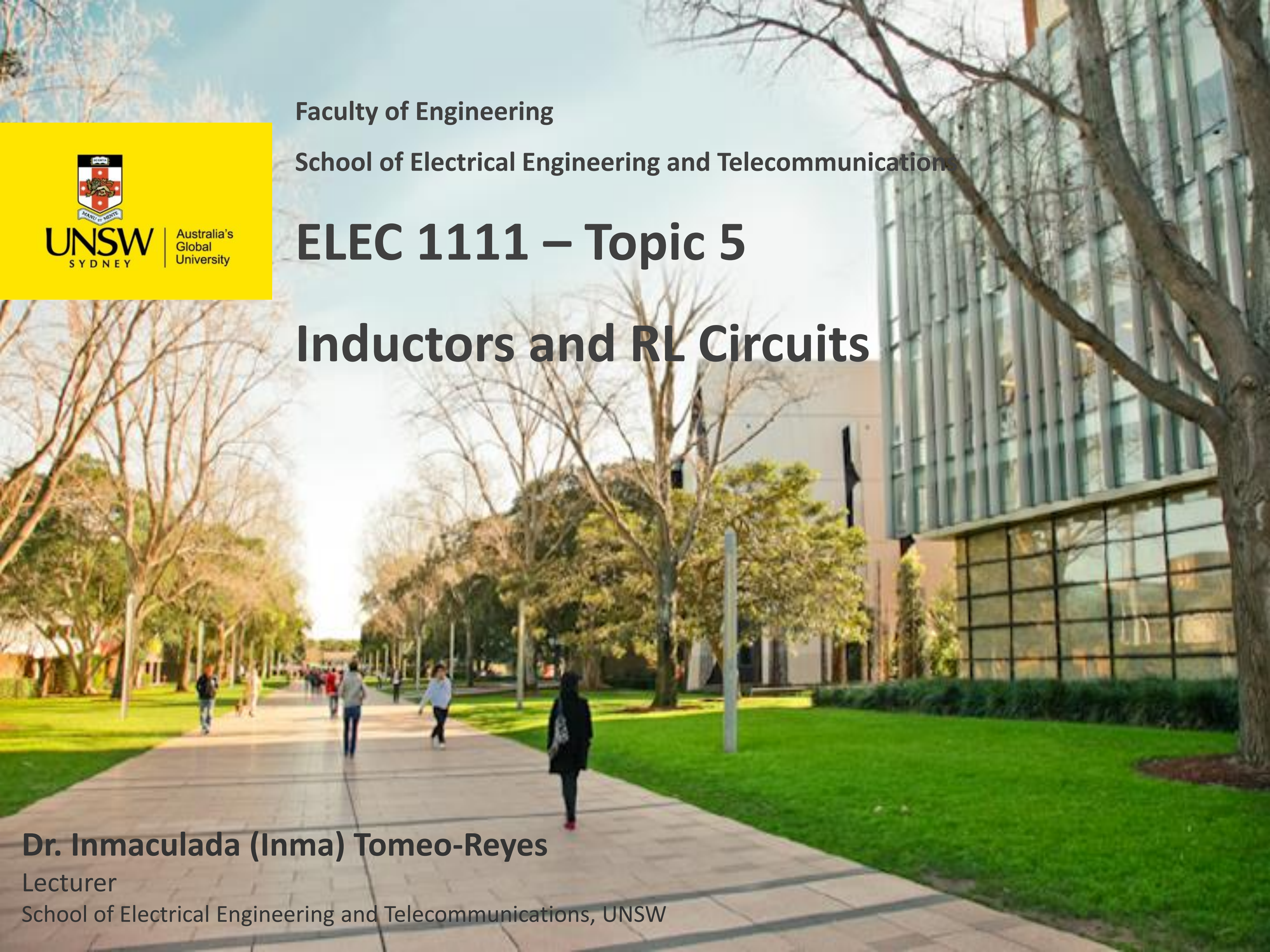

Faculty of Engineering

School of Electrical Engineering and Telecommunications

ELEC 1111 – Topic 5

Inductors and RL Circuits

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Topic 5 Content

This lecture covers:

- Inductors
- Circuit analysis with inductors
- First order circuits with resistors and inductors (RL circuits)
  - Natural response
  - Step response

Corresponds to parts of Chapter 6 and 7 of your textbook

Inductors

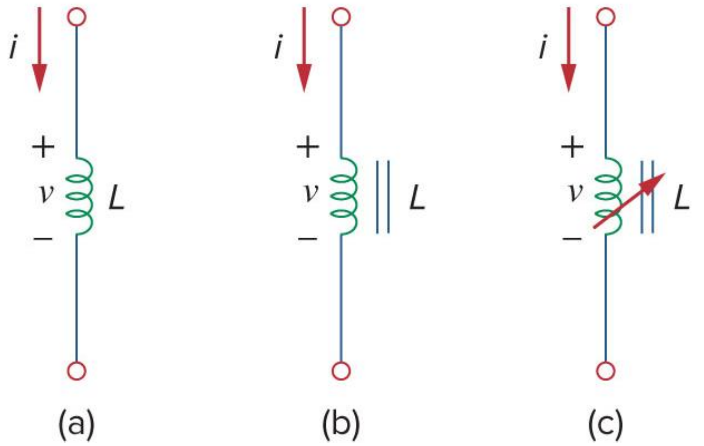
- An **inductor** is a circuit element that stores energy in its magnetic field.
  - An inductor consists of a coil of conducting wire.
  - Any conductor of electric current has inductive properties.
  - The inductive effect is typically enhanced by coiling the wire up.
- If a current is passed through an inductor, the **voltage** across it is **directly proportional** to the **time rate of change** in **current**.

$$\boxed{v = L\frac{di}{dt}}$$

Inductors

- **Inductance** is the property whereby an inductor exhibits opposition to the change of current flowing through it.
- The symbol for inductance is L.
- Inductance is measured in henry, F, which are volts per ampere.

$$1\ \mathrm{H} = \frac{1\ \mathrm{V}}{1\ \mathrm{A}}$$

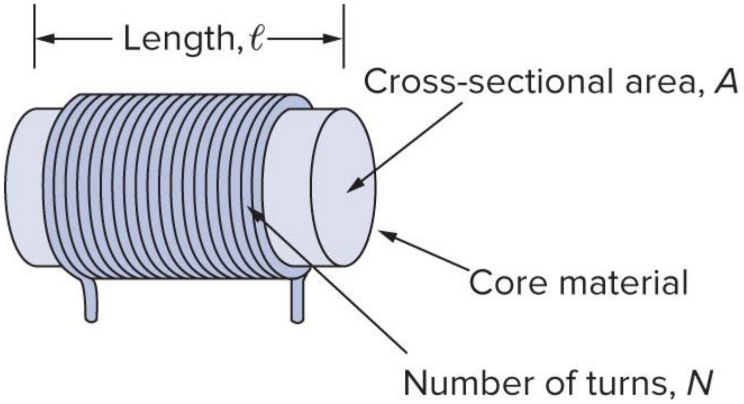


Circuit symbol for fixed inductor, (a) and (b), and variable inductor, (c).

Inductors

- Inductance depends of the **physical dimensions** and **construction** of the inductor.
- For **solenoid** inductors, the inductance is given as follows:
  - N is the **number of turns** of the wire.
  - A is the **cross-sectional** area of the core.
  - ℓ is the **length** of the core.
  - μ is the **permeability** of the core.

$$\boxed{L = \frac{N^2 \mu A}{\ell}}$$



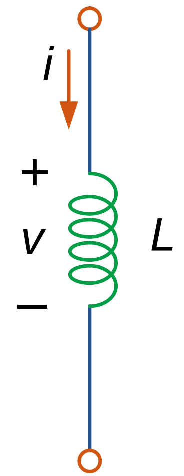
Inductors

- The terms *coil* or *choke* are also used for inductors.
- Like capacitors, inductors are available in different values and types.
- They are described by their **core material.**
  - E.g. the core may be made of iron, steel, plastic, or air.
- Most inductors are rated in microhenry (μH) to tens of henrys (H).
- They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

Inductors

- **Passive sign convention** applies to inductors as well.
  - If $\boldsymbol{v} \times \boldsymbol{i} > \mathbf{0}$, the inductor is being **charged** (absorbing energy).
  - If $\boldsymbol{v} \times \boldsymbol{i} < \mathbf{0}$, the inductor is **discharging** (supplying energy).
- The **inductor's voltage** $\boldsymbol{v}$ is proportional to the **rate of change** of its **current** $\boldsymbol{i}$**,** with **inductance** $\boldsymbol{L}$ as the constant of proportionality, assuming **passive sign convention.**

$$\boxed{v = L\frac{di}{dt}}$$



- Current will then be:

$$\boxed{i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)}$$

where $i(t_0)$ is called the **initial current** or **initial conditions** at time t_0.

Inductors

- **Instantaneous power** delivered to the inductor:

$$\boxed{p = vi = Li\frac{di}{dt}}$$

- The **energy** stored in the **magnetic field** that exists around the coil and wires of the can be then calculated as:

$$w(t) = \int_{t_0}^{t} p(\tau)d\tau = \int_{t_0}^{t} L\frac{di(\tau)}{d\tau} i(\tau)\, d\tau = L\int_{t_0}^{t} i(\tau)di(\tau) = L\frac{i(\tau)^2}{2}\Big|_{t_0}^{t}$$

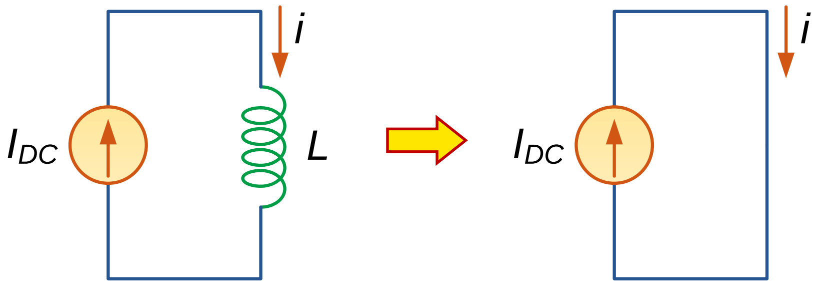
$$w(t) = \frac{1}{2}L(i(t)^2 - i(t_0)^2)$$

If $i(t_0) = 0 \longrightarrow$ $$\boxed{w(t) = \frac{1}{2}Li(t)^2}$$

Properties of inductors

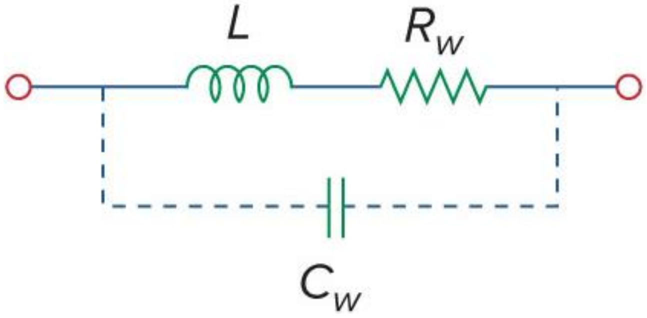
- An inductor acts like a **short circuit** to **DC current**.

  $v = L\frac{di}{dt} = 0$ for constant current.



Properties of inductors

- Like an ideal capacitor, an **ideal inductor** does not dissipate energy.
  - Energy is absorbed from a circuit, stored as magnetic field and then released back to the circuit.
- A **real inductor** has a very small **winding resistance** R_w in series and a very small **winding capacitance** C_w in parallel with both, leading to a slow loss of stored energy.



Inductors in series

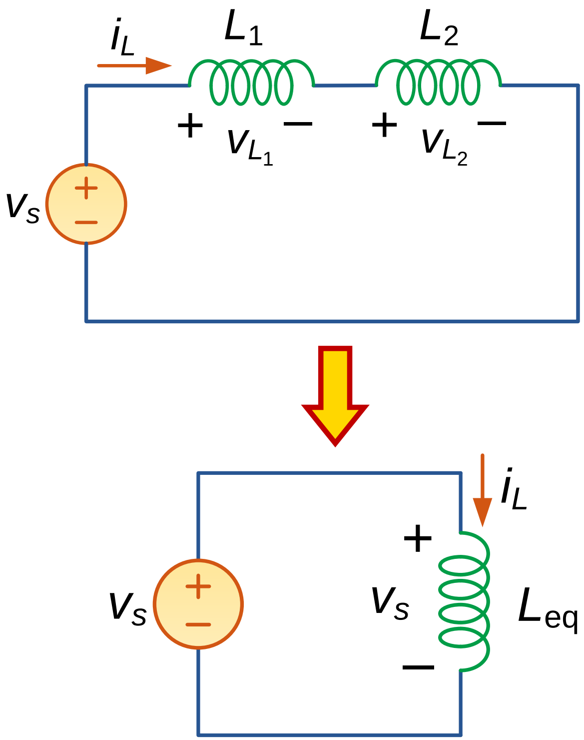
- Similar to resistors, inductors in series or parallel can be combined to simplify a circuit.
- For inductors in series, the current is the same across each inductor.
- Applying KVL and current-voltage relation $v = L\frac{di}{dt}$ for inductors:

$$v_s = v_{L_1} + v_{L_2}$$

$$v_s = L_1\frac{di_L}{dt} + L_2\frac{di_L}{dt}$$

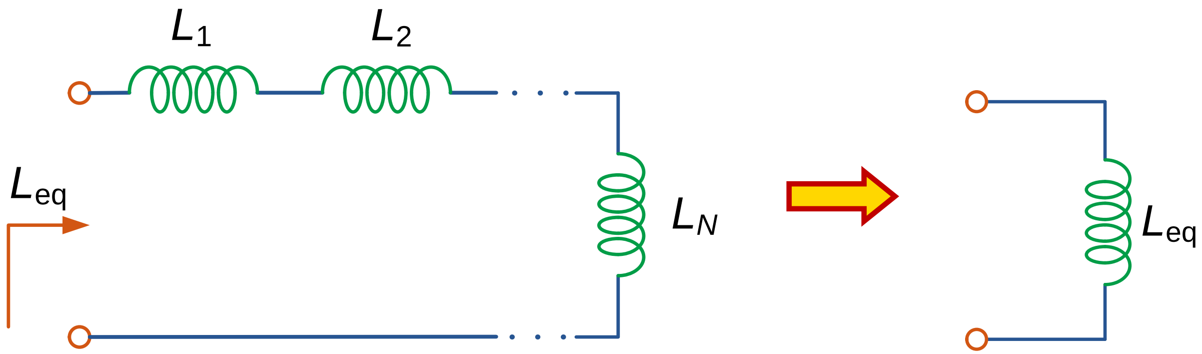
$$v_s = (L_1 + L_2)\frac{di_L}{dt} \quad \rightarrow \quad v_s = L_{\mathrm{eq}}\frac{di_L}{dt}$$

$$\boxed{L_{\mathrm{eq}} = L_1 + L_2}$$



Inductors in series

- The **equivalent inductance** of any number of **inductors in series** is the **sum** of the **individual inductances**.



$$L_{eq} = L_1 + L_2 + \cdots + L_N = \sum_{k=1}^{N} L_k$$

Inductors in parallel

- For inductors in parallel, the voltage is the same across each inductor.
- Applying KCL and voltage-current relation for inductors:

$$i_s = i_{L_1} + i_{L_2}$$

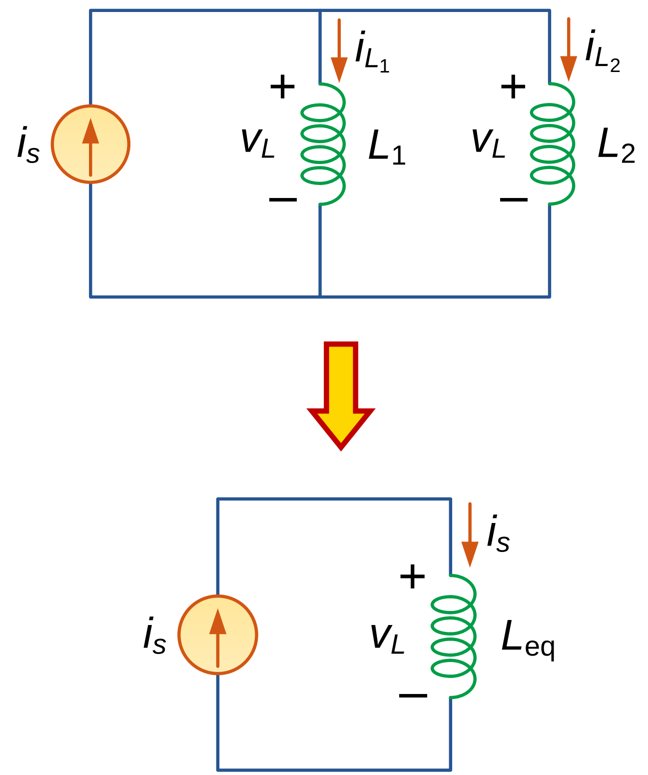
$$i_s = \frac{1}{L_1}\int_{t_0}^{t} v_L(\tau)d\tau + i_{L_1}(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v_L(\tau)d\tau + i_{L_2}(t_0)$$

$$i_s = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)\int_{t_0}^{t} v_L(\tau)d\tau + \left(i_{L_1}(t_0) + i_{L_2}(t_0)\right)$$

$$\leftrightarrow \quad i_s = \frac{1}{L_{\text{eq}}}\int_{t_0}^{t} v_L(\tau)d\tau + i_{L_{\text{eq}}}(t_0)$$

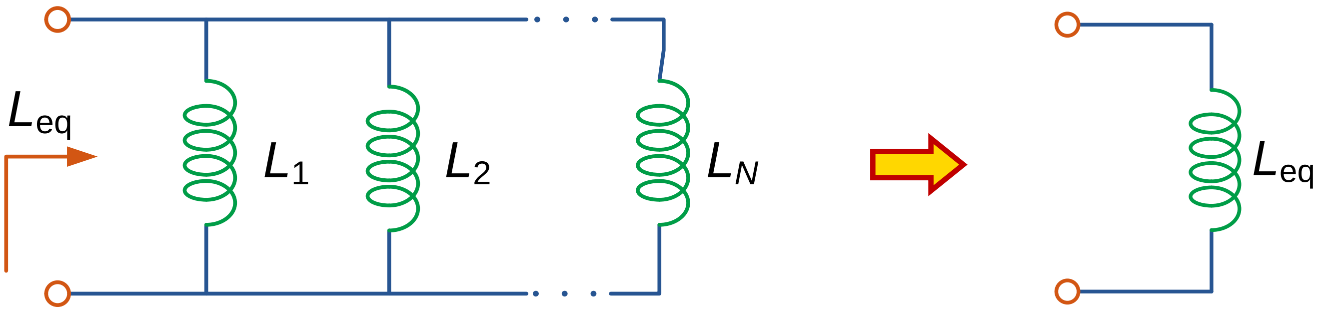
$$\boxed{\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2}} \quad \text{or} \quad \boxed{L_{\text{eq}} = \frac{L_1 L_2}{L_1 + L_2}}$$

$$\text{and} \quad \boxed{i_{L_{\text{eq}}}(t_0) = i_{L_1}(t_0) + i_{L_2}(t_0)}$$



Inductors in parallel

- The **reciprocal** of the **equivalent inductance** of any number of **inductors in parallel** is the **sum** of the **individual reciprocal inductances.**



$$\boxed{\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} \cdots + \frac{1}{L_N}}$$

or

$$\boxed{L_{\text{eq}} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} \cdots + \frac{1}{L_N}}}$$

Characteristics of basic elements

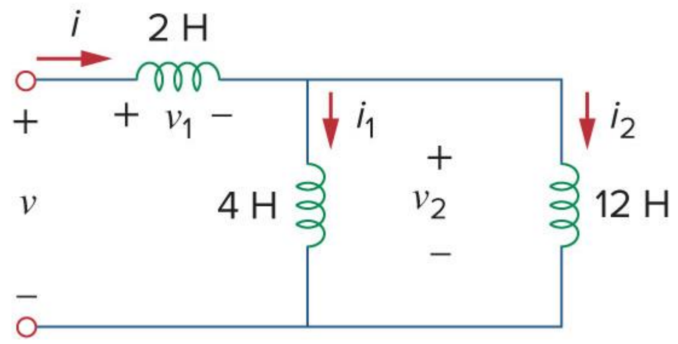
Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

†*Passive sign convention is assumed.*

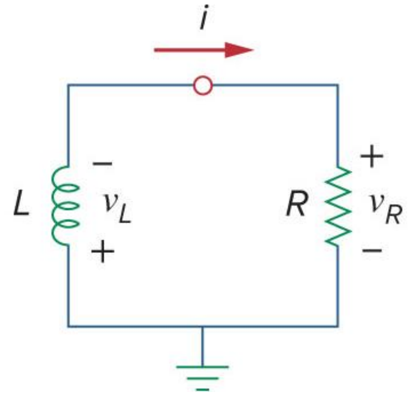
Exercise

In the circuit below, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find $i_1(0)$, $v(t)$, $v_1(t)$ and $v_2(t)$, $i_1(t)$ and $i_2(t)$.



Natural response of RL circuit

Let's consider a circuit with a single inductor, charged to an initial current I_o and connected to a resistor. How will it behave?



We can use KVL to get:

$$v_L + v_R = 0$$

$$L\frac{di}{dt} + Ri = 0$$

Note: We will consider how this inductor was charged later in this lecture.

Note: A circuit characterised by a first order differential equation is called a first order circuit.

Natural response of RL circuit

Solve differential equation:

Rearrange:

$$L\frac{di}{dt} = -iR$$

$$\frac{di}{dt} = \frac{-iR}{L}$$

Separate:

$$\frac{1}{i}di = \frac{-R}{L}dt$$

Integrate:

$$\int \frac{1}{i}di = \frac{-R}{L}\int dt$$

$$\ln(i) = \frac{-R}{L}t + D$$

Solve for $v(t)$

$$i(t) = e^{-\frac{R}{L}t+D}$$

$$i(t) = e^{-\frac{R}{L}t}\, e^{D}$$

$$i(t) = Ae^{-\frac{R}{L}t}$$

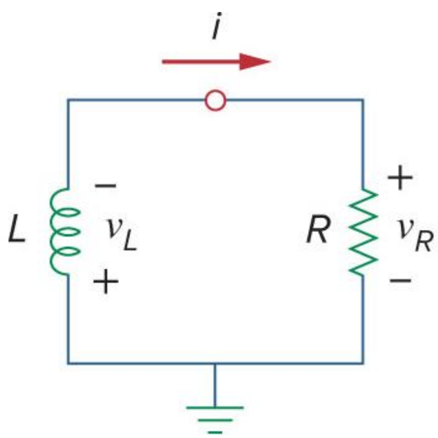
Apply initial conditions

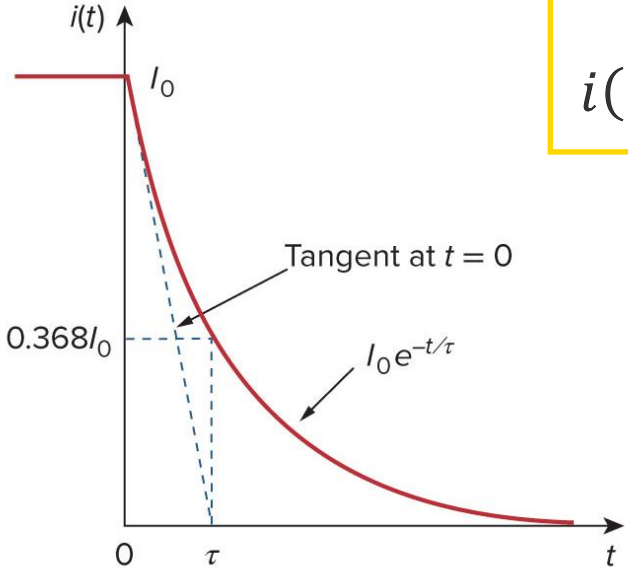
$i(0) = Ae^{0} = A = I_0$

$$\boxed{i(t) = I_0 e^{-\frac{R}{L}t}}$$

Natural response of RL circuit

- There is no need to derive the differential equation solution every time, just use the result.
- The result shows that the **current response** of the RL circuit is an **exponential decay** of the **initial current**.
- As in the case of RC circuits, the **time constant** of this circuit is the time required for the response to **decay** to $\mathbf{1/e}$ (or **36.8%**) of its initial value or to **increase** to $\mathbf{1-1/e}$ (or **63.2%**) of its final value. It is denoted by $\boldsymbol{\tau}$.





$$i(t) = I_0 e^{-\frac{R}{L}t} = I_0 e^{-\frac{t}{\tau}}$$

$$\tau = \frac{L}{R}$$

Natural response of RL circuit

- Follow these steps to find the natural (or source-free response) of RL circuits:

1. Find the **initial current** $i(0) = I_0$ through the inductor **before** it is connected to the resistor.
   - The inductor is assumed to be **fully charged** at the beginning and can be replaced with a **short circuit**.

2. Find the **time constant** $\tau = \frac{L}{R}$.
   – If the circuit has **more than one resistor**, the resistance that we need to find in order to calculate the time constant is the equivalent resistance as seen by the terminals of the inductor, i.e. the **Thevenin equivalent resistance** $\boldsymbol{R = R_{\mathrm{Th}}}$.
   – When possible, this resistance can be obtained by simplification of series or parallel resistances.

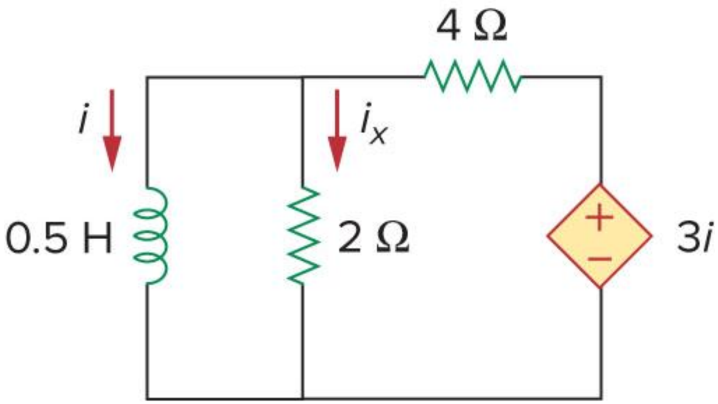
3. Calculate the current through the inductor as $i(t) = I_0 e^{-\frac{t}{\tau}}$.

4. Find any other circuit variable using the inductor's current.

Note: A **switch** which **opens** or **closes** can **remove** part of the circuit or **add** something to it.

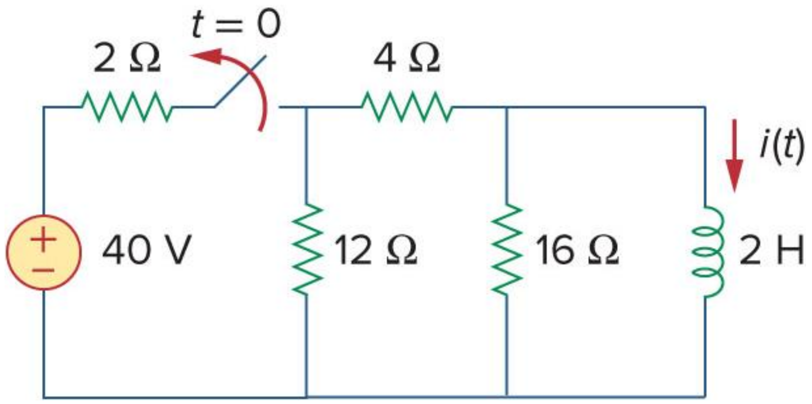
Exercise

Find the currents i and i_x for $t > 0$ if the initial current is $i(0) = 10$ A.



Exercise

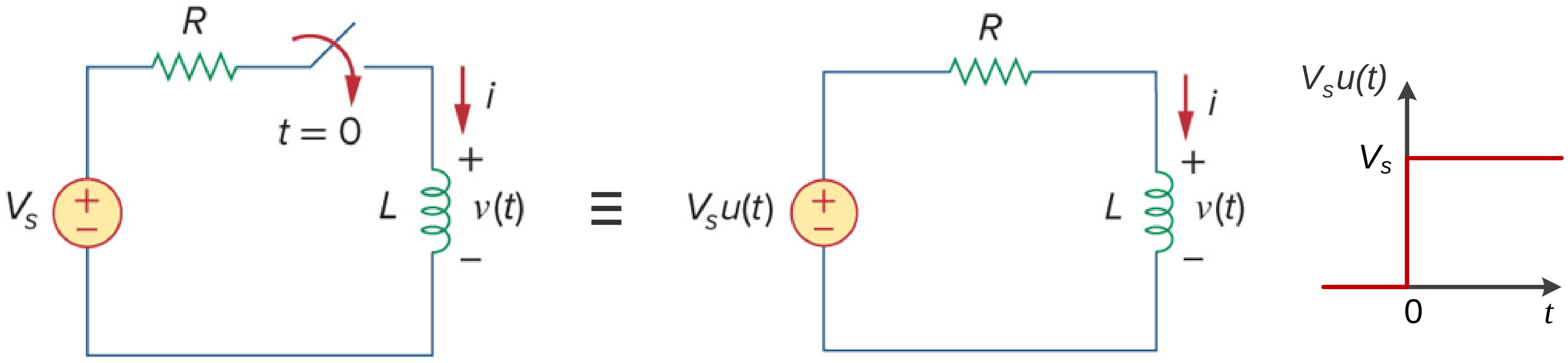
The switch in the circuit has been closed for a long time. At t=0, the switch is opened. Find the current $i(t)$ for $t > 0$.



Step response of RL circuit

- Step response is the response of the circuit due to a sudden application of a DC voltage or current.
  - It is the circuit behaviour when the **excitation/input** is the **step function**, which may be a voltage or a current source.
  - We can model this behavior with a switch opened or closed at $t = t_0$.
- Let's assume that the inductor is initially energised with a current I_0. Since the current of the inductor cannot change instantaneously:

$$\boxed{i(0^-) = i(0^+) = I_0}$$



Note: $i(0^-)$ is the inductor current **just before** switching and $i(0^+)$ **just after** switching.

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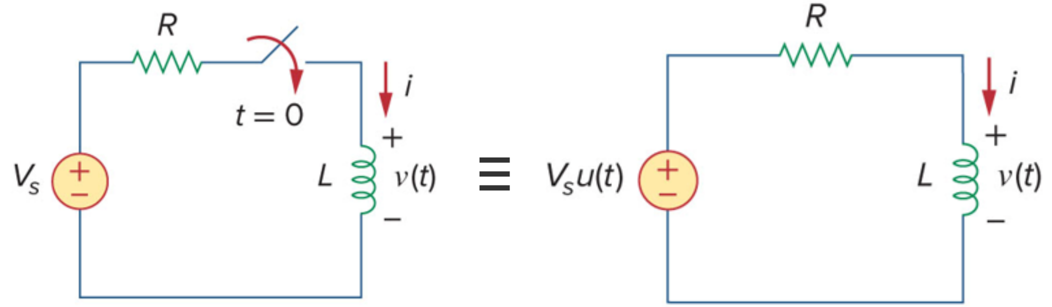
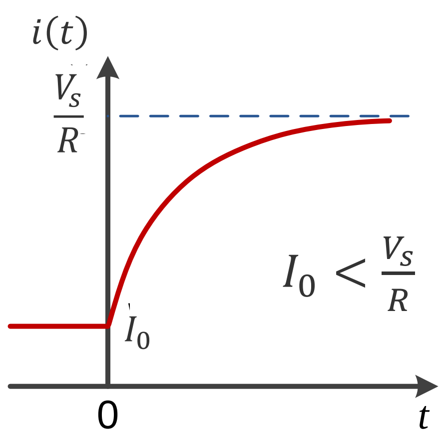
Step response of RC circuit

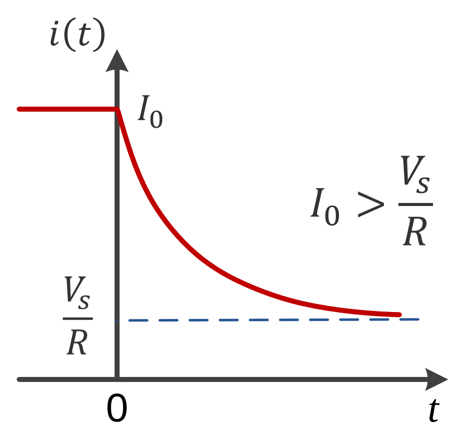
- Following an analogous process to the one shown in Topic 5 for RC circuits, we find that the **current response** of the RL circuit will change from the initial I_o to the value of V_s/R in an **exponential manner**.
  - Depending on the value of initial conditions I_0 and voltage source V_s, the inductor can be charged or discharged.

$$i(t) = \begin{cases} I_0, & t < 0 \\ \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{t}{\tau}}, & t > 0 \end{cases}$$

$$\tau = \frac{L}{R}$$







Step response of RL circuit

- Recall that the **complete response** in **first order circuits** can be described as the sum of **transient** response i_t and **steady-state** response i_{ss}.

$$i = i_t + i_{ss}$$

- Transient response is always a decaying exponential:

$$i_t = Ae^{-\frac{t}{\tau}}, \qquad \tau = \frac{L}{R}$$

- The steady-state response is obtained **a long time after** the switch is closed (about **5$\boldsymbol{\tau}$**).
- Replace inductor with **short circuit** and find the current through it:

$$i_{ss} = \frac{V_s}{R}$$

- Adding the responses:

$$i = Ae^{-\frac{t}{\tau}} + \frac{V_s}{R} \xrightarrow[\text{Apply initial conditions}]{i(0^-) = i(0^+) = I_0} I_0 = A + \frac{V_s}{R} \rightarrow A = I_0 + \frac{V_s}{R}$$

- Therefore: $\boxed{i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-\frac{t}{\tau}}}$

Step response of RL circuit

- More specifically:

1. First stage of steady-state ($t < 0$):

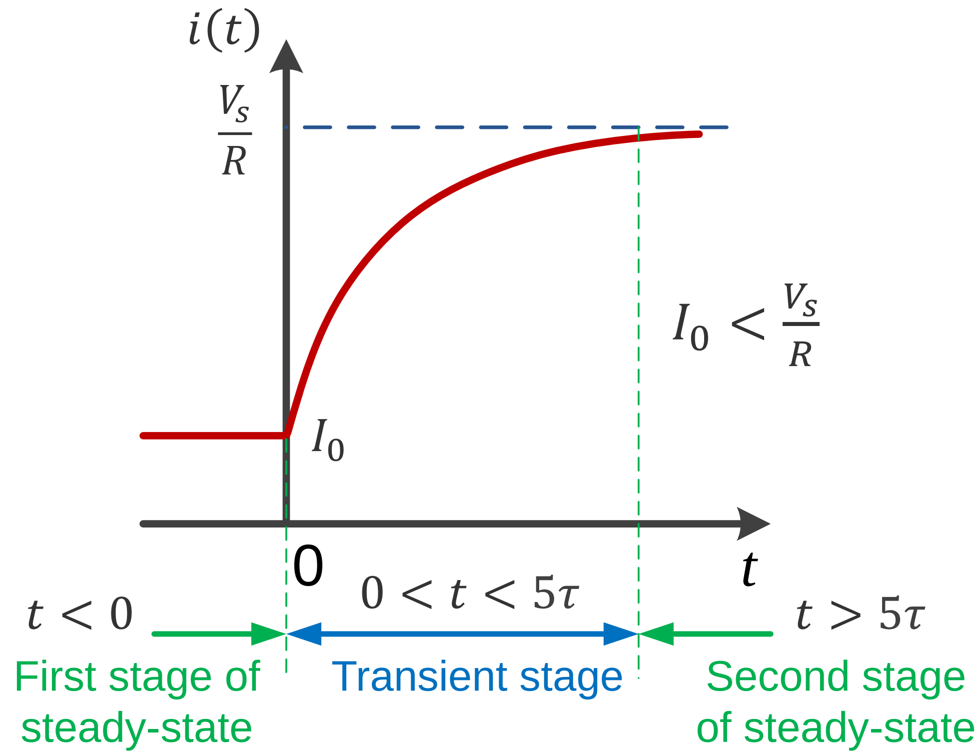
   There has been **no change** in the circuit for a **long time** and the inductor is a **short circuit** with $\boldsymbol{i(0) = I_0}$.

2. Transient stage ($0 < t < 5\tau$):

   The inductor's voltage changes **exponentially** with $\tau = \frac{L}{R}$.

3. Second stage of steady-state ($t > 5\tau$):

   The inductor's current reaches its **final value** or **steady-state value** and becomes **short circuit** again with $\boldsymbol{i(t) = \frac{V_s}{R}}$ when $\boldsymbol{t} \to \infty$ or $\boldsymbol{i(\infty) = \frac{V_s}{R}}$.



$$
\begin{aligned}
&\frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-\frac{t}{\tau}}, \qquad t > 0 \\
i(t) &= i(\infty) + [i(0) - i(\infty)] e^{-\frac{t}{\tau}}, \quad t > 0
\end{aligned}
$$

Step response of RL circuit

- Follow these steps to find step response of RL circuits:

1. Find the **initial current** $i(0)$ at $t = 0$ through the inductor **before any changes** in the circuit ($t < 0$).
   - The inductor is assumed to be a **short circuit.**

2. Find the **final current** $i(\infty)$ at $t \to \infty$ through the inductor **after the changes** in the circuit ($t \geq 0$).
   - The inductor is assumed to be a **short circuit**.

3. Find the **time constant** $\tau = \frac{L}{R_{\mathrm{Th\_\infty}}}$ **after the changes** in the circuit.
   - $R_{\mathrm{Th\_\infty}}$ is Thevenin equivalent resistance **after the changes** ($t \geq 0$).

4. Calculate the current through the inductor as:

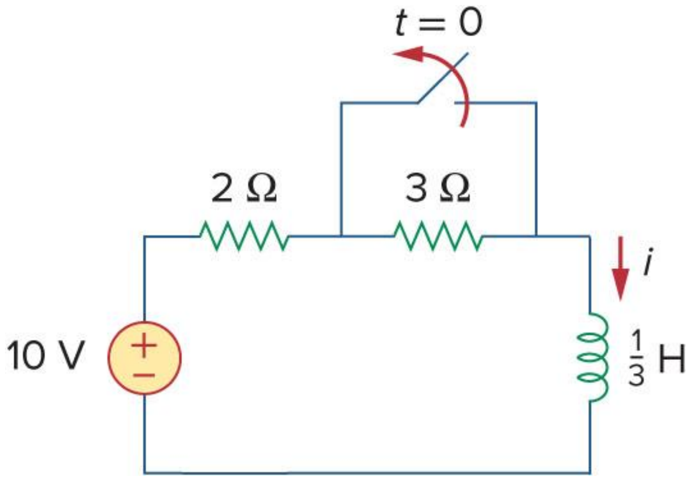
$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}}$$

5. Find any other circuit variable using the inductor's current.

Note: A **switch** which **opens** or **closes** can **remove** part of the circuit or **add** something to it.

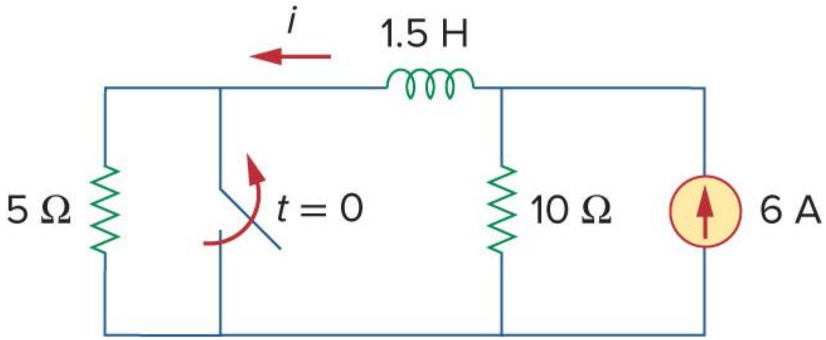
Exercise

Find $i(t)$ in the circuit for $t > 0$ assuming that the switch has been closed for a long time.



Exercise

Find $i(t)$ in the circuit for $t > 0$ assuming that the switch has been closed for a long time.



Questions?