1+c}{2}$ and $w_2^* = \frac{\theta+c}{2}$. To make sure $\{w_1^*, w_2^*\}$ is feasible in region R_3, we substitute them in the constraint

$$\theta \ge \tilde{\theta} \Rightarrow \theta \ge 2c$$

$$\theta \le \hat{\theta} \Rightarrow \text{all range of } \theta$$

Also we require

$$w_1^* \ge w_2^* \Rightarrow \text{all range of } \theta$$

$$w_2^* \ge c \Rightarrow \theta \ge c$$

Therefore, when $2c \le \theta$, $w_1^* = \frac{c+1}{2}$ and $w_2^* = \frac{\theta+c}{2}$, $\pi_m^* = \frac{(\theta+2)\left(\theta+c^2\right)-6c\theta}{4\theta(4-\theta)}$

Case 2: After unconstrained optimization, let's try examining optimal solution obtained in the boundary of R_2 and R_3, i.e., $\theta = \tilde{\theta}$

Since we can equivalently write $\theta = \tilde{\theta}$ as $\theta^2 - \theta\left(1 + w_1 + w_2\right) + 2w_2 = 0$. Substitute $w_2 = \frac{\theta(1-\theta+w_1)}{2-\theta}$ and the resulting objective function is concave of w_1. Therefore we set the first-order condition with respect to w_1 to zero and solve the equation: $w_1^* = \frac{1+c}{2}$. But can we obtain this w_1^*? Examine constraints involving w_1:

$$c \le w_2^* \le w_1 \Rightarrow w_1 \ge \max\left\{\frac{(2-\theta)\,c + \theta^2 - \theta}{\theta}, \frac{\theta}{2}\right\}$$

$$w_1 \le p_1^* \le 1 \Rightarrow w_1 \le 1$$

$$w_2 \le p_2^* \le 1 \Rightarrow p_2^* = w_2$$

This will repeat what we have discussed in Case 2.3.

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