1 Introduction

We derive a special case of Euler's totient function where the resulting formula is only satisfied by twin primes.

Theorem 1. *Let ϕ be Euler's function. Then*

$$\frac{p+q}{4} - \frac{\phi(pq)}{p+q} = 1$$

if and only if $q - p = 2$ where $p, q \in \mathbb{P}$.

We begin with providing a number of prerequisite results used for the proof of Theorem 1.

Lemma 2. *For different primes p and q,*

$$\phi(pq) = \phi(p)\phi(q) = (p-1)(q-1).$$

Proof. Count the multiples of p in the range $1 \leq p \leq 2p \leq \cdots \leq qp$, and the multiples of q in the range $1 \leq q \leq 2q \leq \cdots \leq pq$. There are q and p such numbers, respectively. Both p and q coincide at pq since any positive integer divisible by p and q is divisible by pq and cannot be less than pq. Account for the double-occurence of pq by removing one from the amount of multiples present in both inequalities, viz. $p + q - 1$. Take away this number from pq to find $\phi(pq)$:

$$\begin{aligned} \phi(pq) &= pq - p - q + 1 \\ &= (p-1)(q-1). \end{aligned}$$

$\square$

Lemma 3. *For positive integers m, n where $m < n$ and $n - m = 2$,*

$$mn = \left(\frac{m+n}{2}\right)^2 - 1. \tag{1}$$

Proof. Suppose mn is not prime. Then there are positive integers a and b such that $a < b < mn$ and $mn = ab$. Let $a = (x-1)$ and $b = (x+1)$. Then

$$\begin{aligned} ab &= x^2 - 1 \\ &= \left(\frac{2x}{2}\right)^2 - 1 \\ &= \left(\frac{x + x + 1 - 1}{2}\right)^2 - 1 \\ &= \left(\frac{(x-1) + (x+1)}{2}\right)^2 - 1 \\ &= \left(\frac{m+n}{2}\right)^2 - 1. \end{aligned}$$

On the other hand, suppose $\left(\frac{a+b}{2}\right) = x$. Then

$$\begin{aligned}\left(\frac{a+b}{2}\right)^2 - 1 &= x^2 - 1 \\ &= (x-1)(x+1) \\ &= mn.\end{aligned}$$

If $m < n$, we can conclude that $m = x - 1 < x + 1 = n$. Therefore, $n - m = x + 1 - (x - 1) = 2$. $\square$

2 Completion

We now complete the proof of Theorem 1.

Proof. Begin with the result of Lemma 2:

$$\begin{aligned}\phi(pq) &= (p-1)(q-1) \\ &= pq - p - q + 1 \\ \text{(By 1)} \qquad &= \left(\left(\frac{p+q}{2}\right)^2 - 1\right) - p - q + 1 \\ &= \frac{1}{2}(p+q)\frac{1}{2}(p+q) - p - q \\ \frac{\phi(pq)}{p+q} &= \frac{1}{4}(p+q) - 1 \\ \frac{p+q}{4} - \frac{\phi(pq)}{p+q} &= 1.\end{aligned}$$

$\square$