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Play Calling Strategy in American Football: A Game-Theoretic Stochastic Dynamic Programming Approach

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This manuscript presents a model to assist in the determination of optimal American football play selection for first down and goal situations. A game theoretic approach is embedded within a stochastic dynamic programming formulation, resulting in a mixed strategy satisfying the ex-ante declared objective of maximizing the probability of scoring a touchdown. The methodology provides a quantitative framework to a problem that impacts on team performance and addresses a gap in the literature concerning the application of quantitative methods to sports.

Football is more than the game of brutality it was once thought to be. Gone are the days where size was the essential feature of a winning team. Today's game requires intelligence and strategy, in addition to the physical attributes necessary in all major sports: size, strength, and speed. Moreover, efficient plays and strategies are continually developed and implemented. Given the myriad of possibilities regarding alternative plays and defenses, the application of quantitative methodology may provide a systematic approach to increasing this play calling efficiency.

Much of the quantitative research that has been conducted on American football focuses on the gambling industry and predicting the outcomes of games with respect to the point spreads established by Las Vegas odds-makers. For example, Badarinathi and Kochman (1996) analyze the relationship between scoring margins and point spreads in an effort to provide bettors with a set of criteria for increasing their chances of "beating the market." Quantitative methods have also been used for administrative purposes. For instance, Saltzman and Bradford (1996) present optimal realignments of the league's teams in order to minimize intra-divisional travel. These type of results, however, do not enhance decision making, execution, and performance on the field. In these areas, the importance of mathematical analysis and statistical methodologies are widely recognized and may be utilized as a means to increase performance (Ryan, Francia, & Strawser, 1973).

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Previous applications of quantitative methods to football performance include Irving and Smith's (1976) quadratic regression model to determine under what circumstances a field goal should be attempted. Carter and Machol (1971) utilize empirical results to estimate the probability of scoring a touchdown for a given field position. Porter (1967) applies decision analysis and expectation in determining extra-point strategy. While these results all provide useful information, they do not focus on the simultaneous two-party game theoretic nature and the stochastic elements involved in play selection. This manuscript attempts to bridge this gap by considering an interesting problem: play selection for first-and-goal situations.

The use of dynamic programming in football is not unprecedented; Sackrowitz and Sackrowitz (1996) utilize this optimization methodology in the context of football strategy for the purpose of efficient time management. Results refute the widely accepted principle that a team will increase its chances of winning when using a strategy of ball control or time consumption. In addition, they suggest that teams should make an effort to better understand the principles of optimal strategies as they pertain to play selection: This paper offers one such study.

The remainder of the paper is organized as follows: The theoretical model from which optimal results are derived is presented in Section II. Section III provides an illustrative numerical example from which sample results are obtained. Section IV is by way of concluding comments.

The Stochastic Dynamic Programming Model

Dynamic programs are often used to solve multistage problems where the decision maker has to choose from various strategies dependent upon the state of the system when entering a given stage. When states are not deterministic, but governed by some probability distribution, the formulation is termed *stochastic*. A stochastic dynamic programming formulation is utilized here, where state spaces are defined by the yardage required to score a touchdown and stages represented by down situations. A game theoretic component is embedded within each stage-state pair in order to determine the optimal probability and associated optimal mixed strategy for an offense operating under the *ex-ante* declared objective of maximizing the probability that a touchdown is scored over the sequence of downs.

The specific problem of interest is the determination of a sequence of plays that will maximize the probability of scoring a touchdown over the duration of three plays, or downs. The utilization of a fourth down forms a straightforward extension of the model presented here.

A stochastic model is utilized to handle the probabilistic nature of the resulting yardage gained or lost on any particular play. A dynamic programming approach is utilized, since the problem is multistage in nature, with stages $(i \in \{1,2,3\})$ representing downs, and states $(j \in Z)$ representing the number of yards required to score a touchdown: We make the basic assumption that yardage is restricted to the set Z of positive integers. Due to the assumed field position within 10 yards of the opponent's goal and the 100-yard playing field, Z is a finite set defined on the interval [1, 99]. A game theoretic approach is embedded within this dynamic program due to the conflicting nature of offensive and defensive objectives. We assume that both play callers behave rationally; that is, both play callers will use the minimax criterion in determining their optimal strategy for any given yardage and down situation. Since yardage gained by the offensive team corresponds to yardage lost by the defensive team, each play may be modeled as a zero sum game. The offensive objective is to choose a sequence of plays that will maximize the probability associated with scoring a touchdown over three plays.

Let $r_{mn} = (r_{mn1}, r_{mn2} \ldots r_{mnk})$ denote the vector of k feasible outcomes, stated in terms of yardage gained (or lost) when the defense chooses to run defense m, $m = 1, 2 \ldots M$, and the offense chooses to run play n, $n = 1, 2 \ldots N$. We assume for purposes of clarity that these outcomes are monotonically increasing. $p_{mn} = (p_{mn1}, p_{mn2} \cdots p_{mnk})$ will represent the associated probability vector for r_{mn}, such that the probability of event r_{mnt} occurring is p_{mnt} $\forall t = 1, 2 \ldots k$. Note that negative values in vector r_{mn} would indicate potential outcomes resulting in lost yardage.

Consider a third down play, where j yards are required for a touchdown (i.e., $i = 3$), in state j. Since the offensive team's objective is to score a touchdown on this final stage play, the offensive payoff for any ordered pair of defensive and offensive plays is the probability that a touchdown is achieved on this play. This probability P_{ij} (m, n) may be shown to equal $P_{ij}(m,n) = P_{3j}(m,n) = \sum_{\forall a | r_{mna} \geq j} p_{mna}$. Utilizing these payoffs, denote the optimal offensive mixed strategy for the resulting M x N zero-sum game entering stage i in state j by $O_{ij} = (o_{ij1}, o_{ij2} \ldots o_{ijN})$, where o_{ijn} represents the probability that the offensive will choose to run offensive play n, $n = 1, 2 \ldots N$, when entering stage i and state j. The corresponding optimal defensive mixed strategy is given by $D_{ij} = (d_{ij1}, d_{ij2} \ldots d_{ijM})$, where d_{ijm} represents the probability that the defense will choose to run defensive setup m, $m = 1, 2 \ldots M$. We make the ordinary assumption that offensive and defensive choices are made simultaneously, with the consideration of offensive changes in strategy based on observing the defensive set-up prior to calling a play, or calling an audible, left as an implication for future research. The resulting probability for the optimal mixed strategy vectors O_{ij} and D_{ij} is given by v^*_{ij}. Specifically, v^*_{ij} represents the probability that the offense will score a touchdown if both offense and defense follow their optimal mixed strategies and further represents the payoff for entering stage i in state j in the ensuing dynamic programming formulation. The offense's objective is to maximize the particular probability v_{1j}.

The resulting optimal mixed strategies for first and second down are determined through the use of dynamic programming. For example, assume that the offensive team is entering stage i, $i < 3$, in state j, and wishes to determine the payoff, or probability of scoring a touchdown, for the combination of offensive play n and defensive setup m. This resulting payoff may be shown to equal the following:

$$P_{ij}(m,n) = \sum_{\forall a | r_{mna} \geq j} p_{mna} + \sum_{\forall a | r_{mna} < j} p_{mna} v^*_{i+1,\, j-r_{mna}}, \quad i < 3 \tag{1}$$

The first term represents the probability that a touchdown is achieved on the play being called in stage i. The second term represents the probability of not scoring a touchdown in stage i, but achieving a touchdown on an ensuing play within the remaining stages. Utilizing the payoffs P_{ij} (m, n), the minimax criterion can be employed to find the optimal strategy sets O_{ij} and D_{ij}, as well as the resulting probability v^*_{ij} for each stage and state combination. The minimax criteria is commonly employed in game theoretic settings: The resultant is a strategy for Player A that cannot be exploited by the opponent (Player B), and where any deviation from the optimal strategy by Player B will ordinarily increase the expected payoff to Player A and never decrease it. The resulting formulation for the resulting stochastic dynamic programming problem, from the offensive perspective, is presented below in Equations 2–6:

$$Maximize: v_{1j} \quad \forall j \tag{2}$$

subject to

$$v^*_{ij} = Max\, v_{ij} \quad \forall i > 1 \tag{3}$$

$$\sum_{n=1}^{N} \left[P_{ij}(m,n)\right]\left[o_{ijn}\right] \geq v_{ij} \;\; \forall\; m \tag{4}$$

$$\left[o_{ijn}\right] \geq 0 \;\forall n \tag{5}$$

$$P_{ij}(m,n) = \begin{cases} \sum\limits_{\forall a | r_{mna} \geq j} p_{mna} + \sum\limits_{\forall a | r_{mna} < j} p_{mna} v^*_{i+1,\, j-r_{mna}} & \forall i < 3,\ \forall m,\ \forall n \\ \sum\limits_{\forall a | r_{mna} \geq j} p_{mna} \;\; \text{if } i = 3,\ \ \forall m,\ \forall n \end{cases} \tag{6}$$

A similar dynamic programming formulation may be developed from the defensive perspective. The following numerical example illustrates how the model given by Equations 2–6 can be used to develop offensive play calling strategies for first-and-goal situations.

Illustrative Numerical Example

Assume that two general play options exist for the offense: run and pass, hence, $N = 2$. In turn, the defense may either defend against the run or pass, consequently, $M = 2$. The outcome vectors r_{mn} and associated probability vectors p_{mn} are presented in Table 1. These values were generated by a division I-AA college assistant football coach after reviewing numerous game tapes of first-and-goal situations over the past season, and sorting the data into the classes shown in Table 1. Consider the upper left hand value in Table 1. This cell shows the probability distribution of outcomes when a run offense is employed against a run defense. The result is 0, 1, or 2 yards gained with probabilities .4, .4, and .3, respectively. Utilizing the values in Table 1, together with Equations 2–6, the optimal mixed strategy vectors $O_{ij} = (o_{ijr}, o_{ijp})$ and $D_{ij} = (d_{ijr}, d_{ijp})$ may be determined, where a run or pass is designated by r and p, respectively. These values are presented below in Tables 2 and 3.

Table 1 Outcome and Probability Vectors

Offense	Defense: Run	Defense: Pass
Run	$r_{rr} = (0, 1, 2)$, $p_{rr} = (.4, .3, .3)$	$r_{pr} = (0, 2, 4)$, $p_{pr} = (.2, .5, .3)$
Pass	$r_{rp} = (0, 5, 9)$, $p_{rp} = (.3, .5, .2)$	$r_{pp} = (0, 3, 6)$, $p_{pp} = (.6, .2, .2)$

Table 2 Offensive Mixed Strategy Sets $O_{ij} = (o_{ijr}, o_{ijp})$

Down	Yards-to-go (j)									
(i)	1	2	3	4	5	6	7	8	9	10
3	.60, .40	.38, .62	0, 1.0	.63, .37	0, 1.0	0, 1.0	0, 1.0	0, 1.0	0, 1.0	.50, .50
2	.60, .40	.41, .59	.50, .50	.51, .49	.68, .32	0, 1.0	.75, .25	.68, .32	0, 1.0	.75, .25
1	.60, .40	.43, .57	.50, .50	.52, .48	.60, .40	.67, .33	.67, .33	.61, .39	0, 1.0	.50, .50

Table 3 Defensive Mixed Strategy Sets $D_{ij} = (d_{ijr}, d_{ijp})$

Down	Yards-to-go (j)									
(i)	1	2	3	4	5	6	7	8	9	10
3	.80, .20	.50, .50	0, 1.0	.12, .88	0, 1.0	.50, .50	0, 1.0	0, 1.0	0, 1.0	.50, .50
2	.80, .20	.54, .46	.16, .84	.24, .76	.01, .99	0, 1.0	.10, .90	.08, .92	.08, .92	.03, .97
1	.80, .20	.58, .42	.34, .66	.33, .67	.18, .82	.08, .92	.12, .88	.14, .86	0, 1.0	.31, .69

From Tables 2 and 3 we are able to determine the percentage of time that both the offense and defense should employ run and pass strategies. For example, for a second down situation from the 4-yard line (Stage 2, State 4), Table 2 tells us that the offense should run 51% of the time. Table 3 recommends that the defense guard against the run 24% of the time. The resulting payoffs, stated in terms of the probability of scoring a touchdown for any state and stage combination, henceforth referred to as "yards-to-go" and "down" situations, are presented below in Table 4, and further illustrated in Figure 1.

For the example stated above, where the offense faces a second down from the 4-yard line, Table 4 indicates that a touchdown will be scored either on this play or a subsequent play 56% of the time, provided that both the offense and

Table 4 Probability v_{ij} for Scoring a Touchdown

Down	Yards-to-go (j)									
(i)	1	2	3	4	5	6	7	8	9	10
3	.64	.55	.40	.26	.20	.20	0	0	0	0
2	.87	.80	.67	.56	.43	.40	.21	.17	.12	.06
1	.95	.91	.84	.76	.66	.59	.44	.38	.26	.26

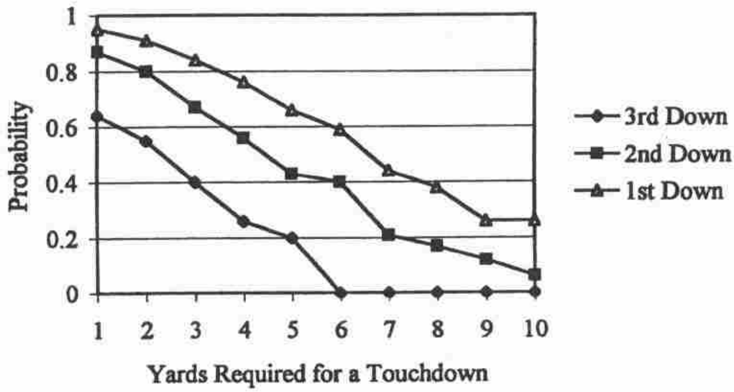

Figure 1 — Touchdown probabilities for various yards-to-go and down combinations.

defense adhere to the recommendations made in Tables 2 and 3. Sample calculations are provided in the appendix, illustrating how the results in Tables 1–4 are determined.

Results from Table 4 indicate that for a first down on the opponent's 10-yard line, the offensive team can expect to score a touchdown 26% of the time. This probability is achieved by adhering to the strategy given in Table 2. For instance, the offensive team should run the ball 50% of the time on first down from the opponent's 10-yard line. If, for example, a gain of 3 yards is achieved on this first down play, then the offense faces second down and 7 yards to go for a touchdown. In this case, Table 2 suggests that the offense should run the ball 75% of the time. By adhering to this strategy, the offense can expect to score a touchdown either on this down or a subsequent down 21% of the time, as shown in Table 4. If 5 yards are gained on this second down play, the offense now faces third and 2 yards to go for a touchdown. From Table 2 it is seen that the offense should run the ball 38% of the time, with a resulting touchdown occurring 55% of the time.

It should be noted that the offense can deviate from the optimal minimax play breakdown given in Table 2; however, the resulting probability of scoring a touchdown will decrease provided that the defensive team continues to follow its own optimal strategy as shown in Table 3. Deviations from the offensive optimal strategies may increase the probability of scoring a touchdown only in those situations where the defense deviates from their optimal strategies. In general, this type of deviation is not recommended, under the assumption that play callers act rationally and adhere to the strategies suggested in Tables 2 and 3.

Results have also been presented for first down situations where the ball is within the 10-yard line of the opposition. As expected (and supported by the results of Table 4), shorter yardage requirements increase the probability that a touchdown will be scored over the sequence of downs. For example, if the offense begins at the opponent's 5-yard line, then the probability of scoring a touchdown over three plays is 66%, whereas this probability increases to 91% if the initial placement of the ball on first down is on the opponent's 2-yard line.

A set of simulations were conducted to illustrate the benefits of employing the optimal strategies suggested in Tables 2 and 3. Six such sets were conducted, each consisting of 50 first-and-ten situations. For the first three sets, offensive plays were selected utilizing a random number generator (RNG) and the optimal strategies, while defensive plays were selected by a Division I-AA college football coach relying on football expertise. For example, if the RNG, which generates a uniformly distributed value between zero and one, returns a value within the interval [0,.41] on a second down and four-to-go situation, then the offense will run. The RNG also generates similar results in order to determine the actual yardage gained once the offense and defense have made choices regarding strategies for each play. The situation was reversed for the final three sets, where offensive plays were selected by the football coach, and defensive plays were selected using a random number generator and the optimal strategies. In all instances the football coach (a) was provided with complete information regarding potential outcomes and outcome probabilities prior to the simulation being conducted, and (b) was able to view and utilize this information during the simulations. Simulation results are presented below in Table 5.

Table 5 Simulation Results

	Set	TDs scored/sequences		TD%
Optimal offense	1	16/50		.32
versus coached	2	10/50		.20
defense	3	23/50		.46
			Total TD%:	.33
Optimal defense	4	7/50		.14
versus coached	5	6/50		.12
offense	6	8/50		.16
			Total TD%:	.14

For Sets 1–3 and 4–6 the strategy of following the results of Tables 2 and 3 outperformed the expertise of the football coach, with one exception (Set 2). In Sets 1–3, the offensive selections based on the optimal strategies achieved an overall scoring percentage of 33%, significantly exceeding the optimal probability of 26% (p *value* $\cong .025$) despite the relatively small sample size. Hence, the use of the optimal strategy for calling offensive plays resulted in 27% more touchdowns when the defensive plays were selected by the football coach than would be expected if the defensive plays chosen adhered to the optimal strategies. Results were even more dramatic when defensive plays were chosen utilizing the optimal strategy taken from Table 3, with offensive plays chosen by the football coach. In these instances (Sets 4–6), the overall touchdown percentage was only 14%, far below the optimal probability of 26% (p value $< .001$). In this case, the use of the optimal strategy has resulted in the football coach scoring only 14% of the time, whereas the expected percentage of touchdowns scored would be 26% had the optimal strategy been adhered to when offensive plays were chosen. Results of these simulations indicate the extent to which the optimal strategies suggested by Tables 2 and 3 may exceed those outcomes achieved by knowledgeable football personnel.

Implementation and Implications

Although the model results presented here provide recommendations based on the data set utilized, practical problems do exist in (a) the accumulation of data and (b) the overall value of the data collected as it concerns accuracy when implemented and applied to actual competition. For example, data specific to an opposing team is either limited in nature or perhaps unavailable until well into or after a contest is played. Even when prior data is available for a competition between two teams, the nature of the game suggests that this data may be imperfect in nature. For example, trades or managerial preferences might lead to the opposition using a different alignment of players than anticipated. Even when precise data from a prior game between two opponents is available, the analysis may still be limited by the high variability involved in small sample analysis.

Larger data sets drawn from games between a number of opponents may generate more complete information, but the degree to which this set of data adequately represents the results between any two particular teams is clearly an open question. If empirical differences in outcomes are relatively small between teams, one may suggest that the model results asymptotically approach optimal strategies as the universe of situations is revealed. However, the comparison of these empirical results between teams requires additional testing in the field using real data collected from actual games.

What is clear is that additional research into data collection and testing, perhaps in scrimmages, is necessary prior to the successful implementation of the model. We do, however, advocate the notion that the model can provide reasonable guidelines regarding the type of offensive and defensive strategies that can be employed. For example, the model might help uncover trends in play selection that might not otherwise be noted, such as running in situations where the previous inclination based on heuristic techniques or instinct suggests passing.

Given that the limitations and issues of implementation are resolved, the net benefits of the model have implications to strategic managerial action in addition to the more obvious implications to on-the-field performance. For example, sports culture has evolved over the past 30 years. Whereas loyalty and allegiance may have previously been salient attributes in fan support, the continued development and emergence of sporting events as a high-profile source of entertainment has led to increased reliance on a team's on-the-field performance as a criteria for attracting fans. Given the disparity in attendance between winning and losing teams in many professional sports and the constant geographical shift for some small market franchises that are unable to attract fans or generate winning records, increased on-the-field performance may have significant benefits in generating revenues.

A similar and interrelated analysis might apply to a team's ability to generate interest in high-profile, marquee players, who oftentimes indicate that a team's ability to (a) compete and contend for a championship and (b) provide lucrative compensation are top attributes when considering a location to play. If we assume that increased performance leads to a larger revenue base through fan support, then the utilization of the model results provided here would increase a team's potential for both attracting and signing marquee players.

Lastly, a third factor related to the two discussed above concerns significant network-cable revenues, which may be impacted by team performance—which, as stated earlier, is directly tied to the successful implementation of quantitative models similar in nature to that presented here. Many networks have caps enforced regarding the maximum number of exposures any team may have, since it is a network's natural inclination to broadcast games involving either large market teams (with a greater geographical viewing base), winning teams, or teams with a large number of marquee players.

The result of these factors collectively is cyclical. For example, consider the following possible scenario: The use of quantitative models improves on-the-field performance thus increasing winning percentage, which in turn increases local fan support and team revenues. The increased winning percentage, fan support, and revenue base then provides a team with a competitive advantage in attracting marquee players. The addition of these players results in further increased performance. These factors cumulatively increase interest from networks, providing additional exposure to fans as well as increased revenues from network resources. In turn, this exposure increases the local geographical fan base, thus increasing local team revenues at the gate, and the cycle begins again.

In conclusion, the financial viability of a professional sports organization may be impacted significantly through the consideration of quantitative models such as that discussed here.

Conclusions and Future Research

This paper considers the determination of optimal play selection for a football team facing a first down within 10 yards of the opponent's goal. The methodology employed incorporates a stochastic dynamic program within which game theoretic components are embedded. The results of the ensuing formulation provide percentage breakdowns for various offensive plays and defensive formations that can be employed. The objective is to determine the sequence of plays that will maximize the probability of scoring a touchdown.

Results are encouraging. For example, when the optimal strategies are employed against a football coach with expertise in play selection, not only does the optimal strategy result in a higher percentage of touchdowns scored, but this percentage exceeds the optimal probability given by the model. This is consistent with the fact that the optimal probability provides a benchmark that may be exceeded when the opponent deviates from their optimal minimax strategy.

Implications for future research involve the generalization of the model to account for any field position. In these cases, care must be exercised in developing the offensive team's objective. Although similar in nature, the multi-objective nature of the problem should be considered, where two objectives may apply: Maximizing the probability of achieving a first down and maximizing expected yardage gained. Other implications for future research include statistical issues of developing cumulative distribution functions for various offensive–defensive plays, where available data is often limited. Simulation may also be considered in evaluating the potential benefits of the model. Finally, it should be noted that the application of this model to American football is only one possible application to which this type of analysis might apply. Game theoretic models, or variants thereof, could be applied to many related athletic competitions that follow similar offensive–defensive structures. For example, one suggested application might involve the batter-pitcher confrontation in baseball, where the pitcher must choose what type of pitch to throw on a given count and the batter must attempt to guess, prior to the pitch being delivered, what type of pitch to anticipate.

Given the lack of quantitative analysis dedicated to play selection in American football, the results here form an interesting and potentially fertile area of continued research. The numerical and simulated results presented demonstrate both the feasibility and practicality of incorporating quantitative methodologies to play selection. The potential benefits are clear, although the application of these techniques is limited by the quality of the data sets that are available and the great uncertainty involved in the estimation of yardage gained for any particular offense when matched against another particular defense. Given the marked increase in costs associated with operating franchises in professional football, the utilization of sound quantitative methodologies provides an alternative through which on-field performance can be enhanced, which in turn may impact both the popularity and success of the organization as a business and sports franchise.

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Appendix

(a) Consider stage $i=3$, state $j=4$, or third down from the opponent's 4-yard line. To find the optimal offensive strategy vector $O_{34}=(o_{34r}, o_{34p})$ and associated security level v^*_{ij}, utilize Equations 3–5, as follows:

$$v^*_{34} = Maximum\ v_{34} \tag{A1}$$

Subject to:

$$\begin{aligned} 0o_{34r} + .7o_{34p} &\geq v_{34} \quad m=r \\ .3o_{34r} + .2o_{34p} &\geq v_{34} \quad m=p \end{aligned} \tag{A2}$$

$$o_{34r}, o_{34p} \geq 0 \tag{A3}$$

The resulting solution for the preceding linear program is $O_{34}=(.63, .37)$, found in Table 2, with $v^*_{34}=.26$, specified in Table 4. The objective (A1) corresponds to the objective (3), whereas constraint sets (A2) and (A3) correspond to constraint sets (4), and (5), respectively.

(b) Consider stage $i=2$, state $j=2$, or, restated, second down and 2 yards to go for a touchdown. In order to ascertain the optimal offensive strategy, utilize equations (3), (5), and (6), as follows:

$$v^*_{22} = Maximum\ v_{22} \tag{A4}$$

Subject to:

$$\begin{aligned} (.3+.4v^*_{32}+.3v^*_{31})o_{22r} + (.7+.3v^*_{32})o_{22p} = (.3+.4(.55)+.3(.64))o_{22r} + (.7+.3(.55))o_{22p} &\geq v_{22} \quad m=r \\ (.8+.2v^*_{32})o_{22r} + (.4+.6v^*_{32})o_{22p} = (.8+.2(.55))o_{22r} + (.4+.6(.55))o_{22p} &\geq v_{22} \quad m=p \end{aligned} \tag{A5}$$

$$o_{22r}, o_{22p} \geq 0 \tag{A6}$$

The resulting solution for the preceding linear program is $O_{22}=(.41, .59)$, found in Table 2, with $v^*_{22}=.80$, specified in Table 4. The objective (A4) corresponds to the objective (3), whereas constraint sets (A5) and (A6) correspond to constraint sets (6) and (5), respectively.