Faculty of Engineering

School of Electrical Engineering and Telecommunications

ELEC 1111 – Topic 7

AC Analysis I

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Alternating current

- Alternating current, or AC, is the **dominant** form of **electrical power** that is delivered to homes and industry.
- An AC current/voltage has **alternating positive** and **negative** values.
- Circuits driven by **sinusoidal** current or voltage sources are called **AC circuits.**
  - A **sinusoid** is a signal that has the form of **sine** or **cosine** function.

Sinusoids

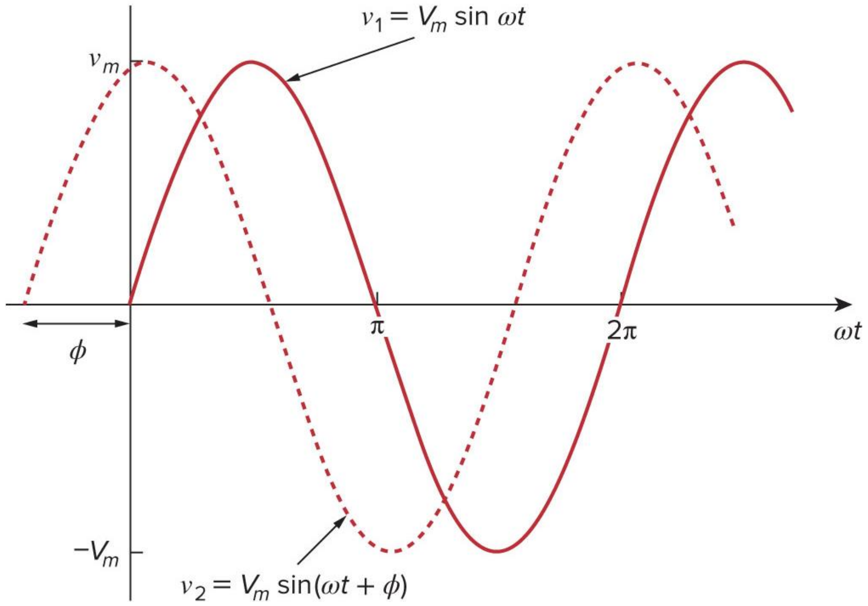
- Sinusoids can be found in many natural phenomena:
  - Pendulum motion.
  - String vibration.
  - Ocean surface ripples.
- Are easy to generate and transfer.
- Are easy to handle mathematically.

Sinusoids

- A sinusoidal voltage can be written as:

$$v(t) = V_m \sin(\omega t + \phi)$$

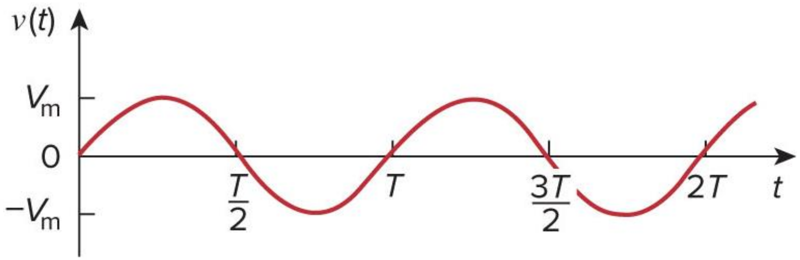
  - V_m: **Amplitude** of the sinusoid.
  - ω: **Angular frequency** in rad/s.
  - $(\omega t + \phi)$: **Argument** of the sinusoid.
  - ϕ: Phase (in degrees or radians).



Sinusoids – Period and frequency

- A **periodic function** is the one that satisfies $f(t) = f(t + nT)$ for all t and for all integers n.
- $v(t)$ repeats itself every T seconds, which is called the **period** of the sinusoid.
  - The period T is the time of **one complete cycle** or the number of seconds per cycle.
  - For a sinusoid:

$$\boxed{T = \frac{2\pi}{\omega}}$$



- The **reciprocal** of the period is the number of cycles per second known as the **cyclic frequency** f of the sinusoid. It is measured in **Hertz** (Hz).

$$\boxed{f = \frac{1}{T}}$$

- Using $T = \frac{2\pi}{\omega}$, **angular frequency** is shown to be proportional to frequency:

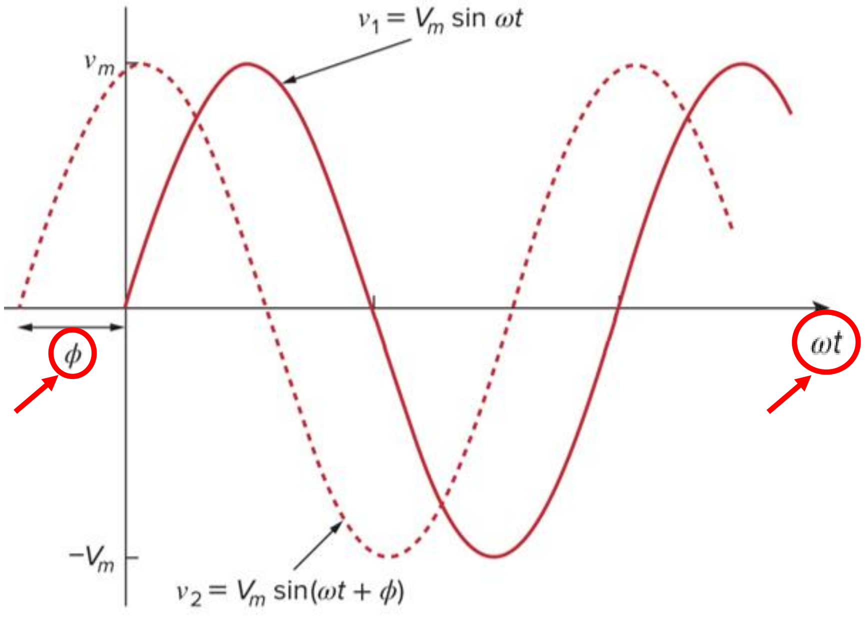
$$\boxed{\omega = 2\pi f}$$

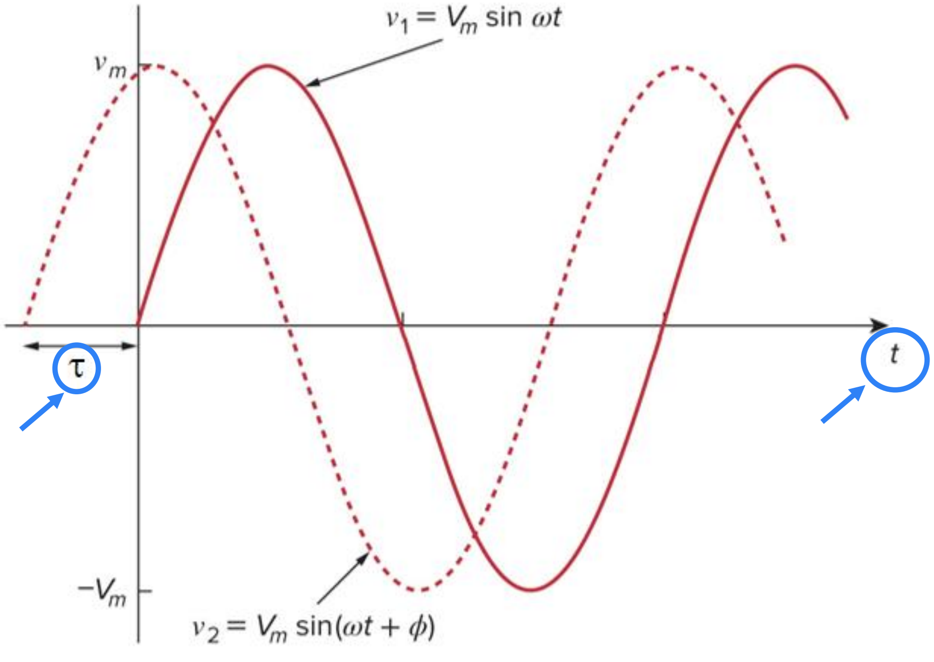
Sinusoids - Phase

- We can have sinusoids with different **phases**:

$$v_1(t) = V_m \sin(\omega t)$$

$$v_2(t) = V_m \sin(\omega t + \phi)$$





$$\boxed{\phi = \frac{\tau}{T} \times 360°}$$

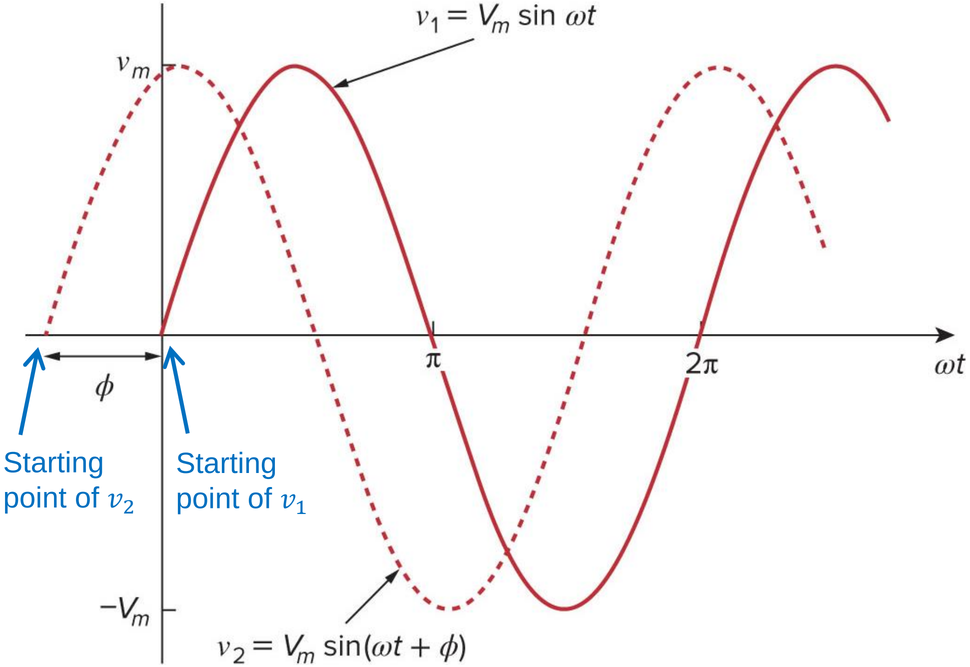
Sinusoids - Phase

- Given two sinusoids with different phases:

$$v_1(t) = V_m \sin(\omega t)$$

$$v_2(t) = V_m \sin(\omega t + \phi)$$

  - If $\phi \neq 0$, v_1 and v_2 are **out of phase**.
  - If $\phi = 0$, v_1 and v_2 are **in phase** (they reach their maxima and minima at exactly the same time).



- It can be assumed that v_1 starts at a **later** time ($t = 0\ s$) compared to v_2 which begins **earlier** at $t = -\frac{\phi}{\omega}\ s$. Thus,

v_1 is said to **lag** v_2

v_2 is said to **lead** v_1

Note: We can compare them in this manner because they have the same **same frequency**.

Sinusoids - Phase

- To evaluate a sinusoidal function at any value of time, it is required that **phase** is given in **radian** rather than in **degree** since **angular frequency** ω is given in **rad/s**.
- E.g. if you are asked to evaluate $v(t) = 50\cos(30t + 10°)$ for $t = 10\text{ ms}$ you can do one of the following:

1. Convert radian to degree:

$$\boxed{\phi° = \phi\text{ rad} \times \frac{180}{\pi}}$$

$$v(0.01) = 50\cos(\underbrace{30 \times 0.01\text{ rad}}_{\left(0.3 \times \frac{180}{\pi}\right) = 17.18°} + 10°)$$

$$\Rightarrow \boxed{v(0.01) = 50\cos(17.18° + 10°) = 44.47\text{ V}}$$

or

2. Convert degree to radian:

$$\boxed{\phi\text{ rad} = \phi° \times \frac{\pi}{180}}$$

$$v(0.01) = 50\cos(0.3\text{ rad} + \underbrace{10°}_{\left(10 \times \frac{\pi}{180}\right) = 0.174\text{ rad}})$$

$$\Rightarrow \boxed{v(0.01) = 50\cos(0.3\text{ rad} + 0.174\text{ rad}) = 44.47\text{ V}}$$

Sinusoids – Sine vs cosine

- Sinusoids may be expressed as **sine** or **cosine** form.
- To **compare** two sinusoids, it is convenient to express both as either sine or cosine functions with **positive** amplitudes, using some **trigonometric identities**:

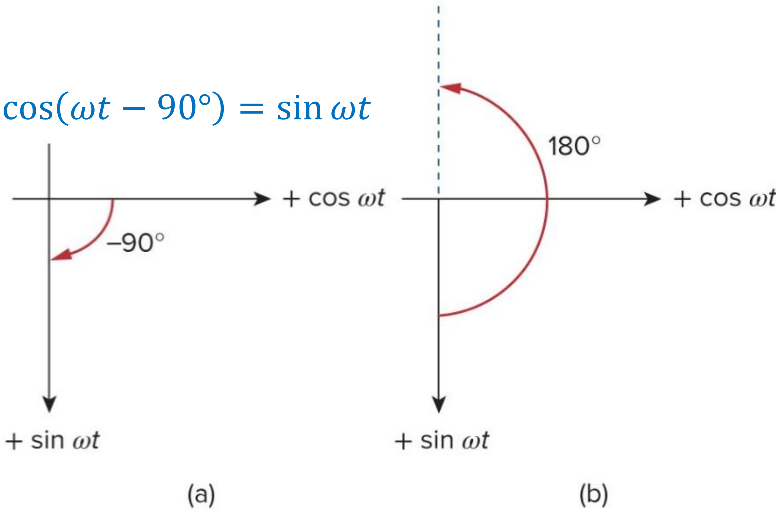
$$\boxed{\begin{aligned}\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \cos A \sin B\end{aligned}}$$

and

$$\boxed{\begin{aligned}\sin(\omega t \pm 180°) &= -\sin \omega t \\ \cos(\omega t \pm 180°) &= -\cos \omega t \\ \sin(\omega t \pm 90°) &= \pm \cos \omega t \\ \cos(\omega t \pm 90°) &= \mp \sin \omega t\end{aligned}}$$

$\sin(\omega t + 180°) = -\sin \omega t$

$\cos(\omega t - 90°) = \sin \omega t$



Two examples of graphical representation for trigonometric identities (phase shift)

Exercise

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50°)$ and $v_2 = 12\sin(\omega t - 10°)$. State which sinusoid is leading.

$\cos(\omega t - 180°) = -\cos\omega t$
$\cos(\omega t - 90°) = \sin(\omega t)$

Review of complex numbers

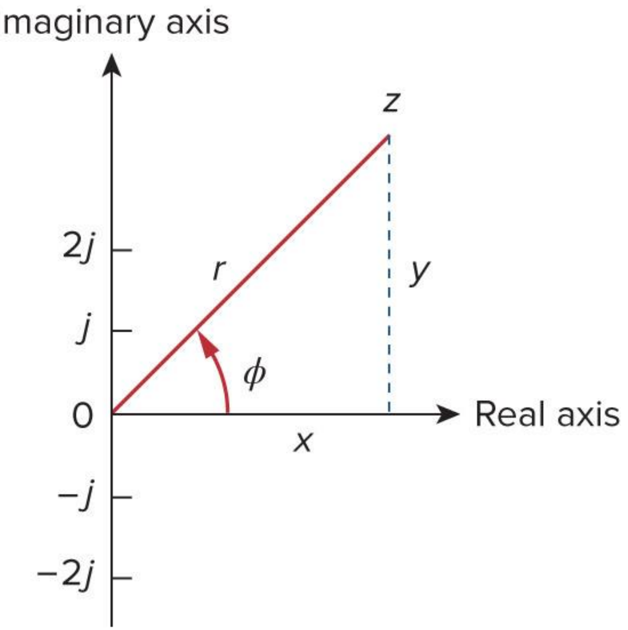
- A powerful method for representing sinusoids is the **phasor**.
- In order to understand how phasors work, we need to review some of the **complex numbers** properties first.
  - A complex number z can be represented in **rectangular** or **Cartesian** form as:

$$\boxed{z = x + jy}$$

  - It can also be written in **polar** or **exponential** form as:

Polar form: $\boxed{z = r\angle\phi}$

Exponential form: $\boxed{z = re^{j\phi}}$



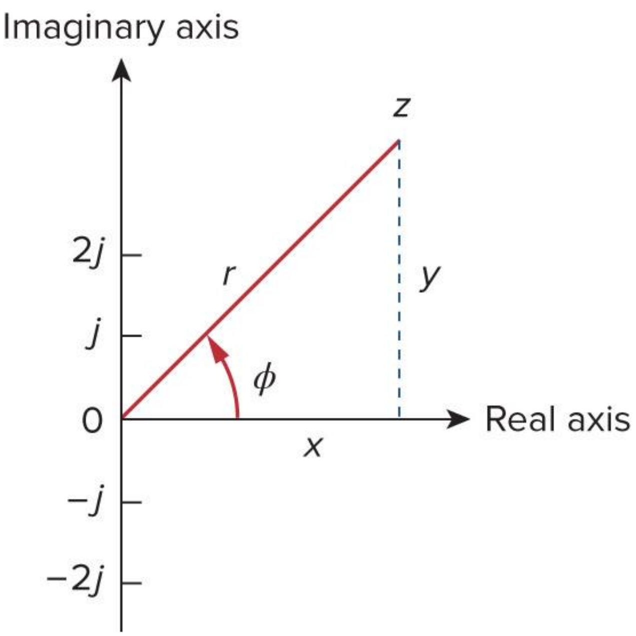
- $j = \sqrt{-1}$: **Unit imaginary** number
- x: The **real** part of z
- y: The **imaginary** part of z
- r: The **magnitude** of z ($|z|$)
- ϕ: The **phase** of z

Review of complex numbers

- **Rectangular** to **polar** form transformation
  when $z = x + jy$:

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} \frac{y}{x}$$



- **Polar** to **rectangular** form transformation
  when $z = r\angle\phi$:

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$\Rightarrow \quad z = r\cos\phi + j\,r\sin\phi$$

Review of complex numbers

Given the complex numbers $z = x + jy = r\angle\phi = re^{j\phi}$,
$z_1 = x_1 + jy_1 = r_1\angle\phi_1 = r_1e^{j\phi_1}$, and $z_2 = x_2 + jy_2 = r_2\angle\phi_2 = r_2e^{j\phi_2}$, some basic properties to be used for phasor analysis are as follows:

- **Addition**:

$$\boxed{z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)}$$

- **Subtraction**:

$$\boxed{z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)}$$

- **Multiplication**:

$$\boxed{z_1z_2 = r_1r_2\angle(\phi_1 + \phi_2) = r_1r_2e^{j(\phi_1+\phi_2)}}$$

- **Division**:

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2) = \frac{r_1}{r_2}e^{j(\phi_1+\phi_2)}}$$

Review of complex numbers

Given the complex numbers $z = x + jy = r\angle\phi = re^{j\phi}$, $z_1 = x_1 + jy_1 = r_1\angle\phi_1 = r_1 e^{j\phi_1}$, and $z_2 = x_2 + jy_2 = r_2\angle\phi_2 = r_2 e^{j\phi_2}$, some basic properties to be used for phasor analysis are as follows:

- **Reciprocal**:

$$\boxed{\frac{1}{z} = \frac{1}{r}\angle(-\phi) = \frac{1}{r}e^{-j\phi}}$$

- **Square Root**:

$$\boxed{\sqrt{z} = \sqrt{r}\angle\frac{\phi}{2} = \sqrt{r}e^{j\frac{\phi}{2}}}$$

- **Complex Conjugate**:

$$\boxed{z^* = x - jy = r\angle - \phi = re^{-j\phi}}$$

$$\boxed{\frac{1}{j} = -j} \quad \text{and} \quad \boxed{zz^* = r^2 = |z|^2}$$

- **Also**: $\boxed{e^{\pm j90^\circ} = \pm j, \quad e^{\pm j180^\circ} = -1}$

Phasor

- The idea of phasor representation is based on **Euler's identity**:

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi}$$

- We can write $\cos\phi$ and $\sin\phi$ as the **real part**, $\mathrm{Re}(\cdot)$, and **imaginary part**, $\mathrm{Im}(\cdot)$, of $e^{j\phi}$:

$$\cos\phi = \mathrm{Re}\left(e^{j\phi}\right)$$
$$\sin\phi = \mathrm{Im}\left(e^{j\phi}\right)$$

- Using the Euler's identity for a given sinusoid $v(t)$ we have:

$$v(t) = V_m\cos(\omega t + \phi) = \mathrm{Re}\left(V_m e^{j(\omega t + \phi)}\right)$$

$$v(t) = \mathrm{Re}\left(V_m e^{j\phi} e^{j\omega t}\right)$$

$$\boxed{v(t) = \mathrm{Re}\left(\mathbf{V} e^{j\omega t}\right)} \Rightarrow \boxed{\mathbf{V} = V_m e^{j\phi} = V_m \angle\phi}$$

$\mathbf{V} = V_m\angle\phi$ is the **phasor representation** of the sinusoid $v(t) = V_m\cos(\omega t + \phi)$

Phasor

- A **phasor** is a **complex number** that represents the **amplitude** and **phase** of a **sinusoid.**
  - For a given **sinusoid** in **cosine form**, $v(t) = V_m \cos(\omega t + \phi)$, the **phasor** is defined as a **complex number** by supressing the time factor $e^{j\omega t}$, and taking the **amplitude** V_m and the **phase** ϕ.
  - To obtain the time-domain representation of a given phasor $\boldsymbol{V}$, use a **cosine function** with the **same magnitude** as the **phasor** and the **argument** ωt plus the **phase** of the phasor.

$$\boxed{v(t) = V_m \cos(\omega t + \phi)} \Longleftrightarrow \boxed{\mathbf{V} = V_m \angle \phi = V_m e^{j\phi}}$$

Time-domain representation

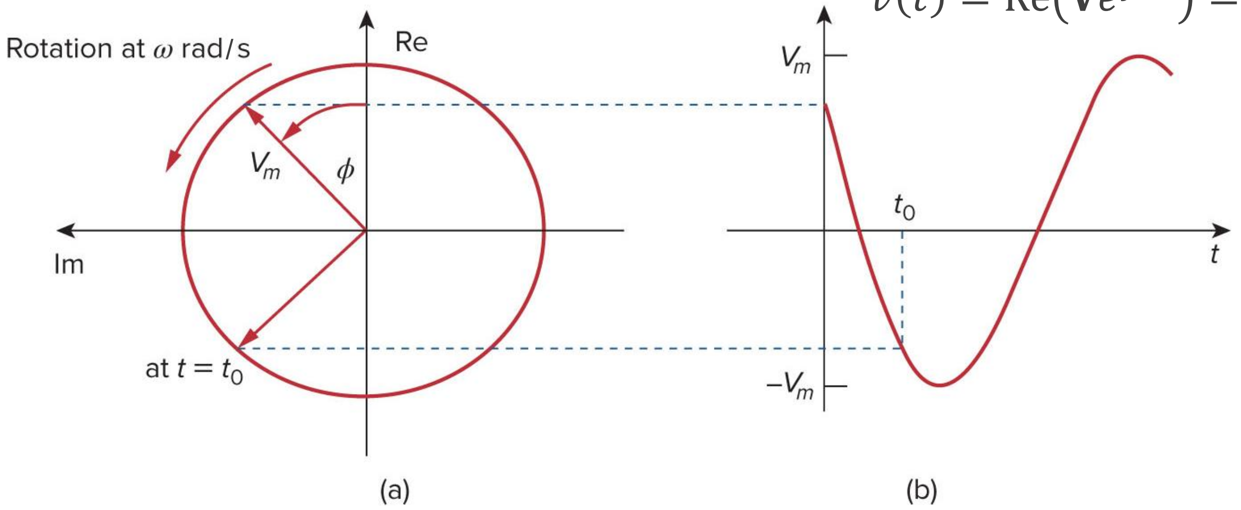
Phasor-domain representation

- **Boldface** letters like $\mathbf{V}$ are used to represent phasors because they are **vector-like** quantities.

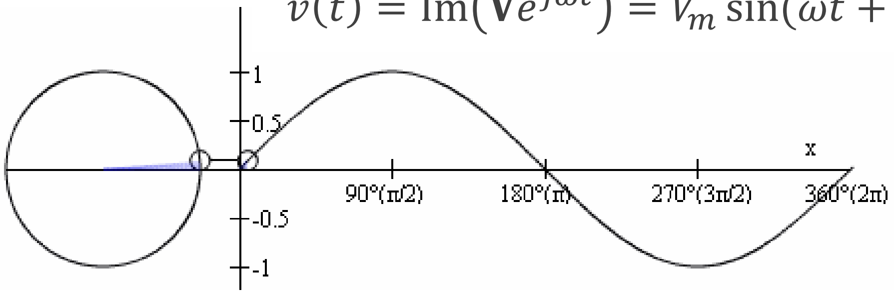
Phasor – Graphical representation

- Consider the plot of $\mathbf{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$
  - As time increases, the vector rotates on a circle of **radius** V_m at an **angular velocity** ω in the **counter clockwise** direction.

$$v(t) = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) = V_m \cos(\omega t + \phi)$$



$$v(t) = \mathrm{Im}\left(\mathbf{V}e^{j\omega t}\right) = V_m \sin(\omega t + \phi)$$

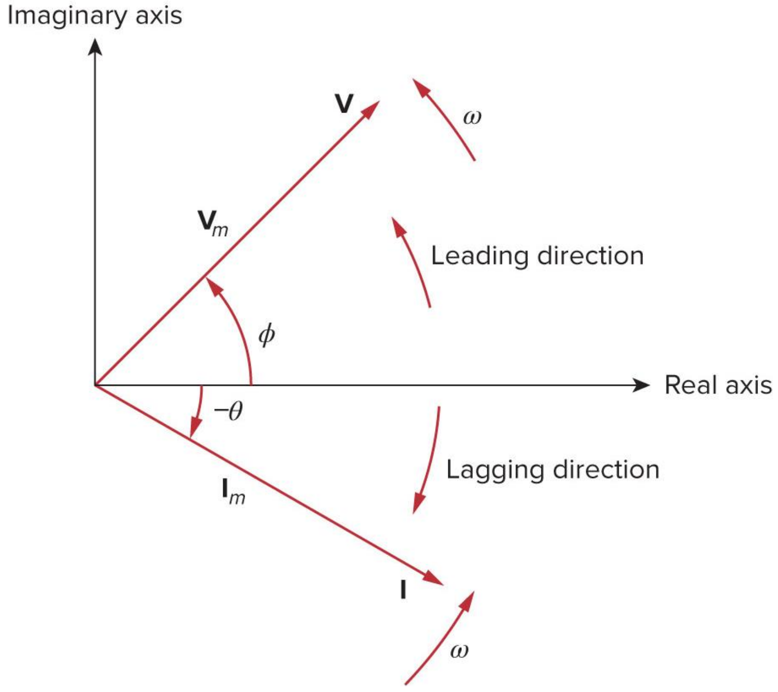


https://giphy.com/gifs/mathematics-sin-pi-NKLdcqhwo2f8A/

Phasor diagram

- A phasor can be expressed as a **vector** with a **magnitude** and a **phase** (direction).
- **Sketching** the phasors in a Cartesian coordinates or complex plane is called **phasor diagram**.
- Phasor diagram is plotted for **constant** and **equal** frequency ω, without being shown explicitly, that is why it is also known as **frequency domain**.

  - The convention for measuring **angle/phase** is from **positive real axis** rotating **counter clockwise**.
  - A **leading** phasor is the one **ahead** of other phasors in **counter clockwise** direction.



Phasor diagram showing $\mathbf{V} = V_m \angle \phi$ and $\mathbf{I} = I_m \angle -\theta$

Sinusoid-phasor transformation

- Phasors are defined in **cosine form** for both **voltage** and **current signals**.
- Any other form of sinusoid **should** be converted to cosine form using trigonometric identities.

Time-domain representation	Phasor-domain representation
$v(t) = V_m \cos(\omega t + \phi)$	$\mathbf{V} = V_m \angle \phi$
$v(t) = V_m \sin(\omega t + \phi)$	$\mathbf{V} = V_m \angle (\phi - 90°)$
$v(t) = -V_m \cos(\omega t + \phi)$	$\mathbf{V} = V_m \angle (\phi \pm 180°)$
$v(t) = -V_m \sin(\omega t + \phi)$	$\mathbf{V} = V_m \angle (\phi + 90°)$
$i(t) = I_m \cos(\omega t + \theta)$	$\mathbf{I} = I_m \angle \theta$
$i(t) = I_m \sin(\omega t + \theta)$	$\mathbf{I} = I_m \angle (\theta - 90°)$
$i(t) = -I_m \cos(\omega t + \theta)$	$\mathbf{I} = I_m \angle (\theta \pm 180°)$
$i(t) = -I_m \sin(\omega t + \theta)$	$\mathbf{I} = I_m \angle (\theta + 90°)$

Sinusoid-phasor transformation

- We can find the relationship between **linear mathematical operations** in **time domain** and their transformation in **phasor domain**.
- **Differentiating** a **sinusoid** is equivalent to **multiplying** its corresponding **phasor** by $\boldsymbol{j\omega}$.
  - Consider a sinusoid and its corresponding phasor:

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \mathbf{V} = V_m \angle \phi$$

Take **time derivative** of the sinusoid:

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi)$$

$$= \omega V_m \cos(\omega t + \phi + 90°)$$

$$= \mathrm{Re}\left(\omega V_m e^{j(\omega t + \phi + 90°)}\right) = \mathrm{Re}\left(\omega \underbrace{V_m e^{j\phi}}_{\mathbf{V}} e^{j\omega t} \underbrace{e^{j90°}}_{j}\right) = \mathrm{Re}\left(j\omega \mathbf{V} e^{j\omega t}\right)$$

$$\boxed{\frac{dv}{dt}} \quad \Longleftrightarrow \quad \boxed{j\omega\mathbf{V}}$$

(Time domain) (Phasor domain)

Sinusoid-phasor transformation

- **Integrating** a **sinusoid** is equivalent to **dividing** its corresponding **phasor** by $\boldsymbol{j\omega}$.
  - Consider the same sinusoid and its corresponding phasor:

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

Take **time integral** of the sinusoid:

$$\int v\,dt = \frac{1}{\omega} V_m \sin(\omega t + \phi)$$

$$= \frac{1}{\omega} V_m \cos(\omega t + \phi - 90°)$$

$$= \mathrm{Re}\left(\frac{1}{\omega} V_m e^{j(\omega t + \phi - 90°)}\right) = \mathrm{Re}\left(\frac{1}{\omega} \underbrace{V_m e^{j\phi}}_{\mathbf{V}} e^{j\omega t} \underbrace{e^{-j90°}}_{-j=\frac{1}{j}}\right) = \mathrm{Re}\left(\frac{1}{j\omega} \mathbf{V} e^{j\omega t}\right)$$

$$\boxed{\int v\,dt} \quad \Leftrightarrow \quad \boxed{\frac{1}{j\omega}\mathbf{V}}$$

(Time domain) (Phasor domain)

Some notes on sinusoids and phasors

- There are some conceptual differences between sinusoids and their corresponding phasors.

$$\boxed{v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi}$$

1. $v(t)$ is the **instantaneous** or **time-domain** representation of a sinusoid, while $\mathbf{V}$ is the **phasor** or **frequency-domain** representation of the sinusoid.
2. $v(t)$ is **time dependent**, while $\mathbf{V}$ is **constant**.
3. $v(t)$ is always a **real function** with **no** complex term, while $\mathbf{V}$ is generally a **complex number**.
4. Keep in mind that **phasor analysis** on one or multiple sinusoids applies **only** when the **frequency is constant** and they are of the **same frequency**.
   - **Note:** Phasor analysis for a circuit with **multiple frequencies** is possible but requires the use of **superposition**, and at **each stage** you have a **different set of phasors.**

Exercise

Transform these sinusoids to phasors:

a) $i = 6\cos(50t - 40°)\ \text{A}$

b) $v = -4\sin(30t + 50°)\ \text{V}$

c) $i_1 = -8\cos(16t + 15°)\ \text{A}$

! $v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$

Exercise

- Find the sinusoids represented by these phasors:
  a) $\mathbf{I} = -3 + j4\ \mathrm{A}$
  b) $\mathbf{V} = j8e^{-j20°}\ \mathrm{V}$
  c) $\mathbf{V_1} = -25\angle 40°\ \mathrm{V}$

Cartesian to polar transformation: $z = \sqrt{x^2 + y^2}\angle \tan^{-1}({}^{y}/{}_{x})$

Exercise

Given $i_1(t) = 4\cos(\omega t + 30°)$ A and $i_2(t) = 5\sin(\omega t - 20°)$ A, find $i_1(t) + i_2(t)$.

Polar to Cartesian transformation $z = r\cos\phi + j\,r\sin\phi$
Cartesian to Polar transformation: $z = \sqrt{x^2 + y^2}\angle\tan^{-1}(^y/_x)$

Exercise

If $v_1(t) = -10\sin(\omega t - 30°)$ V and $v_2(t) = 20\cos(\omega t + 45°)$ V, find $v(t) = v_1(t) + v_2(t)$

- For practice!
- Answer: $v(t) = 29.77\cos(\omega t + 49.98°)$

Phasor relationships for resistor

- Current and voltage relationships can be **mapped** from **time domain** into **phasor domain** very simply for passive elements like resistors, capacitors and inductors.

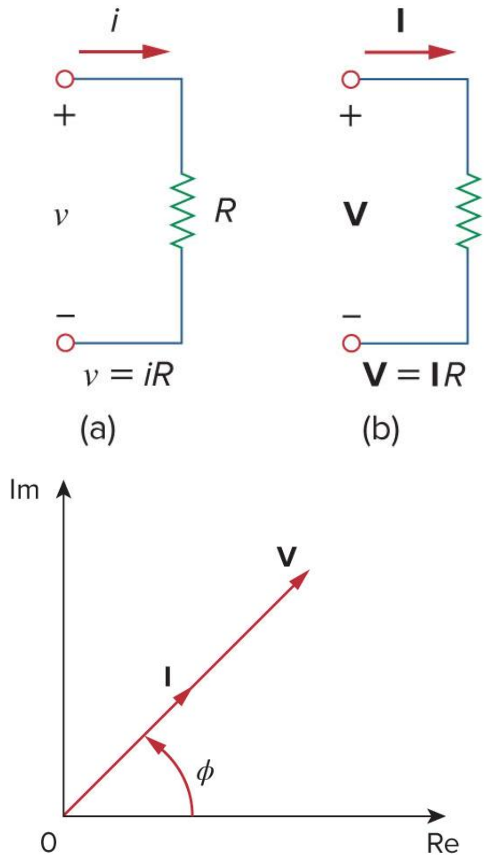
- For a resistor R, if the **current** is given as a **sinusoid** $i = I_m \cos(\omega t + \phi)$ (AC) with the corresponding current phasor of $\mathbf{I} = I_m \angle \phi$, the voltage is obtained using **Ohm's law** as below:

$$v = Ri = RI_m \cos(\omega t + \phi)$$

The phasor representation of the voltage is:

$$\mathbf{V} = RI_m \angle \phi = R\mathbf{I}$$

Therefore:



$$\boxed{\mathbf{V} = R\mathbf{I}}$$

Voltage and **current** of a **resistor** are **in phase** and related via **Ohm's law** in **phasor domain.**

Phasor relationships for inductor

- For an inductor L, if the **current** is given as a **sinusoid** $i = I_m \cos(\omega t + \phi)$ (AC) with the corresponding current phasor of $\mathbf{I} = I_m \angle \phi$, the voltage is obtained as below:

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

Recall that $-\sin a = \cos(a + 90°)$:

$$v = \omega L I_m \cos(\omega t + \phi + 90°)$$

The voltage phasor is: $\mathbf{V} = \omega L I_m \angle(\phi + 90°)$

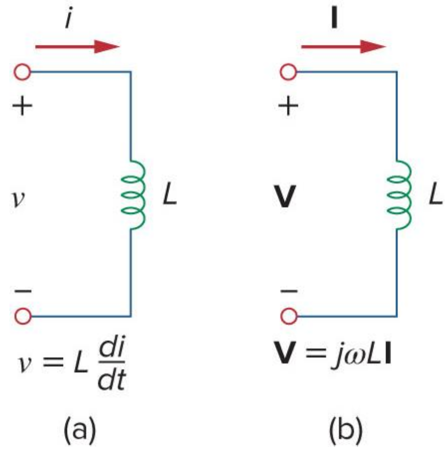
Recall that $1\angle 90° = e^{j90°} = j$:

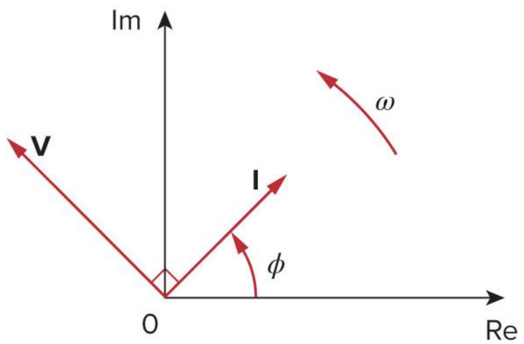
$$\mathbf{V} = \omega L I_m e^{j\phi} e^{j90°} = j\omega L I_m \angle \phi = j\omega L \mathbf{I}$$

Therefore:

$$\boxed{\mathbf{V} = j\omega L \mathbf{I}}$$

- **Voltage** and **current** of an **inductor** are **90° out of phase.**
- The **inductor current "<u>lags</u>"** its **voltage** by **90°**.





Phasor relationships for capacitor

- For a capacitor C, if the **voltage** is given as a **sinusoid** $v = V_m \cos(\omega t + \phi)$ (AC) with the corresponding voltage phasor of $\mathbf{V} = V_m \angle \phi$, the current is obtained as below:

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$

$$i = \omega C V_m \cos(\omega t + \phi + 90°)$$

The current phasor is:

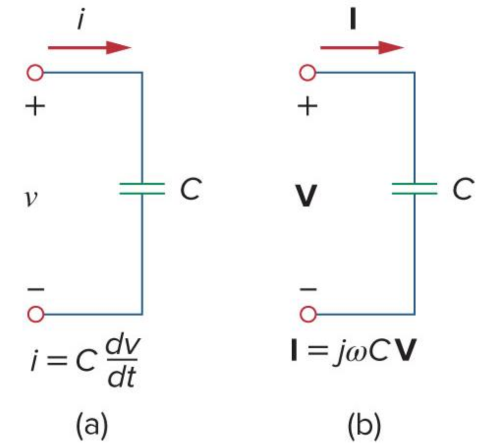
$$\mathbf{I} = \omega C V_m \angle(\phi + 90°)$$

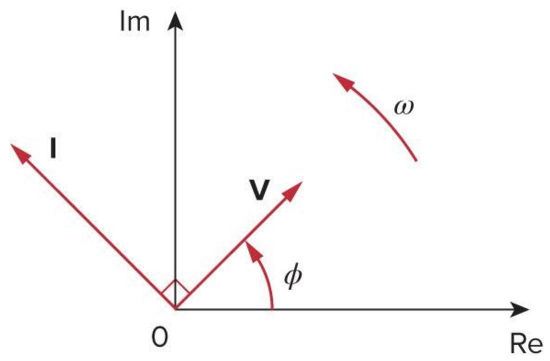
$$\mathbf{I} = \omega C V_m e^{j\phi} e^{j90°} = j\omega C V_m \angle\phi = j\omega C\mathbf{V}$$

$$\mathbf{I} = j\omega C\mathbf{V}$$

Therefore:

$$\boxed{\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}}$$





- **Voltage** and **current** of a **capacitor** are **90° out of phase.**
- The **capacitor current "<u>leads</u>"** its **voltage** by **90°**.

Passive elements phasors

- Summary of time-domain and phasor-domain representations of the passive circuit elements:

Element	Time-domain representation	Phasor-domain representation
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I}$

Impedance

- **Ohm's law can** be extended to capacitors and inductors in **phasor domain**.
- The **<u>impedance</u>** $\mathbf{Z}$ of a circuit element is the **ratio** of the **phasor voltage** $\mathbf{V}$ to the **phasor current** $\mathbf{I}$, measured in **ohms** ($\mathbf{\Omega}$).

Recall the **voltage-current** relations for the three passive elements in **phasor domain**:

$$\boxed{\mathbf{V} = R\mathbf{I}} \qquad \boxed{\mathbf{V} = j\omega L\mathbf{I}} \qquad \boxed{\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}}$$

Resistor | Inductor | Capacitor

Write these expression in terms of the ratio of the phasor voltage to the phasor current:

$$\boxed{\frac{\mathbf{V}}{\mathbf{I}} = R} \qquad \boxed{\frac{\mathbf{V}}{\mathbf{I}} = j\omega L} \qquad \boxed{\frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}}$$

Resistor | Inductor | Capacitor

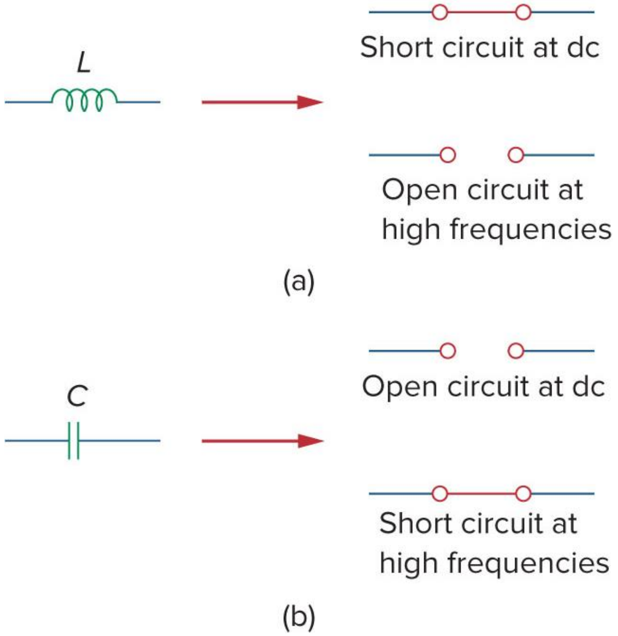
- These **impedances** represent **Ohm's law** in the **phasor domain:**

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}} \quad \text{or} \quad \boxed{\mathbf{V} = \mathbf{Z}\mathbf{I}}$$

Impedance

- Given the impedances of resistors, inductors, and capacitors, only the impedance of the **resistor** is a **constant real value.**
- The impedance of the **inductor** and **capacitor** are **frequency-dependent** complex values.

$$\mathbf{Z}_R = R \qquad \mathbf{Z}_L = j\omega L \qquad \mathbf{Z}_C = \frac{1}{j\omega C}$$



- Consider two **extreme cases** for **angular frequency** ω:

1. At **low** frequencies: $\omega = 0$ $\Rightarrow$ $\mathbf{Z}_L = 0$, $\mathbf{Z}_C \to \infty$

2. At **high** frequencies: $\omega \to \infty$ $\Rightarrow$ $\mathbf{Z}_L \to \infty$, $\mathbf{Z}_C = 0$

- At **low frequencies**, **inductor** acts like a **short circuit** while **capacitor** acts like an **open circuit.**
- At **high frequencies**, **Inductor** acts like an **open circuit** while **capacitor** acts like a **short circuit.**

Impedance

- The **impedances** of resistors, inductors, and capacitors **can be combined** together as one impedance.
- **Impedance**, as a **complex quantity**, can be expressed in **rectangular** or **polar** forms.
  - $R = \mathrm{Re}(\mathbf{Z})$: **Resistance.**
  - $X = \mathrm{Im}(\mathbf{Z})$: **Reactance.**
  - The impedance Z, resistance R, and reactance X, are all measured in ohms (Ω).

$$\boxed{\mathbf{Z} = R + jX} \iff \boxed{\mathbf{Z} = |\mathbf{Z}|\angle\theta}$$

P → R $\quad \boxed{R = |\mathbf{Z}|\cos\theta} \quad \boxed{X = |\mathbf{Z}|\sin\theta}$

R → P $\quad \boxed{|\mathbf{Z}| = \sqrt{R^2 + X^2}} \quad \boxed{\theta = \tan^{-1}\left(\frac{X}{R}\right)}$

Impedance

- **Inductors** and **capacitors** have **pure imaginary** impedances:

$$\boxed{\mathbf{Z}_L = j\omega L}$$

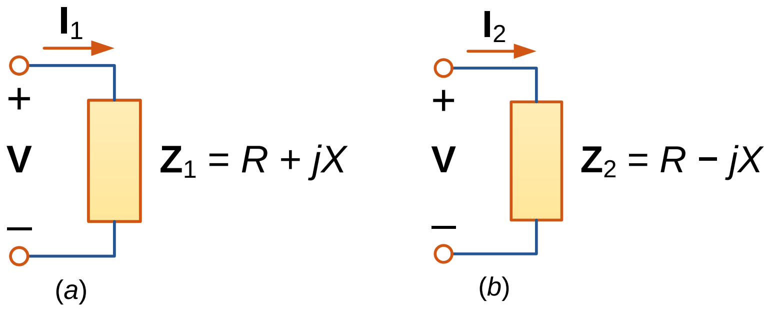
$$\boxed{\mathbf{Z}_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}}$$

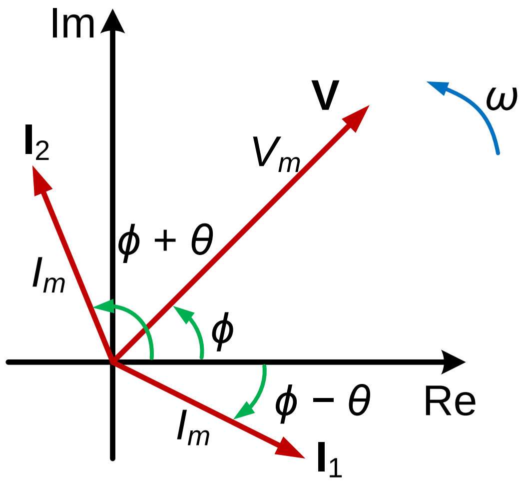
- Also, an **impedance** in general is expressed as a **complex number** with either **positive** or **negative** imaginary part:

$$\boxed{\mathbf{Z} = R \pm jX}$$

$\mathbf{Z} = R + jX$: **Inductive** or **lagging** (current **lags** voltage, $\mathbf{I}_1$).

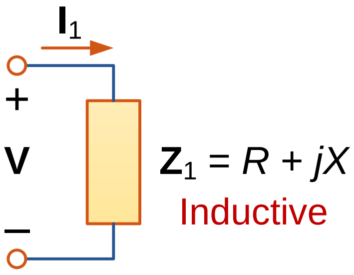
$\mathbf{Z} = R - jX$: **Capacitive** or **leading** (current **leads** voltage, $\mathbf{I}_2$).



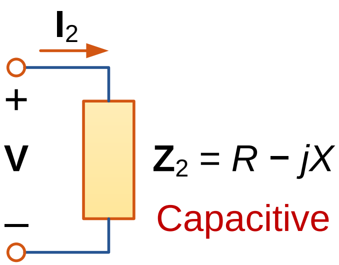


Impedance

Proof: Consider the following circuits with equal input voltage phasor $\mathbf{V} = V_m\angle\phi$, and complex conjugate impedances $\mathbf{Z}_1 = R + jX$ and $\mathbf{Z}_2 = R - jX$ ($|\mathbf{Z}_1| = |\mathbf{Z}_2|$).



(a)



(b)

Based on Ohm's law:

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{V_m\angle\phi}{R + jX} = \frac{V_m\angle\phi}{\sqrt{R^2 + X^2}\angle\tan^{-1}\left(\frac{X}{R}\right)}$$

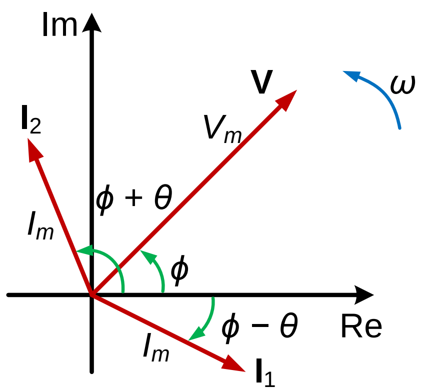
$$\mathbf{I}_1 = \frac{V_m}{\underbrace{\sqrt{R^2 + X^2}}_{|\mathbf{Z}_1|}}\angle(\phi - \underbrace{\tan^{-1}\left(\frac{X}{R}\right)}_{\theta}) = \underbrace{\frac{V_m}{|\mathbf{Z}_1|}}_{I_m}\angle(\phi - \theta) = I_m\angle(\phi - \theta)$$

and:

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{V_m\angle\phi}{R - jX} = \frac{V_m\angle\phi}{\sqrt{R^2 + X^2}\angle\tan^{-1}\left(\frac{-X}{R}\right)}$$

$\tan^{-1}(-a) = -\tan^{-1}(a)$
odd function

$$\mathbf{I}_2 = \frac{V_m}{\underbrace{\sqrt{R^2 + X^2}}_{|\mathbf{Z}_2| = |\mathbf{Z}_1|}}\angle(\phi + \underbrace{\tan^{-1}\left(\frac{X}{R}\right)}_{\theta}) = \underbrace{\frac{V_m}{|\mathbf{Z}_1|}}_{I_m}\angle(\phi + \theta) = I_m\angle(\phi + \theta)$$



Admittance

- **Impedance** represents the **resistance** of a circuit element to the **flow** of **sinusoidal current.**
  - Impedance is not a phasor, it is just a complex number.
- **Admittance** is simply the **reciprocal** of the impedance. It is measured in **siemens (S).**

$$\boxed{\mathbf{Y} = \frac{1}{\mathbf{Z}}} \quad \text{or} \quad \boxed{\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}}$$

- The impedance and admittance of resistors, inductors, and capacitors can be obtained as follows:

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Admittance

- Admittance $\mathbf{Y}$ is also a **complex quantity**.
  - $G = \mathrm{Re}(\mathbf{Y})$: **Conductance.**
  - $B = \mathrm{Im}(\mathbf{Y})$: **Susceptance.**
  - Admittance Y, conductance G, and susceptance B, are all measured in **siemens** (S).

$$\boxed{\mathbf{Y} = G + jB}$$

- The admittance $\mathbf{Y} = G + jB$ and impedance $\mathbf{Z} = R + jX$ components are related:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} \quad \rightarrow \quad G + jB = \frac{1}{R + jX}$$

To find the relationship, cancel j in the denominator (i.e. multiply the fraction by the **complex conjugate** of $R + jX$, since $\mathbf{Z}\mathbf{Z}^* = |\mathbf{Z}|^2$).

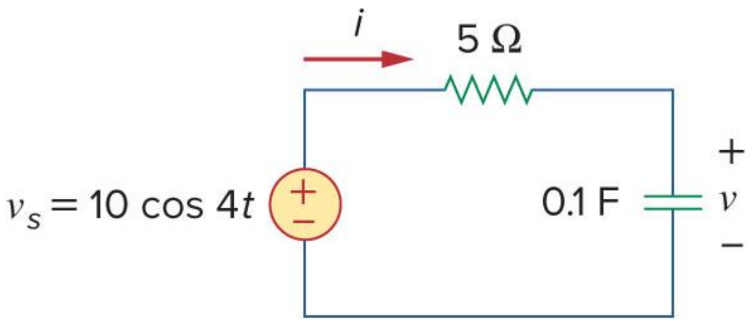
$$G + jB = \frac{1}{R + jX} \times \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} \quad \rightarrow \quad \boxed{G = \frac{R}{R^2 + X^2}} \quad \boxed{B = \frac{-X}{R^2 + X^2}}$$

- Similarly, $\mathbf{Z}$ components are related to $\mathbf{Y}$:

$$\boxed{R = \frac{G}{G^2 + B^2}} \quad \boxed{X = \frac{-B}{G^2 + B^2}}$$

Exercise

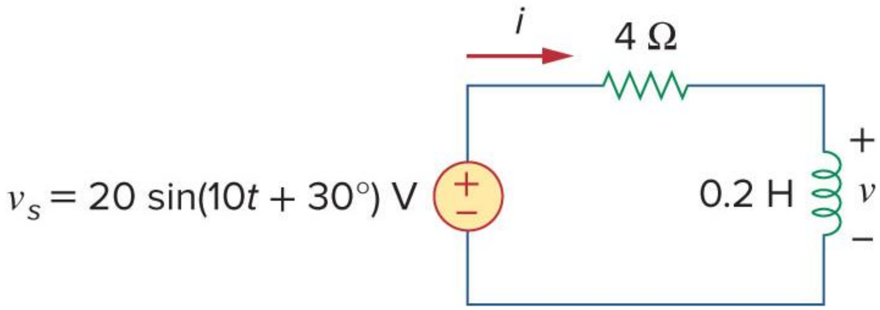
Find $v(t)$ and $i(t)$ in the circuit below.



Exercise

Find $v(t)$ and $i(t)$ in the circuit below

- For practice!
- Answer: $v(t) = 8.944\cos(10\text{t} + 3.43°)\ \text{V}$
  $i(t) = 4.472\cos(10\text{t} - 86.57°)\ \text{A}$



Kirchhoff’s laws

- **KVL** and **KCL** applies to **sinusoidal AC circuits** in the **same way** as in DC circuits by using the **sum** of **voltage phasors** for **KVL** and the **sum** of **current phasors** for **KCL.**
- For a single loop circuit with **voltages** $v_1, v_2, \ldots, v_n$, we can apply **KVL**:

$$v_1 + v_2 + \cdots + v_n = 0$$

  – Assuming sinusoidal steady-state conditions, all voltages in the circuit can be written in cosine form:

$$V_{m1}\cos(\omega t + \phi_1) + V_{m2}\cos(\omega t + \phi_2) + \cdots + V_{mn}\cos(\omega t + \phi_n) = 0$$

  – For each sinusoid, we can use the corresponding phasor in the sum:

$$V_{mk}\cos(\omega t + \phi_k) \Leftrightarrow \mathbf{V}_k = V_{mk}\angle\phi_k \text{ for } k = 1, 2, \ldots, n$$

  Therefore:

$$\boxed{\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0}$$

- Using the same procedure for **currents** in a single node and applying **KCL** using the current phasors:

$$I_{mk}\cos(\omega t + \theta_k) \Leftrightarrow \mathbf{I}_k = I_{mk}\angle\theta_k \text{ for } k = 1, 2, \ldots, n \;\rightarrow\; \boxed{\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0}$$

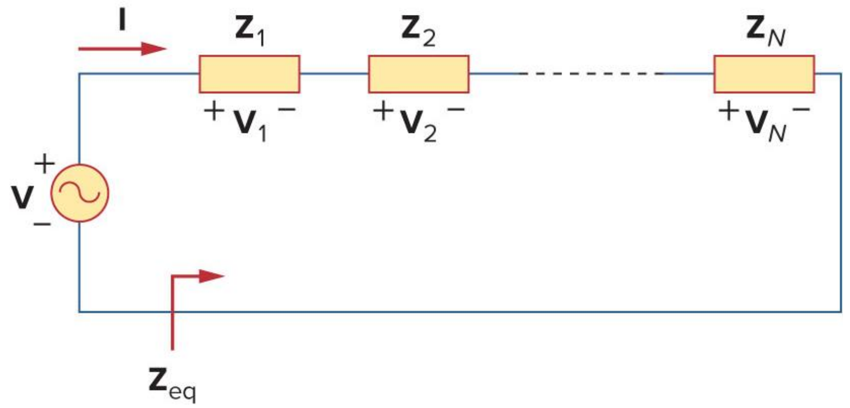
Impedances in series

- The **equivalent impedance** $\mathbf{Z}_{\text{eq}}$ of **series-connected** impedances is the **sum** of the **individual** impedances (**same** as **series** connection of **resistors)**.
  - Consider *N* **series-connected impedances** as shown in the circuit below.
  - The **same** current $\mathbf{I}$ flows through the impedances.
  - Apply KVL in phasor domain:

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n)$$

- The **equivalent impedance** $\mathbf{Z}_{\text{eq}}$ at the input terminals is:

$$\mathbf{Z}_{\text{eq}} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n$$



$$\boxed{\mathbf{Z}_{\text{eq}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n}$$

Voltage division

- **Voltage division** for **impedances** works the **same** way as voltage division for **resistors.**
  - For **two impedances** connected **in series** as shown in the circuit, the current $\mathbf{I}$ is obtained using Ohm's law:

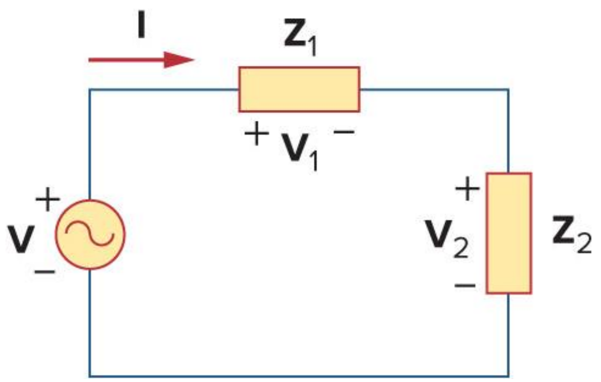
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

  - Also, $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$. Therefore, each voltage is obtained as follows:

Voltage Division Principle

$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}}$$

$$\boxed{\mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}}$$



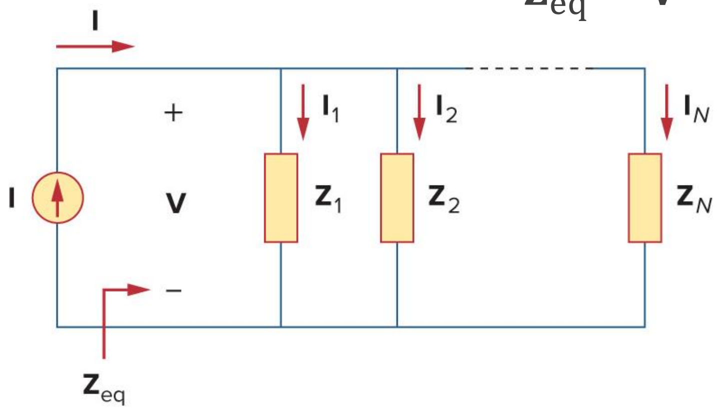
Impedances in parallel

- The **equivalent impedance** $\mathbf{Z}_{\text{eq}}$ of **parallel-connected** impedances is the **reciprocal** of the **sum** of the **individual reciprocal** impedances (**same** as **parallel** connection of **resistors).**
  - Consider *N* **parallel-connected impedances** as shown in the circuit below.
  - The **voltage** across each impedance is **same.**
  - Apply KCL in phasor domain at the top node:

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}\right)$$

- The **equivalent impedance** $\mathbf{Z}_{\text{eq}}$ at the input terminals is:

$$\frac{1}{\mathbf{Z}_{\text{eq}}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}$$



$$\boxed{\frac{1}{\mathbf{Z}_{\text{eq}}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}}$$

OR (using **admittance**):

$$\boxed{\mathbf{Y}_{\text{eq}} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n}$$

Current division

- **Current division** for **impedances** works the **same** way as current division for **resistors.**
  - For **two impedances** connected **in parallel** as shown in the circuit, the equivalent impedance $\mathbf{Z}_{\mathrm{eq}}$ is obtained as:

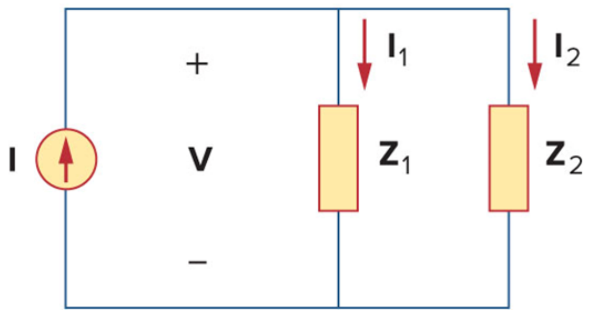
$$\mathbf{Z}_{\mathrm{eq}} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

  - Also, $\mathbf{V} = \mathbf{Z}_{\mathrm{eq}}\mathbf{I} = \mathbf{Z}_1\mathbf{I}_1 = \mathbf{Z}_2\mathbf{I}_2$. Therefore, each current is obtained as follows:

Current Division Principle

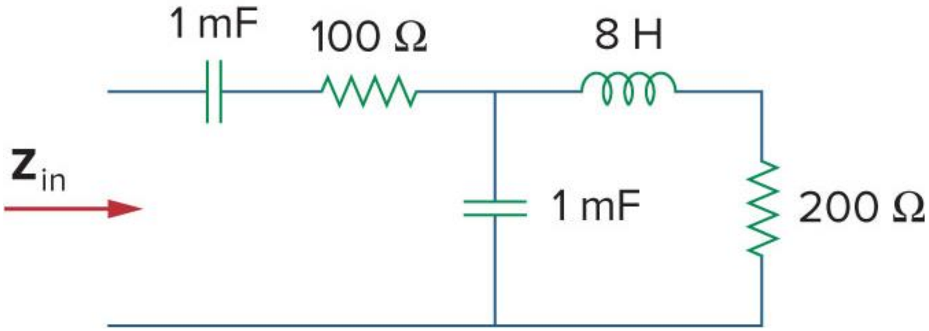
$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_1 \| \mathbf{Z}_2}{\mathbf{Z}_1}\mathbf{I} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}}$$

$$\boxed{\mathbf{I}_2 == \frac{\mathbf{Z}_1 \| \mathbf{Z}_2}{\mathbf{Z}_2}\mathbf{I} = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}}$$



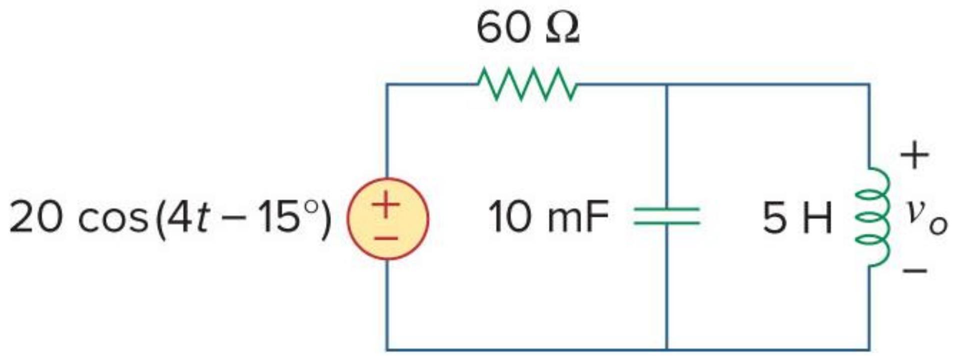
Exercise

Determine the input impedance of the circuit shown below assuming the circuit operates at $f = \frac{20}{2\pi}$ Hz.



Exercise

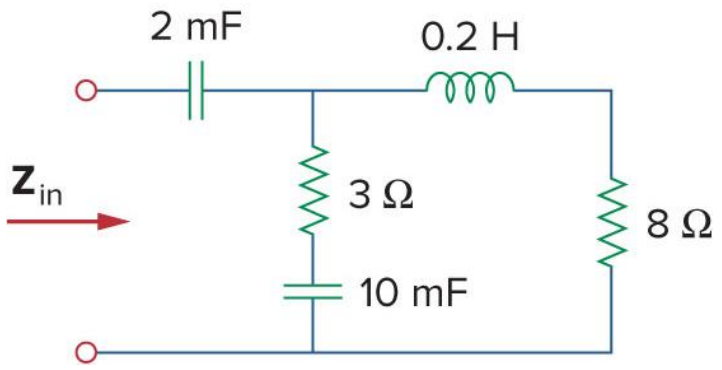
Find the output voltage $v_o(t)$ in the circuit below.



Exercise

Find the input impedance of the circuit shown below assuming the circuit operates at $\omega = 50\ \text{rad}/s$

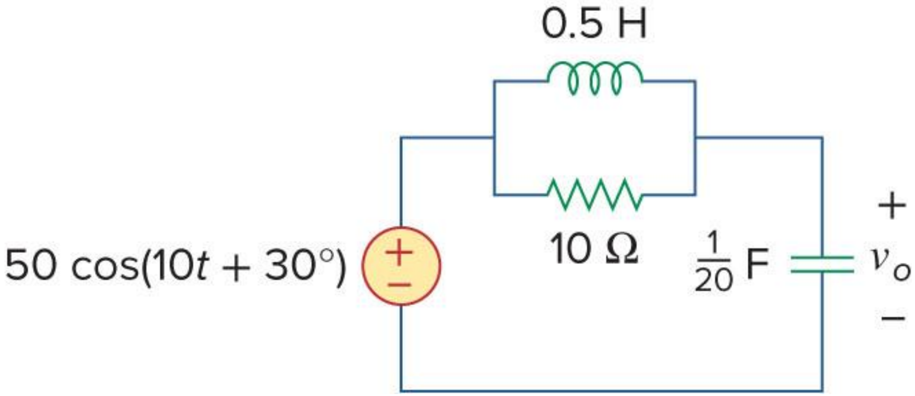
- For practice!
- Answer: $\mathbf{Z}_{\text{in}} = 3.22 - j11.07\ \Omega$



Exercise

Calculate the output voltage $v_o(t)$ in the circuit below.

- For practice!
- Answer: $v_o(t) = 35.36\cos(10t - 105°)\ \mathrm{V}$



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Questions?