Faculty of Engineering

School of Electrical Engineering and Telecommunications

ELEC 1111 – Topic 3
Circuit Theorems

Dr. Inmaculada (Inma) Tomeo-Reyes

Lecturer

School of Electrical Engineering and Telecommunications, UNSW

Topic 3 Content

This lecture covers:

- Linearity and Superposition
- Source Transformation
- Thevenin's and Norton's theorems
- Maximum Power Transfer

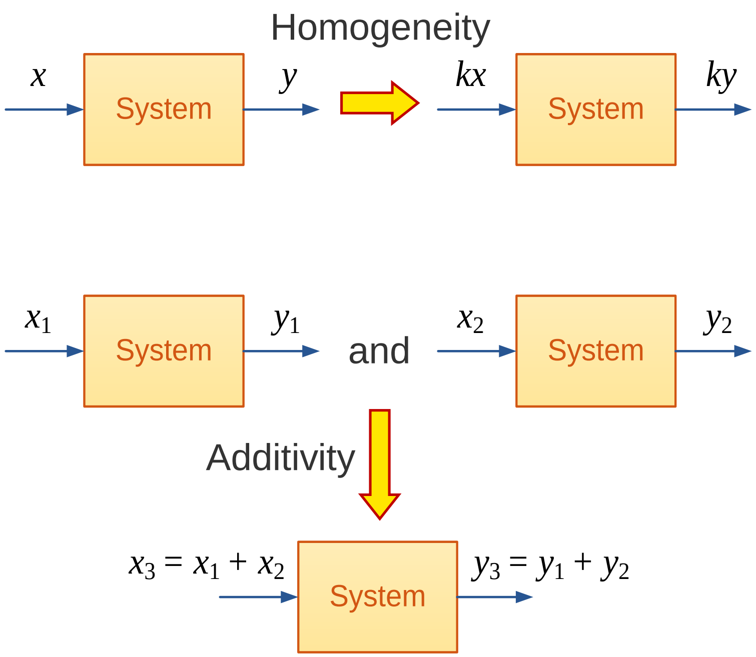
Corresponds to Chapter 4 of your textbook

Linear property

- A system is called **linear** if it follows two main properties:
  - **Homogeneity:**

    If the input (x) is multiplied by a constant (k), the output/response (y) is multiplied by the **same** constant.
  - **Additivity:**

    The response to the sum of the inputs is the sum of the individual responses to each input.

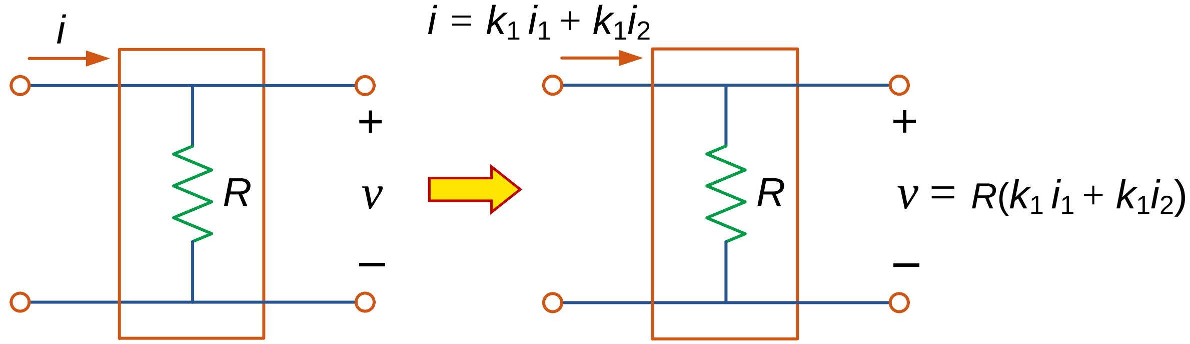


- Together, the response to the **linear combination** of inputs is equal to the **linear combination** of individual responses to each input.

k_1 and k_2 are constant

Linear property

- Resistor is a **linear electrical element** since the voltage-current relationship satisfies both the **homogeneity** and **additivity** properties.



$$v = R(k_1 i_1 + k_2 i_2)$$

$$v = k_1 \underbrace{R i_1}_{v_1} + k_2 \underbrace{R i_2}_{v_2}$$

$$v = k_1 v_1 + k_2 v_2$$

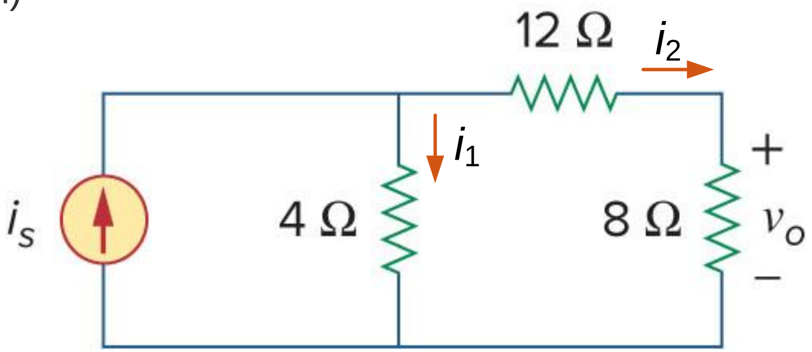
Linear property

A **linear circuit** is one whose **output** is **linearly related** (or directly proportional) to its **input**.

Example: Find the voltage v_o when $i_s = 30\ \text{A}$ and $i_s = 60\ \text{A}$.

$$i_2 = \frac{4 \parallel 20}{20} i_s = \frac{20/6}{20} i_s = \frac{1}{6} i_s$$ (current division)

$$v_o = 8i_2 = 8 \times \frac{1}{6} i_s = \frac{4}{3} i_s$$ (Ohm's law)



If $i_s = \boxed{30\ \text{A}}$ ⇒ $v_o = \frac{4}{3} i_s = \boxed{40\ \text{V}}$

If $i_s = \boxed{60\ \text{A}}$ ⇒ $v_o = \frac{4}{3} i_s = \boxed{80\ \text{V}}$

If $i_s = \boxed{k_1 30 + k_1 60\ \text{A}}$ ⇒ $v_o = \frac{4}{3} i_s = \frac{4}{3}(k_1 30 + k_1 60) = \boxed{k_1 40 + k_1 80\text{V}}$

Superposition

If there are **two** or **more independent** sources in a circuit, there are different ways to solve for the circuit parameters:

- Nodal or mesh analyses.
- **Superposition**.
  - It is based on the **linear property** of circuits.
  - Only **one independent** source is considered at a time.
  - The **rest** of the independent sources are **set to zero** (turned off).
  - **Dependent** sources are **left intact** since they are controlled by circuit variables.

The **superposition** principle states that the voltage across (or current through) an element in a **linear circuit** is the **algebraic sum** of the voltages across (or currents through) that element due to each **independent source** acting **alone**.

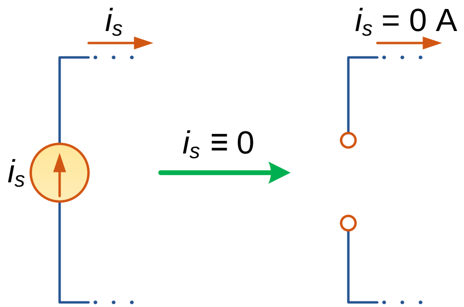
Superposition

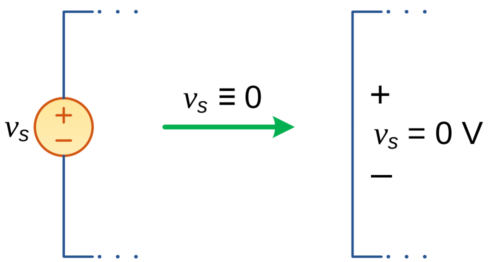
Steps to apply superposition principle:

1. Turn off all **independent** sources except **one** by setting them to **zero**.
2. Find the output (voltage or current in question) using methods covered in Topics 2 and 3.
3. Repeat step 1 for each of the other independent sources.
4. Find the total contribution by **adding** algebraically all the contributions due to the **independent** sources.

Turning off a **current source** means replacing it with an **open circuit**

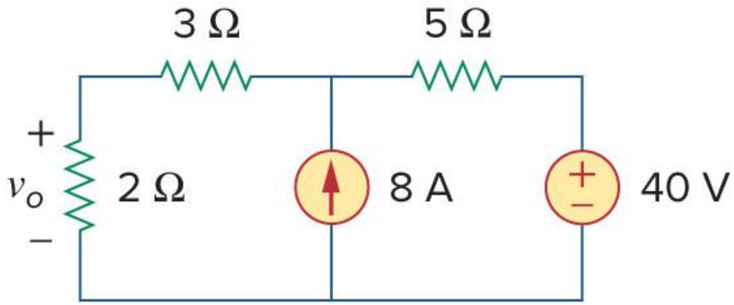
Turning off a **voltage source** means replacing it with a **short circuit**





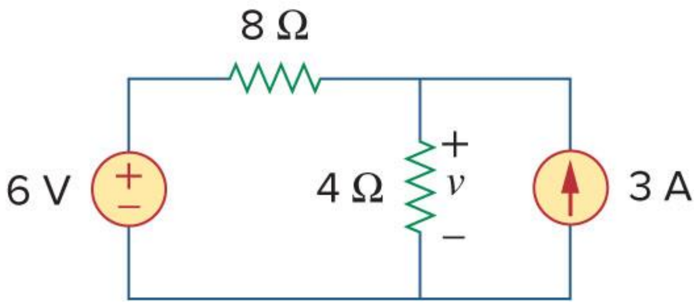
Exercise

Using the superposition theorem, find v_o in the circuit below.



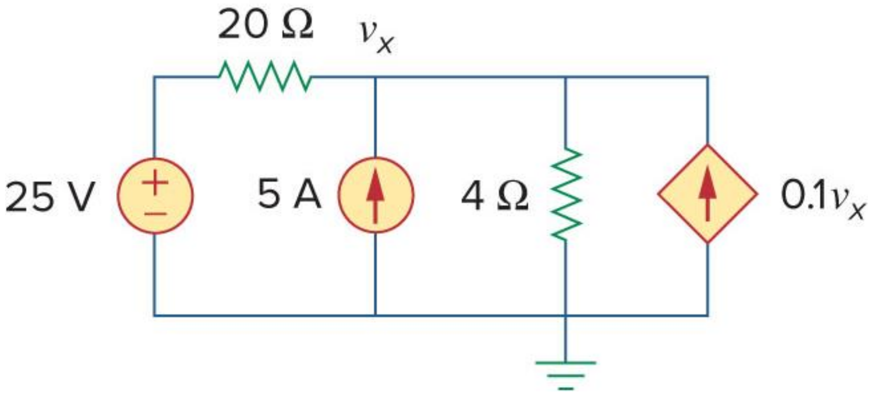
Exercise

Using the superposition theorem, find v in the circuit below.



Exercise

Using the superposition theorem, find v_x in the circuit below.

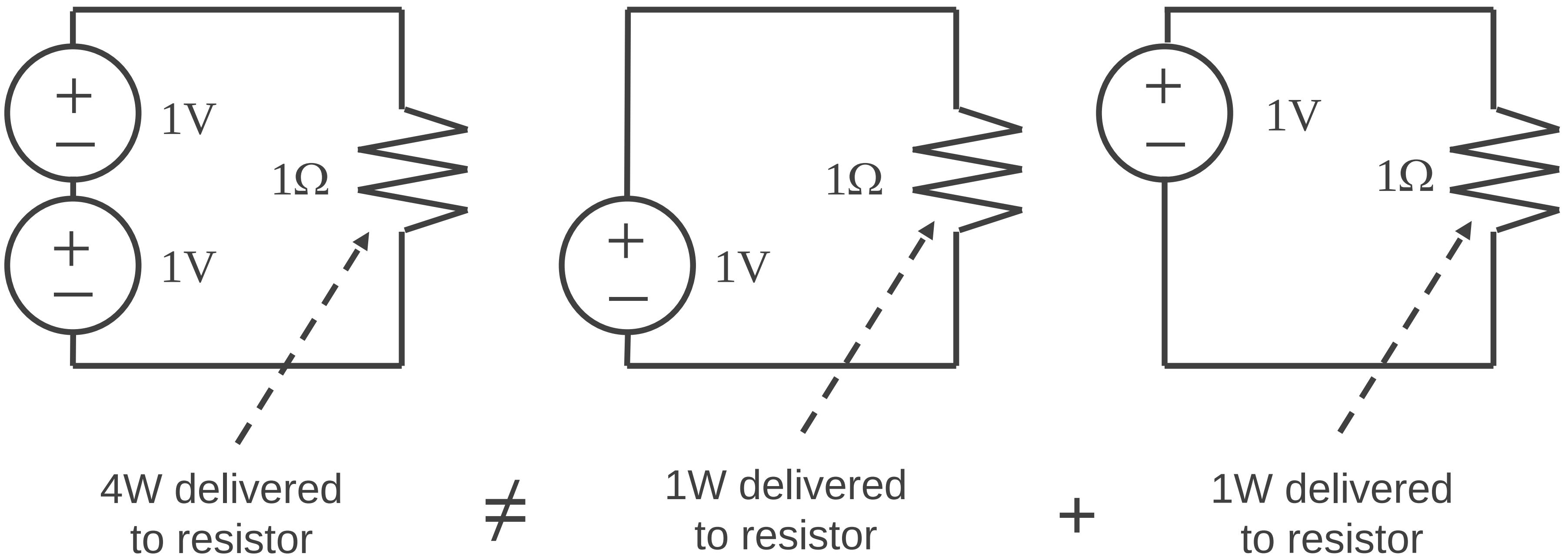


Superposition

Superposition is only applicable to **linear responses.**

E.g. power in resistors is a nonlinear response.

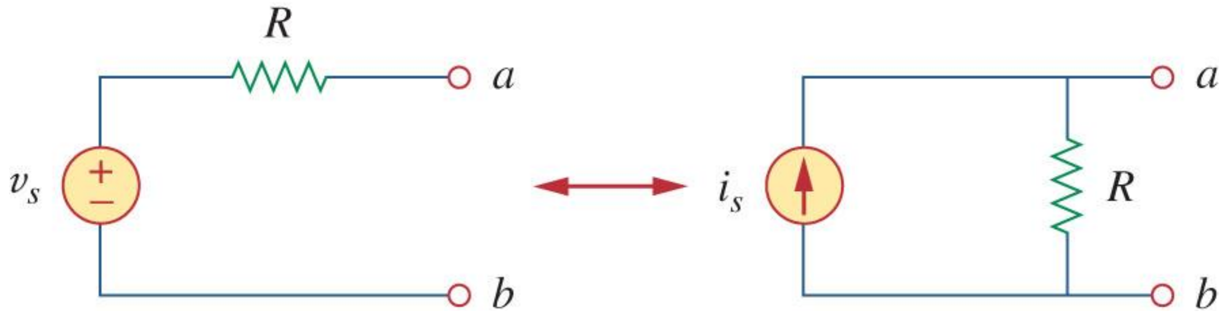
$$p = vi = i^2 R = \frac{v^2}{R}$$



Source transformation

- It is possible to transform a source from one form to another. This can be useful for **simplifying** circuits.
- The principle behind this transformation is the concept of **equivalence**, where two circuits are called equivalent if their $\boldsymbol{v}$**-**$\boldsymbol{i}$ **characteristics** are **identical** at the terminal.
- The **direction** of current and **polarity** of voltage should follow the **passive sign convention**.

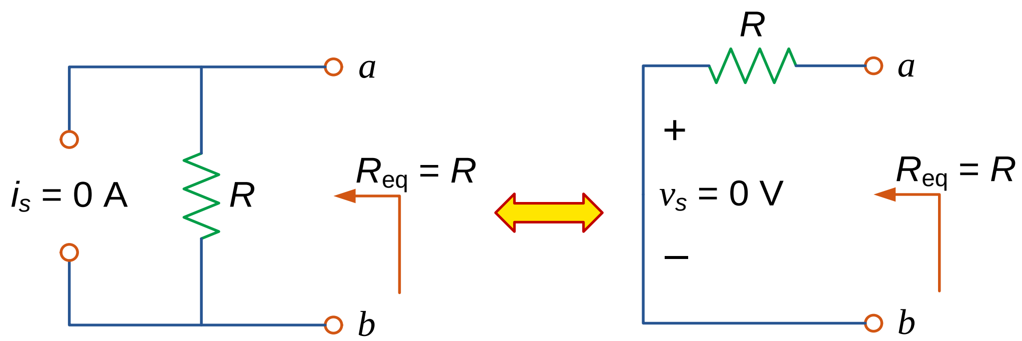
A **source transformation** is the process of **replacing** a voltage source $\boldsymbol{v_s}$ in **series** with a resistor $\boldsymbol{R}$ by a current source $\boldsymbol{i_s}$ in **parallel** with a resistor $\boldsymbol{R}$ or vice versa



Source transformation

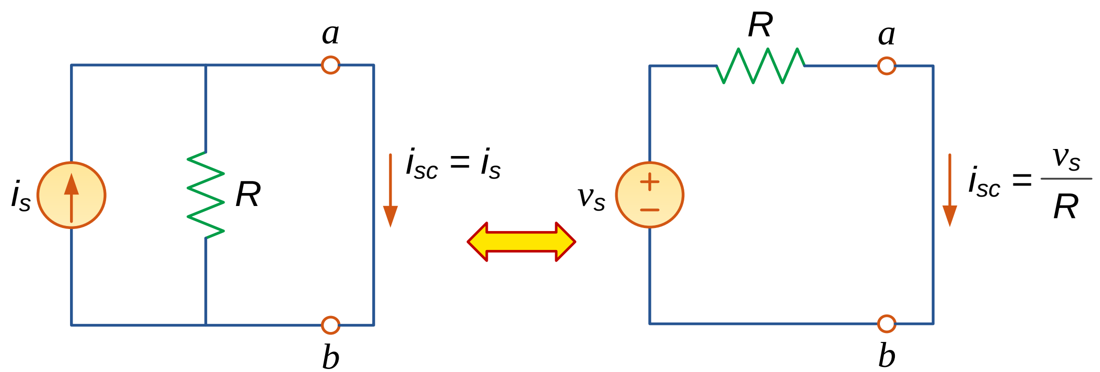
Terminal equivalency

- If we turn off the sources, the **equivalent resistance** (R_{eq}) at terminal *a*-*b* must be the **same**.



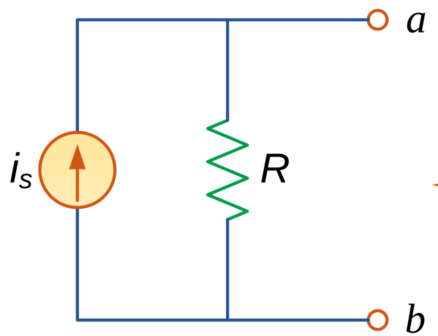
Resistors are the same

- **Short-circuit current** (i_{sc}) flowing from *a* to *b* must be the **same**.

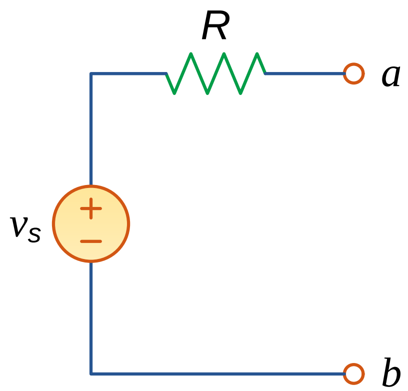


$$i_{sc} = i_s = \frac{v_s}{R}$$

Source transformation

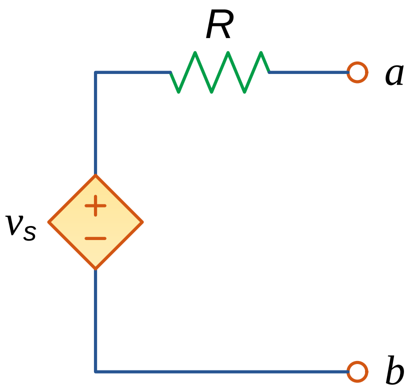


$$i_s = \frac{v_s}{R} \quad \text{or} \quad v_s = R i_s$$

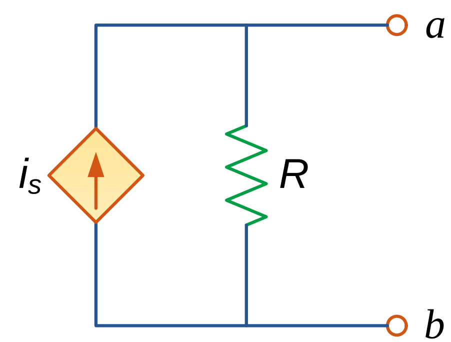


Source transformation

- Source transformation can also be applied to **dependent** sources, provided that the dependent **variable** is handled **carefully**.



$$i_s = \frac{v_s}{R} \quad \text{or} \quad v_s = Ri_s$$

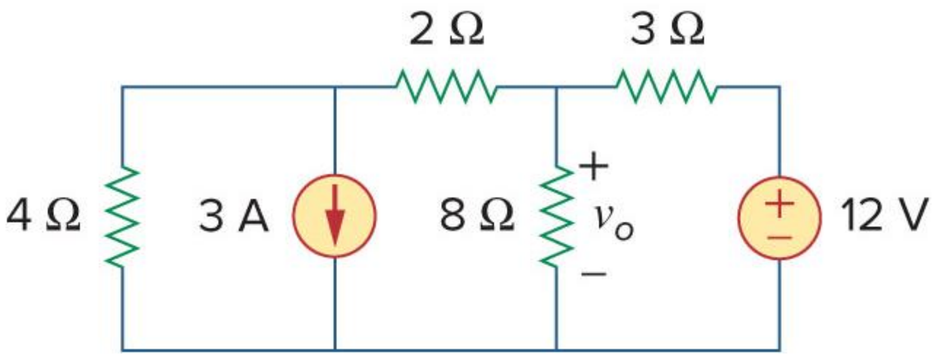


Some limitations:

- Source transformation is **not** possible when $\boldsymbol{R = 0}$ for an **ideal voltage source** (for a practical voltage source $\boldsymbol{R \neq 0}$)
- Source transformation is **not** possible when $\boldsymbol{R = \infty}$ for an **ideal current source** (for a practical current source $\boldsymbol{R \neq \infty}$)

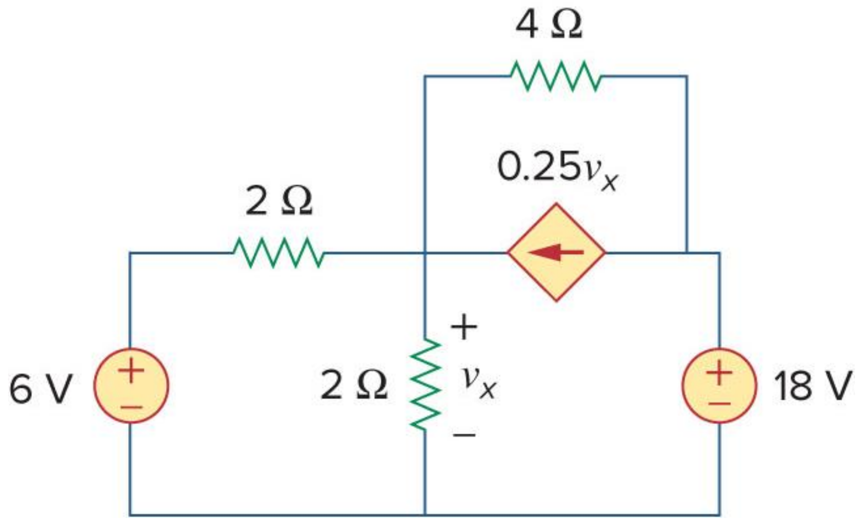
Exercise

Use source transformation to find v_0 in the circuit below.



Exercise

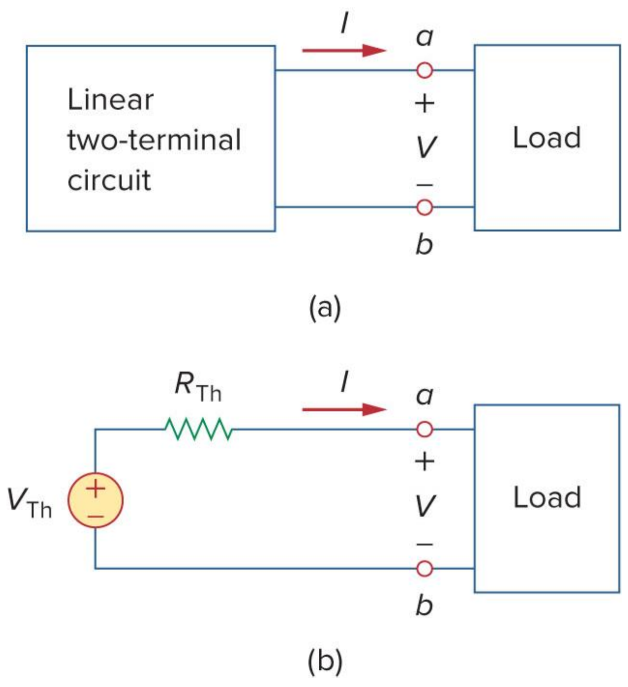
Use source transformation to find v_x in the circuit below.



UNSW SYDNEY

Thevenin's theorem

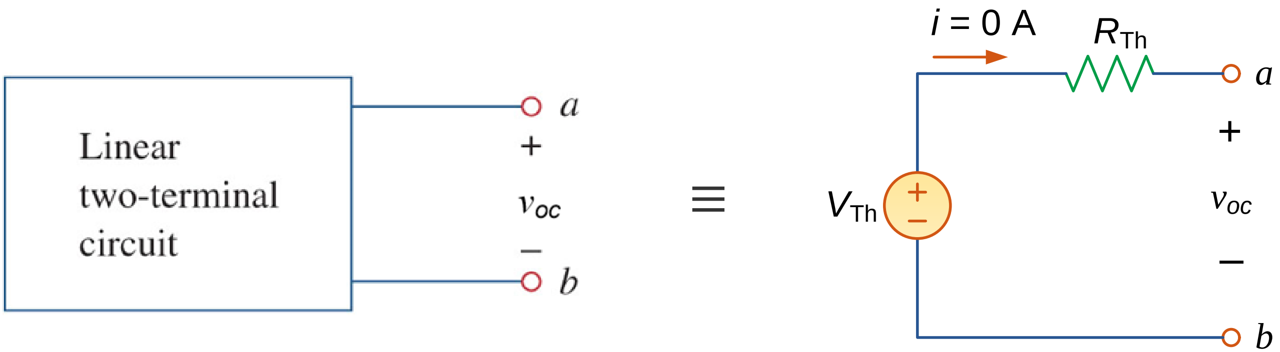
- In many circuits, one element can be variable (e.g. house hold appliances consuming different powers − hairdryer, fridge, etc.). This variable element is called the **load**.
- Every time the load changes, the circuit would have to be analyzed again.
- Thevenin's theorem provides a technique to simplify the analysis by **replacing the fixed part** of the circuit with an equivalent one known as **Thevenin equivalent circuit**.



Thevenin's theorem states that a **linear** two-terminal circuit (Fig. (a)) can be replaced by an **equivalent circuit** consisting of a **voltage source** $\boldsymbol{V_{\mathrm{Th}}}$ in **series** with a **resistor** $\boldsymbol{R_{\mathrm{Th}}}$ (Fig. (b))

Thevenin's theorem

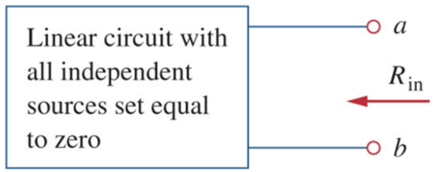
- The voltage source's value, known as **Thevenin voltage** $\boldsymbol{V_{\mathrm{Th}}}$, is equal to the **open-circuit voltage** at the terminals.

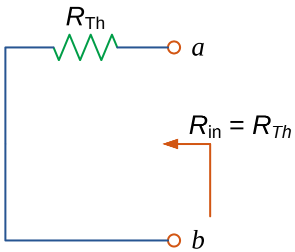


$$\boxed{V_{\mathrm{Th}} = v_{oc}}$$

Thevenin's theorem

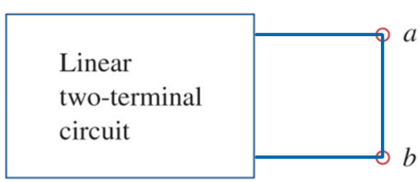
- R_{Th} can be obtained using different methods:
  1. Input **resistance** measured at the terminal pair when **all independent sources** are **turned off** (not valid if there is any dependent source).

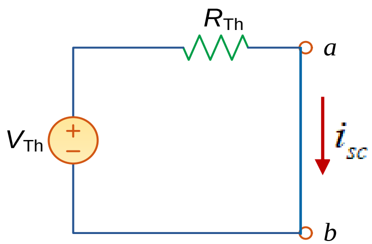




$$\boxed{R_{\text{Th}} = R_{\text{eq}} = R_{\text{in}}}$$

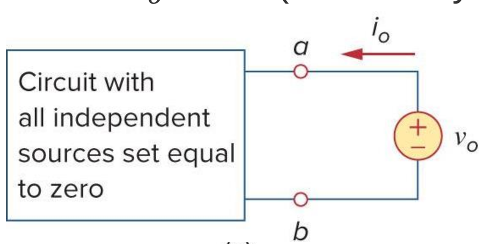
  2. Ratio of the open-circuit voltage to the short-circuit current at the terminal pair (not valid if there are <u>only</u> dependent sources).



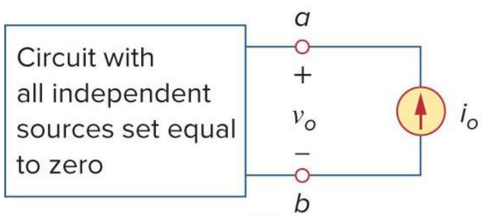


$$\boxed{R_{\text{Th}} = \frac{V_{Th}}{i_{sc}} = \frac{V_{Th}}{i_N}}$$

  3. Turn off independent sources and (a) attach a voltage source $\boldsymbol{v_o}$ to the terminals $\boldsymbol{a}$-$\boldsymbol{b}$ and find the resulting current $\boldsymbol{i_o}$, or (b) attach a current source $\boldsymbol{i_o}$ and find the resulting voltage $\boldsymbol{v_o}$. For simplicity: $v_o = 1\ \text{V}$ or $i_o = 1\ \text{A}$ (valid always).



(a)

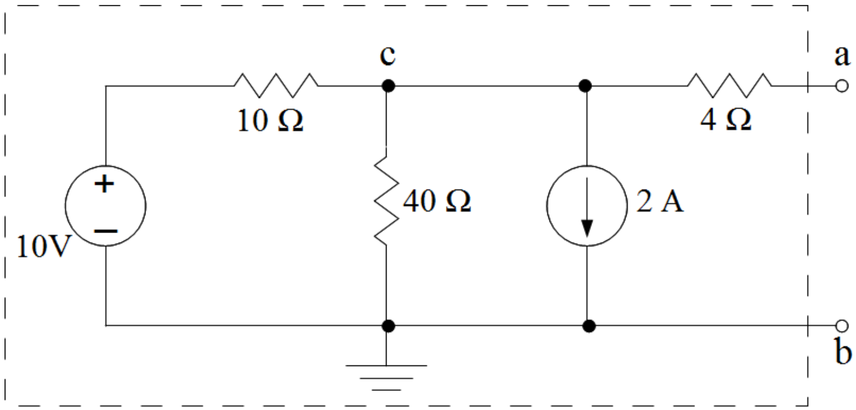


(b)

$$\boxed{R_{\text{Th}} = \frac{v_o}{i_o}}$$

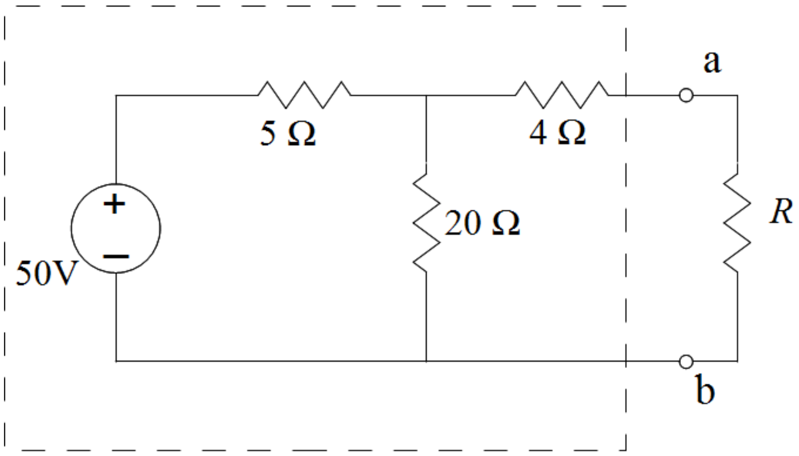
Exercise

Calculate the Thevenin equivalent of the circuit, as seen from terminals a-b.



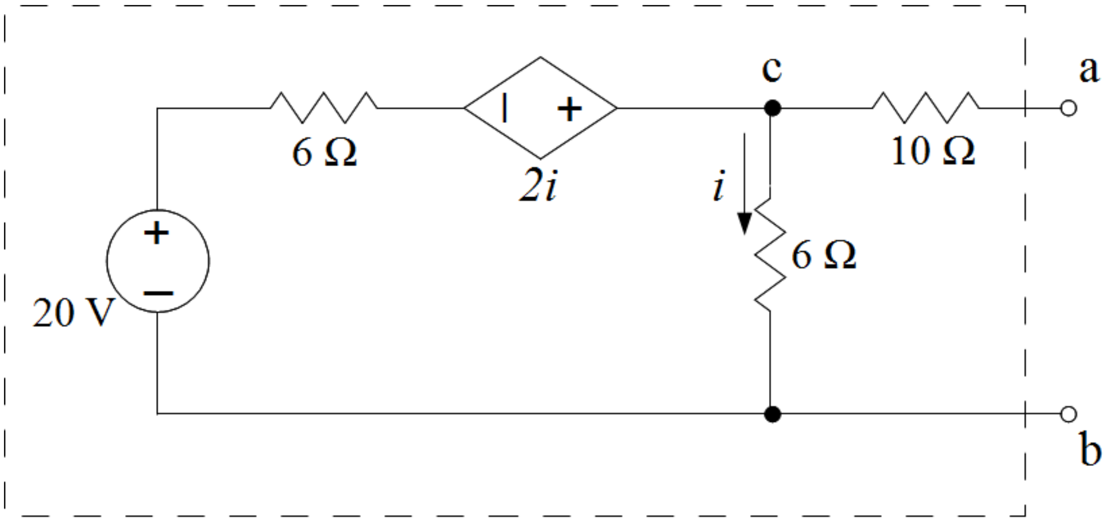
Exercise

Replace the circuit in the box with its Thevenin equivalent circuit.



Exercise

Calculate the Thevenin equivalent of the circuit, as seen from terminals a-b.



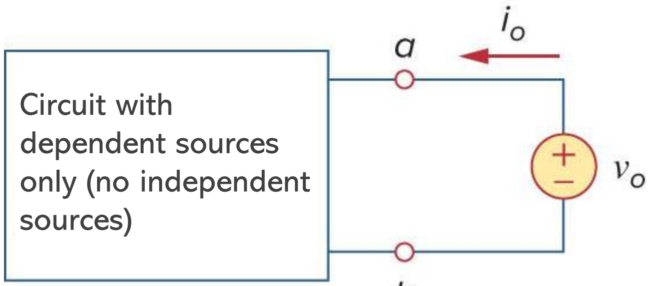
Thevenin's theorem

When we have a circuit with **dependent sources but not independent ones**, we have a “dead network”, i.e. all voltages and currents are equal to zero.

- Thevenin voltage V_{Th}:

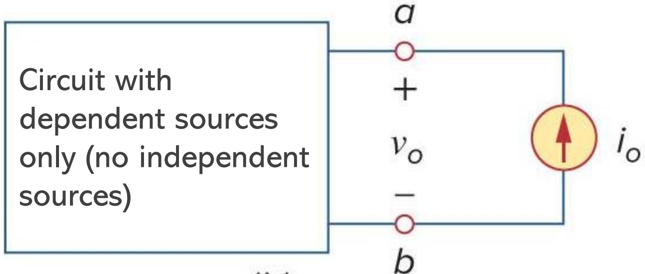
$$V_{\text{Th}} = v_{oc} = 0$$

- Thevenin resistance R_{Th}:
  - All currents and voltages in the circuit are equal to zero, including i_{sc}.
  - The only possible method to calculate R_{Th} in this case is method 3, i.e. attach a **voltage source $\boldsymbol{v_o}$** to the terminals $\boldsymbol{a}$-$\boldsymbol{b}$ and find the resulting **current $\boldsymbol{i_o}$** (Fig. (a)), or attach a **current source $\boldsymbol{i_o}$** and find the resulting **voltage $\boldsymbol{v_o}$** (Fig. (b)).
  - Note: In this case it is possible for R_{Th} to be negative ($R_{\text{Th}} < 0$). This implies that the circuit is supplying power.



(a)

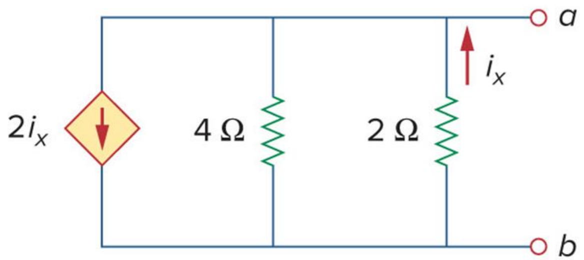
$$\boxed{R_{\text{Th}} = \frac{v_o}{i_o}}$$



(b)

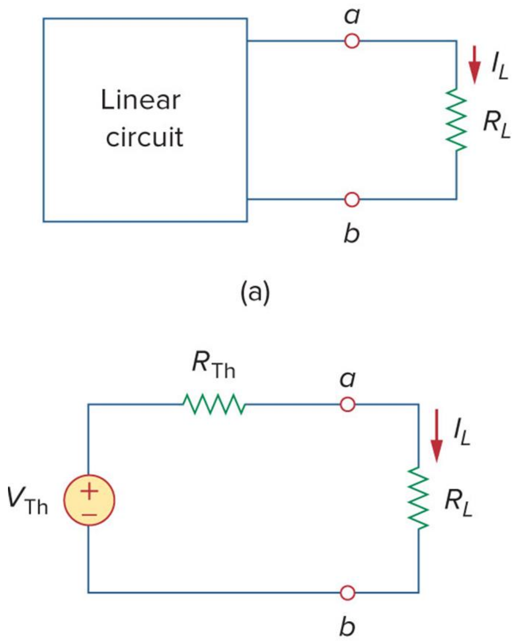
Exercise

Find the Thevenin equivalent of the circuit below at the terminals a-b.



Thevenin's theorem

- Thevenin's theorem is a powerful technique in circuit analysis with variable loads.
- It allows to **simplify** a large linear circuit.
- The **equivalent circuit** behaves externally **exactly the same** as the original circuit.
- The current through the **load** $\boldsymbol{R_L}$ (**load current** $\boldsymbol{I_L}$) and the voltage across the load (**load voltage** $\boldsymbol{V_L}$) is obtained using simple voltage division or KVL/KCL.



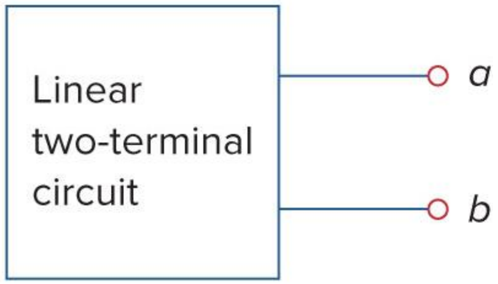
$$I_{\mathrm{L}} = \frac{V_{\mathrm{Th}}}{R_{\mathrm{Th}} + R_L}$$

$$V_{\mathrm{L}} = \frac{R_L}{R_{\mathrm{Th}} + R_L} V_{\mathrm{Th}}$$

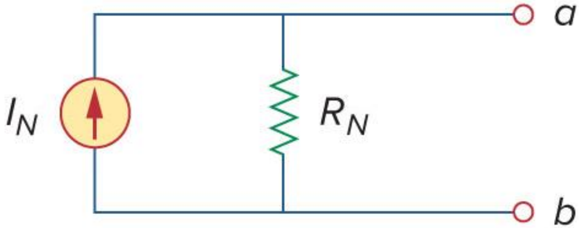
Norton’s theorem

- Norton’s theorem is the **dual** form of Thevenin’s theorem.
- It provides a similar technique to **simplify** the analysis by **replacing a linear circuit** with an equivalent one known as **Norton’s equivalent circuit**.

Norton’s theorem states that a **linear** two-terminal circuit (Fig. (a)) can be replaced by an **equivalent circuit** consisting of a **current source** $\boldsymbol{I_N}$ in **parallel** with a **resistor** $\boldsymbol{R_N}$ (Fig. (b)).



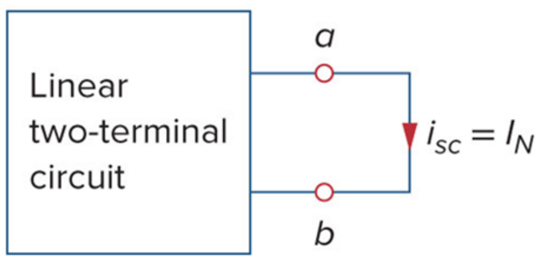
(a)

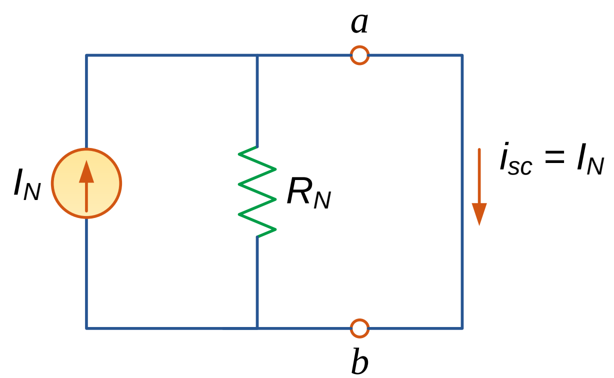


(b)

Norton’s theorem

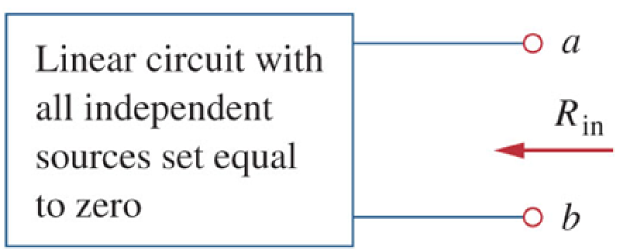
- The current source’s value, known as Norton current $\boldsymbol{I_N}$, is equal to the short-circuit current at the terminals.

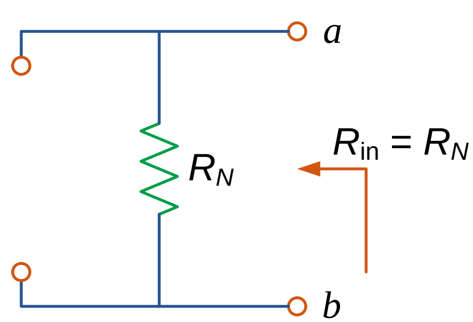




$$\boxed{I_N = i_{sc}}$$

- The **Norton resistance $\boldsymbol{R_N}$** is **same** as the **Thevenin resistance $\boldsymbol{R_{\mathrm{Th}}}$,** which is the **input resistance** measured at the terminals when **all independent sources** are **turned off**.

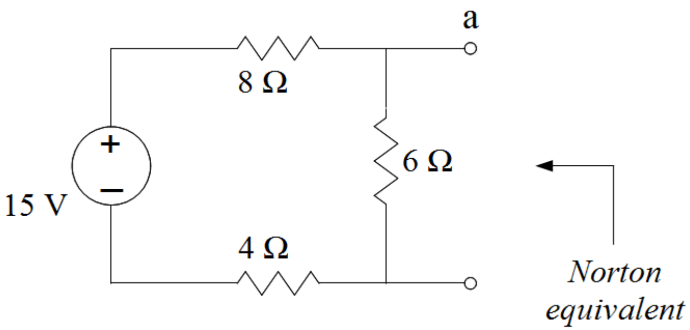




$$\boxed{R_{\mathrm{N}} = R_{\mathrm{eq}} = R_{\mathrm{in}}}$$

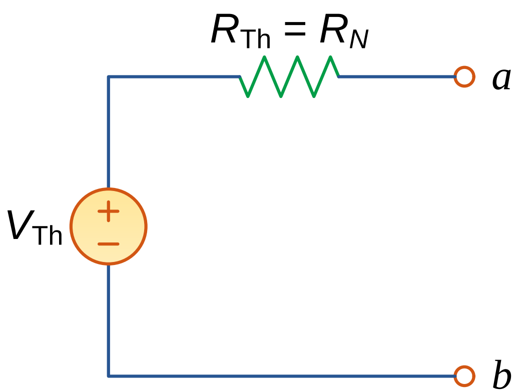
Exercise

Calculate the Norton equivalent of the following circuit:

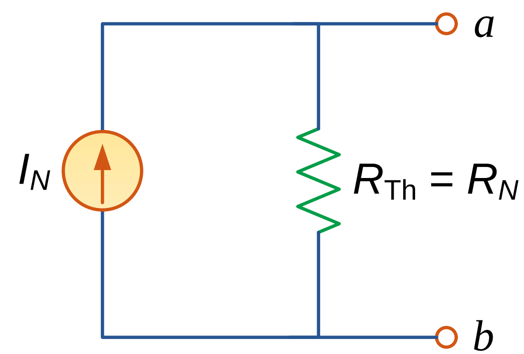


Thevenin-Norton transformation

- Thevenin's and Norton's theorems are **related** to each other through **source transformation**.
- Thevenin and Norton resistances are **exactly the same** and equal to **input resistance** measured from the terminals of the circuit.



$$\boxed{R_{\mathrm{Th}} = R_N}$$

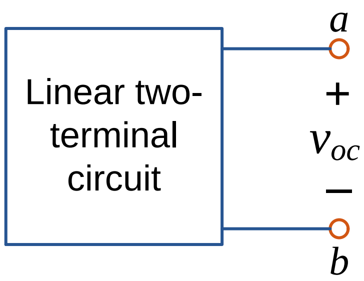


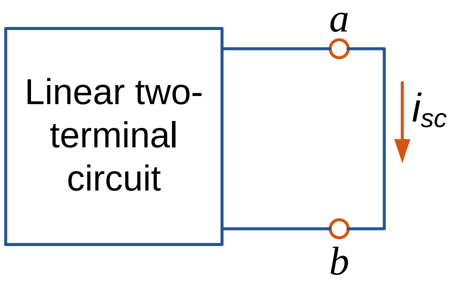
$$\boxed{V_{\mathrm{Th}} = R_N I_N} \longleftrightarrow \boxed{I_N = \frac{V_{\mathrm{Th}}}{R_{\mathrm{Th}}}}$$

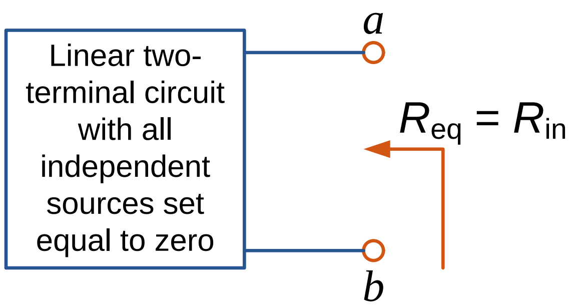
Thevenin-Norton transformation

- **Dependent** sources in Norton equivalent circuit are handled in **the same way** as Thevenin equivalent circuit.
- Since V_{Th} , I_N and $R_{\mathrm{Th}} = R_N$ are related based on source transformation, finding the Thevenin or Norton equivalent circuit requires **two** of the following:

  1. The **open-circuit voltage** v_{oc} across terminals a and b.
  2. The **short-circuit current** i_{sc} at terminals a and b.
  3. The **equivalent or input resistance** $R_{\mathrm{eq}} = R_{\mathrm{in}}$ at terminals a and b when all **independent** sources are **turned off**.



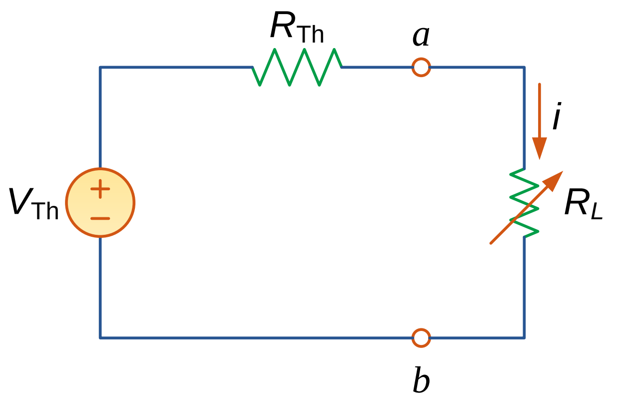




$$V_{\mathrm{Th}} = v_{oc}$$
$$I_N = i_{sc}$$
$$R_{\mathrm{Th}} = \frac{v_{oc}}{i_{sc}} = R_N$$

Maximum power transfer

- In many practical applications, a circuit is designed to provide **maximum power** to a **load**.
- Unlike ideal sources, **internal resistance** of real sources **restricts** the amount of power that can be transferred to a load.
- The power consumption of that internal resistance is known as **power loss**.
- Using the **Thevenin equivalent circuit** we can find the **maximum power** that can be transferred to a load based on **fixed** V_{Th} and R_{Th} and **variable** R_L.



$$p(R_L) = R_L i^2 = R_L \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2$$

Maximum power transfer

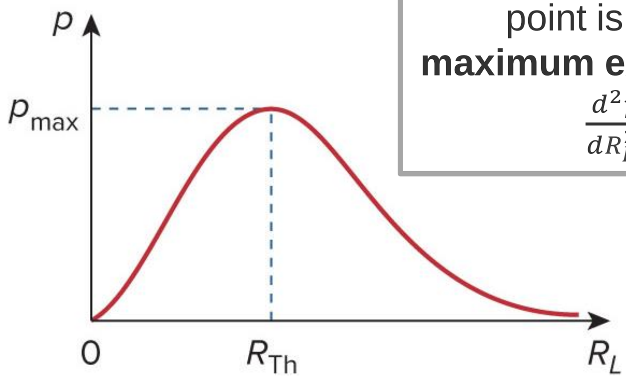
- To find for which value of R_L the power would be maximum, differentiate p with respect to R_L and set the result to zero.

$$\frac{dp}{dR_L} = {V_{Th}}^2 \left[\frac{(R_{\mathrm{Th}} + R_L)^2 - 2R_L(R_{\mathrm{Th}} + R_L)}{(R_{\mathrm{Th}} + R_L)^4}\right]$$

$$= {V_{Th}}^2 \left[\frac{(R_{\mathrm{Th}} + R_L - 2R_L)}{(R_{\mathrm{Th}} + R_L)^3}\right] = 0 \quad \Rightarrow \quad (R_{\mathrm{Th}} + R_L - 2R_L) = 0$$

- **Maximum power** is transferred to the load when the **load resistance R_L** is **equal** to the **Thevenin resistance R_{Th}** as seen from the load terminals

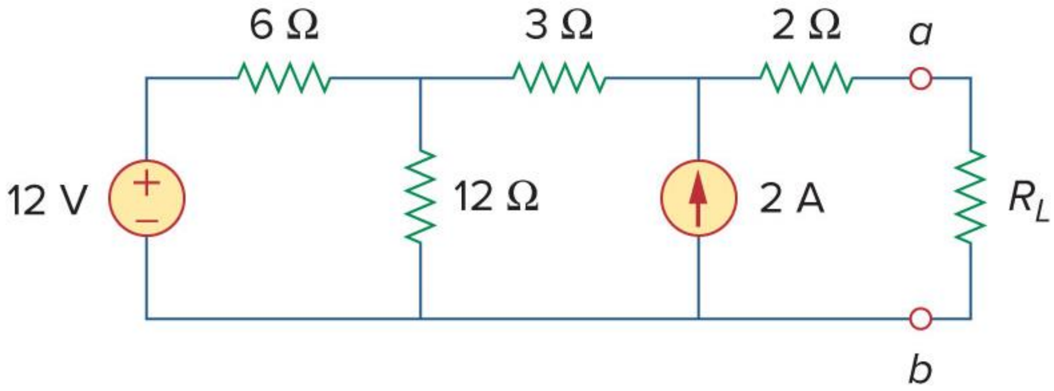
$$\boxed{R_L = R_{\mathrm{Th}}} \Rightarrow \boxed{p_{\max} = \frac{{V_{\mathrm{Th}}}^2}{4R_{\mathrm{Th}}}} \quad \boxed{v_L = \frac{V_{\mathrm{Th}}}{2}}$$



It can be shown that this point is the global **maximum extremum** since $\frac{d^2p}{dR_L^2} < 0$

Exercise

Find the value of load resistance R_L for maximum power transfer in the circuit below, and then find the maximum power p.



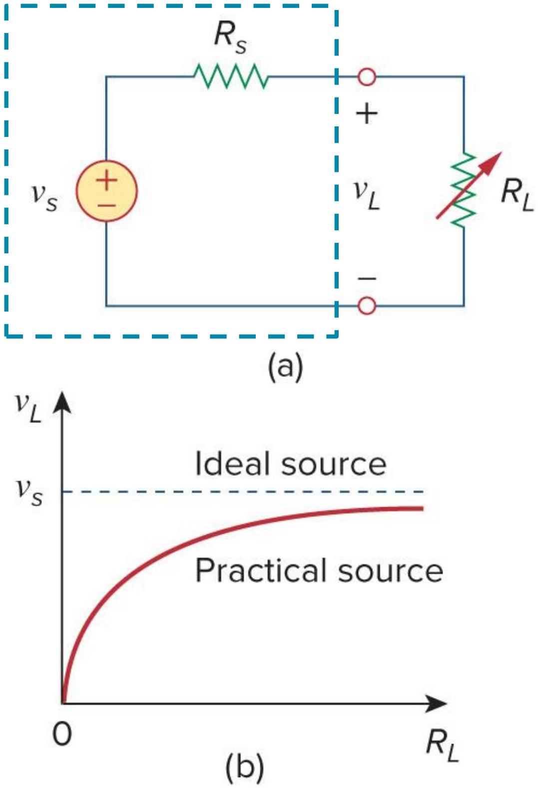
Source Modeling – Voltage source

Thevenin and Norton equivalent circuits can be useful in modeling **realistic** voltage and current sources that deliver maximum power to a load.

- With a load connected to the voltage source, the terminal voltage drops in magnitude, which is known as **loading effect**.
- The internal resistance R_s of the voltage source v_s in series with the load R_L acts as **voltage divider**.

$$\boxed{v_L = \frac{R_L}{R_s + R_L} v_s}$$

- The load voltage will be constant if R_s is **zero** or **very small** compared to the load $R_s \ll R_L$.
- Without load, $R_L \to \infty$, $v_{oc} = v_s$ which is known as **unloaded voltage source**.



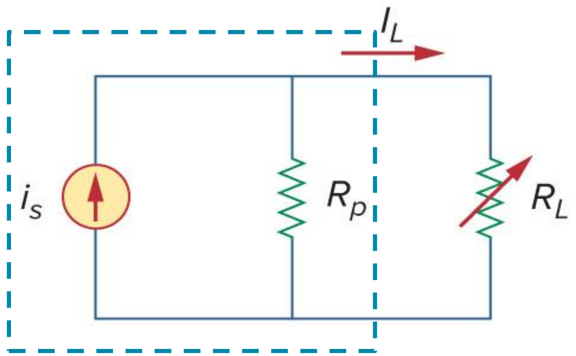
Note that $v_{oc} = v_s = V_{\mathrm{Th}}$, Thevenin voltage, they are all equivalent.

Source Modeling – Current source

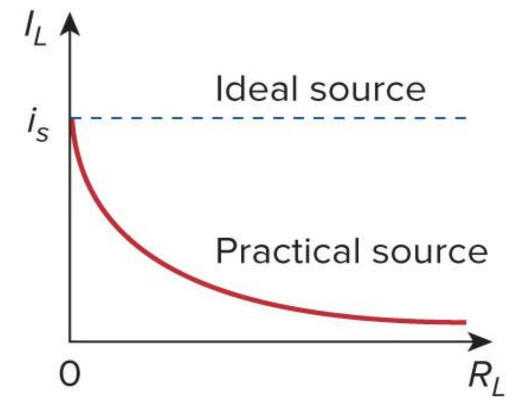
- Similarly, with a load connected to the current source, the terminal current drops in magnitude, which is known as **loading effect**.
- The internal resistor R_p in parallel with the load R_L acts as **current divider**.

$$\boxed{i_L = \frac{R_p}{R_p + R_L} i_s}$$

- The load current will be constant if R_p is **infinitive** or **very large** compared to the load $R_p \gg R_L$.
- Without load, $R_L = 0$, $i_{sc} = i_s$ which is known as **unloaded current source**.



(a)



(b)

Note that $i_{sc} = i_s = I_N$, Norton current, they are all equivalent.

Source Modeling

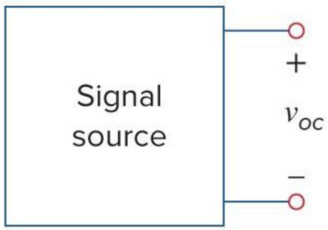
- The following steps can be followed to determine the unloaded source voltage v_s and internal resistance R_s of **voltage source** model in **practice**:
  1. Measure the open-circuit voltage (Fig. (a)).

$$\boxed{v_s = v_{oc}}$$

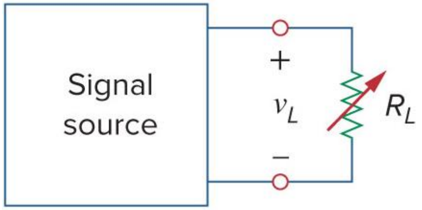
  2. Connect a variable load R_L (Fig. (b)) and adjust its resistance until $v_L = \frac{v_s}{2}$.
  3. Disconnect R_L and measure its resistance as it should be **equal** to internal resistance R_s based on maximum power transfer principle.

$$\boxed{v_L = \frac{v_s}{2} \quad \Rightarrow \quad R_s = R_L = R_{\mathrm{Th}}}$$

- Use source transformation to identify i_s and R_p for current source model.



(a)

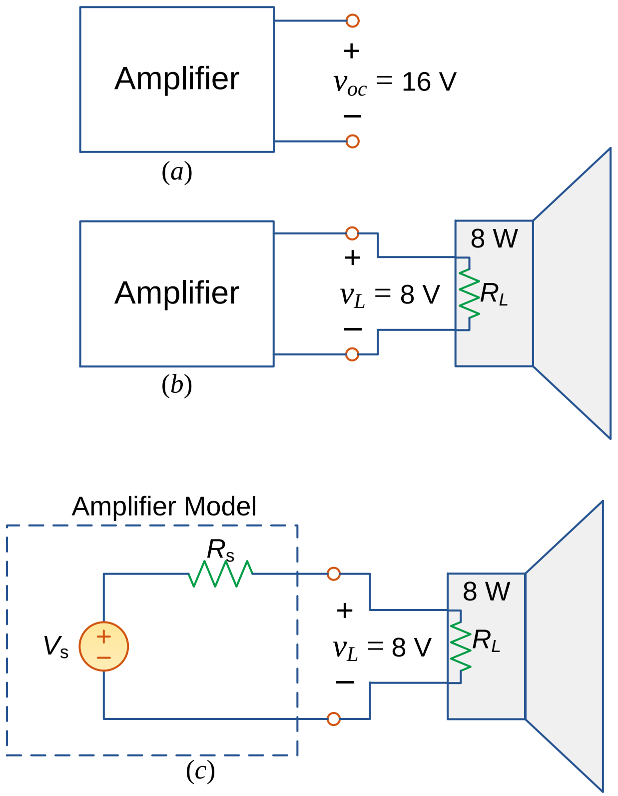


(b)

Note: In the laboratory, you will be doing a similar experiment but with constant load R_L to identify v_s and R_s and not necessarily following maximum power transfer principle.

Exercise

The open-circuit voltage across a certain amplifier is 16 V. The voltage drops to 8 V when an 8-W speaker is connected to the amplifier. Determine the internal resistance of the amplifier and calculate the load voltage when a 24-Ω speaker is used instead.



Questions?