School of Electrical Engineering & Telecommunications

ELEC1111

Topic 5: Inductors and RL Circuits

1. For the circuits shown in Fig. 1 and 2,

   a) Find v_c, i_L, and the energy stored in the capacitor and inductor in Fig. 1 under DC conditions.

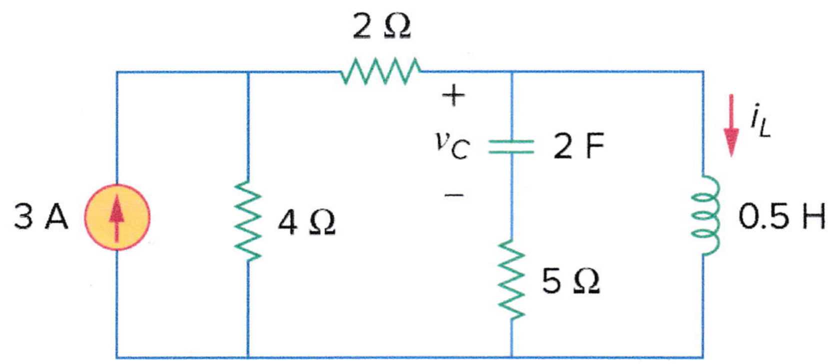


Fig. 1.

   b) Find v_c, i_L, i, and the energy stored in the capacitor and inductor in Fig. 2 under DC conditions.

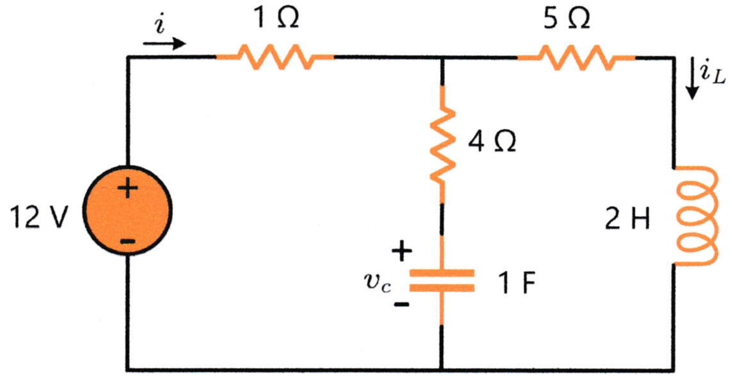


Fig. 2.

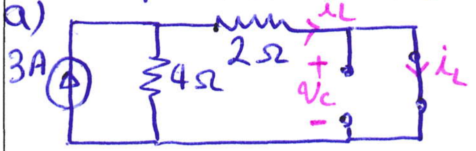
Answer:

a) $v_c = 0\ \text{V}$, $i_L = 2\ \text{A}$, $w_c = 0\ \text{J}$, and $w_L = 1\ \text{J}$

b) $v_c = 10\ \text{V}$, $i_L = i = 2\ \text{A}$, $w_c = 50\ \text{J}$, and $w_L = 4\ \text{J}$

Solution: Draw the circuit in DC conditions by replacing the capacitor with open-circuit and the inductor with short circuit

a)



Using current division

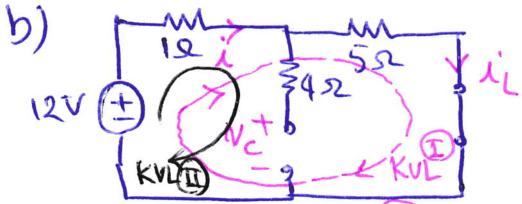
$i_L = \frac{4}{4+2} \times 3 = 2A$

$V_c = 0$ parallel with short circuit

$W_L = \frac{1}{2} L i_L^2 = \frac{1}{2} \times 0.5 \times 4 = 1J$

$W_C = \frac{1}{2} C V_c^2 = 0J$

b)



No current is going through 4-Ω restor.

Thus $i = i_L = \frac{12}{1+5} = 2A$ (Simple KVL (I))

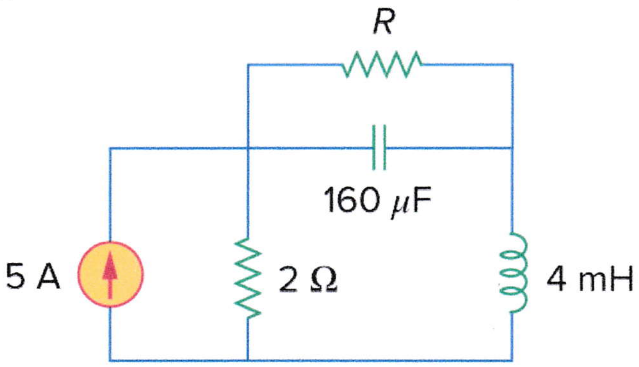
KVL (II): $-12 + i + V_c \rightarrow V_c = 10V$

* V_c is also the same as the voltage across 5-Ω resistor $\rightarrow V_c = 5 \times i_L = 10V$

$W_L = \frac{1}{2} L i_L^2 = \frac{1}{2} \times 2 \times 4 = 4J$

$W_C = \frac{1}{2} C V_c^2 = \frac{1}{2} \times 1 \times 100 = 50J$

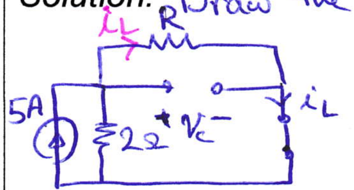
2. In the following circuit, Calculate the value of R that will make the energy stored in the capacitor the same as that stored in the inductor under steady-state conditions.



Answer: $R = 5\ \Omega$,

Hint: Find the capacitor and inductor energies in term of R.

Solution: Draw the circuit under steady-state conditions



Using current division $\rightarrow i_L = \frac{2}{2+R} \times 5 = \frac{10}{2+R}$ A

$V_C = R i_L = \frac{10R}{2+R}$

$W_L = \frac{1}{2} L i_L^2 = \frac{1}{2} \times 4 \times 10^{-3} \times \left(\frac{10}{2+R}\right)^2$

$= 2 \times 10^{-3} \times \frac{100}{(R+2)^2}$

$W_C = \frac{1}{2} C V_c^2 = \frac{1}{2} \times 160 \times 10^{-6} \times \left(\frac{10R}{R+2}\right)^2$

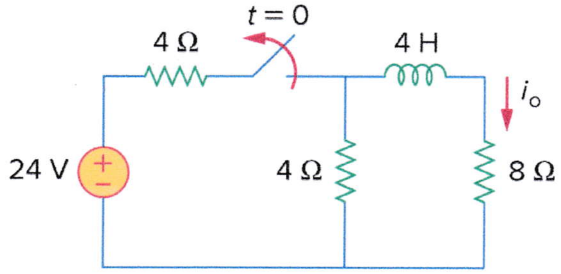
$= 80 \times 10^{-6} \times \frac{100R^2}{(R+2)^2}$

$\rightarrow W_L = W_C \rightarrow \frac{2 \times 10^{-3} \times \cancel{100}}{\cancel{(R+2)^2}} = \frac{80 \times 10^{-6} \times \cancel{100} R^2}{\cancel{(R+2)^2}}$

$\Rightarrow R^2 = \frac{1}{40 \times 10^{-3}} = \frac{100}{4} = 25$

$\Rightarrow R = \pm\sqrt{25} \longrightarrow \boxed{R = 5\Omega}$ only the positive vale is acceptable.

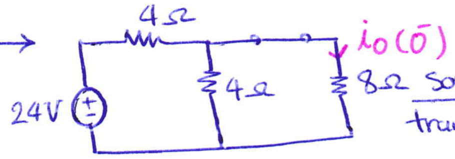
3. The switch in the following circuit has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i_o(t)$ for all time (i.e., for both $t < 0$ and $t > 0$), and sketch the current $i_o(t)$ as a function of time showing all critical points in the sketch.



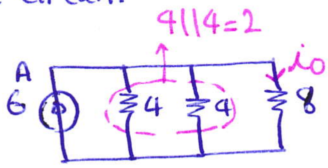
Answer: $i_o(t) = \begin{cases} 1.2\text{ A} & t \le 0 \\ 1.2e^{-3t}\text{ A} & t \ge 0 \end{cases}$

Solution: Draw the circuit before changes in the circuit.

$t<0 \rightarrow$



Source transformation $\rightarrow$

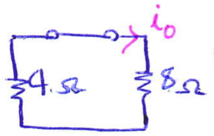


Using current division $\rightarrow$ $i_o = \frac{2}{2+8}\times 6 = \boxed{1.2A} \rightarrow i(0^-) = 1.2A = i(0^+) = I_0$

Current of inductor cannot change suddenly.

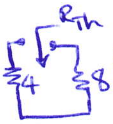
After changes

$t>0$ switch is opened



Source-free response $\boxed{i_o(t) = I_0 e^{-t/\tau}}$

$\tau = \frac{L}{R_{Th\text{-}\infty}} \rightarrow$

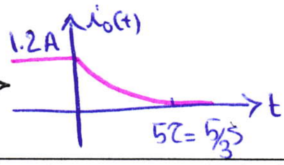


$R_{Th} = 4+8 = \boxed{12\Omega}$

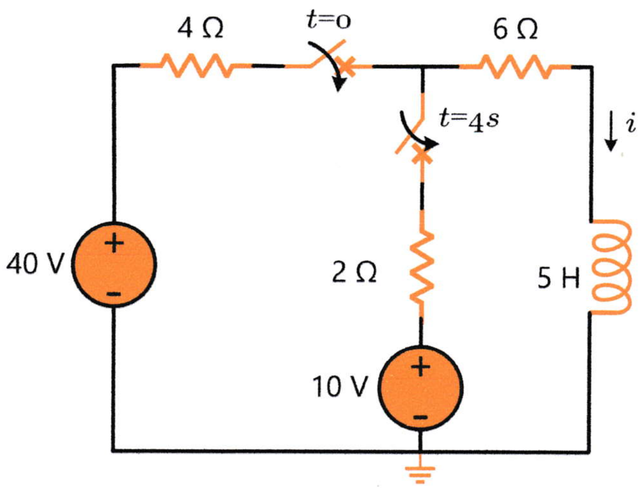
$L=4H \rightarrow \tau = 4/12 = 1/3\text{ s}$

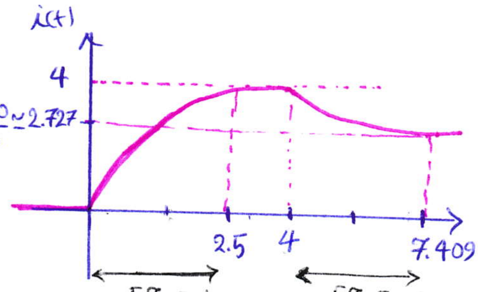
Note the current is a continuous function

$\Rightarrow i_o(t) = \begin{cases} 1.2\text{ A} & t \le 0 \\ 1.2e^{-3t} & t \ge 0 \end{cases} \rightarrow$



4. In the following circuit, at $t = 0$, switch 1 is closed (next to 4-Ω resistor), and switch 2 is closed 4 seconds later. Find $i(t)$ for all time (i.e., for both $t < 0$ and $t > 0$), and calculate $i(t)$ at $t = 2$ s and $t = 5$ s. Sketch $i(t)$ waveform showing all critical points in the sketch.





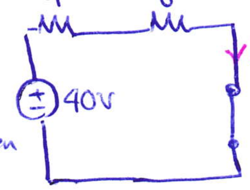
Answer: $i(t) = \begin{cases} 0 \text{ A}, & t \le 0 \\ 4(1 - e^{-2t}) \text{ A}, & 0 \le t \le 4 \\ 2.727 + 1.273e^{-1.466(t-4)}, \text{A} & t \ge \end{cases}$

Hint: Use time shift property explained at the end of Topic 4.

Solution: You need to solve this problem in three (3) time intervals. $t \le 0$, $0 \le t \le 4$, and $t \ge 4$

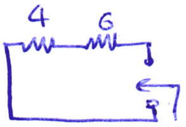
* Befor first change $t < 0$ → Both switches are open, thus no source is connected to the inductor ⇒ $i(0^-) = i(0^+) = i(0) = 0\,A$

* After the first change $0 \le t \le 4$ S_1 is closed, S_2 is still open



The assumption is that S_1 can be remained in closed position forever.

$\rightarrow i(\infty) = \dfrac{40}{4+6} = 4\,A$ $\xrightarrow{\text{time Constant}}$ $\tau = L/R_{Th_\infty}$ →

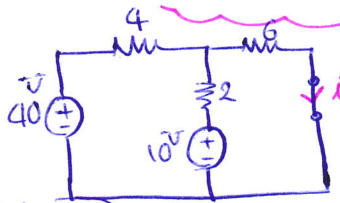


$R_{Th_\infty} = 4+6 = 10\,\Omega$

R_{Th_∞} $\tau = \dfrac{5}{10} = \dfrac{1}{2}\,s$

Thus $i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 4(1 - e^{-2t})\,A$ $0 \le t \le 4$

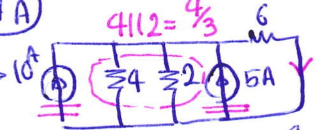
* After the second change $t \ge 4$ Both S_1 and S_2 are now closed



Note that in this case, the initial value of i is obtained from previous interval as i cannot change suddenly

$i(4) = i(4^-) = 4(1 - e^{-2t})\big|_{t=4} \cong 3.998 \simeq 4\,A$

for $i(\infty)$ → Source transformation →



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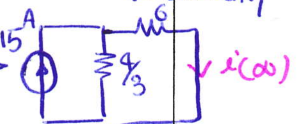
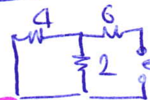


Using current division $i(\infty) = \dfrac{4/3}{6 + 4/3} \times 15 = \dfrac{30}{11}\,A$ → $\tau = L/R_{Th_\infty}$ ⇒

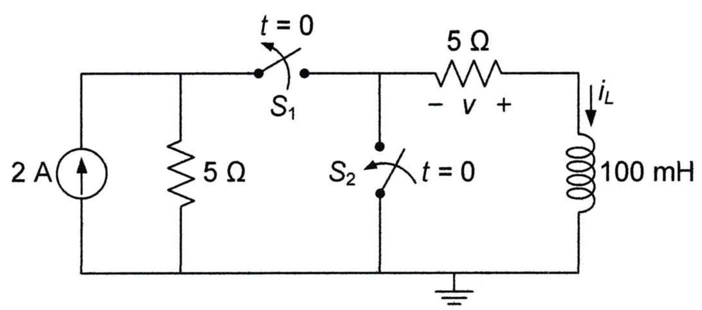


$R_{Th_\infty} = 2\|4 + 6 = \dfrac{22}{3}\,\Omega$

$i(t) = i(\infty) + (i(4) - i(\infty))e^{-\frac{(t-4)}{\tau}}$ $t \ge 4$ → $i(t) = \dfrac{30}{11} + \dfrac{14}{11}e^{-\frac{22}{15}(t-4)}$ $t \ge 4$ ⇒ $\tau = \dfrac{15}{22}\,s$

Note that if changes happen at a time other than zero, use time shift.

5. (**Final Exam – S1, 2016**) Consider the circuit shown below, where switch S_1 has been closed for a long time and switch S_2 has been open for a long time. At $t = 0$, switch S_1 opens and switch S_2 closes. Derive an expression for the inductor current $i_L(t)$ and $v(t)$ for $t > 0$.

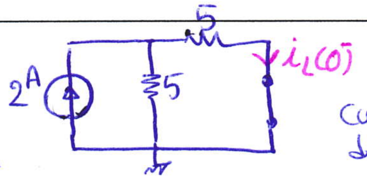


Answer: $i_L(t) = \; 1e^{-50t}u(t)$ A, $v(t) = -5e^{-50t}u(t)$ V

Solution:

Before changes $t<0$

S_1: closed, S_2: opened

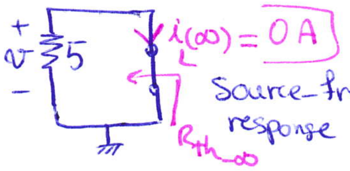


$i(0^-) = i(0^+) = \boxed{1A} = I_0$

Current division $i_L = \frac{5}{5+5} \times 2 = \boxed{1A}$

After changes

$t>0$ S_1: opend
S_2: closed



$i_L(\infty) = \boxed{0\,A}$

Source-free response →

$\Rightarrow i(t) = I_0 e^{-t/\tau} \rightarrow \tau = L/R_{th\_\infty} = \frac{100\times10^{-3}}{5} = 0.02 = \frac{1}{50}\,s$

Thus → $V(t) = -5 \times i_L(t) = -5e^{-50t} \quad t>0$

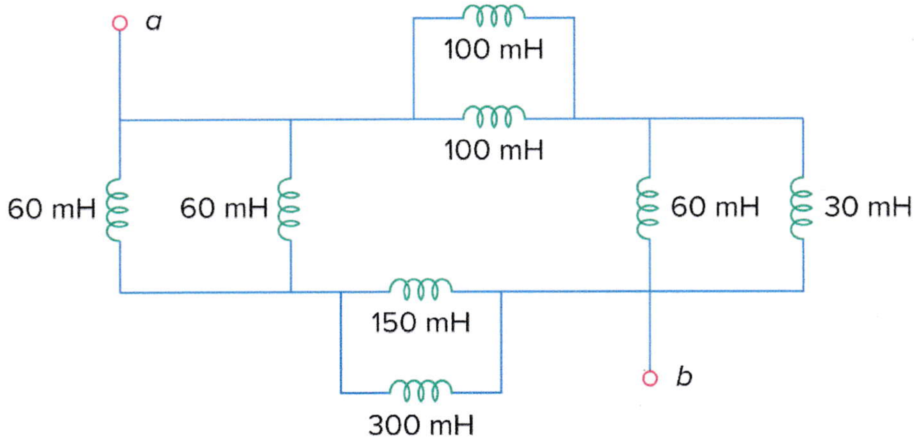
based on given polarity

$\Rightarrow i(t) = 1e^{-50t} \quad t>0$

$= 1e^{-50t}u(t)$

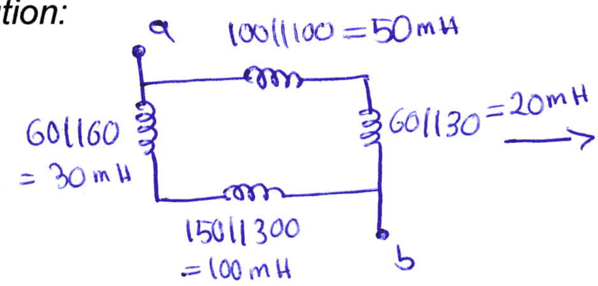
You can use step function

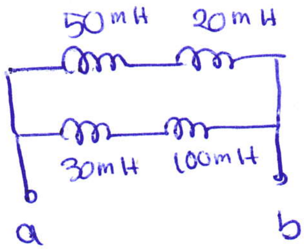
6. Find the equivalent inductance as seen from the terminals a-b in the circuit below.



Answer: $L_{eq} = 45.5\,\text{mH}$

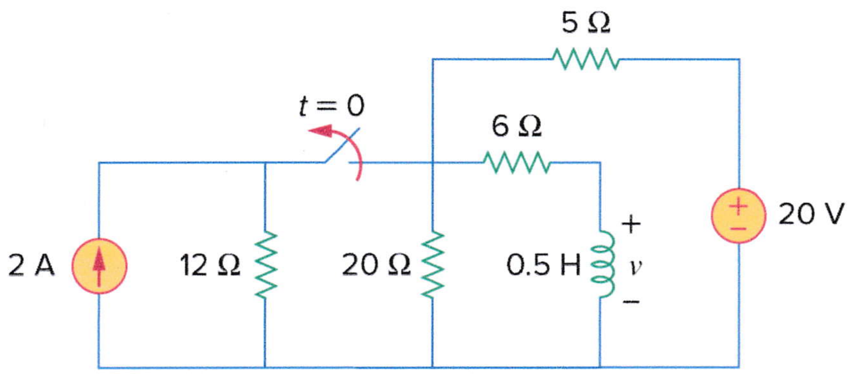
Solution:





$\Rightarrow L_{eq} = (50+20)\,||\,(30+100) = \frac{70\times130}{70+130} = 45.5\,mH$

7. Find the voltage across inductor $v(t)$ for $t > 0$.



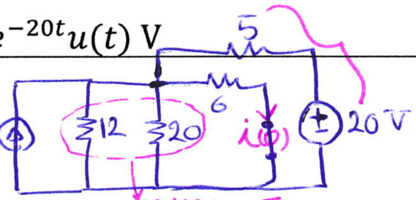
Answer: $v(t) = -4e^{-20t}u(t)$ V

Solution:

Before changes $t<0$

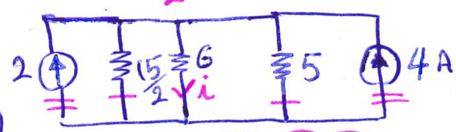
Switch closed →

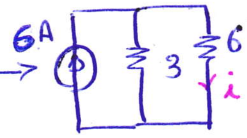
Use source transformation



$12 \| 20 = \frac{15}{2}$

$V = L\frac{di}{dt}$ → So find the current through inductor first

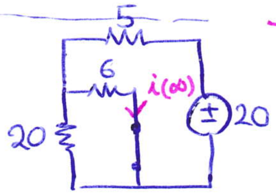




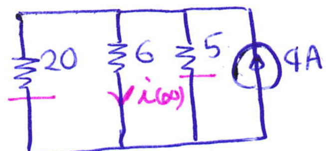
current division ⇒ $i = \frac{3}{3+6} \times 6 = 2\,A = i(0^-)$

→ $i(0) = i(0^-) = i(0^+) = 2A$

After changes $t>0$

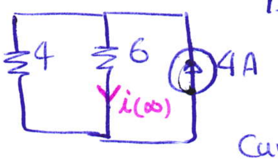


source transformation



$20 \| 5 = 4$

→



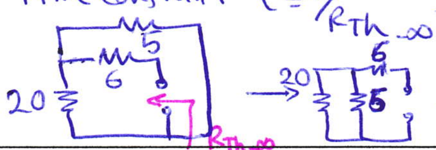
$i(\infty) = \frac{4}{4+6} \times 4 = \boxed{1.6\,A}$

Current division

⇒ $i(t) = i(\infty) + (i(0) - i(\infty))e^{-t/\tau}$

$= 1.6 + (2 - 1.6)e^{-t/\frac{1}{20}} = 1.6 + 0.4e^{-20t}$ $t>0$

Time constant $\tau = L/R_{Th,\infty}$ → $\tau = \frac{0.5}{10} = \frac{1}{20}$



→ $R_{Th,\infty} = 6 + (5 \| 20)$

$= 6 + 4 = 10\,\Omega$

Thus $v = L\frac{di}{dt} = 0.5 \times 0.4 \times (-20)e^{-20t}$

→ $v(t) = -4e^{-20t}$ $t>0$

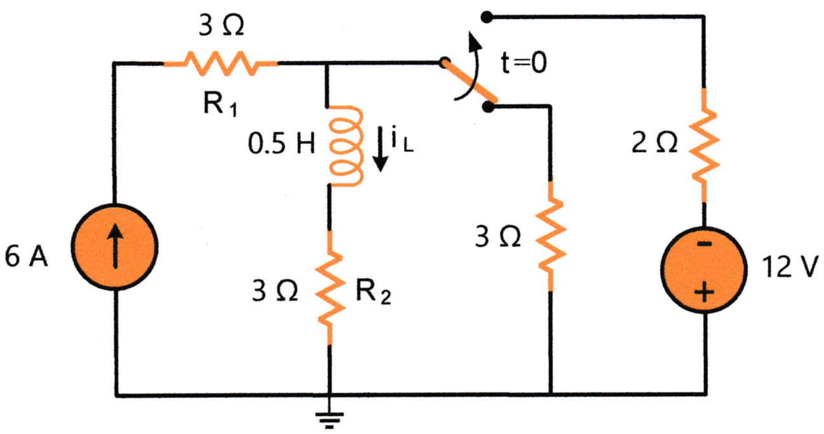
8. (**Final Exam – S1 2017**) In the circuit below,

a) Find the energy stored in the inductor under steady-state when the switch is in the open position (connected to 3-Ω resistor).

b) If the switch has been in the open position for a long time and closes at $t = 0$, derive an analytical expression for the current $i_L(t)$ through the inductor for $t > 0$.

c) Plot the current through the inductor as a function of time.

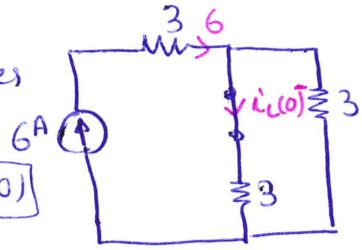
d) Derive an analytical expression for the voltages across resistors R_1 and R_2 (v_{R_1} and v_{R_2}) as a function of time for $t > 0$.



Answer:
a) $w_L(0) = 2.25\ \mathrm{J}$
b) $i_L(t) = 3e^{-10t}u(t)\ \mathrm{A}$
d) $v_{R_1}(t) = 18\ \mathrm{V}$, $v_{R_2}(t) = 9e^{-10t}u(t)\ \mathrm{V}$

Solution:

a) $W_L(0) = \frac{1}{2} L i_L^2(0) \longrightarrow$ Before changes $t<0$

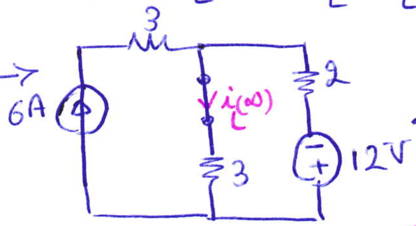


Current division $\longrightarrow$ $i_L = \frac{3}{3+3} \times 6 = \boxed{3A = i_L(0^-) = i_L(0^+) = i_L(0)}$

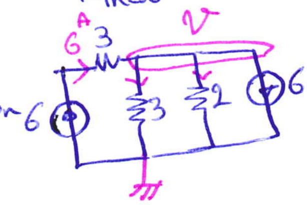
$\Rightarrow W_L(0) = \frac{1}{2} \times 0.5 \times 3^2 = \frac{9}{4} = 2.25\mathrm{J}$

b) $i_L(0^+) = 3A$ $\quad i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-t/\tau}$ $\quad \tau = \frac{L}{R_{Th-\infty}}$

After changes $t>0$ $\longrightarrow$

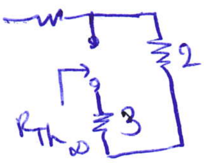


$\xrightarrow{\text{Source}}$ transformation



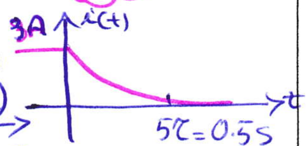
Nodal Analysis : $\cancel{6} = \frac{v}{3} + \frac{v}{2} + \cancel{6} \longrightarrow \boxed{v = 0V} \Rightarrow i_L(\infty) = 0A$ current sources cancel eachother

$R_{Th-\infty}$



$\boxed{R_{Th-\infty} = 2+3 = 5\Omega} \longrightarrow \tau = \frac{L}{R_{Th-\infty}} = \frac{0.5}{5} = \frac{1}{10}s$

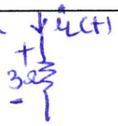
$\Rightarrow i_L(t) = 0 + (3-0)e^{-t/\frac{1}{10}} = 3e^{-10t}$ $t>0$ (Source-free) response c)



d) $v_{R_1} = 3 \times 6 = 18V \longrightarrow$ Constant current flowing through $3\text{-}\Omega$ resistor

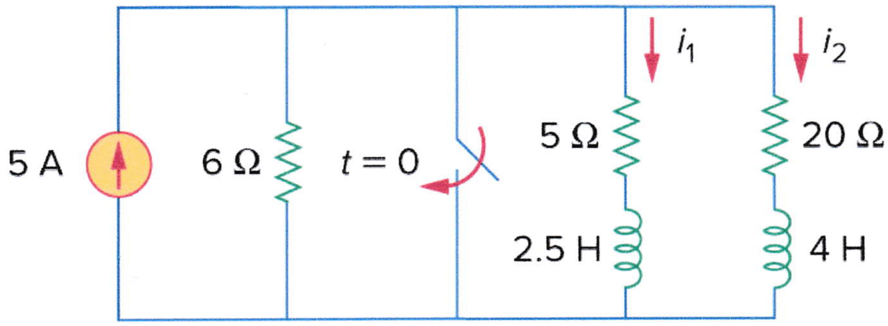
$v_{R_2} = 3 \times i_L(t) = 9e^{-10t}$ $t>0$ or $9e^{-10t}u(t)$

Passive sign convention



Note that any function $f(t)$ valid for only $t>0$ Can be written as $f(t)u(t)$ showing that it only exists after $t>0$ where $u(t)$ is unit step function.

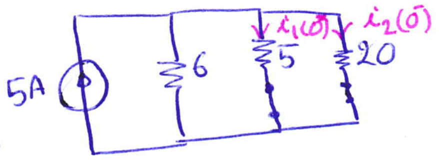
9. Derive an expression for $i_1(t)$ and $i_1(t)$ for $t > 0$.



Answer: $i_1(t) = 2.4e^{-2t}u(t)$ A, $i_2(t) = 0.6e^{-5t}u(t)$ A

Solution:

Befor changes
$t<0$
Switch: opened



current division

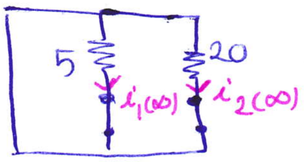
for $i_1 \rightarrow$ 5A is divided between 5Ω and (20||6)Ω

$$i_1 = \frac{20\|6}{5+20\|6}\times 5 = \frac{\frac{20\times 6}{20+6}}{5+\frac{20\times 6}{20+6}}\times 5 = \frac{120\times 5}{5\times 26+120} = \frac{12\cancel{0}\times 5}{25\cancel{0}} = \frac{12}{5} = 2.4A = i_1(0^-)$$

for $i_2 \rightarrow$ 5A is divided between 20Ω and (5||6)Ω

$$i_2 = \frac{5\|6}{20+5\|6}\times 5 = \frac{\frac{30}{11}\times 5}{20+\frac{30}{11}} = \frac{150}{220+30} = \frac{150}{250} = 0.6A = i_2(0^-)$$

After changes
$t>0$ switch: closed



Note that both inductors will discharge through their resistors and shortcircuit

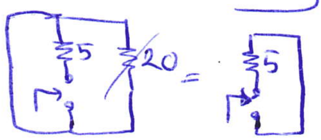
$i_1(\infty) = i_2(\infty) = 0$

Source-free response

$i_1(t) = i_1(0^+)e^{-t/\tau_1}$

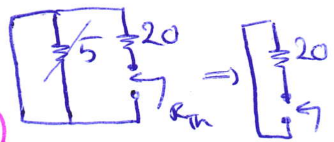
$i_2(t) = i_2(0^+)e^{-t/\tau_2}$ $\rightarrow$ $\boxed{i_1(0^+) = i_1(0^-) = 2.4A \quad i_2(0^+) = i_2(0^-) = 0.6A}$

$\tau_1 = \frac{L_1}{R_{Th_1}}$ $\rightarrow$ from L_1 terminals



$\boxed{R_{Th_1} = 5\Omega}$ $\rightarrow$ $\tau_1 = \frac{2.5}{5} = 1/2\,s$

$\tau_2 = \frac{L_2}{R_{Th_2}}$ $\rightarrow$ from L_2 terminals



$\boxed{R_{Th_2} = 20}$ $\rightarrow$ $\tau_2 = 4/20 = 1/5\,s$

$i_1(t) = 2.4e^{-2t}$ $t>0$

$i_2(t) = 0.6e^{-5t}$ $t>0$

or write them with u(t)