ELEC 1111 - Electric Circuits

Math Handbook

Scope of this document

The scope of this Handbook is to provide a collection of the basic mathematical formulas, tables and methods that will be valuable for those taking the ELEC1111 - Electric Circuits course at the School of Electrical Engineering and Telecommunications, UNSW Australia.

The purpose of this handbook is not be exhaustive and detailed but a quick reference and problem solving companion for the students while studying for the course.

Trigonometrical Formulas

$$\sin \omega t = \cos(\omega t - 90^\circ), \tag{1}$$

$$\cos \omega t = \sin(\omega t + 90^\circ), \tag{2}$$

$$\sin \omega t = \sin(180^\circ - \omega t), \tag{3}$$

$$\sin(\omega t \pm \theta) = \cos\theta \sin \omega t \pm sin\theta \cos \omega t, \tag{4}$$

$$\cos(\omega t \pm \theta) = \cos\theta \cos \omega t \mp sin\theta \sin \omega t, \tag{5}$$

$$\sin^2 \omega t = \frac{1 - cos2\omega t}{2} \tag{6}$$

$$\cos^2 \omega t = \frac{1 + cos2\omega t}{2} \tag{7}$$

$$\sin(-\omega t) = -\sin \omega t \tag{8}$$

$$\cos(-\omega t) = \cos \omega t \tag{9}$$

$$\sin 2a = 2 \sin a \cos a \tag{10}$$

$$\cos 2a = \cos^2 a - \sin^2 a \tag{11}$$

$$\sin^2 a + \cos^2 a = 1 \tag{12}$$

$$A \cos \omega t + B \sin \omega t = (\sqrt{A^2 + B^2}) \cos(\omega t - \theta), \tag{13}$$

where $\theta = \tan^{-1}\frac{B}{A}$ when A > 0 and where $\theta = 180° \tan^{-1}\frac{B}{A}$ when A < 0.

$$\cos\alpha\cos(\alpha+\beta) = \frac{1}{2}\cos\beta + \frac{1}{2}cos(2\alpha+\beta), \tag{14}$$

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha \tag{15}$$

Matrices

A rectangular array of numbers

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{16}$$

is called a matrix. The numbers inside the matrix a_{ij} are known as the elements of the matrix, with the first subscript i denoting the row and the second subscript j denoting the column of the element.

A matrix with m rows and n columns is a matric of order (m, n) or also called an $m \times n$ matrix. When the number of rows is equal to the number of columns $(m = n)$, the matrix is called *square* of order n. $m \times n$ matrices are usually denoted with bold capital letters, e.g. $\mathbf{A}$.

A matrix consisting of only one column is known as a column matrix or a column vector. There are usually denoted with lower case bold letters (e.g. $\mathbf{x}$) as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{17}$$

When the elements of follow the relationship of $a_{ij} = a_{ji}$, the matrix is called a symmetrical matrix. For example,

$$\mathbf{Z} = \begin{bmatrix} 5 & -4 & 2 \\ -4 & 1 & 7 \\ 2 & 7 & -4 \end{bmatrix} \tag{18}$$

is a symmetrical 3×3 matrix.

Addition of matrices

Addition of matrices is possible for matrices of the same order and the sum is obtained by adding the corresponding elements. The elements of $\mathbf{C}$ in the addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{19}$$

are given by

$$c_{ij} = a_{ij} + b_{ij}. \tag{20}$$

The process is commutative so $\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ and associative $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Multiplication of matrices

The multiplication of two matrices **AB** requires that the number of columns of $mathbfA$ is equal to the number of rows of **B**. If $mathbfA$ is of order $m \times n$ and $mathbfB$ is of order $n \times p$ the product of the multiplication is a matrix with an order of $m \times p$. The elements of the matrix

$$\mathbf{C} = \mathbf{AB} \tag{21}$$

are found by multiplying the i^{th} row of **A** with the j^{th} column of **B** and summing these products so that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}bkj \tag{22}$$

In general, multiplication of matrices is not commutative $\mathbf{AB} \neq \mathbf{BA}$.

Determinants and Cramer's Rule

Determinants

The determinant of a matrix is a numerical value. In the case of a 2×2 matrix,

$$\mathbf{Z} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{23}$$

the determinant is defined as

$$\Delta = a_{11}a_{22} - a_{12}a_{21} \tag{24}$$

The method of obtaining Δ uses the diagonal rule. The determinant of the matrix is the difference of the product down the diagonal to the right and the product of th elements down the diagonal to the left.

The determinant of a 3×3 matrix is

$$\Delta = \left| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right] | \tag{25}$$

and is equal to

$$\Delta = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}) - (a_{13}a_{22}a_{31} + a_{23}a_{32}a_{a1} + a_{33}a_{21}a_{12}) \tag{26}$$

In general, the determinant is calculated in terms of co-factors and minors. The determinant of a submatrix of **A** obtained by deleting from **A** the i^{th} row and the j^{th} column is called the minor of the element a_{ij} and is denoted as m_{ij}. The cofactor c_{ij} is a minor with an associated sign, so that

$$c_{ij} = (-1)^{(i+j)}m_{ij} \tag{27}$$

The determinant of an $n \times n$ matrix is then

$$\Delta = \sum_{j=1}^{n} a_{ij}c_{ij} \tag{28}$$

for a selected value of i. Alternatively, the calculation can be performed by using the j^{th} column

$$\Delta = \sum_{i=1}^{n} a_{ij}c_{ij} \tag{29}$$

for a selected value of j.

Cramer's Rule

For a system of simultaneous equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \tag{30}$$

written in a matrix form

$$\mathbf{Ax} = \mathbf{b} \tag{31}$$

Cramer's rule states that the solution for x_k, the k^{th} element of the vector $\mathbf{x}$, is

$$x_k = \frac{\Delta_k}{\Delta}, \tag{32}$$

where Δ is the determinant of $\mathbf{A}$ and Δ_k is the determinant Δ with the k^{th} column replaced by the column vector $\mathbf{b}$.

Complex Numbers

It is important to note that in Electrical Engineering i is commonly used to denote the current. In order to avoid confusion, we denote the $i$maginary unit as j so that,

$$j = \sqrt{-1} \tag{33}$$

An imaginary number is defined as the product of the imaginary unit j with a real number jb. A complex number is defined the sum of a real number and an imaginary number, so that

$$z = x + jy \tag{34}$$

where x and y are real numbers.

The complex number can be represented on a rectangular coordinate place called the complex plane, which has a real and an imaginary axis. An alternative way to express a complex number c is in an exponential form as

$$z = re^{j\theta} \tag{35}$$

where

$$r = \sqrt{(x^2 + y^2)} \tag{36}$$

and

$$\theta = \tan^{-1}\frac{y}{x} \tag{37}$$

when $x > 0$. When $x < 0$, $\theta = 180° - \tan^{-1}\frac{y}{-x}$.

The number r is called the magnitude of z, also denoted as $|z|$. The polar form of the complex number z is

$$z = |z|\angle\theta = r\angle\theta. \tag{38}$$

Also note that $x = r\cos\theta$ and $y = r\ sin\theta$

Mathematical Operations

1. Conjugate z^* of a complex number $z = x + jy$

$$z^* = x - jy = r\angle - \theta \tag{39}$$

2. Addition of $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{40}$$

3. Subtraction of $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y2) \tag{41}$$

4. Multiplication of $z_1 = x_1 + jy_1 = r_1\angle\theta_1$ and $z_2 = x_2 + jy_2 = r_2\angle\theta_2$

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + y_1 x_2) \tag{42}$$

   or in polar form

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2) \tag{43}$$

5. Division of $z_1 = x_1 + jy_1 = r_1\angle\theta_1$ and $z_2 = x_2 + jy_2 = r_2\angle\theta_2$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2) \tag{44}$$

6. Reciprocal of $z = x + jy = r\angle\theta$

$$\frac{1}{z} = \frac{1}{r}\angle(-\theta) \tag{45}$$

7. Square root of $z = x + jy = r\angle\theta$

$$\sqrt{z} = \sqrt{z}\angle(\theta/2) \tag{46}$$

As a rule of thumb, it is easier to add and subtract complex numbers in rectangular form and to multiply or divide them in polar form.

Sinusoids

We define a **periodic function** as a function that satisfies

$$f(t) = f(t + nT) \tag{47}$$

for all t and all integer n values.

A **sinusoid** (either sin or cos) is such a periodic function and is described by

$$v(t) = V_m \sin \omega t, \tag{48}$$

where V_m is the amplitude of the sinusoid, ω (omega and not w!!!) is the angular frequency, and ωt is the argument of the sinusoid.

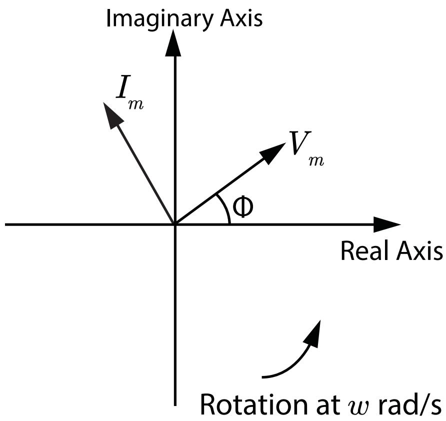

Figure 1: Phasor diagram with a voltage V_m and a current I_m phasor. Note that the current phasor is *leading* the voltage phasor in this example.

Phasors & Phasor Diagrams

A phasor is a **complex number** that represents the amplitude and phase of a sinusoid. This means that complicated calculations between trigonometrical functions can be made significantly simpler as calculations between complex numbers (refer to the previous section with complex numbers).

The representation of one or multiple phasors as complex numbers in a graphic manner, is called the phasor diagram, as shown in the following Figure.

Mean and RMS Values

Mean Value

Mean or average value $\bar{F}$ of a waveform $f(t)$ is the average value of the waveform taken over a specific time interval. This is calculated as

$$\bar{F} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t)dt \tag{49}$$

In periodic waveforms, the average is taken over the period T as

$$\bar{F} = \frac{1}{T} \int_{t}^{t+T} f(t)dt \tag{50}$$

RMS Value

The **effective or root-mean-square** (RMS) value, which is sometimes denoted with a *tilde* above the value as $\tilde{F}$, of a waveform $y(t)$ is calculated as

$$\tilde{F} = \sqrt{\frac{1}{T} \int_{t}^{t+T} (f(t))^2 dt} \tag{51}$$

Note the importance of the RMS value of a waveform in electrical engineering, as the RMS value of a voltage or current is the value of the ac voltage or current that develops the same power over a resistor as a dc voltage or current.