## Question 3 by Dan Nguyen (z5206032)

Given a positive integer, $n$.

Given a decimal digit, $k$.

Let there be a set, $S_i$, of all $i$-digit integers of $\mathbb{N}$. The size of $S_i$ is $10^i - 10^{i-1}$.

Consider the case where $k$ appears an odd number of times.

Define $P(i)$ as the problem of $k$ appearing an odd number of times in $S_i$ where $p(i)$ as the solution to $P(i)$.

Define $Q(i)$ as the problem of $k$ appearing an even number of times in $S_i$ where $q(i)$ as the solution to $Q(i)$.

For each $1 \leq i \leq n$, solve for $P(i)$ and $Q(i)$ using dynamic programming where the recurrences are respectively:

$$p(i) = 9 \times p(i-1) + q(i-1)$$

$$q(i) = 9 \times q(i-1) + p(i-1)$$

The base cases for $k = 0$ are:

$$p(1) = 0, \; q(1) = 9$$

The base cases for $k > 0$ are:

$$p(1) = 1, \; q(1) = 8$$

The final answers for $P$ and $Q$ respectively are:

$$p(n) = 9 \times p(n-1) + q(n-1)$$

$$q(n) = 9 \times q(n-1) + p(n-1)$$

The order of solving $P$ and $Q$ are important i.e. subproblems with lesser $i$ must be solved first.

There are $n$ subproblems which are solved in $O(n)$. The time complexity of solving each subproblem is constant. Therefore, the overall time complexity is $O(n)$ as required.