Question 5 by Dan Nguyen (z5206032)

Consider the following theorems.

Thereom 1 – Big O Notation:

$$f(n) = O(g(n)) \text{ if } \exists\ C, N > 0 \text{ such that } 0 \leq f(n) \leq Cg(n)\ \forall\ n \geq N$$

Thereom 2 – Big Ω Notation:

$$f(n) = \Omega(g(n)) \text{ if } \exists\ c, N > 0 \text{ such that } 0 \leq cg(n) \leq f(n)\ \forall\ n \geq N$$

Thereom 3 – Θ Notation:

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$

The general method to showing that a given $f(n)$ and $g(n)$ satisfies the above Theorems, is to rearrange the inequality with $f(n)/g(n)$ or $g(n)/f(n)$ in the middle of the inequality so it is bounded by zero on the left-hand side and some constant on the right-hand side. The Theorem satisfaction criteria are thus:

- If $f(n)/g(n)$ converges, then it will satisfy Theorem 1. It can be said that $f(n)$ does not grow faster than $g(n)$.
- If $f(n)/g(n)$ does not converge, then it is divergent and does not satisfy Theorem 1. There is no need to check the type of divergence.
- If $g(n)/f(n)$ converges, then it will satisfy Theorem 2. It can be said that $f(n)$ does not grow slower than $g(n)$.
- If $g(n)/f(n)$ does not converge, then it is divergent and does not satisfy Theorem 2. There is no need to check the type of divergence.

Part A

The following pair of equations is given:

$$f(n) = n^{1+log(n)} \tag{1}$$

$$g(n) = nlog(n) \tag{2}$$

Theorem 1

Substituting Equations 1 and 2 into the Theorem 1 inequality yields:

$$0 \leq n^{1+log(n)} \leq Cnlog(n) \tag{3}$$

Rearranging and simplifying Equation 3 yields:

$$0 \leq \frac{n^{log(n)}}{log(n)} \leq C \tag{4}$$

Equations 1 and 2 satisfies Theorem 1 if:

$$\lim_{n \to \infty} \frac{n^{log(n)}}{log(n)} = L, L \in [0, C] \tag{5}$$

Check if $f(n)/g(n)$ has an indeterminate form by considering the limits of n to infinity of the numerator and denominator, respectively:

$$\lim_{n \to \infty} n^{log(n)} \to \infty$$
$$\lim_{n \to \infty} log(n) \to \infty$$

Since both limits approach infinity, $f(n)/g(n)$ is an indeterminate form and L'Hôpital's rule can be applied to solve Equation 5:

$$\lim_{n \to \infty} \frac{n^{log(n)}}{log(n)} = \lim_{n \to \infty} \frac{2n^{log(n)-1}log(n)}{\frac{1}{n}} = \lim_{n \to \infty} 2n^{log(n)}log(n) \to \infty$$

Since the limit, L, approaches infinity, it can be said that $f(n)$ grows asymptotically faster than $g(n)$. Therefore, $f(n) \notin O(g(n))$.

Theorem 2

Substituting Equations 1 and 2 into the Theorem 2 inequality yields:

$$0 \leq cnlog(n) \leq n^{1+log(n)} \tag{6}$$

Rearranging and simplifying Equation 6 yields:

$$0 \leq \frac{log(n)}{n^{log(n)}} \leq \frac{1}{c} \tag{7}$$

Equations 1 and 2 satisfies Theorem 2 if:

$$\lim_{n \to \infty} \frac{log(n)}{n^{log(n)}} = L, L \in [0, \frac{1}{c}] \tag{8}$$

From checking the limits of the numerator and denominator of $f(n)/g(n)$, $g(n)/f(n)$ is known to be an indeterminate form. Applying L'Hôpital's rule to solve Equation 8:

$$\lim_{n \to \infty} \frac{log(n)}{n^{log(n)}} = \lim_{n \to \infty} \frac{\frac{1}{n}}{2n^{log(n)-1}log(n)} = \lim_{n \to \infty} \frac{1}{2n^{log(n)}log(n)} = 0$$

Since the limit, L, is zero, it can be said that $f(n)$ grows no slower than $g(n)$. Therefore, $f(n) \in \Omega(g(n))$.

Case

Since Equations 1 and 2 does not satisfy Theorem 1 and satisfies Theorem 2, this problem is classified as a case II.

Part B

The following pair of equations is given:

$$f(n) = n^{1+\frac{1}{2}cos(\pi n)} \tag{9}$$

$$g(n) = n \tag{10}$$

Theorem 1

Substituting Equations 9 and 10 into the Theorem 1 inequality yields:

$$0 \leq n^{1+\frac{1}{2}cos(\pi n)} \leq Cn \tag{11}$$

Rearranging and simplifying Equation 11 yields:

$$0 \leq n^{\frac{1}{2}cos(\pi n)} \leq C \tag{12}$$

Equations 9 and 10 satisfies Theorem 1 if:

$$\lim_{n\to\infty} n^{\frac{1}{2}cos(\pi n)} = L, L \in [0, C] \tag{13}$$

From Equation 13, the square root term dominates the function, thus the lower and upper bounds of Equation 13 are, respectively:

$$\lim_{n\to\infty} n^{-\frac{1}{2}} \leq \lim_{n\to\infty} n^{\frac{1}{2}cos(\pi n)} \leq \lim_{n\to\infty} n^{\frac{1}{2}}$$

Solving the left-hand side and right-hand side limits shows that L approaches zero and infinity simultaneously and is boundedly divergent:

$$\lim_{n\to\infty} n^{-\frac{1}{2}} = 0$$

$$\lim_{n\to\infty} n^{\frac{1}{2}} \to \infty$$

Therefore, $f(n) \notin O(g(n))$.

Theorem 2

Substituting Equations 9 and 10 into the Theorem 2 inequality yields:

$$0 \leq cn \leq n^{1+\frac{1}{2}cos(\pi n)} \tag{14}$$

Rearranging and simplifying Equation 14 yields:

$$0 \leq \frac{1}{n^{\frac{1}{2}cos(\pi n)}} \leq \frac{1}{c} \tag{15}$$

However Inequality 15 has the same bounds as Inequality 12. Thus L will approach zero and infinity simultaneously and is boundedly divergent.

Therefore, $f(n) \notin \Omega(g(n))$.

Case

Since Equations 9 and 10 neither satisfies Theorem 1 and Theorem 2, this problem is classified as a case IV.

Part C

The following pair of equations is given:

$$f(n) = log_2(n^{log(nlog(n))}) \tag{16}$$

$$g(n) = (log(n))^2 \tag{17}$$

Theorem 1

Substituting Equations 16 and 17 into the Theorem 1 inequality yields:

$$0 \leq log_2(n^{log(nlog(n))}) \leq C(log(n))^2 \tag{18}$$

Rearranging and simplifying Equation 18 (refer to Appendix A for working out) yields:

$$0 \leq \frac{1}{log(2)} + \frac{log(log(n))}{log(2)log(n)} \leq C \tag{19}$$

Equations 16 and 17 satisfies Theorem 1 if:

$$\lim_{n\to\infty} \frac{1}{log(2)} + \frac{log(log(n))}{log(2)log(n)} = L, L \in [0, C] \tag{20}$$

Solving Equation 20 for L:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{log(2)} + \frac{log(log(n))}{log(2)log(n)} &= \lim_{n\to\infty} \frac{1}{log(2)} + \lim_{n\to\infty} \frac{log(log(n))}{log(2)log(n)} \\
&= \frac{1}{log(2)} + 0 \\
&= \frac{1}{log(2)}
\end{aligned}$$

Since there exists a limit, $L = 1/log(2)$, it can be said that $f(n)$ grows no faster than $g(n)$. Therefore, $f(n) \in O(g(n))$.

Theorem 2

Substituting Equations 16 and 17 into the Theorem 2 inequality yields:

$$0 \leq c(log(n))^2 \leq log_2(n^{log(nlog(n))}) \tag{21}$$

Rearranging and simplifying Equation 21 (refer to Appendix B for working out) yields:

$$0 \leq \frac{1}{log(2)} + \frac{log(log(n))}{log(2)log(n)} \leq C \tag{22}$$

Equations 16 and 17 satisfies Theorem 2 if:

$$\lim_{n\to\infty} \frac{log(2)log(n)}{log(n) + log(log(n))} = L, L \in [0, C] \tag{23}$$

Check Equation 23 for indeterminate form:

$$\lim n \to \infty log(2)log(n) \to \infty$$
$$\lim n \to \infty log(n) + log(log(n)) \to \infty$$

Since both limits approach infinity, $g(n)/f(n)$ is an indeterminate form and L'Hôpital's rule can be applied to solve Equation 23:

$$\begin{aligned}
\lim_{n\to\infty} \frac{log(2)log(n)}{log(n) + log(log(n))} &= \lim_{n\to\infty} \frac{\frac{log(2)}{n}}{\frac{1}{n} + \frac{1}{nlog(n)}} \\
&= \lim_{n\to\infty} \frac{\frac{log(2)}{n}}{\frac{log(n)+1}{nlog(n)}} \\
&= \lim_{n\to\infty} \frac{log(2)log(n)}{log(n)+1} \\
&= \lim_{n\to\infty} log(2) \lim_{n\to\infty} \frac{log(n)}{log(n)+1} \\
&= log(2) \times 1 \\
&= log(2)
\end{aligned}$$

Since there exists a limit, $L = log(2)$, it can be said that $f(n)$ grows no slower than $g(n)$. Therefore, $f(n) \in \Omega(g(n))$.

Case

Since Equations 16 and 17 satisfies both Theorem 1 and Theorem 2 (and by extension Theorem 3), this problem is classified as a case III.

APPENDIX A Part C - $f(n)/g(n)$ simplification

$$\begin{aligned}
\frac{log_2(n^{log(nlog(n))})}{(log(n))^2} &= \frac{log(nlog(n))log_2(n)}{(log(n))^2} \\
&= \frac{log(nlog(n))log(n)}{(log(n))^2 log(2)} \\
&= \frac{log(nlog(n))}{log(2)log(n)} \\
&= \frac{log(n) + log(log(n))}{log(2)log(n)} \\
&= \frac{log(n)}{log(2)log(n)} + \frac{log(log(n))}{log(2)log(n)} \\
&= \frac{1}{log(2)} + \frac{log(log(n))}{log(2)log(n)}
\end{aligned}$$

APPENDIX B Part C - $g(n)/f(n)$ simplification

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\begin{aligned}
\frac{(log(n))^2}{log_2(n^{log(nlog(n))})} &= \frac{(log(n))^2}{log(nlog(n))log_2(n)} \\
&= \frac{(log(n))^2 log(2)}{log(nlog(n))log(n)} \\
&= \frac{log(2)log(n)}{log(n) + log(log(n))}
\end{aligned}
$$