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School of Electrical Engineering & Telecommunications

ELEC1111 Tutorial

Topic 8: AC Circuits I

1. For the following pairs of sinusoids, find the phasors corresponding to each pair and then determine which one leads and by how much. Also, sketch each pair in phasor diagram using rectangular form of the phasors.

   a) $v(t) = 50\cos(4t - 60°)$ V and $i(t) = 17\sin(4t + 50°)$ A
   b) $v_1(t) = 4\cos(314t + 10°)$ V and $v_2(t) = -20\cos(314t)$ V
   c) $i_1(t) = 13\cos(2t) + 5\sin(2t)$ A and $i_2(t) = 15\cos(2t - 11.8°)$ A
   d) $v(t) = 50\cos(20t + 70°)$ V and $i(t) = 40\cos(25t + 20°)$ V

Answer:

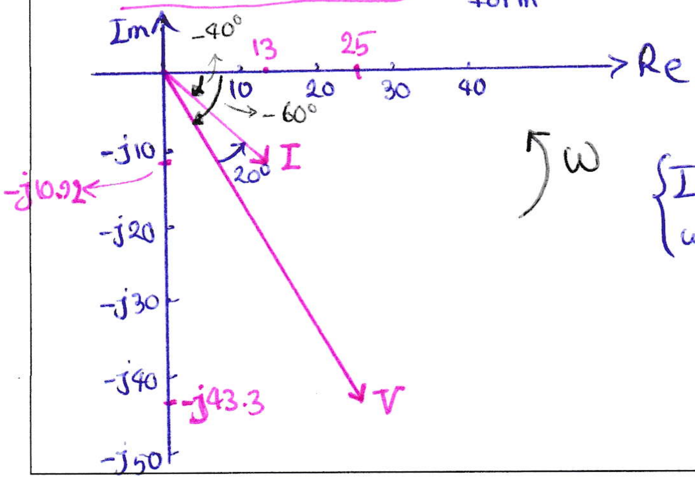
a) $\mathbf{V} = 50\angle(-60°)$ V, $\mathbf{I} = 17\angle(-40°)$ A. i leads v by 20°
b) $\mathbf{V}_1 = 4\angle(10°)$ V, $\mathbf{V}_2 = 20\angle(\pm180°)$ V. v_2 leads v_1 by 170° <u>or</u> v_1 leads v_2 by 190°
c) $\mathbf{I}_1 = 13.92\angle(-21.03°)$ A, $\mathbf{I}_2 = 15\angle(-11.8°)$ A. i_2 leads i_1 by 9.23°
d) Not comparable! Why?

Solution: Transform all sinusoids to Cosineform first

a) $v(t) = 50\cos(4t - 60°)$ V $\longrightarrow$ $\boxed{V = 50\angle -60°}$ $\longrightarrow$ transform to rectangular for for phasor diagram: $V = 50\cos(-60°) + j50\sin(-60°) = \boxed{25 - j43.3\text{ V}}$

$i(t) = 17\sin(4t + 50°)$ A $= 17\cos(4t + 50° - 90°) = 17\cos(4t - 40°)$ A

$\Rightarrow \boxed{I = 17\angle -40°\text{ A}}$ $\xrightarrow[\text{form}]{\text{rectangular}}$ $I = 17\cos(-40°) + j17\sin(-40°) = \boxed{13.02 - j10.92\text{ A}}$



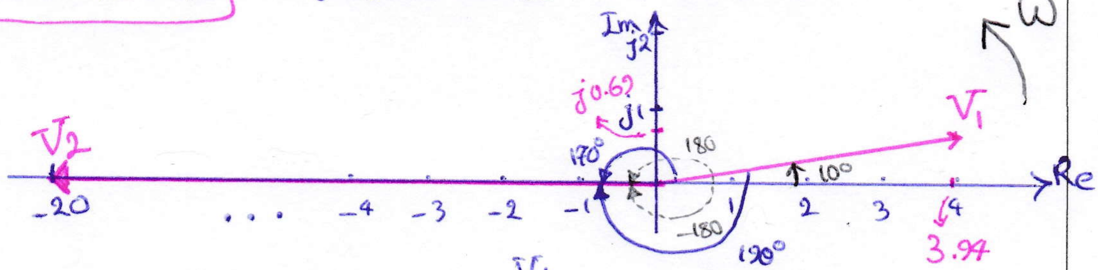
$\left\{\begin{array}{l}\text{I leads V by } 20° \text{ thus } i \text{ leads } v \\ \text{with the same phase } (20°)\end{array}\right.$

b) $v_1 = 4\cos(314t + 10^\circ)\,V \longrightarrow \boxed{V_1 = 4\angle 10^\circ}$ $\xrightarrow[\text{form}]{\text{rectangular}}$ $4\cos(10^\circ) + j4\sin(10^\circ)$

$= \boxed{3.94 + j0.69\ V}$

$v_2 = -20\cos(314t)\,V = 20\cos(314t \pm 180^\circ)$ both are correct!

$\longrightarrow \boxed{V_2 = 20\angle \pm 180^\circ}$ it is better to use either 180° or -180°.



{ if we use $+180^\circ$ for $V_2 \longrightarrow V_2$ leads V_1 by $\underline{170^\circ}$ (preferable)

{ if we use -180° for $V_2 \longrightarrow V_1$ leads V_2 by 190°

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c) $i_1 = 13\cos 2t + 5\sin 2t\ A = 13\cos 2t + 5\cos(2t - 90^\circ)$

$\longrightarrow I_1 = 13\angle 0^\circ + 5\angle -90^\circ = 13 + 5\cos(-90^\circ) + j5\sin(-90^\circ)$

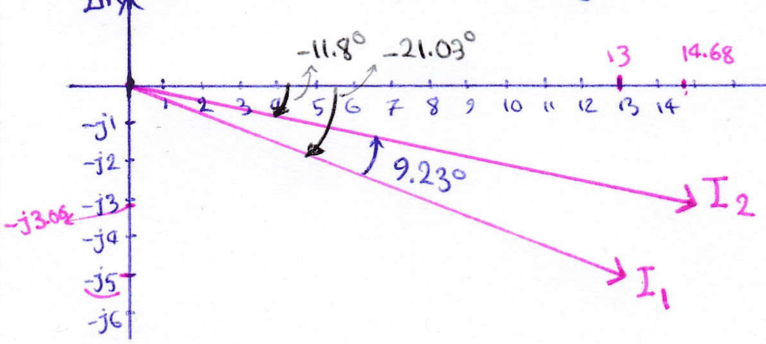
Note: $\boxed{13\angle 0^\circ \equiv 13}$ $\qquad = \boxed{13 - 5j\ A}$

Note: $\boxed{5\angle -90^\circ \equiv -j5}$

Transfor to Polar form $\longrightarrow I_1 = \sqrt{13^2 + 5^2}\angle \tan^{-1}\left(\frac{-5}{13}\right) = \boxed{13.92\angle -21.03^\circ\ A}$

$i_2 = 15\cos(2t - 11.8^\circ)\ A \longrightarrow \boxed{I_2 = 15\angle -11.8^\circ\ A}$

$\xrightarrow{\text{rectangular}} I_2 = 15\cos(-11.8^\circ) + j15\sin(-11.8^\circ) = \boxed{14.68 - j3.06\ A}$



I_2 leads I_1 by 9.23

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d) They cannot be compared in phasor domain since they have different frequencies $\longrightarrow 50\cos(20t + 70^\circ)$ $\quad 40\cos(25t + 20^\circ)$

$\omega_v = 20\,\text{rad/s} \neq \omega_i = 25\,\text{rad/s}$

2. Evaluate the following complex numbers and express your results in both polar and rectangular forms.

a) $\dfrac{(5\angle 10°)(10\angle(-40°))}{(4\angle(-80°))(-6\angle 50°)}$

b) $\dfrac{2+j3}{1-j6}+\dfrac{7-j8}{-5+j11}$

c) $\begin{vmatrix} 2+j3 & -j2 \\ 2e^{-j\pi/2} & 8-j5 \end{vmatrix}$

d) $\dfrac{60e^{j45°}}{7.5-j10}+j2$

Answer:

a) Polar form: $2.083\angle \pm 180°$, rectangular forms: -2.083
b) Polar form: $1.284\angle 173.2°$, rectangular forms: $-1.275+j0.152$
c) Polar form: $37.696\angle 21.8°$, rectangular forms: $35+j14$
d) Polar form: $6.786\angle 95.741°$, rectangular forms: $-0.678+j6.752$

Solution:

a) $\dfrac{(5\angle 10°)\times(10\angle -40°)}{4\angle -80° \times(-6\angle 50°)} = \dfrac{5\times 10\angle(10-40°)}{4\times(-6)\angle(-80°+50°)} = \dfrac{50\angle -30}{-24\angle -30}$

$= -\dfrac{50}{24}\angle 0 = -2.083 \xrightarrow{\text{polar}} 2.083\angle \pm 180°$ either 180° or −180° one is enough!

b) $\dfrac{2+j3}{1-j6}+\dfrac{7-j8}{-5+j11}$ method 1 using Complex Conjugate for division $ZZ^* = |Z|^2$ if $Z = x+jy \rightarrow = x^2+y^2$ also $j^2=-1$

$\Rightarrow \dfrac{(2+j3)(1+j6)}{1+6^2}+\dfrac{(7-j8)(-5-j11)}{5^2+11^2} = \dfrac{2+j12+j3+j^2 18}{37}+\dfrac{-35-j77+j40+j^2 88}{146}$

$= \dfrac{2-18+j(12+3)}{37}+\dfrac{-35-88+j(40-77)}{146} = \dfrac{-16+j15}{37}+\dfrac{-123-j37}{146}$

$= -\dfrac{16}{37}+j\dfrac{15}{37}-\dfrac{123}{146}-j\dfrac{37}{146} = \boxed{-1.275+j0.152}$

to polar form $\rightarrow \sqrt{(1.275)^2+(0.152)^2}\angle \tan^{-1}\left(\dfrac{0.152}{-1.275}\right) = 1.284\angle 173.2°$

Method 2 using polar form transformation first

$2+j3 \triangleq 3.6\angle 56.3°$ $7-j8 = 10.63\angle -48.81°$
$1-j6 \triangleq 6.08\angle -80.53°$ $-5+j11 = 12.08\angle 114.44°$

→ next page →

$$\Rightarrow \frac{3.6\angle 56.3^\circ}{6.08\angle -80.53^\circ} + \frac{10.63\angle -48.81^\circ}{12.08\angle 114.44^\circ} = 0.592\angle 136.83^\circ + 0.88\angle -163.25$$

back to rectangular form $\rightarrow -0.432 + j0.405 + (-0.842 - j0.253) = -1.275 + j0.152$

c) Determinant: $(2+j3)(8-j5) - (-j2)(2e^{-j\pi/2})$ Note: $e^{-j\pi/2} = e^{-j90} = 1\angle -90 = -j$

$$\Rightarrow 16 - j10 + j24 - j^2 15 - j^2 4 = 16 + 15 + 4 + j(24 - 10)$$

$= 35 + j14$ $\longrightarrow$ Polar form: $\sqrt{35^2 + 14^2} \angle \tan^{-1}\left(\frac{14}{35}\right) = 37.69\angle 21.8^\circ$

d) Polar form

$$\frac{60\angle 45^\circ}{\sqrt{7.5^2 + 10^2}\angle \tan^{-1}\left(\frac{-10}{7.5}\right)} + j2 = \frac{60\angle 45^\circ}{12.5\angle -53.13^\circ} + j2 = \frac{60}{12.5}\angle(45^\circ + 53.13^\circ) + 2j$$

$$= 4.8\angle 98.13^\circ + 2j \xrightarrow[\text{form}]{\text{rectangular}} 4.8\cos(98.13^\circ) + j4.8\sin(98.13^\circ) + 2j$$

$$= -0.678 + j4.752 + j2 = -0.678 + j6.752$$

polar form $\rightarrow \sqrt{(0.678)^2 + (6.752)^2} \angle \tan^{-1}\frac{6.752}{-0.678} = 6.786\angle 95.74^\circ$

3. Obtain the sinusoids corresponding to each of the following phasors.

a) $\mathbf{V}_1 = 60\angle 15^\circ$ V, $\omega = 1$ rad/s
b) $\mathbf{V}_2 = 6 + j8$ V, $\omega = 40$ rad/s
c) $\mathbf{I}_1 = 2.8e^{-j\pi/3}$ A, $f = 50$ Hz

Answer:

a) $v_1(t) = 60\cos(t + 15^\circ)$ V
b) $v_2(t) = 10\cos(40t + 53.13^\circ)$ V
c) $i_1(t) = 2.8\cos(314.16t - 60^\circ)$ A

Solution: Pick the magnitude as amplitude with the given frequency in rad/s and the phase as the phase of the sinusoid. (*) Assumption from phasor to sinusoid is always Cosine form

a) $v_1(t) = 60\cos(t + 15^\circ)$

b) $V_2 = \sqrt{6^2 + 8^2}\angle \tan^{-1}\left(\frac{8}{6}\right) = 10\angle 53.13^\circ \Rightarrow v_2(t) = 10\cos(40t + 53.13^\circ)$

c) $I_1 = 2.8\angle -60^\circ$ but $\omega = 2\pi f = 2 \times 3.1416 \times 50 = 314.16$ rad/s

$$\Rightarrow i_1(t) = 2.8\cos(314.16t - 60^\circ)$$

4. Find $v(t)$ and $i(t)$ in the following integrodifferential equations using phasor approach.

   a) $v(t) + \int v(t)\,dt = 10\cos(t)$

   b) $10\int i(t)\,dt + \frac{di(t)}{dt} + 6i(t) = 5\cos(5t + 22°)$

Answer:

a) $v(t) = 7.071\cos(t + 45°)$ V

b) $i(t) = 745\cos(5t - 4.56°)$ mA

Hint: use the properties of differentiation and integration in time domain and their equivalents in phasor domain ,i.e., $\frac{dv(t)}{dt} \Leftrightarrow j\omega\mathbf{V}$ and $\int v(t)\,dt \Leftrightarrow \frac{\mathbf{V}}{j\omega}$, then solve the equations for the phasor of the main variables by writing the equations in phasor domain assuming that $\mathbf{V}$ is the phasor of $v(t)$ and $\mathbf{I}$ is the phasor of $i(t)$.

"phasor domain can be used to solve differential equations"

Solution:

a) $\xrightarrow{\text{phasor domain}} V + \frac{V}{j\omega} = 10\angle 0 \xrightarrow{\omega = 1\,\text{rad/s}} V - jV = 10$

$\rightarrow V(1-j) = 10 \rightarrow V = \frac{10}{1-j} = \frac{10(1+j)}{\underbrace{1^2+1^2}_{2}} = \boxed{5 + j5\ \text{V}}$

$\xrightarrow[\text{form}]{\text{Polar}} \sqrt{5^2+5^2}\angle\tan^{-1}\left(\frac{5}{5}\right) = \sqrt{50}\angle 45° = \boxed{7.071\angle 45°}\,\text{V}$

Thus $\Rightarrow v(t) = 7.071\cos(t + 45°)$ V

the solution to the equation

b) $\xrightarrow{\text{phasor domain}} \frac{10I}{j\omega} + j\omega I + 6I = 5\angle 22° \xrightarrow{\omega = 5\,\text{rad/s}}$

$\Rightarrow -j\frac{10}{5}I + j5I + 6I = 5\angle 22° \rightarrow I(6 + j3) = 5\angle 22°$

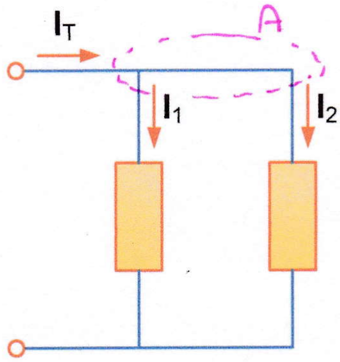
$\Rightarrow I = \frac{5\angle 22°}{6+j3} = \frac{5\angle 22°}{\sqrt{6^2+3^2}\angle\tan^{-1}\left(\frac{3}{6}\right)} = \frac{5\angle 22°}{6.708\angle 26.58°}$

$\Rightarrow I = \frac{5}{6.708}\angle(22 - 26.58) = 0.745\angle -4.56°$ A

Thus $\Rightarrow i(t) = 745\cos(5t - 4.56°)$ mA

* Steady-state solution of a differential equation to a sinusoid input is always a sinusoid with the same frequency. *

5. (**Final Exam – S1, 2015**) In the following figure, $i_1 = 100\sin(50t + 100°)$ A and $i_T = 50\sin(50t - 40°)$ A. Determine phasors $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_T$ and draw the phasor diagram showing $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_T$. Also, determine which one lags and by how much.



Answer: $\mathbf{I}_1 = 100\angle 10°$ A, $\mathbf{I}_2 = 141.98\angle(-156.92°)$ A and $\mathbf{I}_T = 50\angle(-130°)$ A. $\mathbf{I}_2$ lags $\mathbf{I}_T$ by $26.92°$ and lags $\mathbf{I}_1$ by $166.92°$, also $\mathbf{I}_T$ lags $\mathbf{I}_1$ by $140°$.

Solution:

$i_1 = 100\cos(50t + 100° - 90°) = 100\cos(50t + 10°) \longrightarrow I_1 = 100\angle 10°$ A

$i_T = 50\cos(50t - 40 - 90°) = 50\cos(50t - 130°) \longrightarrow I_T = 50\angle -130°$ A

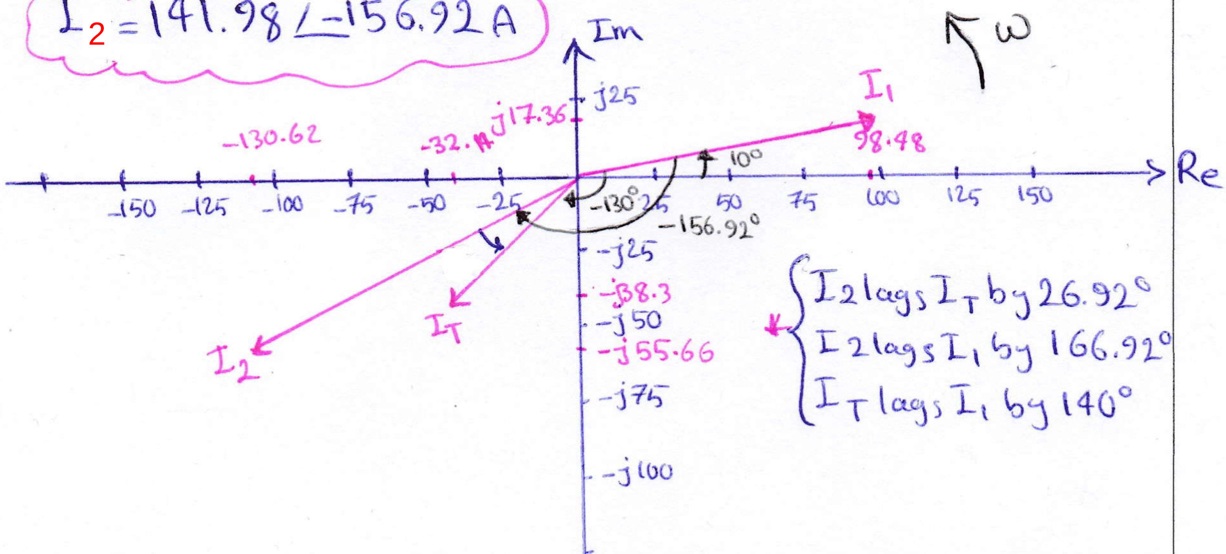
rectangular form $\longrightarrow$ $I_1 = 100\cos(10°) + j100\sin(10°) = \boxed{98.48 + j17.36 \text{ A}}$

$I_T = 50\cos(-130°) + j50\sin(-130°) = \boxed{-32.14 - j38.3 \text{ A}}$

KCL: @ node A $\quad I_T = I_1 + I_2 \longrightarrow I_2 = I_T - I_1 = -32.14 - j38.3 - (98.48 + j17.36)$

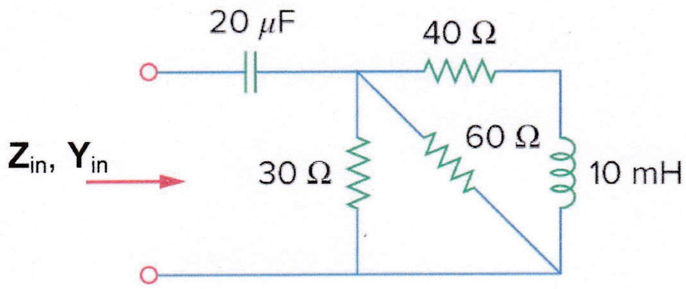
$\Rightarrow \boxed{I_2 = -130.62 - j55.66 \text{ A}}$ $\xrightarrow{\text{Polar form}}$ $\sqrt{130.62^2 + 55.66^2}\ \angle \tan^{-1}\left(\frac{55.66}{-130.62}\right)$

$I_2 = 141.98\angle -156.92$ A



I_2 lags I_T by $26.92°$
I_2 lags I_1 by $166.92°$
I_T lags I_1 by $140°$

6. For the following circuit,
   a) Find the input impedance $\mathbf{Z}_{\text{in}}$ and input admittance $\mathbf{Y}_{\text{in}}$ with their corresponding resistance, reactance, conductance and susceptance if $\omega = 10^3$ rad/s.
   b) Is the circuit *inductive* or *capacitive*?



Answer:

a) $\begin{cases} \mathbf{Z}_{\text{in}} = R + jX = 13.51 - j48.92\ \Omega = 50.75\angle(-74.56°)\ \Omega \\ \mathbf{Y}_{\text{in}} = \frac{1}{\mathbf{Z}_{\text{in}}} = G + jB = 5.24 + j18.99\ \text{mS} = 19.704\angle 74.56°\ \text{mS} \end{cases}$

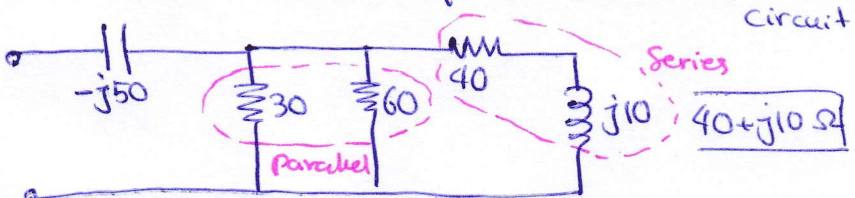
b) Capacitive.

Solution: First find all the individual impedances:

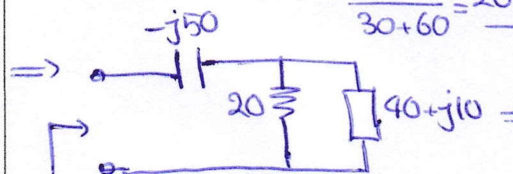
$$20\mu F \longrightarrow \frac{1}{j\omega C} = \frac{-j}{10^3 \times 20\times10^{-6}} = \frac{-j}{2\times10^{-2}} = -j\frac{100}{2} = \boxed{-j50\ \Omega}$$

$$10\,mH \longrightarrow j\omega L = j\times10^3\times10\times10^{-3} = \boxed{j10\ \Omega}$$

Then, redraw the circuit in phasor domain: (It is important to draw the circuit in phasor domain in exam)



$$\frac{30\times60}{30+60} = 20\ \Omega$$



$$\Longrightarrow Z_{in} = 20 \| (40+j10) - j50$$

$$Z_{in} = \frac{20\times(40+j10)}{20+40+j10} - j50 = \frac{80\cancel{0}+j20\cancel{0}}{6\cancel{0}+j1\cancel{0}} - j50 = \frac{(80+j20)(6-j)}{6^2+1^2} - j50$$

$$= 13.51 + j1.08 - j50 = 13.51 - j48.92\ \Omega$$

$\begin{cases} R = 13.51 \\ X = \underline{-48.92} \end{cases}$ → negative reactance ⇒ Capacitive Circuit

$$Y_{in} = \frac{1}{Z_{in}} = \frac{1}{13.51 - j48.92}$$

polar form → $= \dfrac{1}{\sqrt{13.51^2+48.92^2}\ \angle \tan^{-1}\left(\frac{48.92}{13.51}\right)} = \dfrac{1}{50.75\angle -74.56} = 19.704\angle 74.56°\ \text{mS}$

rectangular form → $Y_{in} = 19.704\cos(74.56°) + j\,19.704\sin(74.56°) = \underbrace{5.24}_{G} + j\underbrace{18.99}_{B}\ \text{mS}$

$\begin{cases} G = 5.24 \\ B = 18.99 \end{cases}$

7. Calculate $\mathbf{I}_o$ in Fig. 1 and $v_o(t)$ in Fig. 2.

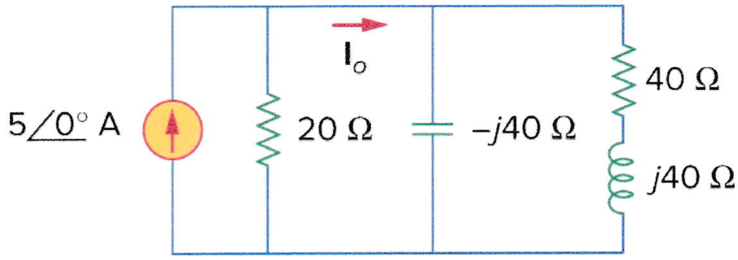


Fig. 1

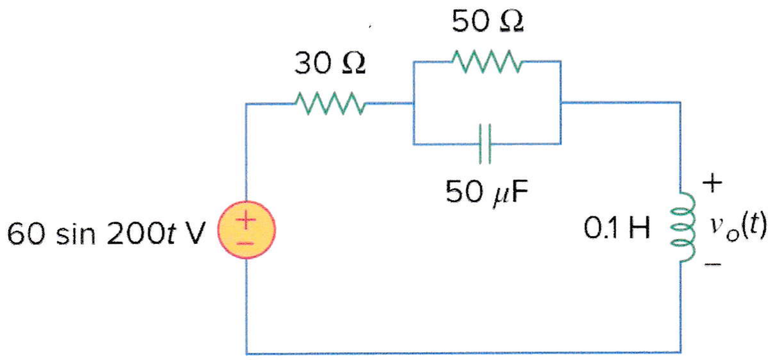


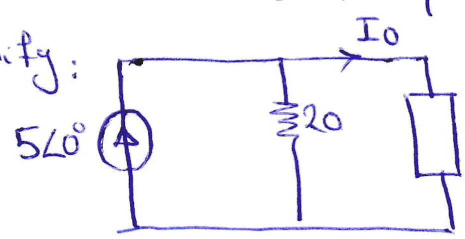
Fig. 2

Answer: Fig. 1: $\mathbf{I}_o = 1.386\angle 33.69°\ \mathrm{A}$, and Fig. 2: $v_o(t) = 17.14\cos(200t)\ \mathrm{V}$

Solution:

Fig 1: The circuit is already in phasor domain

Simplify:



$(40+j40)\|(-j40) = 40-j40$

$$\rightarrow (40+j40)\|(-j40) = \frac{(40+j40)\times(-j40)}{40+\cancel{j40}-\cancel{j40}} = \frac{-j\,\cancel{1600}^{\,40} - \cancel{j^2}^{\,-1}\,\cancel{1600}^{\,40}}{\cancel{40}} =$$

$$= \boxed{40-j40\ \Omega}$$

Thus: Io is obtained via current division:

$$I_o = \frac{2\cancel{0}}{2\cancel{0}+4\cancel{0}-j4\cancel{0}} \times 5\angle 0° = \frac{10\angle 0°}{6-j4} = \frac{10\angle 0°}{\sqrt{6^2+4^2}\angle \tan^{-1}(-4/6)} = \frac{10\angle 0°}{7.21\angle -33.69°}$$

$$\Rightarrow I_o = \frac{10}{7.21}\angle +33.69° = 1.38\angle 33.69°\ \mathrm{A}$$

→ next page for Fig2 →

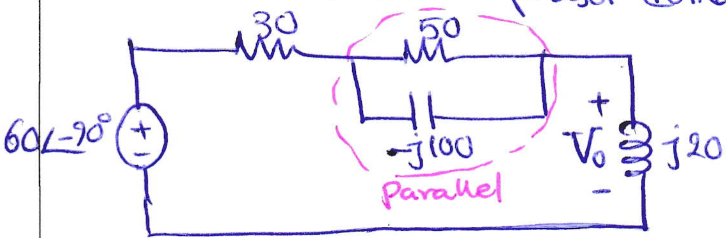
Fig 2: Find all impedances and phasors first:

$$60\sin 200t = 60\cos(200t - 90^\circ) \longrightarrow \boxed{60\angle -90^\circ} \qquad \omega = 200\ \text{rad/s}$$

$$50\,\mu F \longrightarrow \frac{1}{j\omega C} = \frac{-j}{200\times 50\times 10^{-6}} = \boxed{-j100\,\Omega}$$

$$0.1\ H \longrightarrow j\omega L = j200\times 0.1 = \boxed{j20\,\Omega}$$

Redraw the circuit in phasor domain;

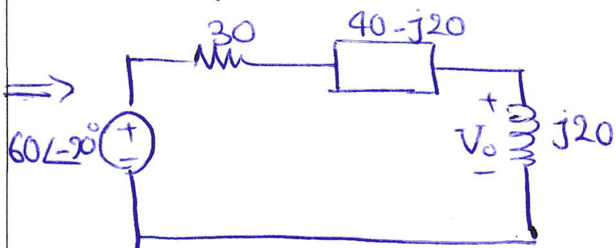


Find Vo which is the phasor of $v_o(t)$

$$50 \parallel (-j100) = \frac{50\times(-j100)}{\underset{50(1-j2)}{50 - j100}} = \frac{-j100}{1-j2} = \frac{-j100(1+j2)}{\underset{5}{1+4}}$$

$$= \underset{-j20-j^2 40}{-j20(1+j2)} = \boxed{40 - j20\,\Omega}$$

$\Longrightarrow$



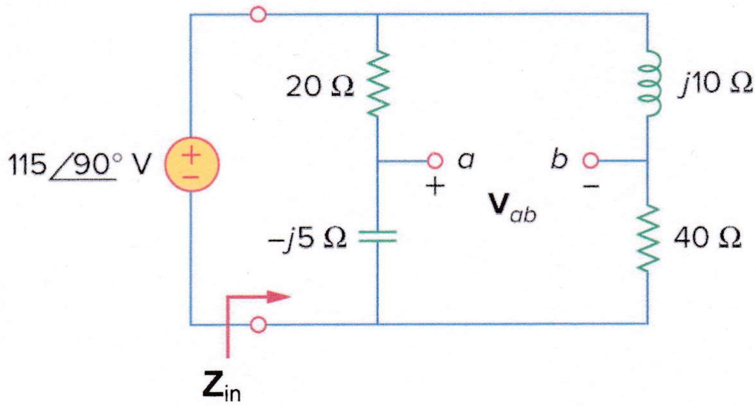
Vo is obtained via Voltage division

$$V_o = \frac{j20}{30+40-\cancel{j20}+\cancel{j20}}\times 60\angle -90^\circ = \frac{j20}{7\cancel{0}}\times 60\angle -20^\circ = 0.285\angle 90^\circ \times 60\angle -90^\circ$$

$$= 17.14\angle 0^\circ\ V$$

Thus: $v_o(t) = 17.14\cos(200t)\ V$

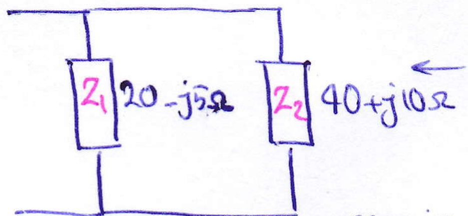
8. For the circuit given below, calculate input impedance $\mathbf{Z}_{\text{in}}$ and $\mathbf{V}_{ab}$.



Answer: $\mathbf{Z}_{\text{in}} = 14.069 - j1.172\ \Omega = 14.117\angle(-4.76°)\ \Omega$, and $\mathbf{V}_{ab} = 101.4\angle(-90°)\ \text{V} \equiv 101.4\angle 270°\ \text{V}$

Solution: The circuit is already in phasor domain

For Z_{in}:



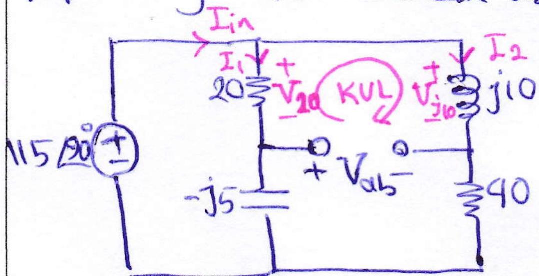
{40Ω in series with j10Ω
{20Ω in series with -j5Ω

$$\Rightarrow Z_{in} = (20-j5)\,||\,(40+j10) = \frac{(20-j5)(40+j10)}{20-j5+40+j10} = \frac{800-j200+j200-j^2 50}{60+j5}$$

$$= \frac{\cancel{850}^{170}}{\cancel{60+j5}_{12\ \ 1}} = \frac{170}{12+j} = 14.069 - j\,1.172\ \Omega$$

in polar form: $Z_{in} = \sqrt{14.069^2 + 1.172^2}\ \angle \tan^{-1}\left(\frac{-1.172}{14.069}\right) = 14.117\angle -4.76°\ \Omega$

* Finding V_{ab}: we can use two approaches: long method and quick method



First we need to write KVL either in the upper mesh or lower mesh

KVL: $-V_{ab} - V_{20} + V_{j10} = 0$

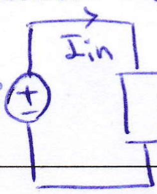
$\Rightarrow \boxed{V_{ab} = V_{j10} - V_{20}}$

(*) Now to find V_{j10} and V_{20} we can use two approaches:

First long method: $V_{20} = 20 I_1$ and $V_{j10} = j10\, I_2 \longrightarrow \boxed{I_1 + I_2 = I_{in}}$

So we need to find I_{in} to calculate I_1 and I_2 using current division

To find I_{in}, we can use Z_{in}: that we calculated befor



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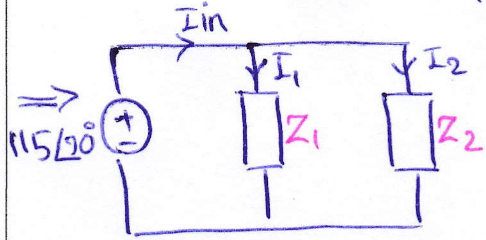
$$\rightarrow \quad I_{in} = \frac{115\angle 90^\circ}{Z_{in}} = \frac{115\angle 90^\circ}{14.117\angle -4.76^\circ} = \boxed{8.145\angle 94.76^\circ\ \text{A}} \quad (\mathrm{I})$$

$\Rightarrow$



$Z_1 = 20 - j5\ \Omega$

$Z_2 = 40 + j10$

$$\Rightarrow \quad I_1 = \frac{40+j10}{60+j5} \times I_{in}$$

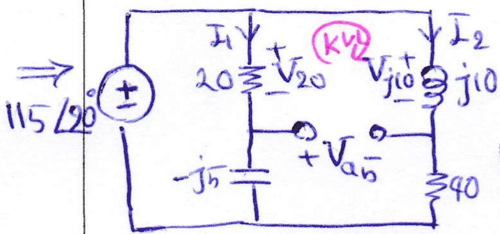
$$I_2 = \frac{20-j5}{60+j5} \times I_{in}$$

Simplify $I_1 = \frac{8+j2}{12+j} I_{in}$

$I_2 = \frac{4-j}{12+j} I_{in}$

No need to find their numerical values as you will see in KVL equation

$\Rightarrow$



$$\xrightarrow{KVL} V_{ab} = V_{j10} - V_{20} = j10 I_2 - 20 I_1$$

$$= j10\left(\frac{4-j}{12+j} I_{in}\right) - 20\left(\frac{8+j2}{12+j} I_{in}\right)$$

factor out I_{in}

$$\Rightarrow \quad V_{ab} = \left[\frac{j40 - j^2 10 - 160 - j40}{12+j}\right] I_{in} = \boxed{\frac{-150}{12+j} I_{in}}$$

$$\xrightarrow{\text{use } (\mathrm{I})} \quad V_{ab} = 101.4\angle -90^\circ \quad \text{or} \quad 101.4\angle 270^\circ\ \text{V}$$

Second approach is to us voltage division to directly find V_{j10} and V_{20} (This one is suggested by a student in a tutorial class)

Note that the $20\,\Omega$ and $-j5\,\Omega$ branch is in parallel with the source as well as parallel with $j10\,\Omega$ and $40\,\Omega$

quick method $\Rightarrow \begin{cases} \text{Find } V_{20} \text{ using voltage division between } 20\,\Omega,\ -j5\,\Omega \text{ and } 115\angle 90^\circ\ \text{V} \\ \text{Find } V_{j10} \text{ using voltage division between } j10\,\Omega,\ 40\,\Omega \text{ and } 115\angle 90^\circ\ \text{V} \end{cases}$

$$\begin{cases} V_{20} = \dfrac{20}{20-j5} \times 115\angle 90^\circ \\ V_{j10} = \dfrac{j10}{40+j10} \times 115\angle 90^\circ \end{cases}$$

simplify $V_{20} = \frac{4}{4-j} \times 115\angle 90^\circ$

$V_{j10} = \frac{j}{4+j} \times 115\angle 90^\circ$

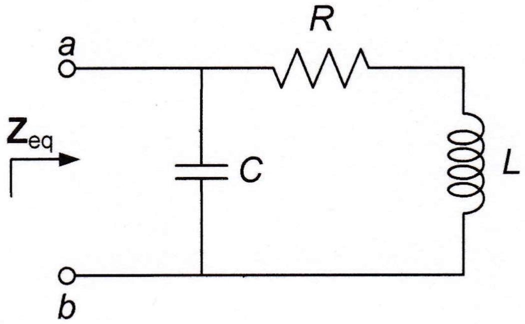
$$\Rightarrow \quad V_{ab} = V_{j10} - V_{20} = \left(\frac{j}{4+j} - \frac{4}{4-j}\right) \times 115\angle 90^\circ$$

$$= \frac{\overbrace{j(4-j)}^{j4 - j^2} - \overbrace{4(4+j)}^{-16-j4}}{\underbrace{(4+j)(4-j)}_{16+1}} \times 115\angle 90^\circ$$

$$= -\frac{15}{17} \times 115\angle 90^\circ = 101.4\angle -90^\circ\ \text{V}$$

Challenging Problem

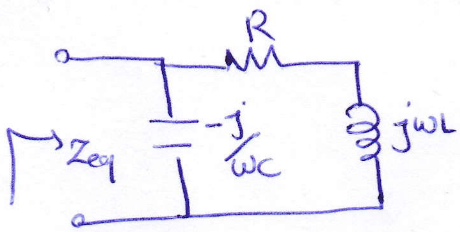
9. (**Final Exam – S1, 2016**) Consider the circuit shown in the figure below, where $L = 27\ \mathrm{mH}$, $C = 22\ \mu\mathrm{F}$ and $R = 6\ \Omega$. Find the angular frequency ω for which the impedance $\mathbf{Z}_{\mathrm{eq}}$ between terminals *a-b* is purely resistive.



Answer: $\omega = 1.278 \times 10^3\ \mathrm{rad/s}$

Hint: The reactance of $\mathbf{Z}_{eq}$ is a function of ω and it should be zero for pure resistive impedance, or the phase of $\mathbf{Z}_{eq}$ should be zero for pure resistive impedance.

Solution: Redraw the circuit in phasor domain without substituting numerical values:



$Z_{eq} = R(\omega) + jX(\omega) \longrightarrow X(\omega) = 0$

$\quad = |Z_{eq}| \angle Z_{eq} \longrightarrow \angle Z_{eq} = 0$

Both methods results in Z_{eq} being a real number meaning pure resistive impedance.

(*) Let's first try to find $X(\omega)$ (hard and long method) and set it to zero to find the corresponding frequency at which Z_{eq} becomes pure resistive

$$Z_{eq} = \frac{-j}{\omega C} \| (R + j\omega L) = \frac{-\frac{j}{\omega C} \times (R + j\omega L)}{-\frac{j}{\omega C} + R + j\omega L} = \frac{-j(R + j\omega L)}{-j + RC\omega + j\omega^2 LC}$$

(I)

$$\boxed{Z_{eq} = \frac{\omega L - jR}{RC\omega + j(\omega^2 LC - 1)}}$$

multiply by the conjugate of denominator

$$= \frac{(\omega L - jR)(RC\omega - j(\omega^2 LC - 1))}{[RC\omega + j(\omega^2 LC - 1)][RC\omega - j(\omega^2 LC - 1)]}$$

$$Z_{eq} = \frac{RLC\omega^2 - j\omega L(\omega^2 LC - 1) - jR^2 C\omega \overset{2 \to -1}{\cancel{+j}} R(\omega^2 LC - 1)}{(RC\omega)^2 + (\omega^2 LC - 1)^2}$$

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$$Z_{eq} = \underbrace{\frac{RLC\omega^2 - R(\omega^2LC-1)}{(RC\omega)^2+(\omega^2LC-1)^2}}_{R(\omega)} + j\underbrace{\frac{-\omega L(\omega^2LC-1)-R^2C\omega}{(RC\omega)^2+(\omega^2LC-1)^2}}_{X(\omega)}$$

$\Rightarrow$ To have $Z_{eq} = R(\omega) \longrightarrow X(\omega)=0 \longrightarrow$ numerator only needs to be zero

$$\Rightarrow \cancel{-}\omega L(\omega^2LC-1)\cancel{-}R^2C\omega = 0 \longrightarrow \omega(\omega^2L^2C - L + R^2C) = 0$$

$$\Rightarrow \begin{cases} \omega = 0 \rightarrow \text{obvious answer} \\ \text{or} \\ \omega^2L^2C - L + R^2C = 0 \Rightarrow \omega^2L^2C = L - R^2C \end{cases}$$

$$\omega = \sqrt{\frac{L-R^2C}{L^2C}}$$

if the frequency of the input source is chosen in this way, the total impedance from the terminal becomes pure resistive

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(*) This time, we try to find the phase of Z_{eq} and set it to zero. (easy and quick method)

Recall (I) $\rightarrow$ $Z_{eq} = \frac{\omega L - jR}{RC\omega + j(\omega^2LC-1)} \Rightarrow Z_{eq} = \frac{r_1\angle\varphi_1}{r_2\angle\varphi_2} = \underbrace{\frac{r_1}{r_2}}_{|Z_{eq}|}\angle\underbrace{\varphi_1-\varphi_2}_{\angle Z_{eq}}$

where

$\Rightarrow r_1 = \sqrt{(\omega L)^2+R^2}\,,\ r_2 = \sqrt{(RC\omega)^2+(\omega^2LC-1)^2}$

$\boxed{\varphi_1 = \tan^{-1}\frac{-R}{\omega L}}\,,\ \boxed{\varphi_2 = \tan^{-1}\left(\frac{\omega^2LC-1}{RC\omega}\right)}$

to have

$\Rightarrow Z_{eq} = |Z_{eq}| \longrightarrow \angle Z_{eq} = 0 \Rightarrow \varphi_1 - \varphi_2 = 0$

we only need this part

$$\Rightarrow \tan^{-1}\left(\frac{-R}{\omega L}\right) = \tan^{-1}\left(\frac{\omega^2LC-1}{RC\omega}\right) \xrightarrow[\text{must be the same}]{\text{arguments}} \boxed{\frac{-R}{\omega L} = \frac{\omega^2LC-1}{RC\omega}}$$

$$\Rightarrow -R^2C\omega = \omega L(\omega^2LC-1) \longrightarrow \boxed{\omega L(\omega^2LC-1)+R^2C\omega = 0}$$

$$\Rightarrow \omega(\omega^2L^2C - L + R^2C) = 0 \longrightarrow \begin{cases} \omega = 0 \\ \omega^2L^2C - L + R^2C = 0 \Rightarrow \omega = \sqrt{\frac{L-R^2C}{L^2C}} \end{cases}$$

Now $\left.\begin{array}{l} R = 6\,\Omega \\ L = 27\,mH \\ C = 22\,\mu F \end{array}\right\} \rightarrow \omega = \sqrt{\frac{27\times10^{-3} - 36\times22\times10^{-6}}{27^2\times10^{-6}\times22\times10^{-6}}} = 1.278\times10^{3}\ \frac{rad}{s}$