Faculty of Engineering

School of Electrical Engineering and Telecommunications

ELEC 1111 – Topic 2

Kirchhoff's Laws, Nodal & Mesh Analysis

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Topic 2 Content

This lecture covers:

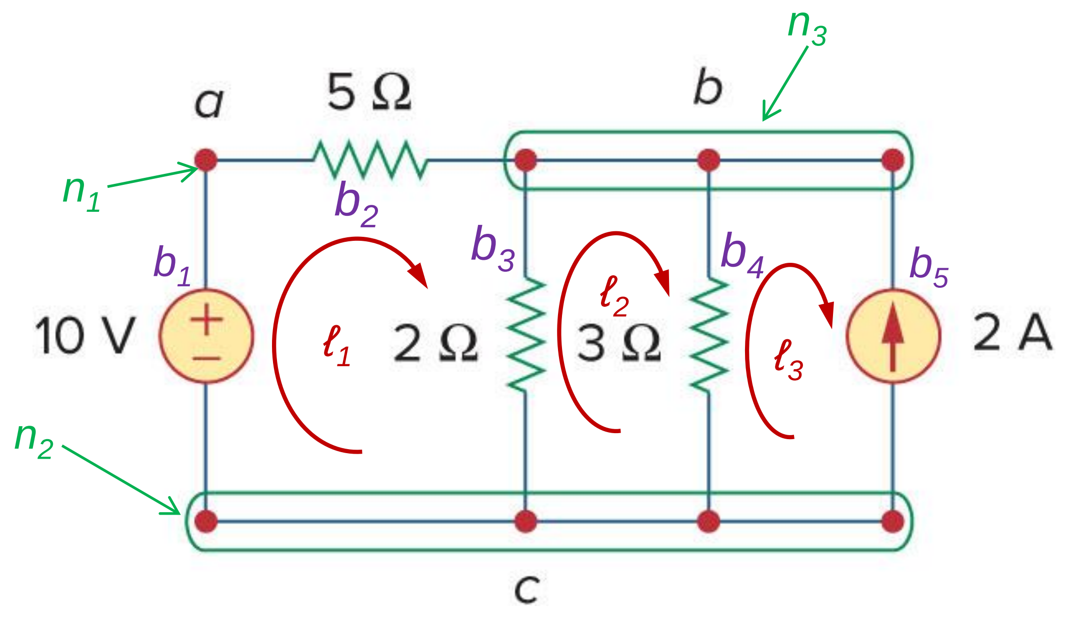
- Nodes, Branches and Loops/Meshes
- Kirchhoff's Laws
- Series and Parallel Connection of Circuit Elements
- Voltage and Current Division
- Nodal Analysis
- Mesh Analysis

Corresponds to Chapters 2 and 3 of your textbook

Nodes, branches and loops/meshes

- Circuit elements can be interconnected in multiple ways.
- To understand this, we need to be familiar with some network topology concepts:
  - A **branch** represents a **single element** such as a voltage source or a resistor.
  - A **node** is the **point of connection** between two or more branches.
  - A **loop** is any **closed path** in a circuit.
  - A **mesh** is a loop that contains **no** other loop.

n_k = node k
b_k = branch k
ℓ_k = loop/mesh k

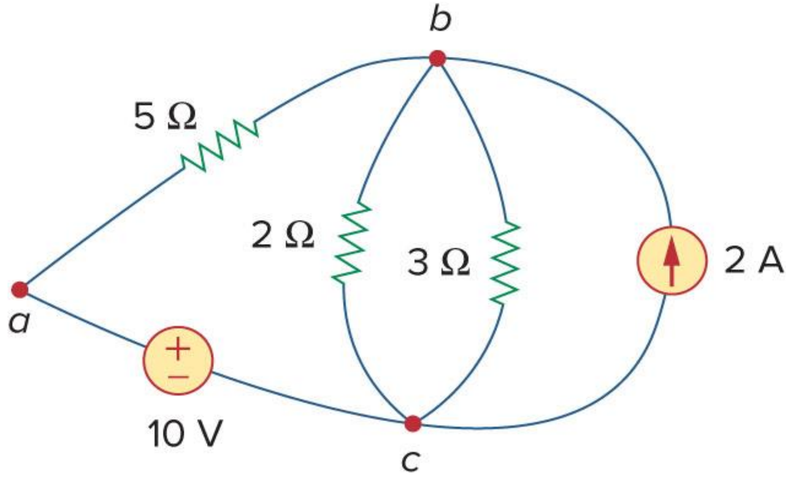


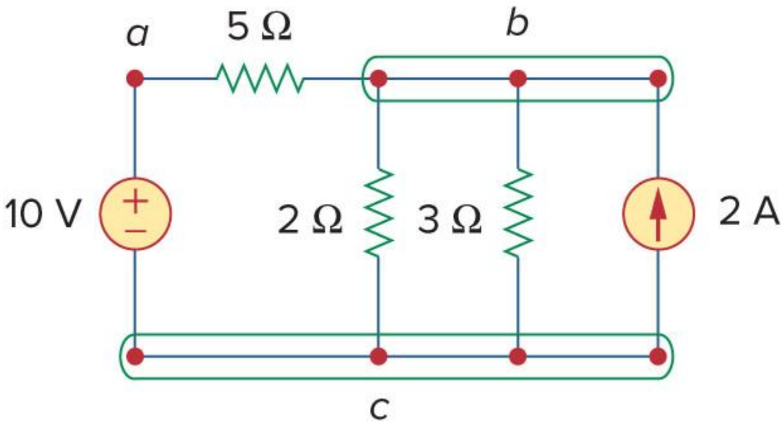
Series and parallel

- Two or more elements are in **series** if they **exclusively share a single node** and consequently carry the **same current.**
- Two or more elements are in **parallel** if they are connected to the **same two nodes** and consequently have the **same voltage** across them.

In the following example:

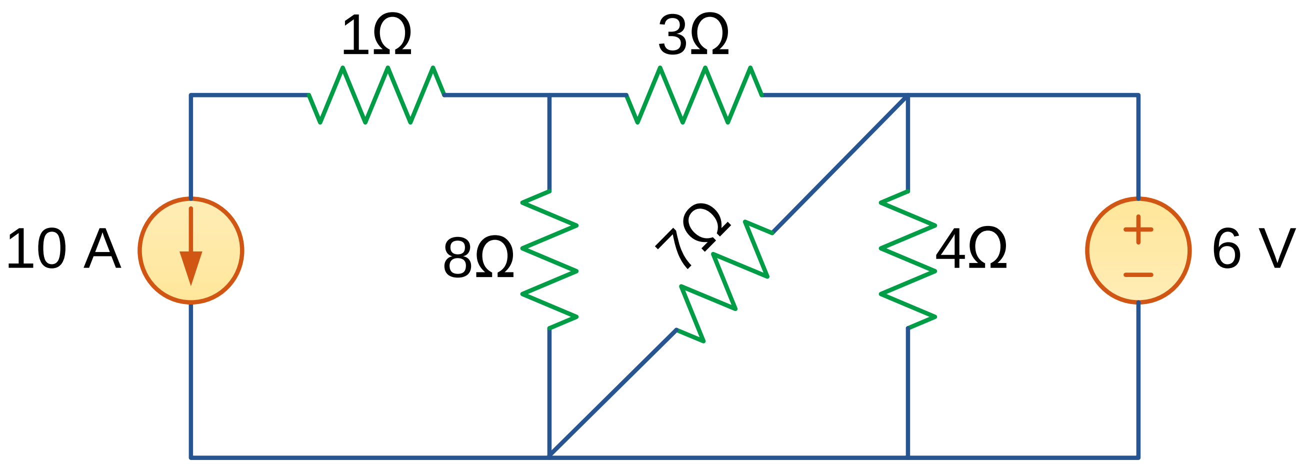
- Series: 5 Ω resistor and 10 V source.
- Parallel: 2 A source and 3 Ω and 2 Ω resistors.





Exercise

- Count the number of branches, nodes, and meshes.
- Identify series and parallel elements.



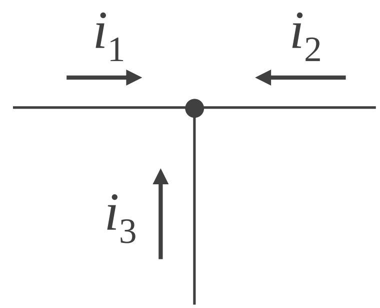
Kirchhoff's Current Law (KCL)

- The **algebraic sum** of **all currents entering a node is zero.**

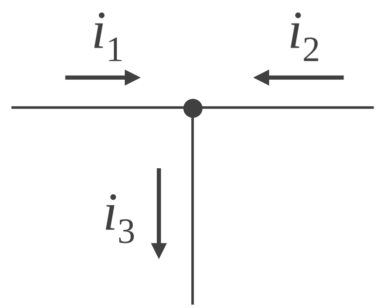
$$\boxed{\sum_{n=1}^{N} i_n = 0}$$

N is the number of branches connected to the node

- You may consider the currents **entering the node** to be **positive** and those **leaving the node** to be **negative** or vice versa.
- You may also consider that the **sum of currents entering the node** is **equal** to the **sum of currents leaving the node.**



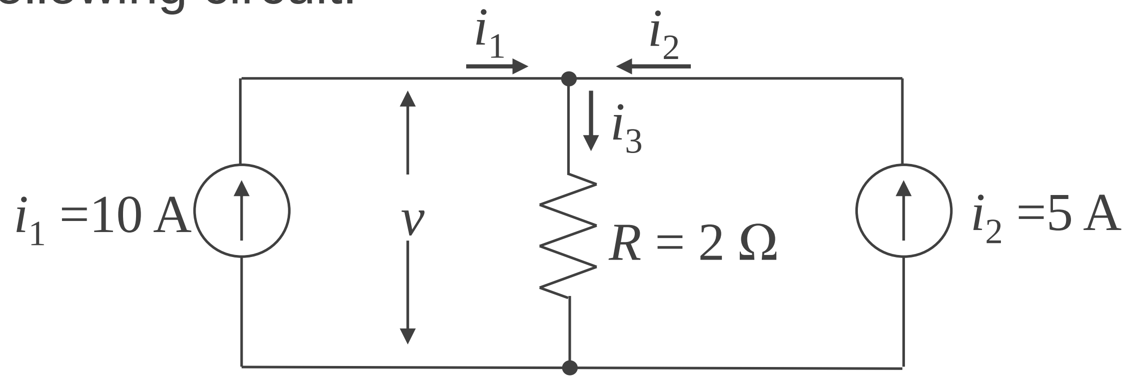
$$i_1 + i_2 + i_3 = 0$$
$$i_1 + i_2 = -i_3$$



$$i_1 + i_2 - i_3 = 0$$
$$i_1 + i_2 = i_3$$

Exercise

Given the following circuit:



- How much current flows through the resistor?

- What is the voltage across the resistor?

Kirchhoff's Voltage Law (KVL)

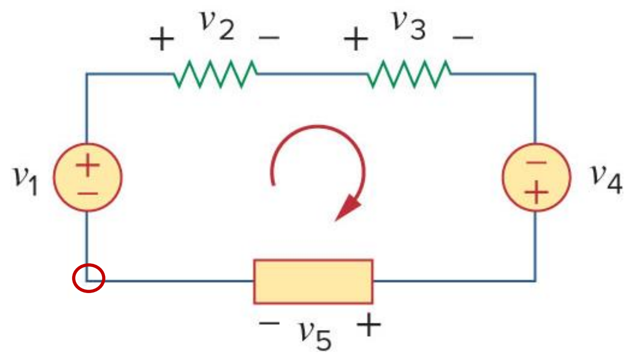
- The **algebraic sum** of **all voltage drops around a closed path** (or a loop) **is zero.**

$$\boxed{\sum_{m=1}^{M} v_m = 0}$$

M is the number of voltage drops in the loop

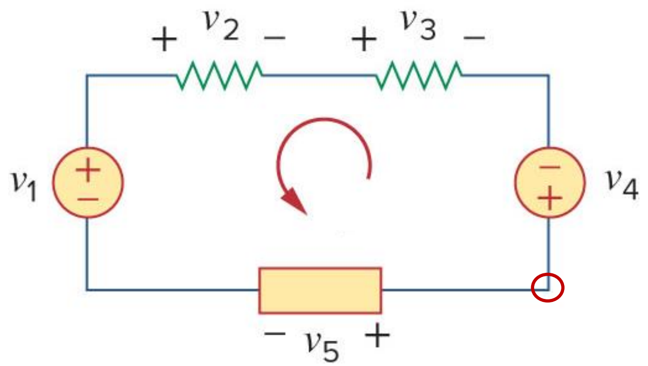
- You may **start from any node** in the loop and go around the loop **clockwise (CW)** or **counterclockwise (CCW).**
- Use the **sign** of the terminal that you **first encounter** as you go around the loop.

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$



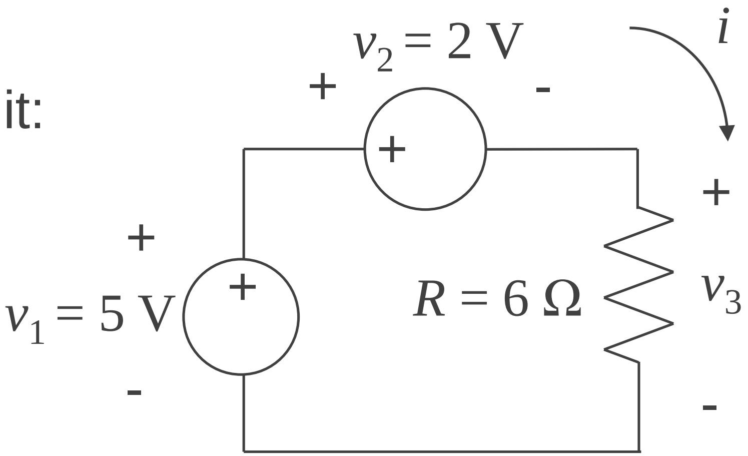
or

$$v_4 - v_3 - v_2 + v_1 - v_5 = 0$$



Exercise

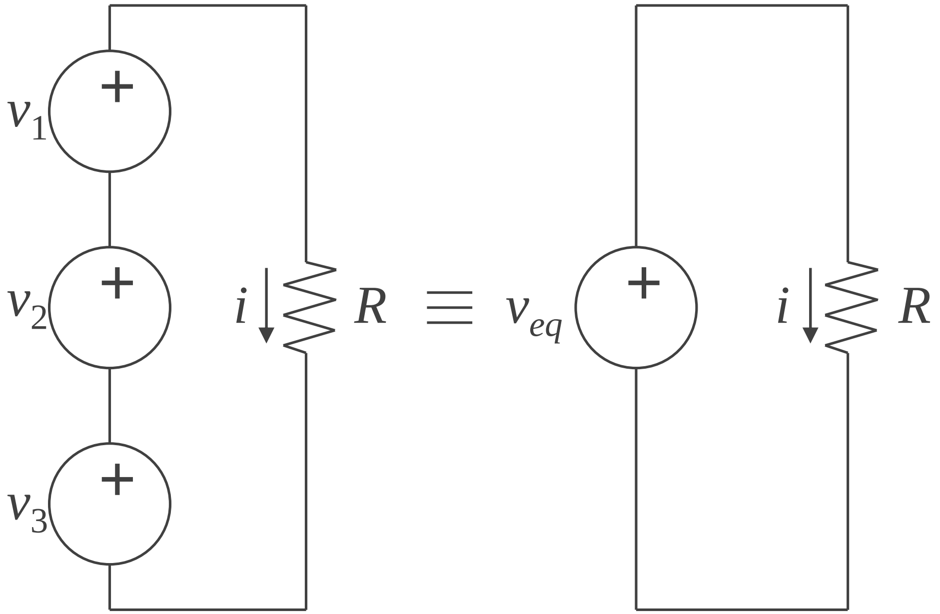
Given the following circuit:



- What is the voltage across the resistor?

- What is the current in the loop?

Voltage sources in series



- Kirchoff's Voltage Law (KVL) gives:

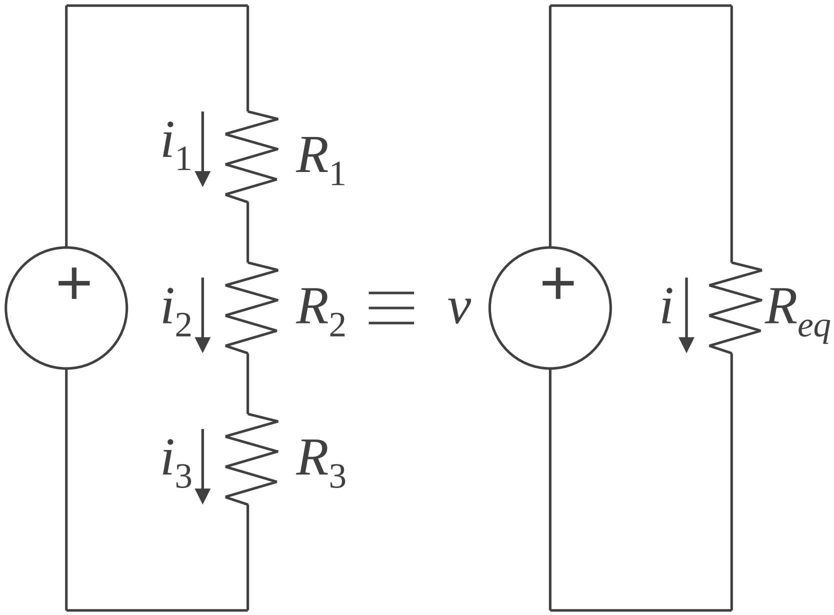
$$-v_1 - v_2 - v_3 + iR = 0$$
$$(v_1 + v_2 + v_3) = iR$$

- Equivalent voltage:

$$v_{eq} = iR$$
$$\boxed{v_{eq} = v_1 + v_2 + v_3}$$

- Voltage sources in series are summed.

Resistors in series



- Current is the same in all resistors:

$$i_1 = i_2 = i_3 = i$$

- KVL gives:

$$-v + iR_1 + iR_2 + iR_3 = 0$$

$$v = i(R_1 + R_2 + R_3)$$

- Equivalent resistance:

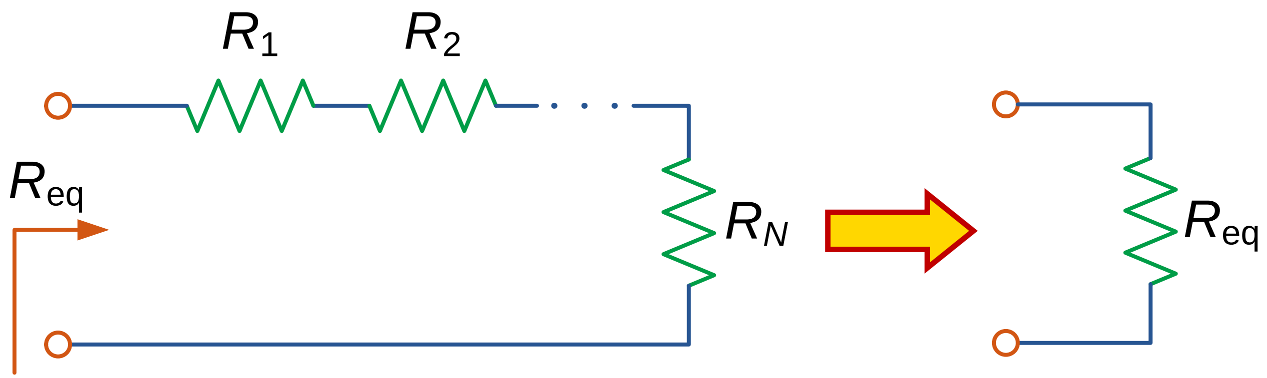
$$v = iR_{eq}$$

$$\boxed{R_{eq} = R_1 + R_2 + R_3}$$

- Resistors in series are summed.

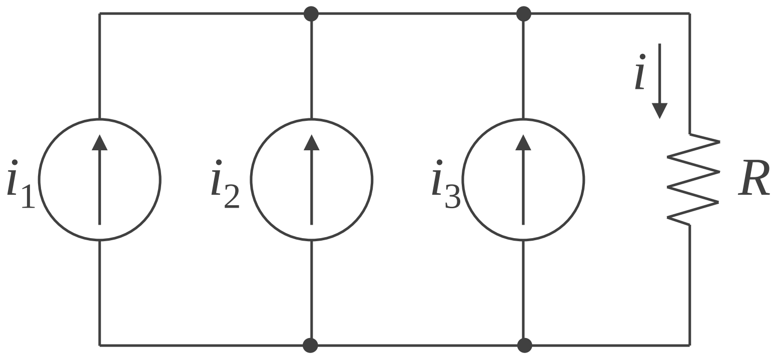
Resistors in series

- The **equivalent resistance** of any number of **series resistors** is the **sum** of the **individual resistances.**



$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n$$

Current sources in parallel



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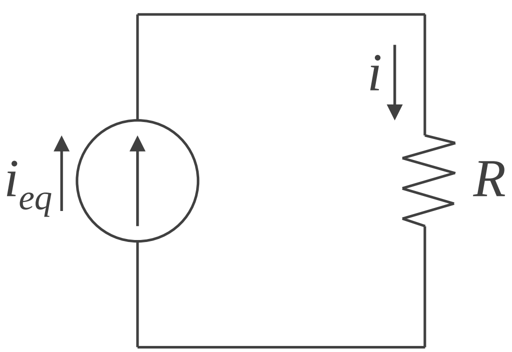


- We can find the current in the resistor by KCL:

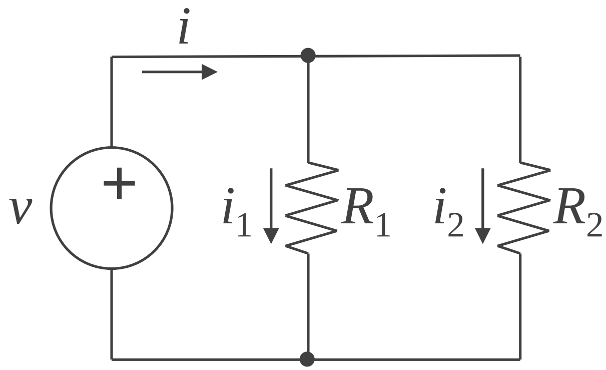
$$i_1 + i_2 + i_3 = i$$

- The equivalent current source would put the same current through the resistor, so:

$$\boxed{i_{eq} = i_1 + i_2 + i_3}$$

- Current sources in parallel are summed.

Resistors in parallel



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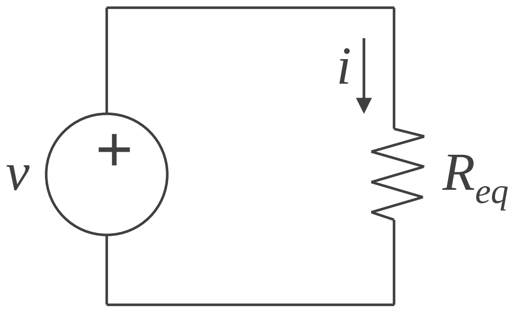


- Voltages across the resistors are the same:

$$v = i_1 R_1 = i_2 R_2$$

Therefore:

$$i_1 = \frac{v}{R_1};\ i_2 = \frac{v}{R_2}$$

- Using KCL:

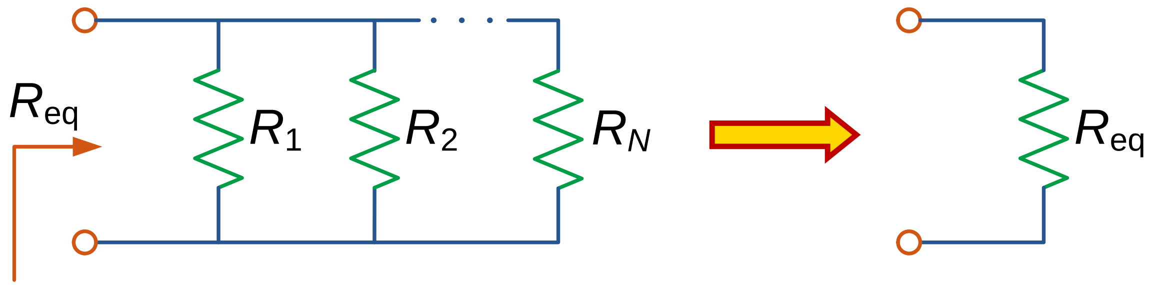
$$i = i_1 + i_2 = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\frac{i}{v} = \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\boxed{R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2}}$$

Resistors in parallel

- The **reciprocal** of the **equivalent resistance** of any number of **parallel resistors** is the **sum** of the **individual reciprocal resistances.**



$$\frac{1}{R_{\mathrm{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \cdots + \frac{1}{R_N}$$

or

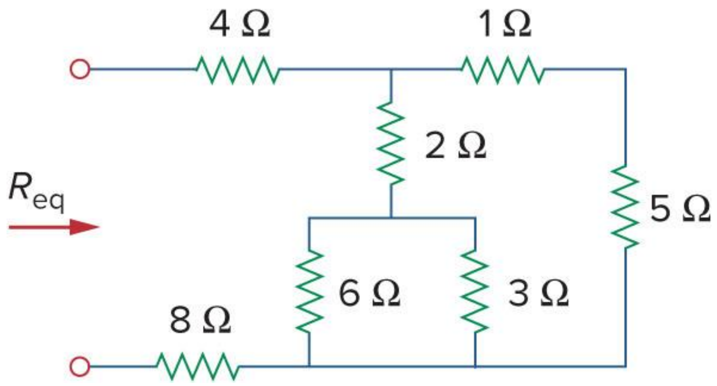
$$G_{\mathrm{eq}} = G_1 + G_2 + \cdots + G_N$$

or

$$R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}}$$

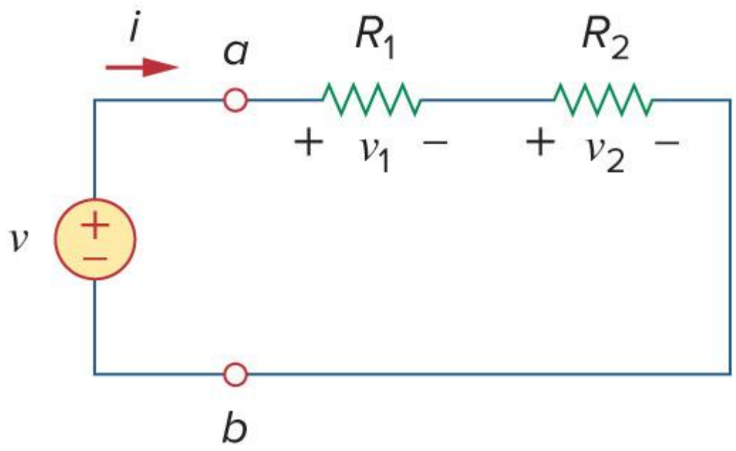
Exercise

Find the equivalent resistance R_{eq}.



Voltage divider

- A voltage divider is a simple circuit that divides a voltage **in proportion** to the **series resistances** (the higher the resistance, the higher the voltage).



- The current through the resistors is:

$$i = \frac{v}{R_1 + R_2}$$

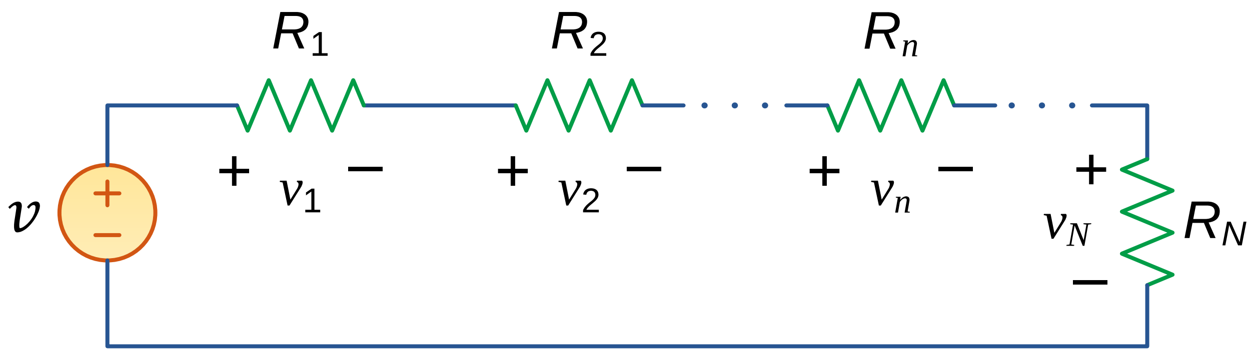
- The voltage across the resistors:

$$v_1 = iR_1; v_2 = iR_2$$

$$\boxed{v_1 = v\frac{R_1}{R_1 + R_2}; v_2 = v\frac{R_2}{R_1 + R_2}}$$

Voltage divider

- In general, the **voltage drop** across the *n*th resistor in a **voltage divider** with *N* **series resistors** is obtained as follows:

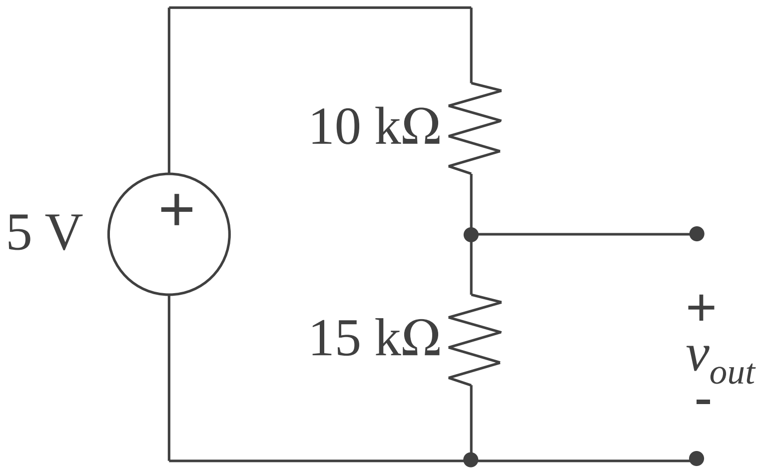


$$\boxed{v_n = v\frac{R_n}{R_1 + R_2 \cdots + R_N} = v\frac{R_n}{R_{eq}}}$$

Remark: As long as we know the input voltage to be divided, this voltage can be provided by any element or circuit, not necessarily a voltage source.

Exercise

- Find the voltage across the 15 kΩ resistor.



$$v_n = v\frac{R_n}{R_{eq}}$$

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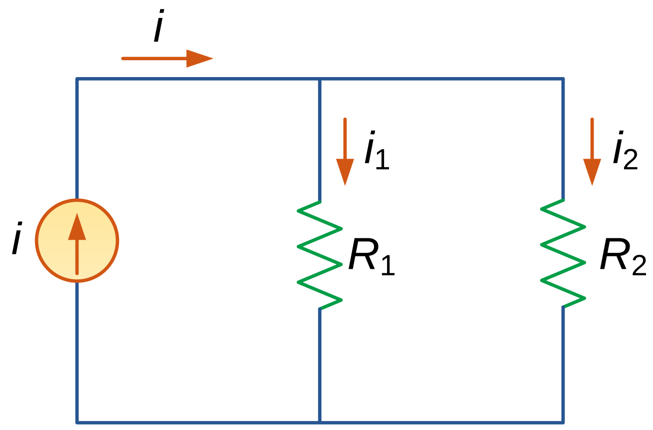
Current divider

- A current divider is a simple circuit that divides the current among **parallel resistors** in **<u>inverse</u> proportion** to their **resistance** (the higher the resistance, the lower the current).



- The voltage across the resistors is:

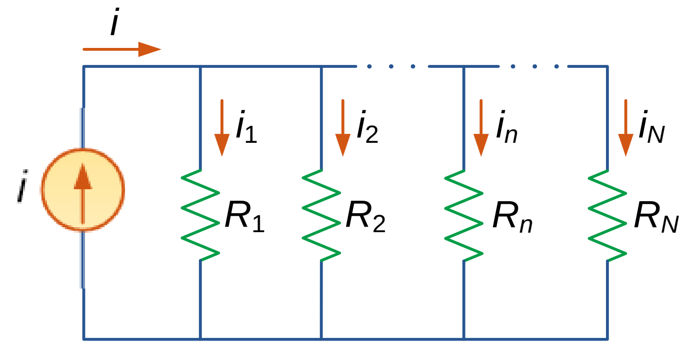
$$v = i(R_1 \| R_2)$$

- The current through each resistor:

$$\boxed{\begin{aligned} i_1 &= \frac{v}{R_1} = i\,\frac{(R_1 \| R_2)}{R_1} \\ i_2 &= \frac{v}{R_2} = i\,\frac{(R_1 \| R_2)}{R_2} \end{aligned}}$$

Current divider

- In general, the **current** through the n^{th} resistor in a **current divider** with N **parallel resistors** is obtained as follows:



$$\boxed{i_n = i\frac{R_1 \| R_2 \| \cdots \| R_N}{R_n} = i\frac{R_{eq}}{R_n}}$$

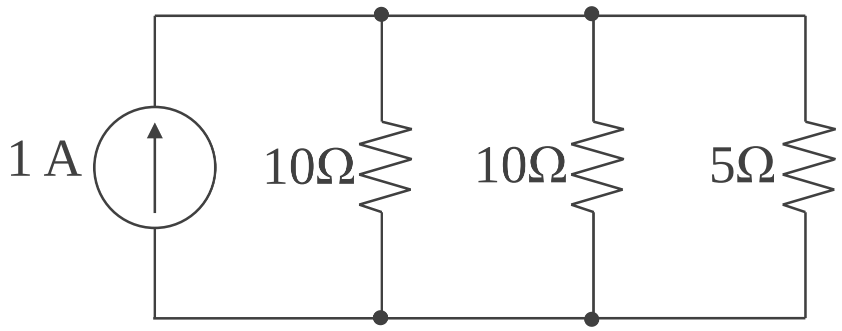
- If conductances are used instead of resistances, then:

$$\boxed{i_n = i\frac{G_n}{G_1 + G_2 \cdots + G_N}, \quad \text{where } G_n = \frac{1}{R_n}}$$

Remark: As long as we know the input current to be divided, this current can be provided by any element or circuit, not necessarily a current source.

Exercise

- Find the current in the 5 Ω resistor.



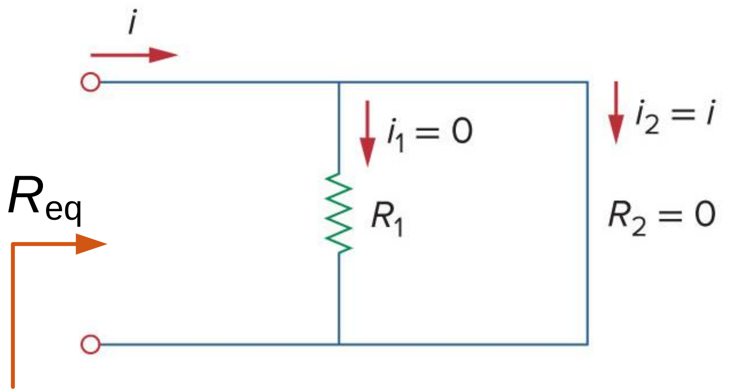
$$i_n = i\frac{R_{eq}}{R_n}$$

Current divider

Specific cases

- **Short circuit:** The entire current i flows through the **smallest resistance** (short circuit), effectively bypassing R_1.

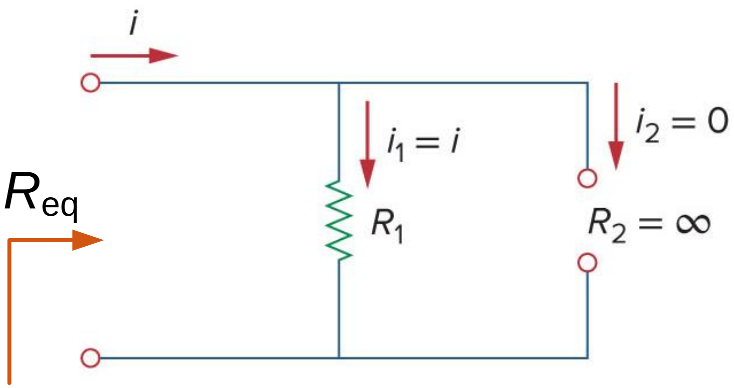
$$\boxed{R_{eq} = 0}$$



(a)

- **Open circuit:** The entire current i flows through the **smallest resistance** (R_1).

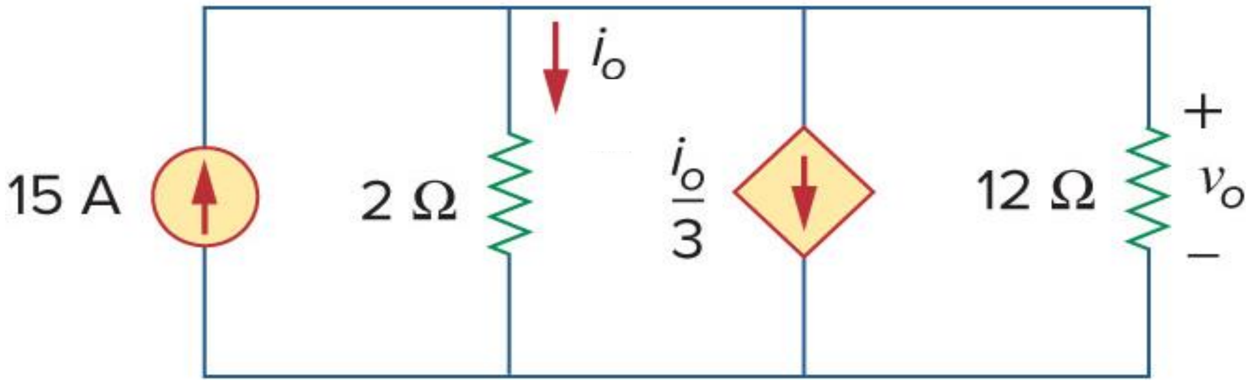
$$\boxed{R_{eq} = R_1}$$



(b)

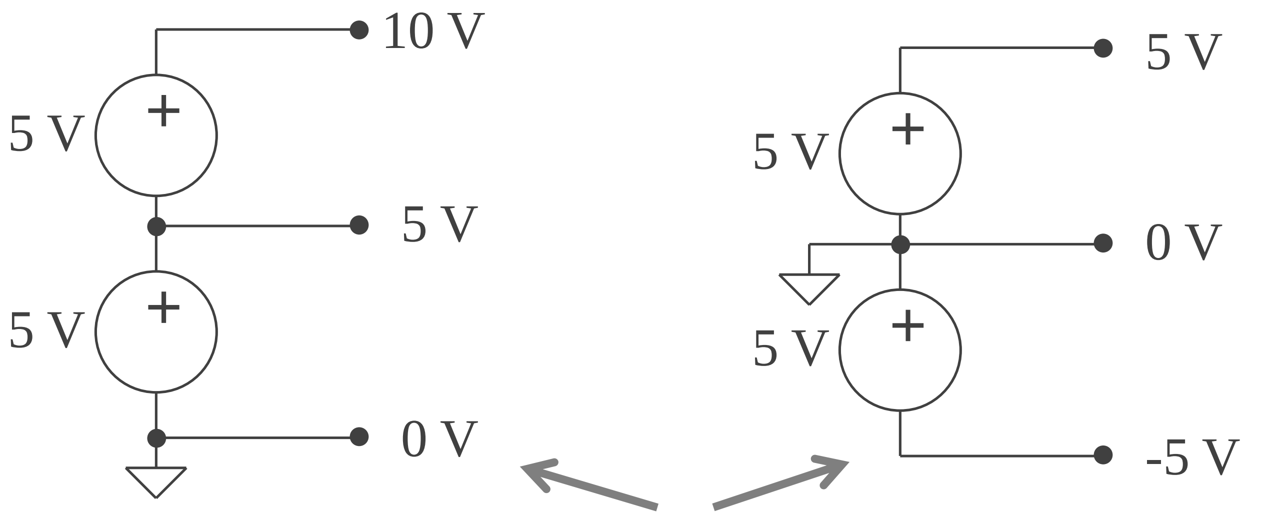
Exercise

Find v_o and i_o in the circuit below.



Circuit ground

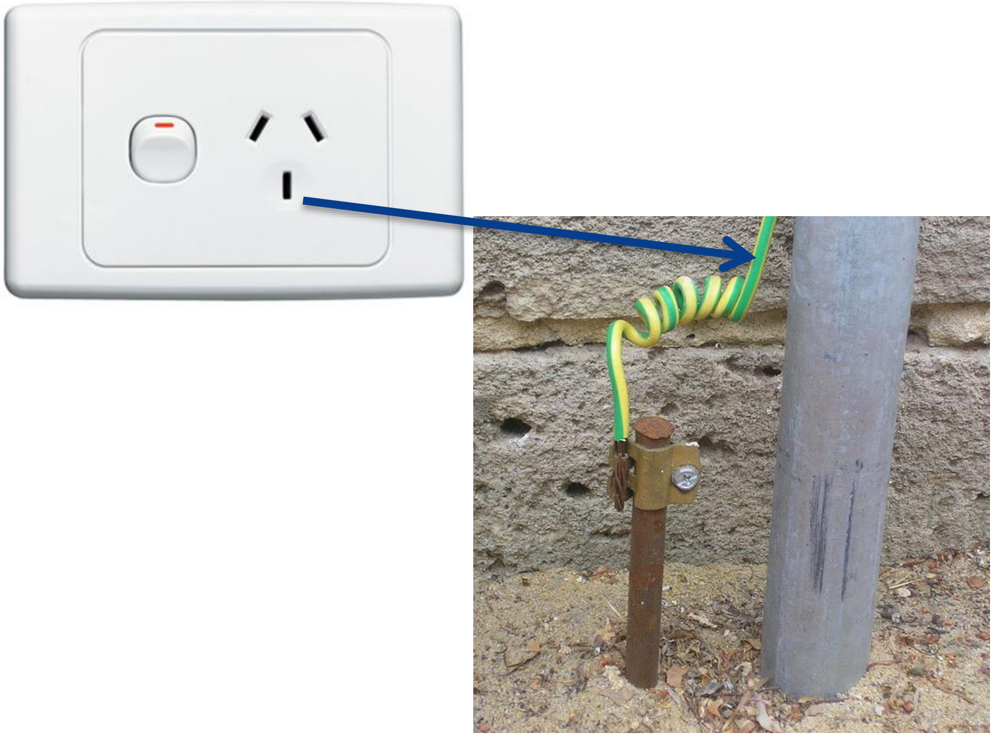
- Voltage is a differential quantity, so we need a reference node, usually at zero volts.
- Any node in a circuit can be defined as zero volts.
- The **zero volt** point is referred to as the **circuit ground**.
- Symbol: 



Different choice of circuit ground leads to different voltages with respect to ground.

Earth

- An earthed ground is literally a connection to the earth, which provides provides an important role in electrical safety.
- Symbol: ⏚



Circuit analysis

- Using Kirchhoff’s laws and Ohm’s law we can analyse any circuit to determine the currents and voltages; however, this might result in many simultaneous equations.
- The challenge of formal circuit analysis is to derive the **smallest set of simultaneous equations** that completely define the operating characteristics of a circuit (voltages and currents).
- Nodal and mesh analysis are two very useful methods for analysing any circuit.
  - They are based on the systematic application of Kirchhoff’s laws.

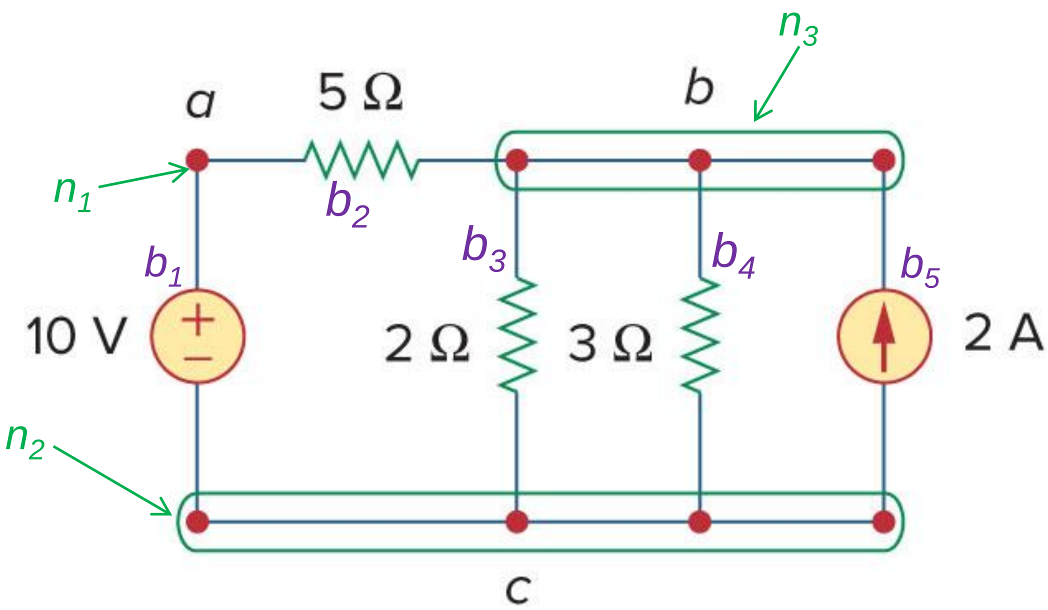
Nodal analysis

Nodes

- A branch represents a single element such as a voltage source or a resistor.
- A node is the point of connection between two or more branches.

n_k = node k

b_k = branch k



Nodal analysis

- It is based on KCL.
- We use **node voltages** (potential of each node) as the **main circuit variables.**
  - Voltages are relative to a reference node.
- Objective: To solve for these node voltages.
  - In general, an *N-node* circuit will need *N*-1 voltages and *N*-1 equations.
  - It will also require the solution of a N-1 system of equations.
  - KCL will be applied at each node except for one – the *reference node*.

Nodal analysis

Reference node

Any node can be chosen as the reference node.

Most common choices are:

- the ground node,
- top or bottom node, or
- node connected to the highest number of branches.

Nodal analysis

- Given a circuit with $\boldsymbol{n}$ nodes, the nodal analysis is accomplished via the following steps:

1. Select a node as the **reference node**.
2. Assign voltages $\boldsymbol{v_1}$, $\boldsymbol{v_2}$,...$\boldsymbol{v_n}$ to the remaining $n-1$ nodes. These voltages are relative to the reference node.
3. Apply **KCL** to each of the $n-1$ non-reference nodes.
   – For resistors, use Ohm's law to express the currents in terms of node voltages. Keep in mind the passive sign convention.
4. Solve the resulting $n-1$ simultaneous equations to obtain the unknown node voltages.

NOTE: Always simplify the circuit before you start doing the analysis.

Nodal analysis

Before starting, **simplify** the circuit (if possible) and identify all the nodes.

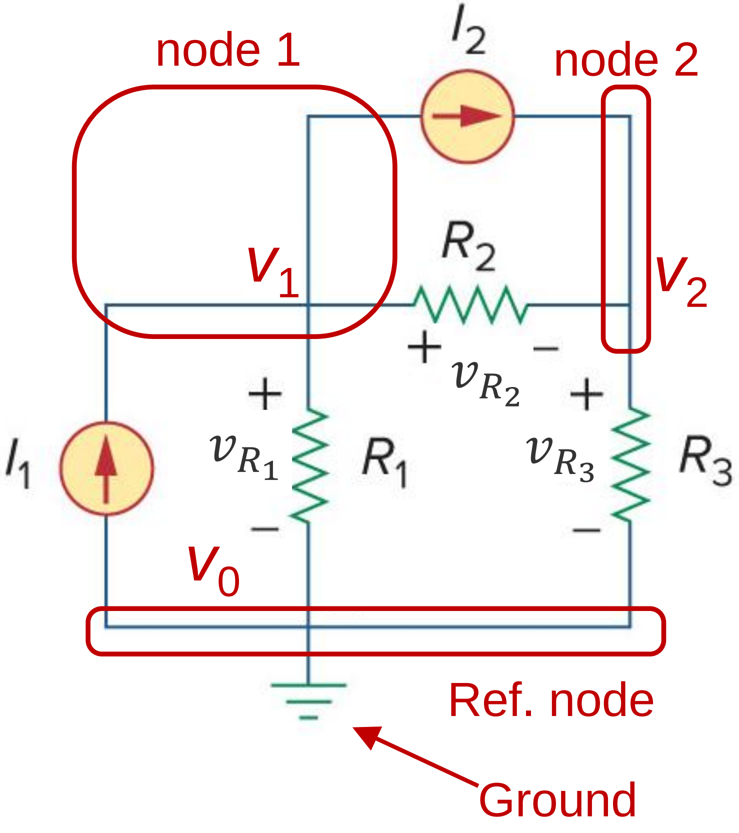
1. Choose **ground** as **reference node** (v_0= 0 V).

2. Assign voltages v_1 and v_2 to nodes 1 and 2.
   - Recall that:

$$v_{R_1} = v_1 - 0$$

$$v_{R_2} = v_1 - v_2$$

$$v_{R_3} = v_2 - 0$$



Nodal analysis

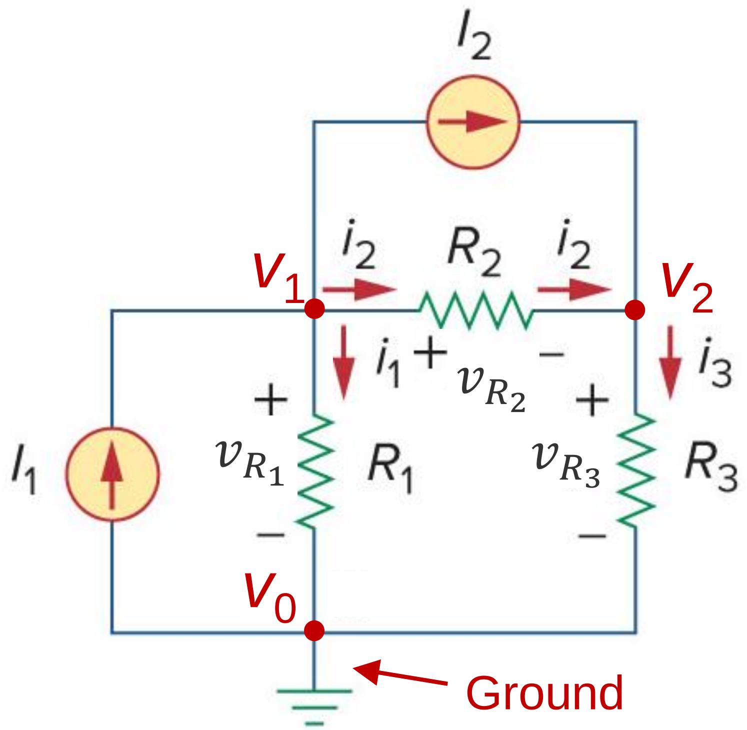
3. Apply KCL to nodes 1 and 2.

   node 1: $I_1 = I_2 + i_1 + i_2$

   node 2: $I_2 + i_2 = i_3$

   – For resistors, use Ohm's law to express the branch currents in terms of node voltages. Keep in mind the passive sign convention.

$$\boxed{i_n = \frac{v_{R_n}}{R_n} = \frac{v_{\text{higher}} - v_{\text{lower}}}{R_n}}$$



$$i_1 = \frac{v_{R_1}}{R_1} = \frac{v_1 - 0}{R_1}$$

$$i_2 = \frac{v_{R_2}}{R_2} = \frac{v_1 - v_2}{R_2}$$

$$i_3 = \frac{v_{R_3}}{R_3} = \frac{v_2 - 0}{R_3}$$

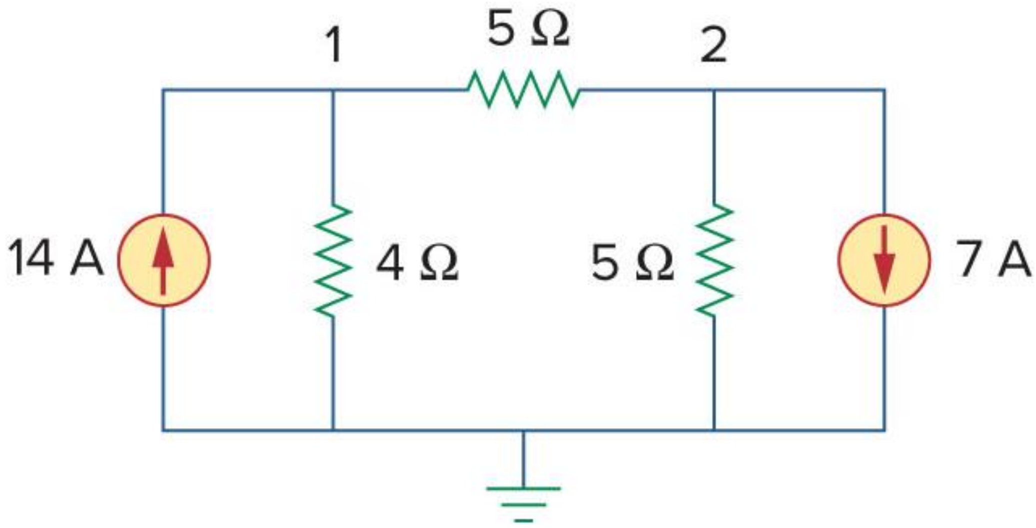
   – Substitute back i_1, i_2, and i_3 into the node equations

$$\begin{cases} I_1 = I_2 + \dfrac{v_1}{R_1} + \dfrac{v_1 - v_2}{R_2} \\ I_2 + \dfrac{v_1 - v_2}{R_2} = \dfrac{v_2}{R_3} \end{cases}$$

4. Solve simultaneous equations (for v_1 and v_2).

Exercise

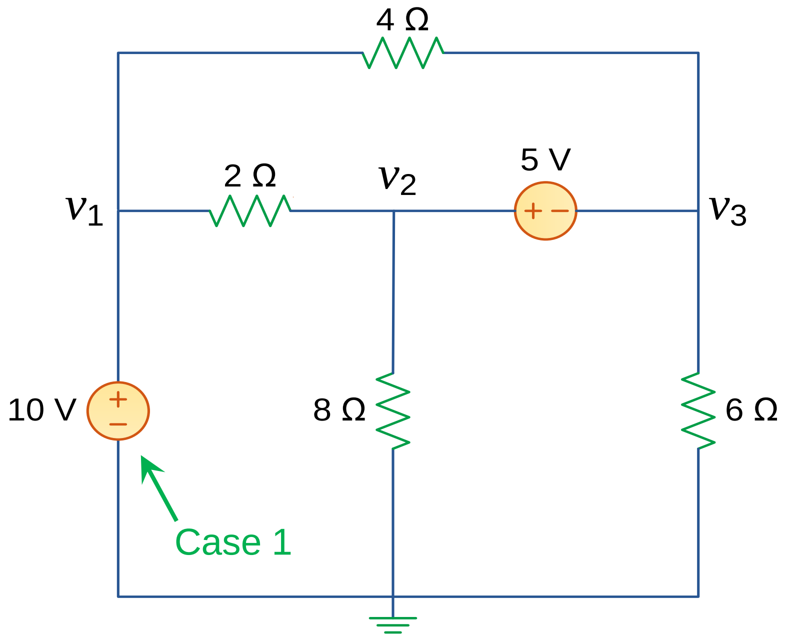
Obtain the node voltages in the circuit below and find the power dissipated in the 4Ω resistor.



Nodal analysis with voltage sources

- Voltage sources generate or dissipate power at a specified voltage with whatever current is required.
  - The voltage is known at the terminals, but the current is not (and Ohm's law does not apply).
- There are 2 cases for nodal analysis with (independent or dependent) voltage sources.
  - **Case 1**: Voltage source is between the reference node and a non-reference node.
    - Set the voltage at the non-reference node to the voltage of the source.

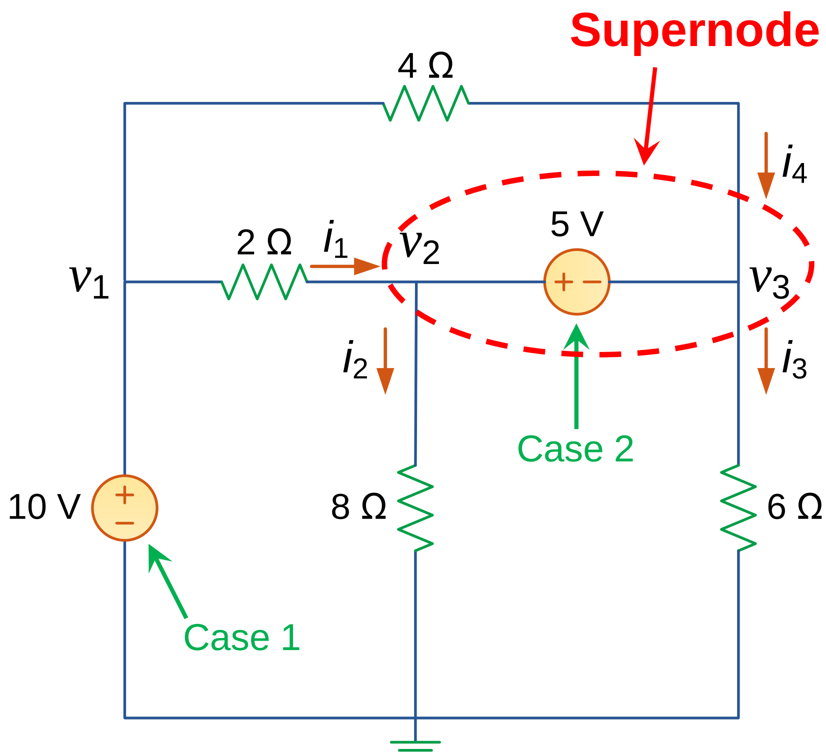
$$v_1 = 10\text{ V}$$



Nodal analysis with voltage sources

- **Case 2**: Voltage source is between two non-reference nodes.
  - The two nodes form a **supernode**.
  - The voltage across the voltage source can be expressed in terms of node voltages (KVL in bottom-right mesh).

$$v_2 - v_3 = 5\ \mathrm{V}$$



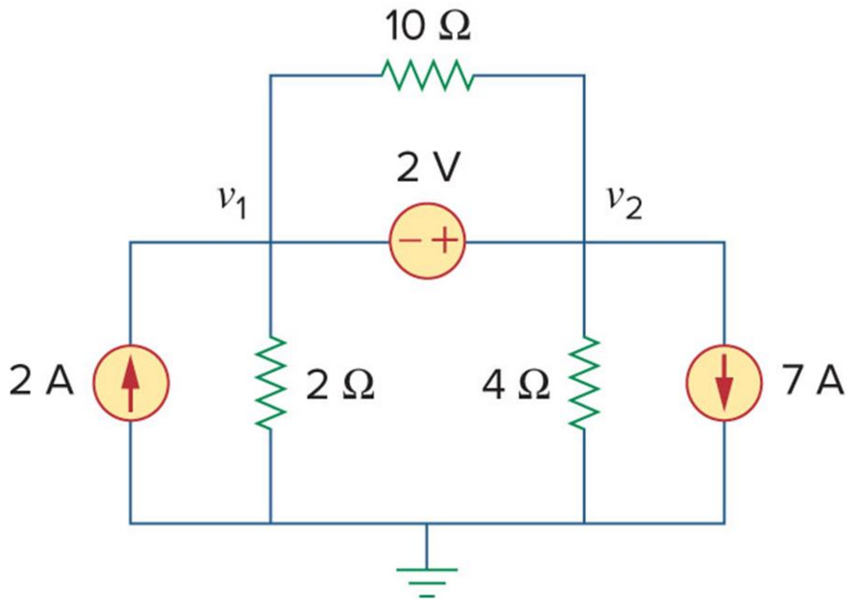
A **supernode** is formed by **enclosing** a **voltage source** connected between **two non-reference** nodes and **any element** connected **in parallel** with it.

Supernode properties

1. The voltage source inside the supernode provides a **constraint equation** needed to solve node voltages.
2. A supernode has **no voltage** of its own.
3. A supernode requires the application of **both KCL** and **KVL**.

Exercise

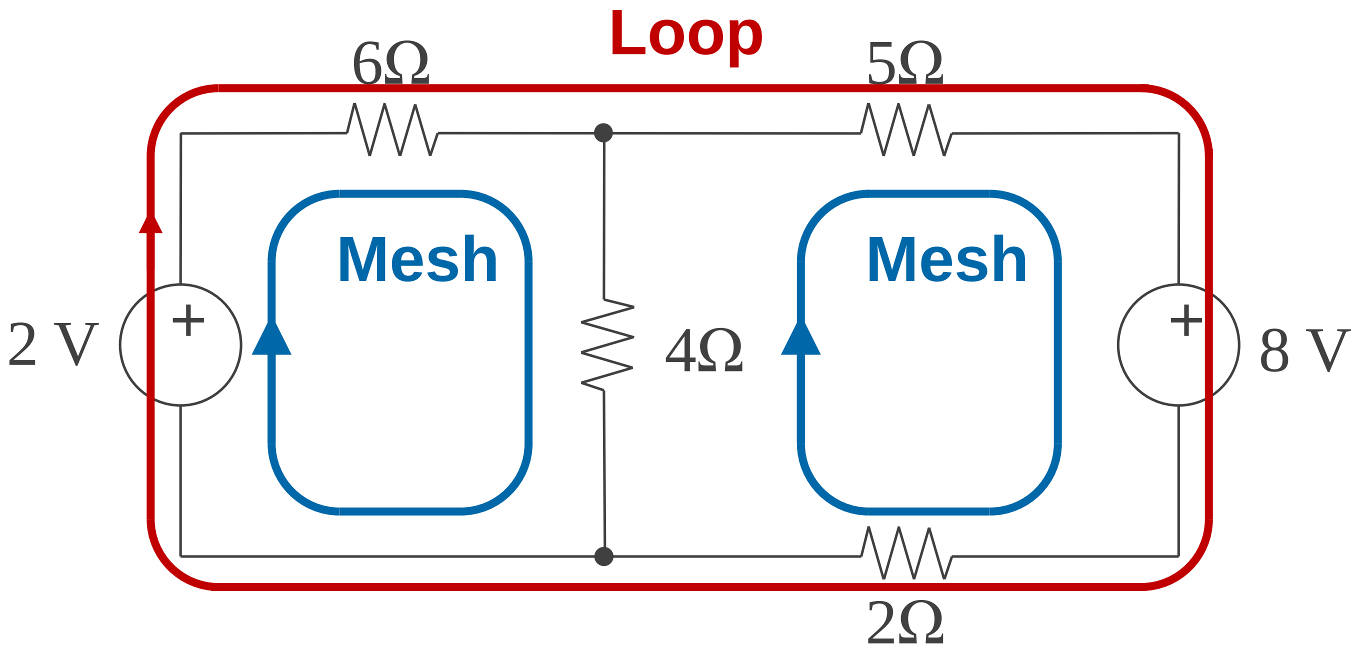
Find the node voltages in the circuit below.



Mesh analysis

Meshes

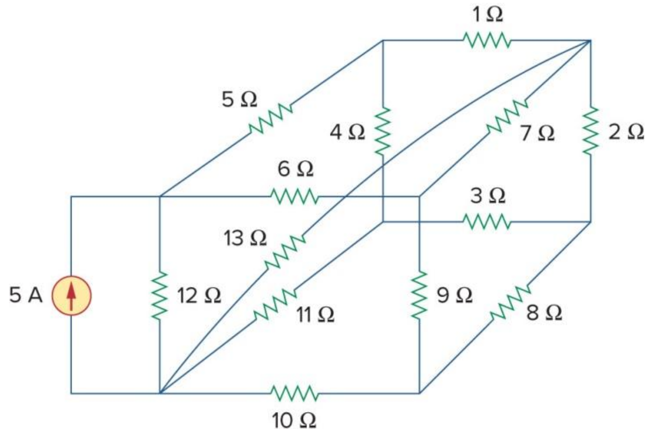
- A loop is any closed path in a circuit.
- A mesh is a loop that contains no other loop.



Mesh analysis

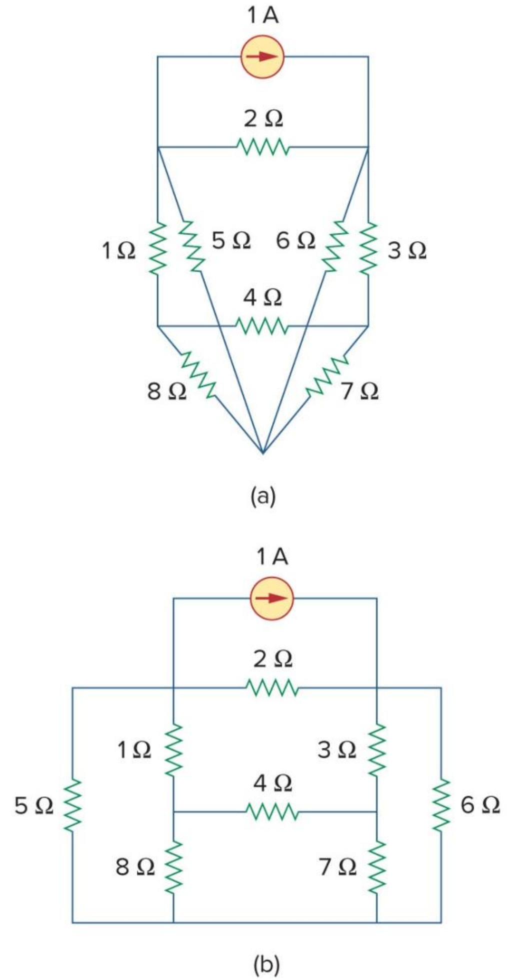
Planar vs non-planar circuits

- A planar circuit is a circuit that can be drawn in a plane with **no crossing branches.**
- Mesh analysis is only applicable to planar circuits.
  - Nodal analysis can be applied to both.



Non-planar circuit.

The branch with the 13Ω resistor prevents the circuit from being drawn without crossing branches.



Planar circuit.

(a) Original, (b) Redrawn circuit.

Mesh analysis

- It is based on KVL.
- We use **mesh currents** instead of element currents as the **main circuit variables**.
  - It reduces the number of equations that must be solved simultaneously.
- Objective: To solve for these mesh currents.
  - In general, an N-mesh circuit will need N currents and N equations.
  - It will also require the solution of a N system of equations.
  - KVL will be applied at each mesh.

Mesh analysis

- Given a circuit with $\boldsymbol{n}$ meshes, the mesh analysis is accomplished via the following steps:

1. Assign mesh currents $\boldsymbol{i_1}$, $\boldsymbol{i_2}, \ldots \boldsymbol{i_n}$ to the n meshes with a direction (generally clockwise).
2. Apply **KVL** to each of the n meshes (following the same direction as mesh currents).
   - For resistors, use Ohm's law to express the voltages in terms of mesh currents.
4. Solve the resulting n simultaneous equations to obtain the unknown mesh currents.

NOTE: Always simplify the circuit before you start doing the analysis.

Mesh analysis

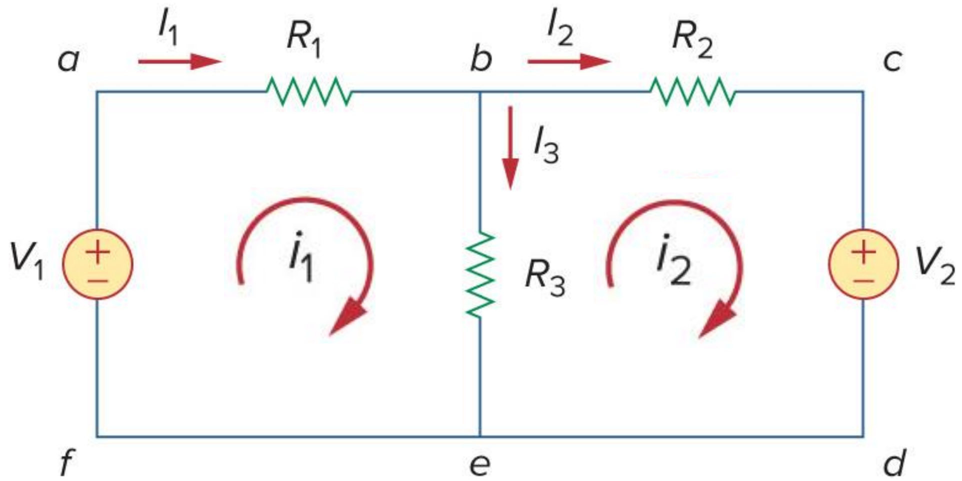
Before starting, **simplify** the circuit (if possible) and identify all the meshes.

1. Assign currents i_1 and i_2 to meshes 1 and 2.
   - Current for each branch is:

$$I_1 = i_1$$

$$I_2 = i_2$$

$$\boxed{I_3 = i_1 - i_2}$$



The current in the **common branch** between two meshes is the **difference** of the **two mesh currents** according to the current **direction** assigned to that branch.

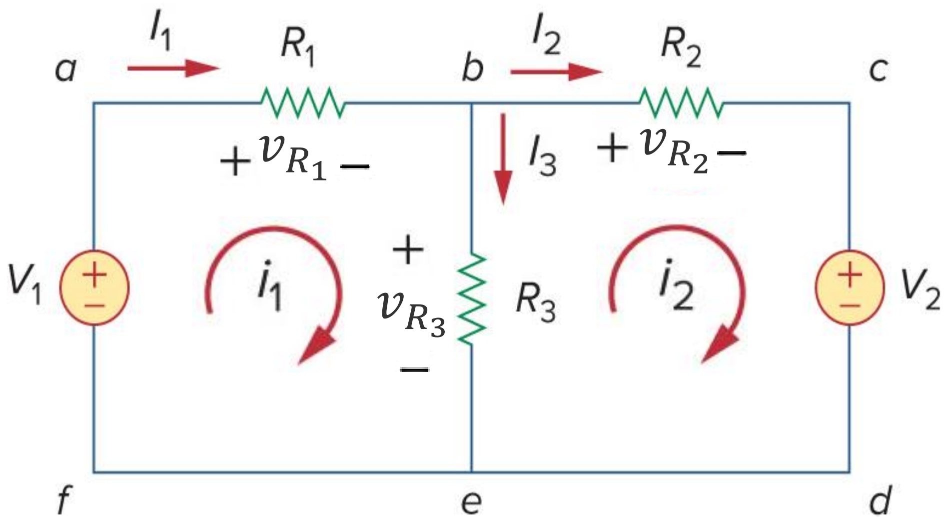
NOTE: You can use ***I*** for **current branch** and ***i*** for **mesh current** or the other way round to distinguish between two types of currents, although most of the time we work directly with the mesh currents.

Mesh analysis

3. Apply KVL to meshes 1 and 2.

mesh 1: $-V_1 + v_{R_1} + v_{R_3} = 0$

mesh 2: $-v_{R_3} + v_{R_2} + V_2 = 0$



– For resistors, use Ohm's law to express the voltages in terms of mesh currents.

$$v_{R_1} = R_1 I_1 = R_1 i_1$$

$$v_{R_2} = R_2 I_2 = R_2 i_2$$

$$v_{R_3} = R_3 I_3 = R_3(i_1 - i_2)$$

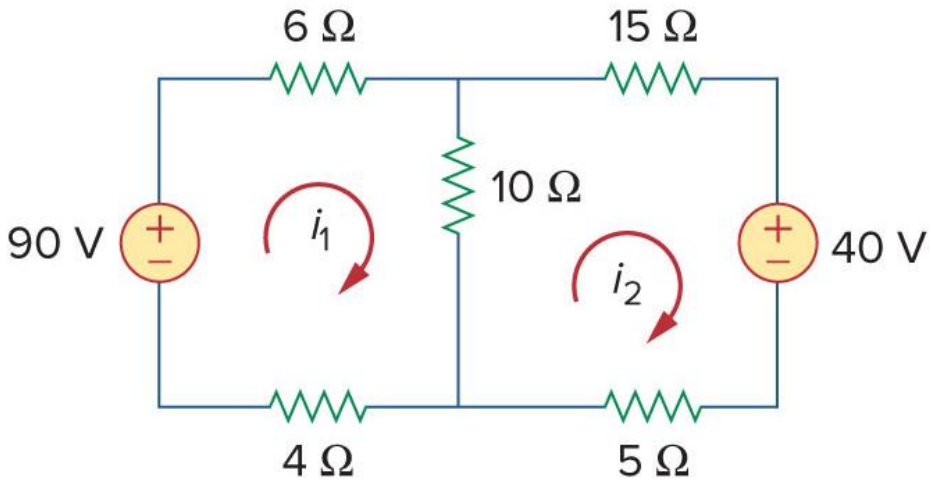
– Substitude back v_{R_1}, v_{R_2}, and v_{R_3} into the mesh equations. $\Rightarrow$

$$\begin{cases} -V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0 \\ -R_3(i_1 - i_2) + R_2 i_2 + V_2 = 0 \end{cases}$$

4. Solve simultaneous equations (for i_1 and i_2).

Exercise

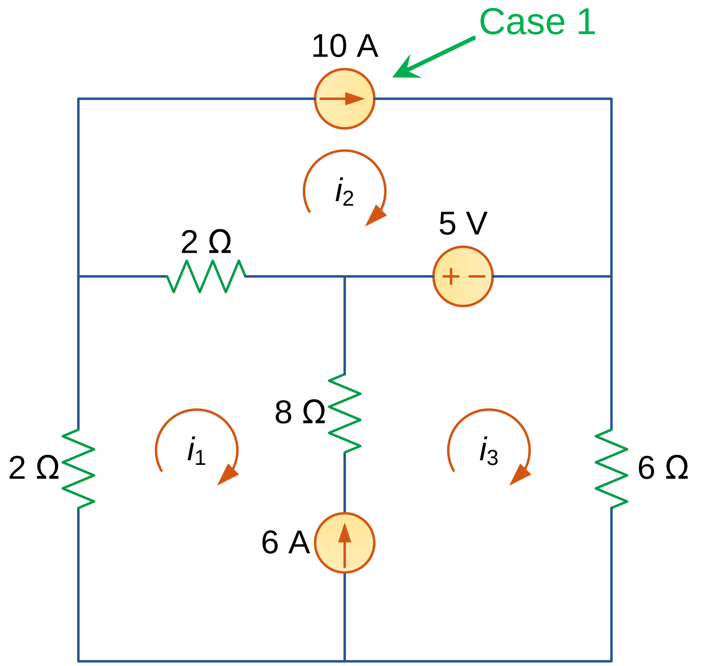
Calculate the mesh currents i_1 and i_2 in the circuit below, and find the power in the 40 V voltage source.



Mesh analysis with current sources

- Current sources generate or dissipate power at a specified current with whatever voltage is required.
  - The current is known at the terminals, but the voltage is not (and Ohm's law does not apply).
- There are 2 cases for mesh analysis with (independent or dependent) current sources.
  - **Case 1**: Current source exists only in one mesh.
    - Set the mesh current to the current of the source.

$$i_2 = 10\ \mathrm{A}$$

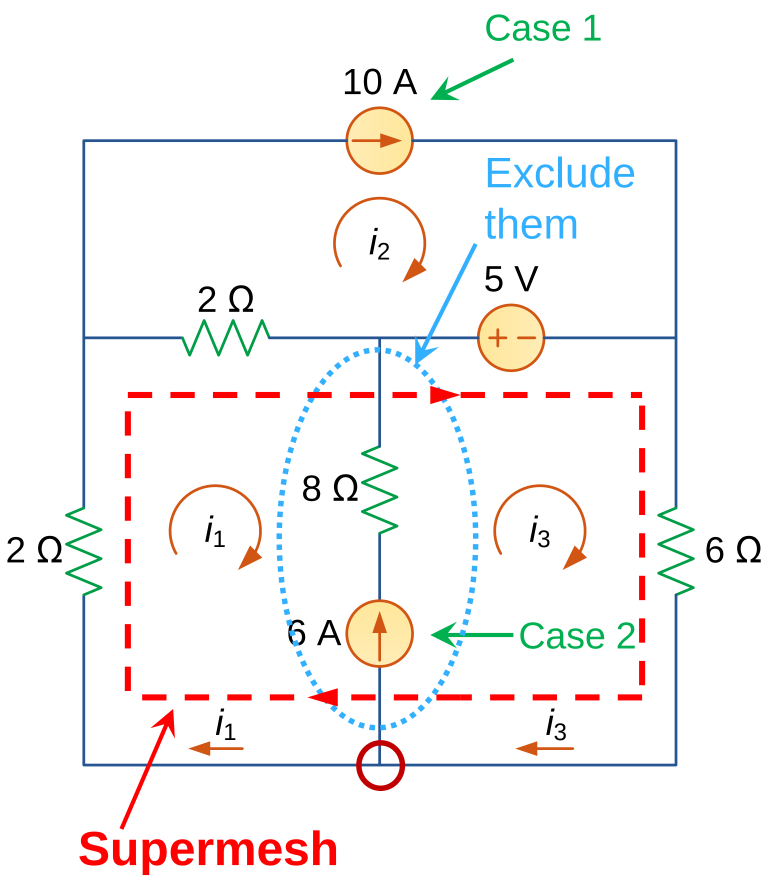


Mesh analysis with current sources

- **Case 2**: Current source is between two meshes.
  - The two meshes form a **supermesh** by excluding the branch with current source.
  - The current through the current source can be expressed in terms of mesh currents (KCL in bottom node).

$$i_3 = i_1 + 6\ \text{A}$$

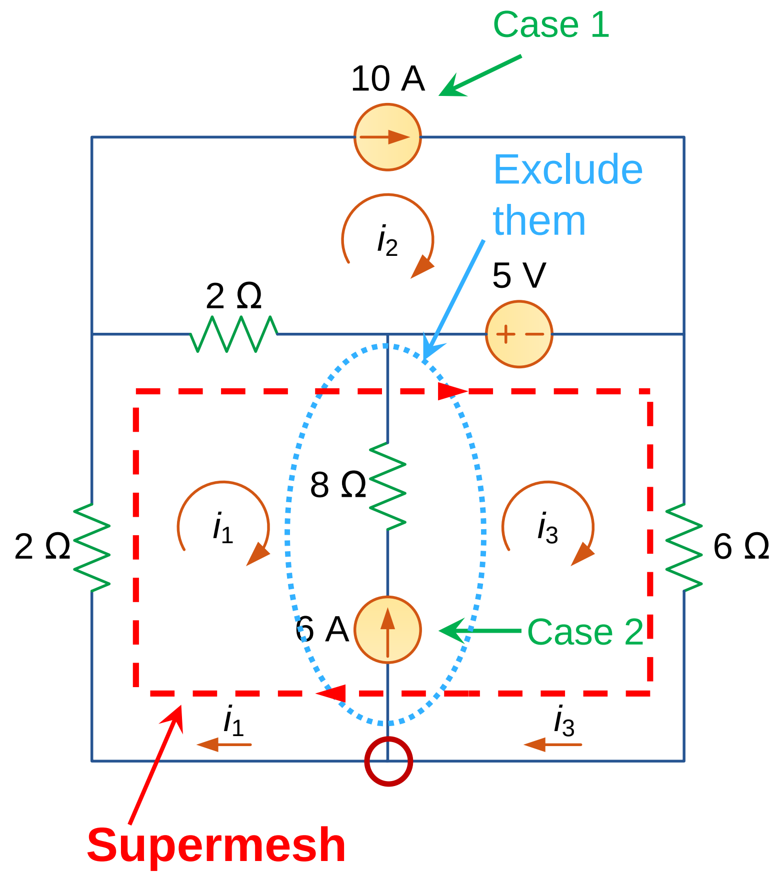
A **supermesh** is formed by **merging** two meshes and **excluding** the shared **current source** and **any element** connected **in series** with it.



Mesh analysis with current sources

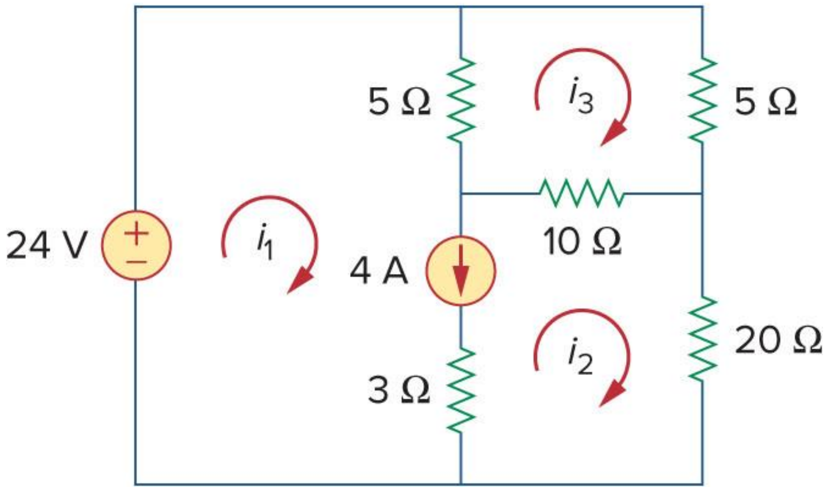
Supermesh properties

1. The current source inside the supermesh provides a **constraint equation** needed to solve mesh currents.
2. A supermesh has **no current** of its own.
3. A supermesh requires the application of **both KVL** and **KCL**.
4. **Intersecting** supermeshes in a circuit must be combined to form a **larger** supermesh.



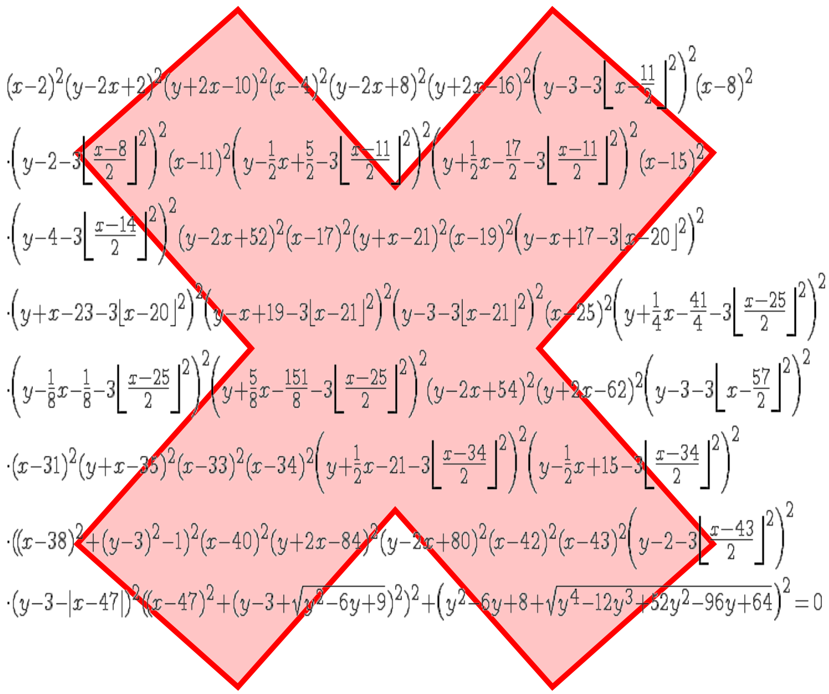
Exercise

In the circuit below, use mesh analysis to determine i_1, i_2 , and i_3.



Nodal analysis vs mesh analysis

- We aim to select the method that results in the smaller number of equations.
  - We prefer **nodal analysis** for circuits with **fewer nodes** than meshes.
  - We prefer **mesh analysis** for circuits with **fewer meshes** than nodes.



Questions?