Faculty of Engineering

School of Electrical Engineering and Telecommunications

# ELEC 1111 – Topic 9

# AC Power

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# Topic 9 Content

This lecture covers:

- AC power.
  - Instantaneous power.
  - Average power.
  - Maximum average power transfer.
- Effective or RMS value.

**Corresponds to the first part of Chapter 11 of your textbook**

# AC power

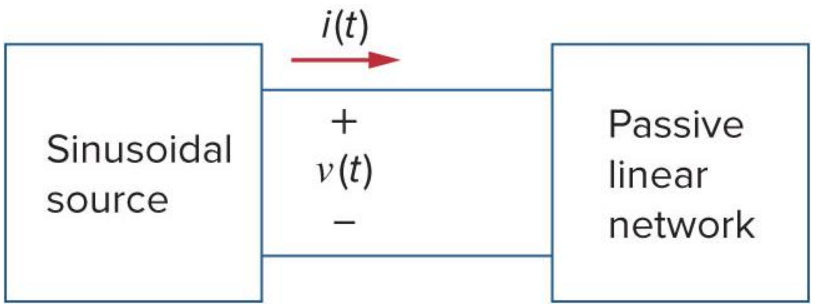
- **Power analysis** is quite important as all electrical, electronic, and communications systems rely on transmission of power (either in AC or DC form) from one point to another.
- The **most common** form of power is **AC power** due to its low cost and convenience in transmission of high-voltage power from the generator to the consumer.
- There are **multiple concepts** for AC power depending on the application of power and how it is being measured.
- In this lecture just **two main concepts** of AC power are introduced:
  - **Instantaneous** power.
  - **Average** power.

https://en.wikipedia.org/wiki/Transmission_tower

# Instantaneous power

- **Instantaneous power** (in watts) is the power at **any instance** of time.
- Instantaneous power $p(t)$ absorbed by an element (or network) is the **product** of the **instantaneous voltage** $v(t)$ across the element (or network) and the **instantaneous current** $i(t)$ through it, assuming **passive sign convention.**

$$\boxed{p(t) = v(t)i(t)}$$



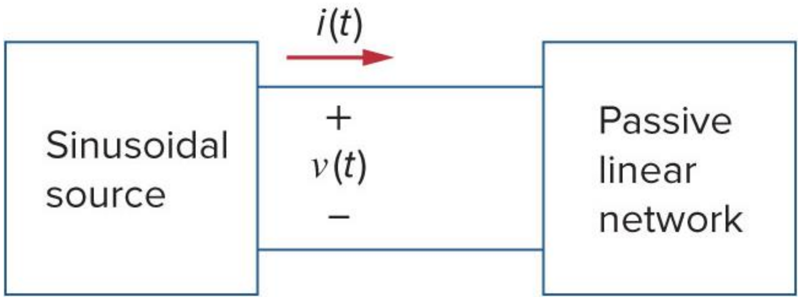

- Instantaneous power is the **rate** at which an element absorbs **energy.**

$$w = \int_0^t p(\tau)d\tau \quad \rightarrow \quad p(t) = \frac{dw}{dt}$$

# Instantaneous power

- Consider the general case where the voltage and current at the terminals of a circuit with passive elements under sinusoidal excitation are as follows:

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$




- The instantaneous power absorbed by the circuit is as follows:

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

Apply the trigonometric identity: $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

$$\boxed{p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)}$$

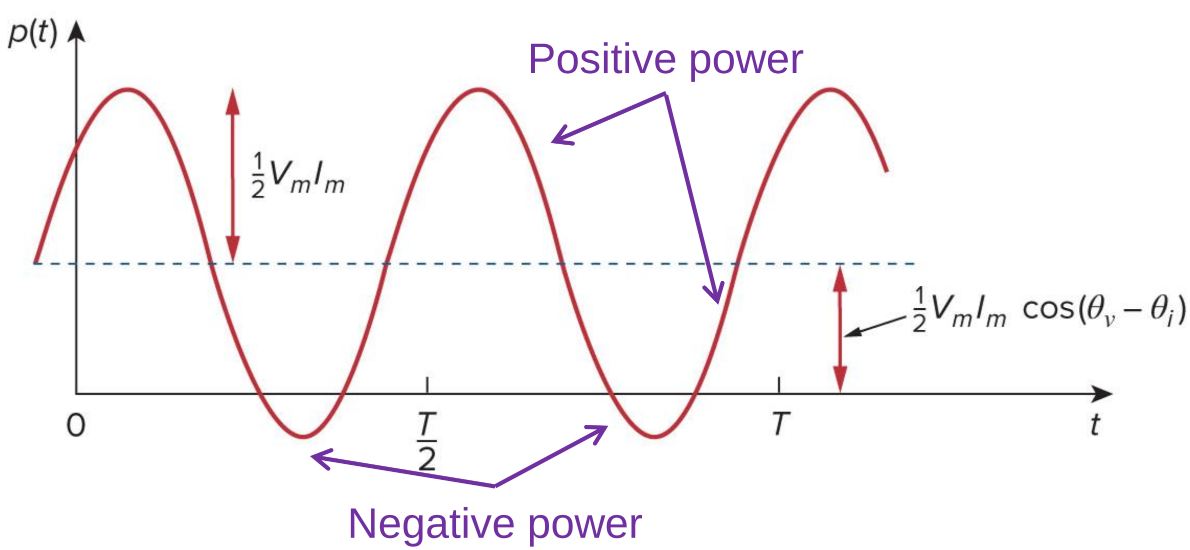
# Instantaneous power

- Instantaneous power has two parts:

$$p(t) = \underbrace{\frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)} + \underbrace{\frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)}$$

Constant term depending **only** on **phase difference** between **voltage and current**

Sinusoidal term with **twice** the **frequency** of voltage or current $T_0 = \frac{T}{2}$




- When $p(t)$ is **positive**, power is **absorbed** by the **circuit** (or supplied by the source).
- When $p(t)$ is **negative**, power is **absorbed** by the **source** (or transferred from the circuit to the source).

  Note: This is possible due to storage elements (**capacitors** and **inductors**).

# Average power

- **Average power** (in watts) is the **average** of the **instantaneous** power over **one period.**
  - Instantaneous power is difficult to measure as it changes with time.
  - The **average power** is more convenient to measure.
  - **Wattmeter**, an instrument for measuring power, provides the **average power.**

$$\boxed{P = \frac{1}{T}\int_0^T p(t)dt}$$

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \underbrace{\frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt}$$

Integrating a sinusoid over an integer multiple of its period is **zero,** $T = 2T_0 = 2 \times \frac{2\pi}{2\omega} = \frac{2\pi}{\omega}$.

The positive area under the sinusoid cancels the negative area

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\underbrace{\frac{1}{T}\int_0^T dt}_{1}$$

$$\boxed{P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

# Average power

- **Average power** $P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$ depends on **amplitudes** and **phases** of current and voltage only.

- **Phasor voltage** $\mathbf{V} = V_m \angle \theta_v$ and **phasor current** $\mathbf{I} = I_m \angle \theta_i$ can be used to calculate the average power as follows:

$$\frac{1}{2}\mathbf{VI}^* = \frac{1}{2} V_m \angle \theta_v \times I_m \angle(-\theta_i) = \frac{1}{2} V_m I_m \angle(\theta_v - \theta_i),$$

where $\mathbf{I}^*$ is the **complex conjugate** of phasor current $\mathbf{I}$.

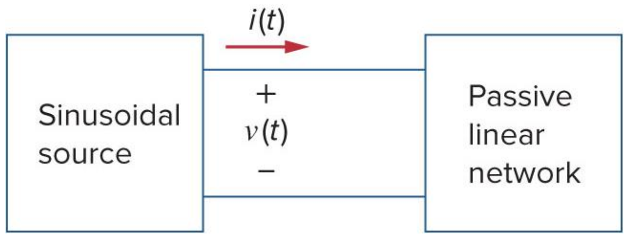
Transform to rectangular form:

$$\frac{1}{2}\mathbf{VI}^* = \underbrace{\frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}_{\text{average power}} + j\frac{1}{2} V_m I_m \sin(\theta_v - \theta_i)$$

$$\Rightarrow \quad \boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{VI}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

# Exercise

Given $v(t) = 120\cos(377t + 45°)\ \mathrm{V}$ and $i(t) = 10\cos(377t - 10°)\ \mathrm{A}$, find the instantaneous and average power absorbed by the passive linear network shown below.



# Average power

- A **resistive load** $\mathbf{Z} = R$ absorbs **average power** at all times in AC circuits.
- When $\theta_v = \theta_i$, the voltage and current are **in phase**. This implies a **purely resistive** circuit or **resistive load**, $\mathbf{V} = R\mathbf{I}$, thus:

$$P = \frac{1}{2}\mathrm{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

$$\boxed{P = \frac{1}{2}V_m I_m} \quad \boxed{P = \frac{1}{2}RI_m^2 = \frac{1}{2}R|\mathbf{I}|^2} \quad \boxed{P = \frac{1}{2}\frac{V_m^2}{R} = \frac{1}{2}\frac{|\mathbf{V}|^2}{R}}$$

Proof of the last two:

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2}\mathrm{Re}[R\mathbf{I} \times \mathbf{I}^*] = \frac{1}{2}R|\mathbf{I}|^2 = \frac{1}{2}RI_m^2}$$

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2}\mathrm{Re}[\mathbf{V} \times \frac{\mathbf{V}^*}{R}] = \frac{1}{2}\frac{|\mathbf{V}|^2}{R} = \frac{1}{2}\frac{V_m^2}{R}}$$

# Average power

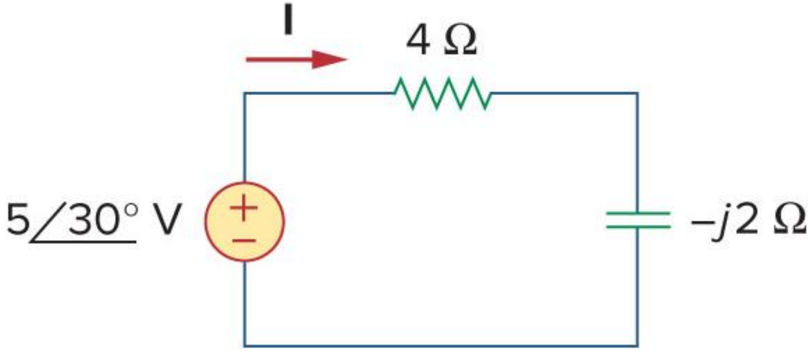
- A **reactive load** $\mathbf{Z} = j\omega L$ or $\mathbf{Z} = 1/\mathrm{j}\omega C$ absorbs **<u>zero</u> average power** in AC circuits.
  - Inductor $L$ and capacitor $C$ absorb zero average power in AC circuits because they **charge and discharge the power** during a full period of the voltage and current.
- When $\theta_v - \theta_i = \pm 90°$, it implies a **purely reactive** circuit or **reactive load,** so:

$$P = \frac{1}{2}\mathrm{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

$$\boxed{P = \frac{1}{2}V_m I_m \cos 90° = 0}$$

# Exercise

For the circuit below, find the average power supplied by the source and the average power absorbed by the resistor.



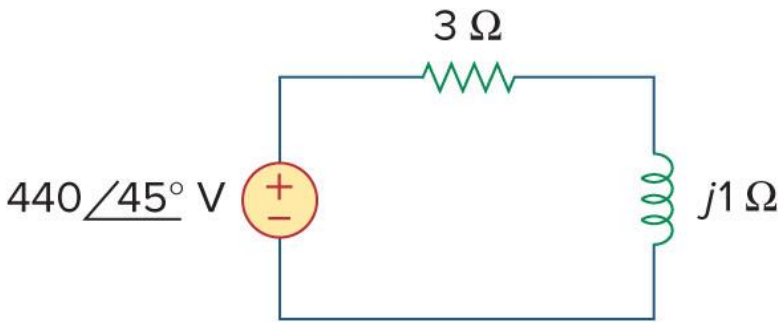
# Exercise

For the circuit below, find the average power supplied by the source and the average power absorbed by the resistor and inductor.

- For practice!
- Answer: $29.04\ \mathrm{kW}$ supplied by the source.

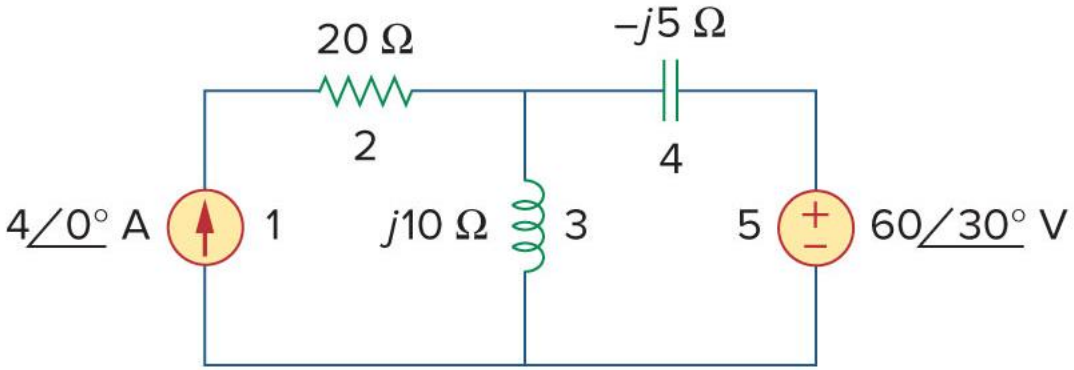
  $29.04\ \mathrm{kW}$ absorbed by the resistor.

  $0\ \mathrm{kW}$ absorbed by the inductor.



# Exercise

Determine the average power generated by each source and the average power absorbed by passive elements in the circuit below.



# Exercise

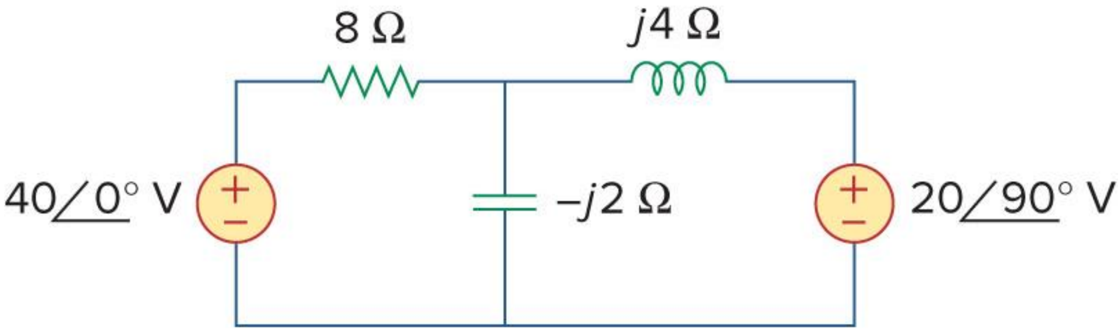
Calculate the average power absorbed/supplied by each of the five elements in the circuit below

- For practice!
- Answer: $P_{40\mathrm{V}} = 60$ W supplied

  $P_{j20\mathrm{V}} = 40$ W supplied

  $P_{8\Omega} = 100$ W absorbed

  $P_{j4\Omega} = P_{-j2\Omega} = 0$ W absorbed

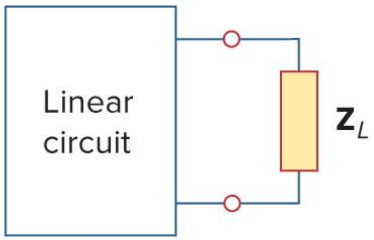


# Maximum average power transfer

- Recall that a DC circuit with resistive elements represented by its **Thevenin equivalent circuit** can transfer **maximum power** if and only if $R_L = R_{\text{Th}}$.
- The results can be extended to AC circuits in a similar fashion

Consider the Thevenin equivalent circuit shown in Fig. (b) with rectangular form for impedances:

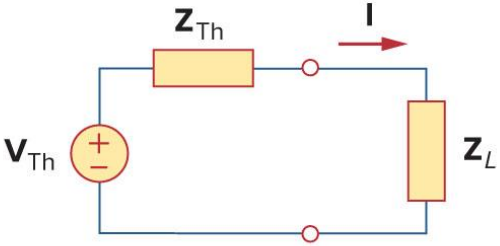
$$\mathbf{Z}_{\text{Th}} = R_{\text{Th}} + jX_{\text{Th}} \text{ and } \mathbf{Z}_L = R_L + jX_L$$



(a)

The current through the load is:

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\text{Th}}}{(R_{\text{Th}} + jX_{\text{Th}}) + (R_L + jX_L)}$$



(b)

The average power delivered to the load is:

$$P = \frac{1}{2} R_L |\mathbf{I}|^2 = \frac{\frac{R_L |\mathbf{V}_{\text{Th}}|^2}{2}}{(R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2}$$

# Maximum average power transfer

- To **maximize the average power** in terms of $R_L$ and $X_L$, set the **partial derivatives** of $P$ with respect to $R_L$ and $X_L$ to **zero** solve them for $R_L$ and $X_L$:

$$P = \frac{1}{2} R_L |\mathbf{I}|^2 = \frac{\frac{R_L |\mathbf{V}_{\mathrm{Th}}|^2}{2}}{(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2}$$

$$\frac{\partial P}{\partial X_L} = \frac{-|\mathbf{V}_{\mathrm{Th}}|^2 R_L (X_{\mathrm{Th}} + X_L)}{[(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2]^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{\mathrm{Th}}|^2 [(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2 - 2R_L(R_{\mathrm{Th}} + R_L)]}{2[(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2]^2}$$

Setting $\frac{\partial P}{\partial X_L}$ and $\frac{\partial P}{\partial R_L}$ to zero results in:

$$\boxed{X_L = -X_{\mathrm{Th}}} \quad \text{and} \quad \boxed{R_L = \sqrt{R_{\mathrm{Th}}^2 + (X_{\mathrm{Th}} + X_L)^2}}$$

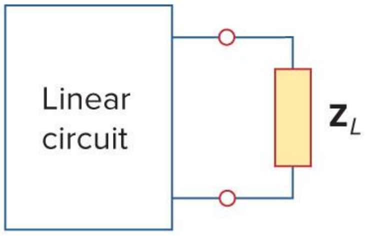
Combining the results:

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\mathrm{Th}} - jX_{\mathrm{Th}} = \mathbf{Z}_{\mathrm{Th}}^*} \Rightarrow \boxed{\begin{cases} R_L = R_{\mathrm{Th}} \\ X_L = -X_{\mathrm{Th}} \end{cases}}$$

# Maximum average power transfer

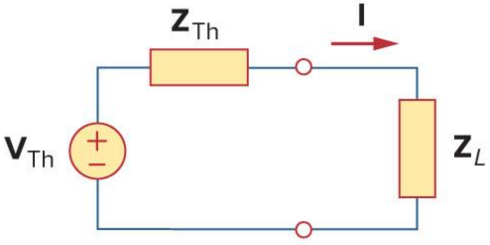
- Setting $R_L = R_{\mathrm{Th}}$ and $X_L = -X_{\mathrm{Th}}$ in the **average power** equation leads to:

$$\boxed{\mathbf{Z}_L = \mathbf{Z}_{\mathrm{Th}}^*} \Rightarrow \boxed{P_{\max} = \frac{|\mathbf{V}_{\mathrm{Th}}|^2}{8R_{\mathrm{Th}}}}$$



(a)

**Maximum power** is transferred to the load when the **load impedance** $\mathbf{Z}_L$ is **equal** to the **complex conjugate** of the **Thevenin impedance** $\mathbf{Z}_{\mathrm{Th}}^*$ as seen from the load terminals.



(b)

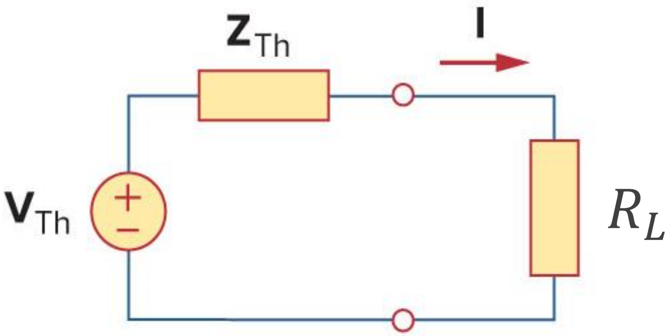
# Maximum average power transfer

- For a **pure resistive load** $\mathbf{Z}_L = R_L$, **maximum average power transfer condition** is given by setting $X_L = 0$ in $R_L = \sqrt{R_{\mathrm{Th}}^2 + (X_{\mathrm{Th}} + X_L)^2}$, as follows:

$$\boxed{R_L = \sqrt{R_{\mathrm{Th}}^2 + {X_{\mathrm{Th}}}^2} = |\mathbf{Z}_{\mathrm{Th}}|}$$

$$\Downarrow$$

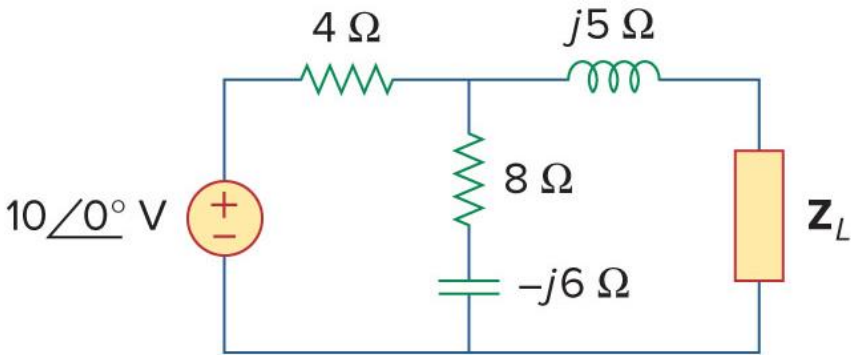
$$\boxed{P_{\max} = \frac{1}{2} R_L |\mathbf{I}_L|^2 = \frac{1}{2} \frac{|\mathbf{V}_L|^2}{R_L}}$$




**Maximum average power** transferred to a **pure resistive load** ($\mathbf{Z}_L = R_L$) can be calculated in an AC circuit using **direct** calculation of **average power** absorbed by a **load resistor.**

# Exercise

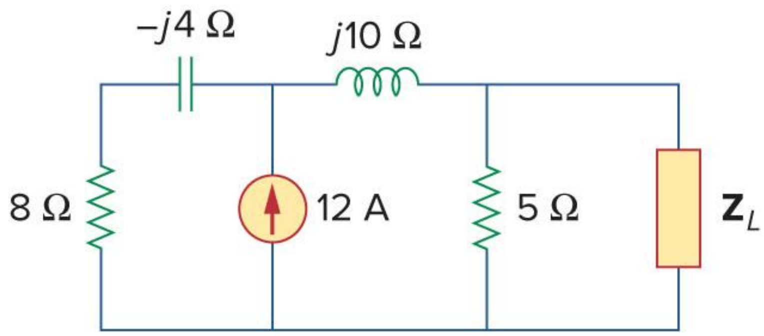
Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit below, and then find the maximum average power.



# Exercise

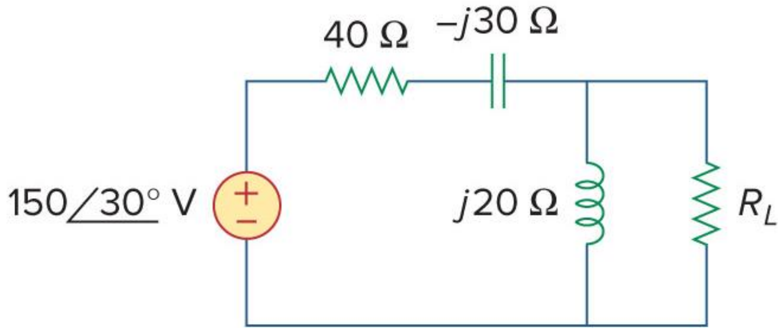
For the circuit shown below, find the load impedance $\mathbf{Z}_L$ that absorbs maximum average power. Calculate the maximum average power.

- For practice!
- Answer: $\mathbf{Z}_L = \mathbf{Z}_{\mathrm{Th}}^* = 3.415 - j0.731\ \Omega$ and $P_{\max} = 51.47$ W absorbed



# Exercise

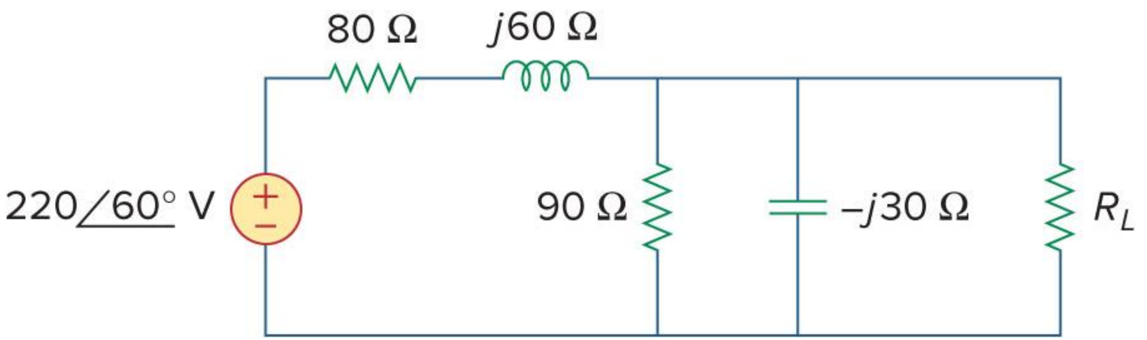
In the circuit below, find the value of $R_L$ that will absorbed the maximum average power, and then calculate that power.



# Exercise

For the circuit shown below, find the load resistance $R_L$ such that it absorbs maximum average power, and then calculate the maximum average power transferred to it.

- For practice!
- Answer: $R_L = |\mathbf{Z}_{\mathrm{Th}}|^2 = 30\ \Omega$ and $P_{\max} = 23.06\ \mathrm{W}$ absorbed



# Effective or RMS value

- When a time-varying source is delivering power to a resistive load, we often want to know the effectiveness of that source on delivering power.
- The **effective value** of a **periodic** current is a **DC** current that can deliver **the same average power** to a **resistor** as the periodic current.

Consider the circuit in Fig. (a). The average power is given by:

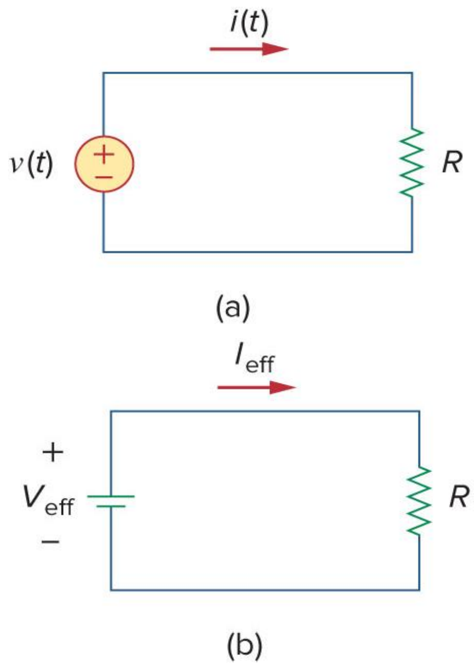
$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T Ri(t)^2 dt = \frac{R}{T}\int_0^T i(t)^2 dt$$

The power absorbed in the DC circuit in Fig. (b) is:

$$P = RI_{\text{eff}}^2$$

The objective is to find $I_{\text{eff}}$ such that it delivers the same average power as the AC circuit. Thus:

$$P = RI_{\text{eff}}^2 = \frac{R}{T}\int_0^T i(t)^2 dt \Rightarrow \boxed{I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i(t)^2 dt}}$$



# Effective or RMS value

- The effective value of the AC voltage is obtained in the same way as the effective current:

$$\boxed{V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v(t)^2 dt}}$$

- The effective values for both current and voltage take the form of the **square root** of the **average (mean)** of the **square** of the periodic signal. This is referred to as the **root mean square**, or **RMS value** for short.

$$I_{\text{rms}} = I_{\text{eff}} \quad \text{and} \quad V_{\text{rms}} = V_{\text{eff}}$$

For any periodic signal x(t) in general, the RMS value is given by:

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x(t)^2 dt}$$

# Effective or RMS value

- The RMS value of the **constant signal** (DC signal) is the **constant itself.**
- For a sinusoid $i(t) = I_m \cos(\omega t + \phi)$ the RMS value is obtained as follows:

Using the trigonometric identity
$\cos^2(a) = \frac{1+\cos(2a)}{2}$

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2(\omega t + \phi)\, dt} = \sqrt{\frac{I_m^2}{T}\left(\underbrace{\int_0^T \frac{1}{2}\, dt}_{=\frac{1}{2}T} + \underbrace{\int_0^T \cos(2\omega t + 2\phi)\, dt}_{\text{= 0, Integration over an integer multiple of period}}\right)}$$

$$\Rightarrow \boxed{I_{\text{rms}} = \frac{I_m}{\sqrt{2}}}$$

- Similarly, for $v(t) = v_m \cos(\omega t + \phi)$ $\Rightarrow$ $\boxed{V_{\text{rms}} = \frac{V_m}{\sqrt{2}}}$

**Note:** These RMS values are only valid for sinusoidal currents and voltages, or sinusoidal signals in general.

**Note:** An example of RMS value is the **household appliances** working with RMS voltage $V_{\text{rms}} = 220\ \text{V}$, which is:

$$v(t) = 220\sqrt{2}\cos(2\pi 50t)\ \text{V}$$

# RMS value and average power

- The **average power** can be determined in terms of RMS values:

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{\sqrt{2} \times \sqrt{2}} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i)$$

$$\boxed{P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)}$$

- Similarly, the **average power** absorbed by a **resistor** $R$ is given by:

$$\boxed{P = V_{\text{rms}} I_{\text{rms}}} \quad \boxed{P = R I_{\text{rms}}^2} \quad \boxed{P = \frac{V_{\text{rms}}^2}{R}}$$

# Questions?