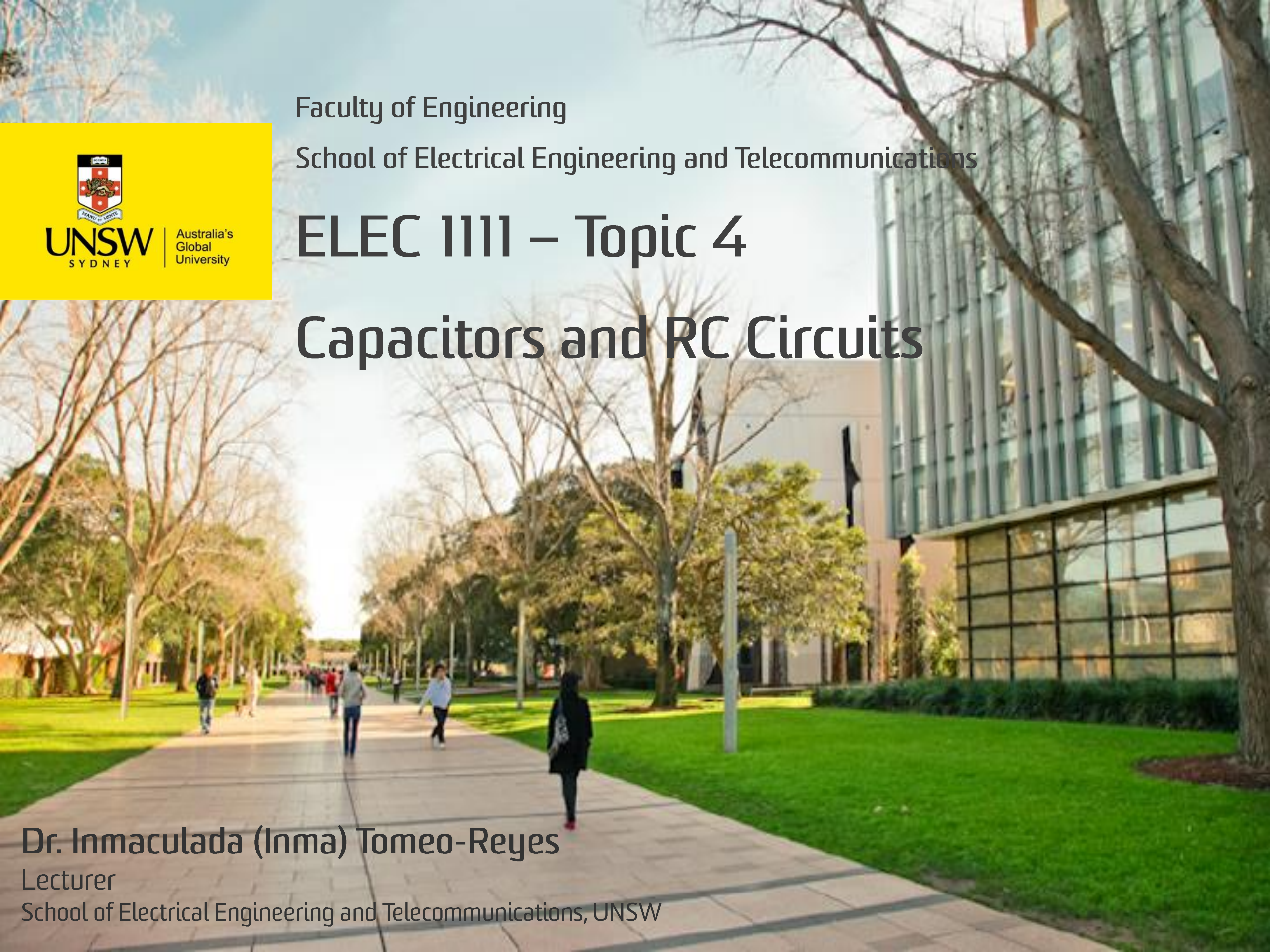

Faculty of Engineering

School of Electrical Engineering and Telecommunications

ELEC 1111 – Topic 4

Capacitors and RC Circuits

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Topic 4 Content

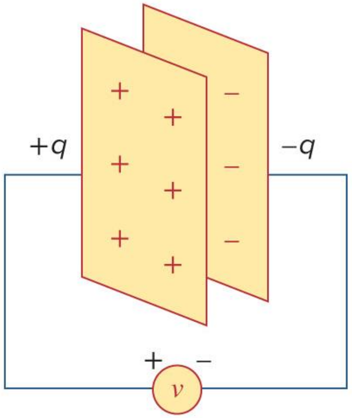
This lecture covers:

- Capacitors
- Circuit analysis with capacitors
- First order circuits with resistors and capacitors (RC circuits)
  - Natural response
  - Step response

Corresponds to parts of Chapters 6 and 7 of your textbook

Capacitors

- A capacitor is a circuit element that **stores energy** in its electric field.
  - It consists of two conducting plates separated by an insulator (or dielectric).
    - The plates are typically aluminum foil.
    - The dielectric is often air, ceramic, paper, plastic, or mica.
- When a voltage source v is connected to the capacitor, the source deposits a **positive charge** $+q$ on one plate and a **negative charge** $-q$ on the other plate.
  - The charges will be equal in magnitude on both plates.
  - The amount of charge is proportional to the voltage.



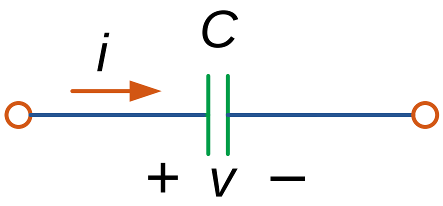
Capacitors

- The ratio of voltage to charge across a capacitor is its *capacitance*.

$$\boxed{q = Cv}$$

- The symbol for capacitance is C.
- Capacitance is measured in farads, F, which are coulombs per volt.

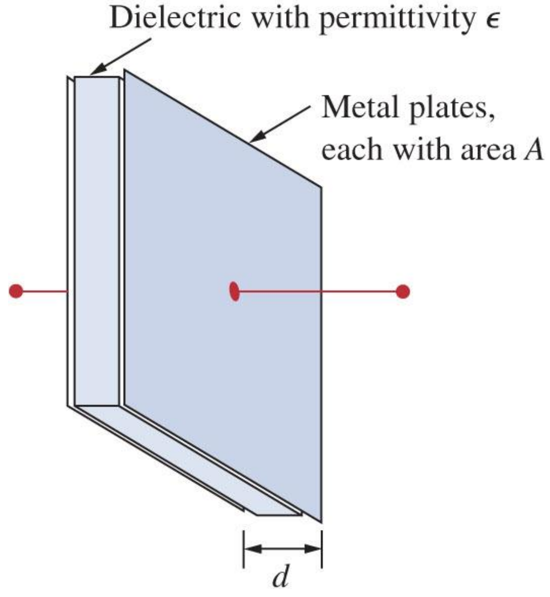
$$C = \frac{q}{v} = \frac{1\ \mathrm{C}}{1\ \mathrm{V}} = 1\ \mathrm{F}$$



Capacitors

- Capacitance depends of the **physical dimensions** and **geometry** of the capacitor.
- For **parallel-plate** capacitors, the capacitance is given as follows:
  - A is the **surface** area of each plate.
  - d is the **distance** between the plates.
  - ε is the **permittivity** of the dielectric.

$$\boxed{C = \frac{\varepsilon A}{d}}$$



Capacitors

- Capacitors are available in different values and types.
- They are described by their **dielectric material.**
  - E.g. thin film capacitors with polyester (Fig. 1(a)), ceramic (Fig. 1(b)), or electrolytic (Fig. 1(c)).
- Most capacitors are rated in picofarad (pF) to microfarad (μF).
- They are used to block DC and pass AC signals, shift phase, store energy, suppress and filter noise, etc.

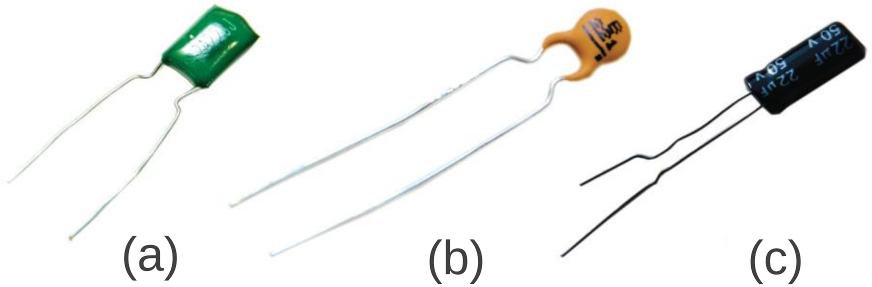
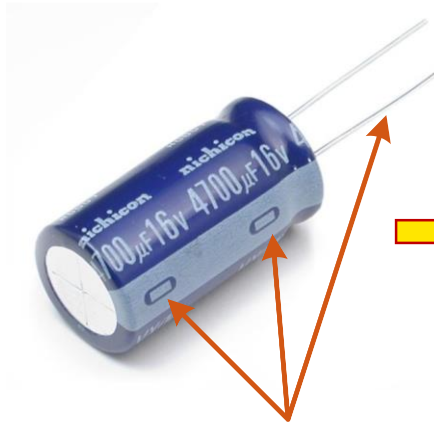


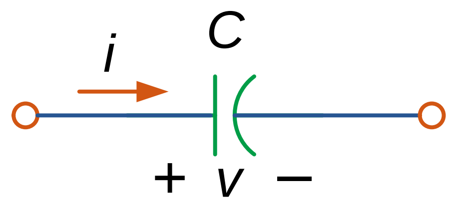
Fig. 1

Capacitors

- **Passive sign convention** applies to capacitors as well.
  - If $\boldsymbol{v} \times \boldsymbol{i} > \mathbf{0}$, the capacitor is being **charged** (absorbing energy).
  - If $\boldsymbol{v} \times \boldsymbol{i} < \mathbf{0}$, the capacitor is **discharging** (supplying energy).
- For capacitors in the range of ***μ*F**, particularly **electrolytic** ones, the **polarity** is **already assigned** (the **negative** sign is marked on the capacitor).



Negative lead



Circuit symbol

Capacitors

- Charge = Capacitance x Voltage

$$q = Cv$$

- Differentiation with respect to time gives:

$$\frac{dq}{dt} = C\frac{dv}{dt}$$

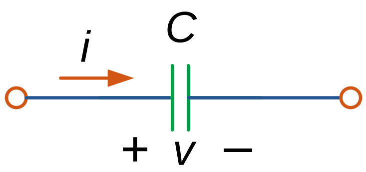
- Current is rate of change of charge, so

$$\boxed{i = C\frac{dv}{dt}}$$

- Voltage will then be:

$$\boxed{v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)}$$

where $v(t_0) = \frac{q(t_0)}{C}$ is called the **initial voltage** or **initial conditions** at time t_0.



Capacitor's current $\boldsymbol{i}$ is proportional to the **rate of change** of its **voltage $\boldsymbol{v}$**, with **capacitance $\boldsymbol{C}$** as the constant of proportionality, assuming **passive sign convention.**

Capacitors

- **Instantaneous power** delivered to the capacitor:

$$\boxed{p = vi = Cv\frac{dv}{dt}}$$

- The **energy** stored in the **electric field** that exists between the plates of the capacitor can be then calculated as:

$$w(t) = \int_{t_0}^{t} p(\tau)d\tau = \int_{t_0}^{t} Cv(\tau)\frac{dv(\tau)}{d\tau}d\tau = C\int_{t_0}^{t} v(\tau)dv(\tau) = C\frac{v(\tau)^2}{2}\bigg|_{t_0}^{t}$$

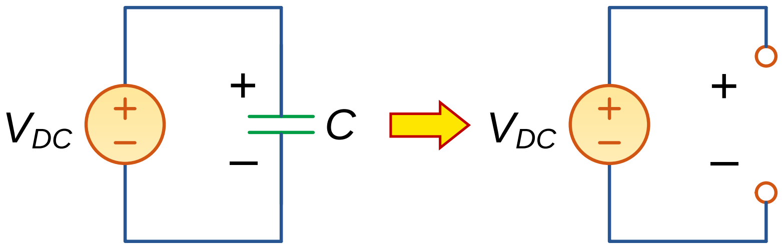
$$w(t) = \frac{1}{2}C(v(t)^2 - v(t_0)^2)$$

If $v(t_0) = 0 \longrightarrow$ $\boxed{w(t) = \frac{1}{2}Cv(t)^2}$

Properties of capacitors

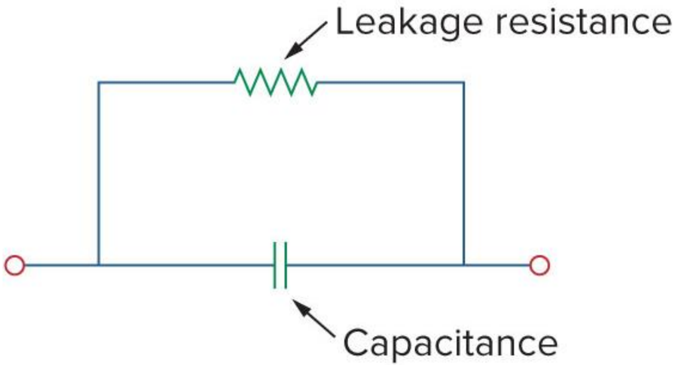
- A capacitor is an **open circuit** to **DC voltage**.

$i = C\frac{dv}{dt} = 0$ for constant voltage.



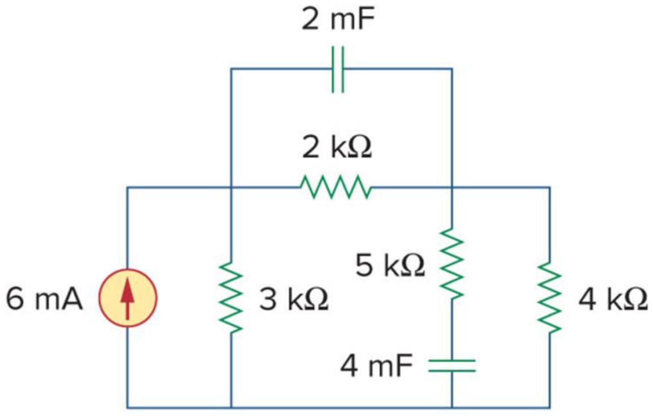
Properties of capacitors

- An **ideal capacitor** does not dissipate energy.
  - Energy is absorbed from a circuit, stored as electric field and then released back to the circuit.
- A **real capacitor** has a parallel-model **leakage resistance**, leading to a slow loss of the stored energy internally.



Exercise

Obtain the energy stored in each capacitor in the circuit below (keep in mind they are under DC conditions)



(a)

$w(t) = \frac{1}{2} C v(t)^2$. Also, $i = C \frac{dv}{dt}$ = 0 for DC voltage/current.

Capacitors in parallel

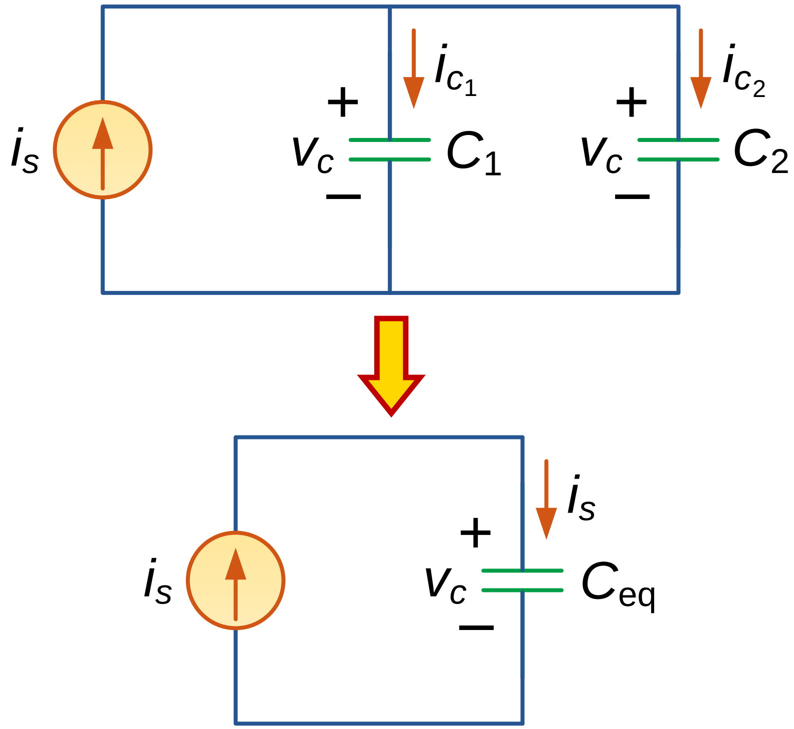
- Similar to resistors, capacitors in series or parallel can be combined to simplify the circuit.
- For capacitors in parallel, the voltage is the same across each capacitor.
- Applying KCL and current-voltage relation $i = C\frac{dv}{dt}$ for capacitors:

$$i_s = i_{c_1} + i_{c_2}$$

$$i_s = C_1\frac{dv_c}{dt} + C_2\frac{dv_c}{dt}$$

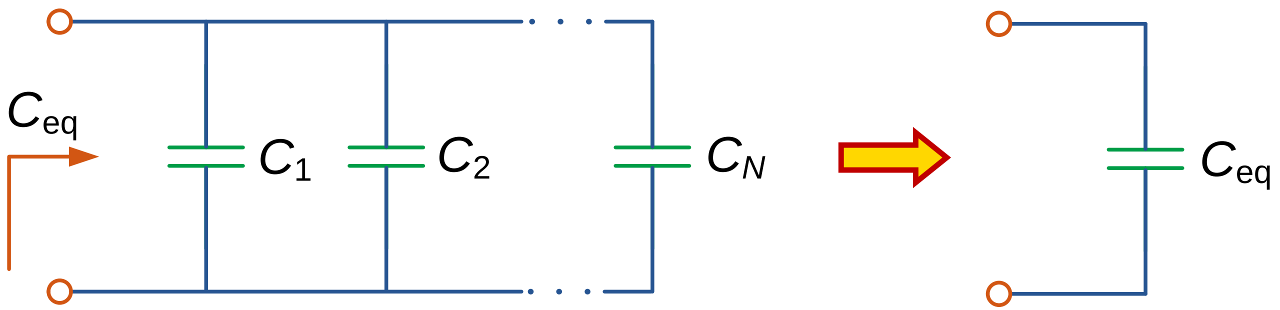
$$i_s = (C_1 + C_2)\frac{dv_c}{dt} \quad \leftrightarrow \quad i_s = C_{\mathrm{eq}}\frac{dv_c}{dt}$$

$$\boxed{C_{\mathrm{eq}} = C_1 + C_2}$$



Capacitors in parallel

- The **equivalent capacitance** of any number of **capacitors in parallel** is the **sum** of the **individual capacitances**.



$$\boxed{C_{eq} = C_1 + C_2 + \cdots + C_N = \sum_{k=1}^{N} C_k}$$

Capacitors in series

- For capacitors in series, the current is the same through each capacitor.
- Applying KVL and voltage-current relation for capacitors:

$$v_s = v_{c_1} + v_{c_2}$$

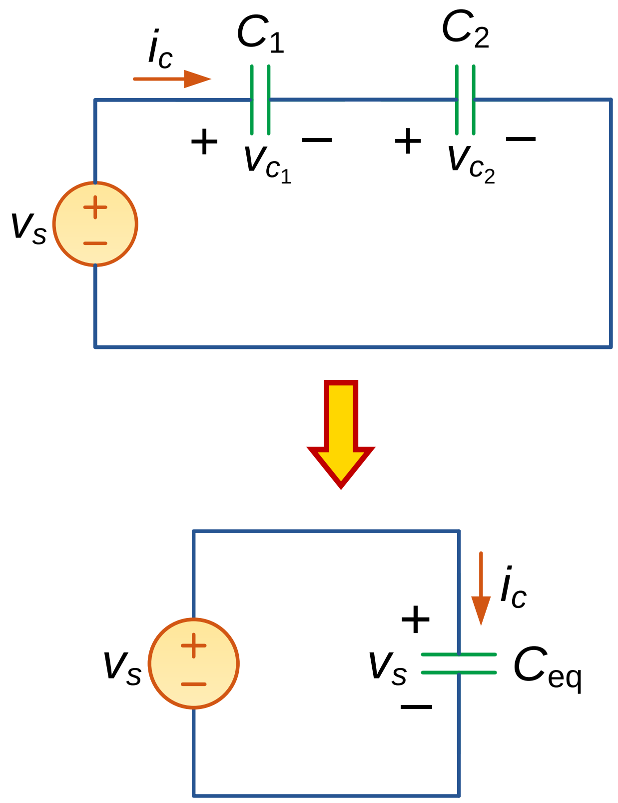
$$v_s = \frac{1}{C_1}\int_{t_0}^{t} i_c(\tau)d\tau + v_{c_1}(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i_c(\tau)d\tau + v_{c_2}(t_0)$$

$$v_s = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_{t_0}^{t} i_c(\tau)d\tau + \left(v_{c_1}(t_0) + v_{c_2}(t_0)\right)$$

$$\leftrightarrow \quad v_s = \frac{1}{C_{\text{eq}}}\int_{t_0}^{t} i_c(\tau)d\tau + v_{c_{\text{eq}}}(t_0)$$

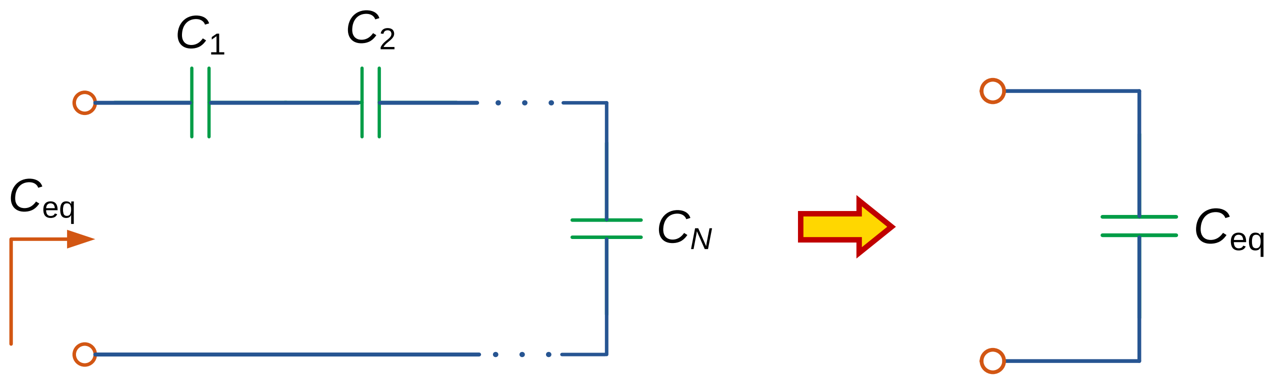
$$\boxed{\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}} \quad \text{or} \quad \boxed{C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}}$$

and $\boxed{v_{C_{\text{eq}}}(t_0) = v_{c_1}(t_0) + v_{c_2}(t_0)}$



Capacitors in series

- The **reciprocal** of the **equivalent capacitance** of any number of **capacitors in series** is the **sum** of the **individual reciprocal capacitances.**



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \cdots + \frac{1}{C_N}$$

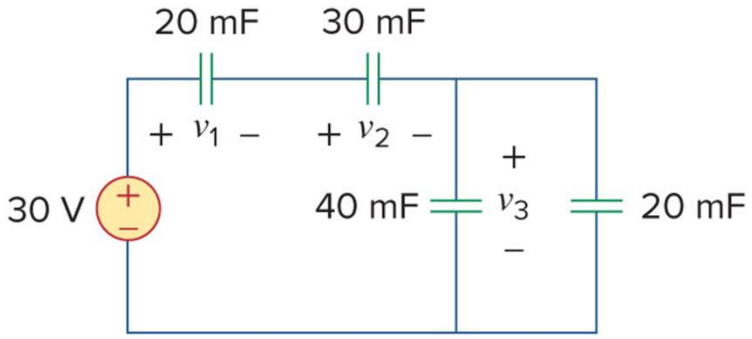
or

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} \cdots + \frac{1}{C_N}}$$

NOTE: Although the voltage across each capacitor is different for different values of capacitance, the **charge** across the plates is **equal** because charge is the integral of current, and current is the same through all of them.

Exercise

Find the voltage across and the charge on each capacitor in the given circuit.

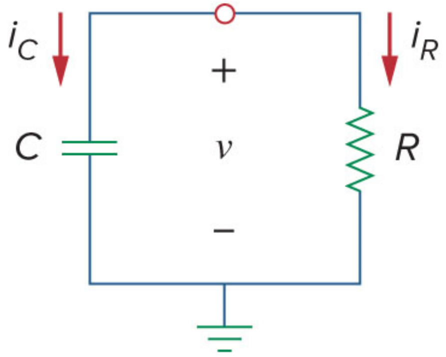


Circuit response

- The **response** of a circuit is the way it reacts to an excitation.
  - **Natural response →** Behaviour of the circuit (in terms of voltage or current) due to **initial energy stored** and **physical characteristics.**
  - **Forced response →** Behaviour of the circuit (in terms of voltage or current) due to **external sources** and **excitation**.

Natural response of RC circuit

Let's consider a circuit with a single capacitor, charged to an initial voltage V_o and connected to a resistor. How will it behave?



We can use KCL to get:

$$-i_c - i_R = 0 \quad \rightarrow \quad -i_c = i_R$$

$$-C\frac{dv}{dt} = \frac{v}{R}$$

Note: We will consider how this capacitor was charged later in this lecture.
Note: A circuit characterised by a first order differential equation is called a first order circuit.

Natural response of RC circuit

Solve differential equation:

Rearrange:

$$-C\frac{dv}{dt} = \frac{v}{R}$$

$$\frac{dv}{dt} = \frac{-v}{RC}$$

Separate:

$$\frac{1}{v}dv = \frac{-1}{RC}dt$$

Integrate:

$$\int \frac{1}{v}dv = \frac{-1}{RC}\int dt$$

$$\ln(v) = \frac{-t}{RC} + D$$

Solve for $v(t)$

$$v(t) = e^{\frac{-t}{RC}+D}$$

$$v(t) = e^{\frac{-t}{RC}}\, e^{D}$$

$$v(t) = Ae^{\frac{-t}{RC}}$$

Apply initial conditions

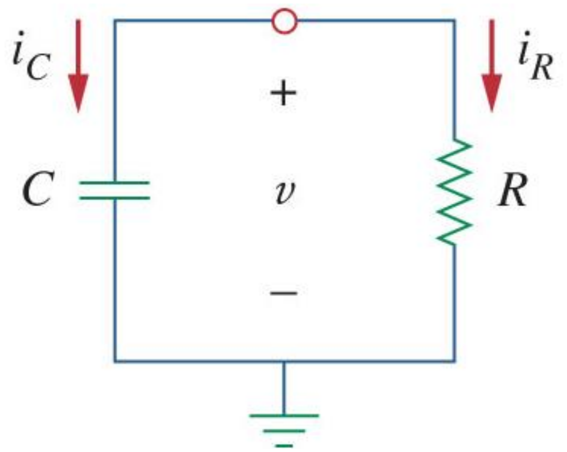
$v(0) = Ae^{0} = A = V_0$

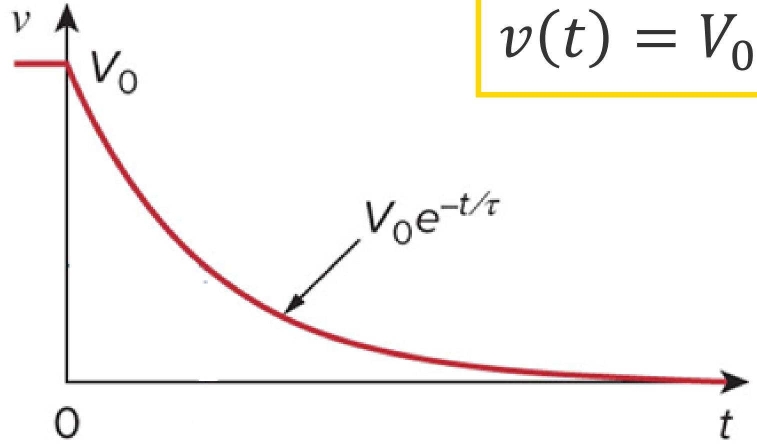
$$\boxed{v(t) = V_0 e^{\frac{-t}{RC}}}$$

Natural response of RC circuit

- There is no need to derive the differential equation solution every time, just use the result.
- The result shows that the **voltage response** of the RC circuit is an **exponential decay** of the **initial voltage**.

$$\boxed{v(t) = V_0 e^{-\frac{t}{RC}}}$$





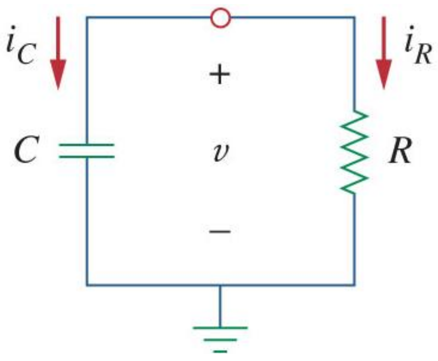
Time constant

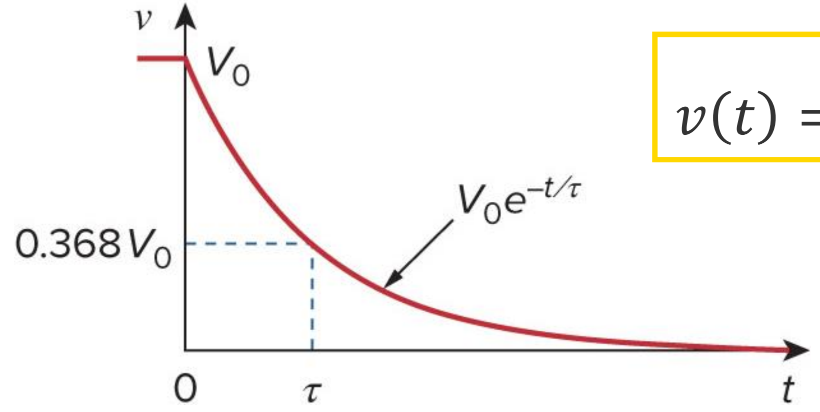
- The **speed** at which the voltage **decays** depends on the **coefficient** of t in the power of exponential function, which can be expressed in terms of the *time constant.*

> The **time constant** of a circuit is the time required for the response to **decay** to $\mathbf{1/e}$ (or **36.8%**) of its initial value or to **increase** to $\mathbf{1-1/e}$ (or **63.2%**) of its final value. It is denoted by $\boldsymbol{\tau}$.

- This implies that at $t=\tau$, the voltage should be $0.368V_0$:

$$V_0 e^{-\frac{\tau}{RC}} = V_0 e^{-1} = 0.368V_0 \ \rightarrow\ \tau = RC$$





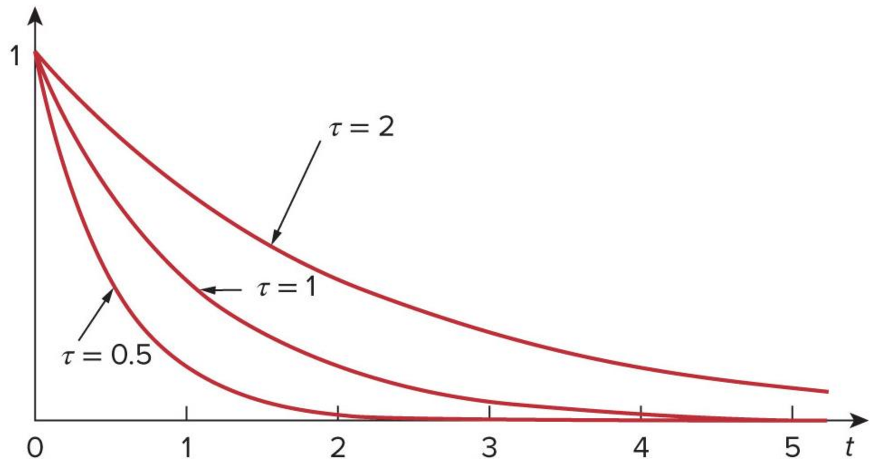
$$\boxed{v(t) = V_0 e^{-\frac{t}{RC}} = V_0 e^{-\frac{t}{\tau}}}$$

$$\boxed{\tau = RC}$$

Time constant

- For every time interval of τ, the voltage is reduced to 36.8 percent of its previous value.
  - A circuit with a **small time constant** has a **fast response** and **vice versa.**
- After **5 time constants**, the voltage $v(t)$ on the capacitor is less that **one** percent of its initial value V_0.

It takes $\boldsymbol{5\tau}$ for an RC circuit to reach its **final state** or **steady-state** (either fully charged or fully discharged).



t	$\frac{v(t)}{V_0} = e^{\frac{-t}{\tau}}$
τ	0.36788
2τ	0.13534
3τ	0.04974
4τ	0.01832
$\boldsymbol{5\tau}$	0.00674

Natural response of RC circuit

- Follow these steps to find the natural (or source-free response) of RC circuits:

1. Find the **initial voltage** $v(0) = V_0$ across the capacitor **before** it is connected to the resistor.
   - The capacitor is assumed to be **fully charged** at the **beginning** and can be replaced with an **open circuit.**

2. Find the **time constant** $\tau = RC$.
   - If the circuit has **more than one resistor**, the resistance that we need to find in order to calculate the time constant is the equivalent resistance as seen by the terminals of the capacitor, i.e. the **Thevenin equivalent resistance** $\boldsymbol{R = R_{\mathrm{Th}}}$.
   - When possible, this resistance can be obtained by simplification of series or parallel resistances.

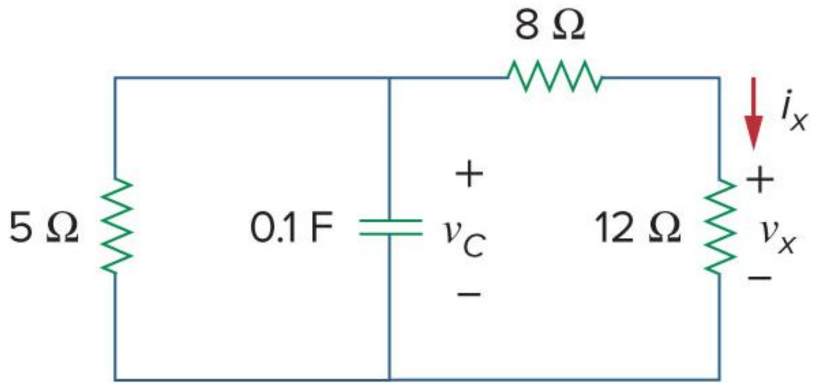
3. Calculate the voltage across the capacitor as $v(t) = V_0 e^{-\frac{t}{\tau}}$.

4. Find any other circuit variable using the capacitor's voltage.

Note: A **switch** which **opens** or **closes** can **remove** part of the circuit or **add** something to it.

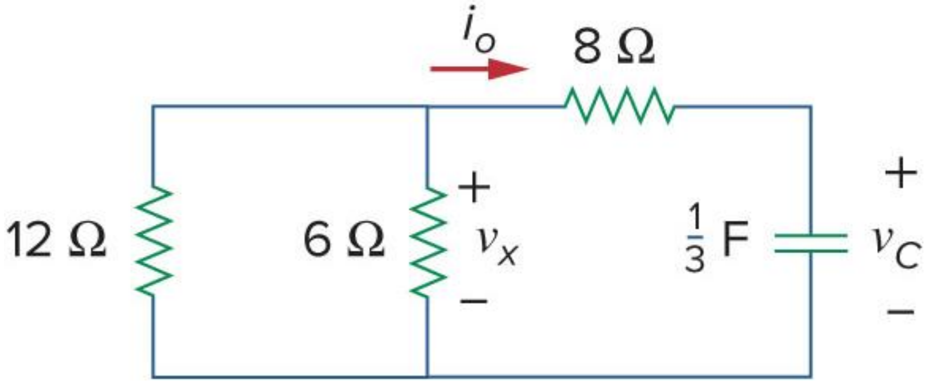
Exercise

Find the voltages v_c, v_x, and the current i_x for $t > 0$ if the initial voltage is $v_c(0) = 15$ V.



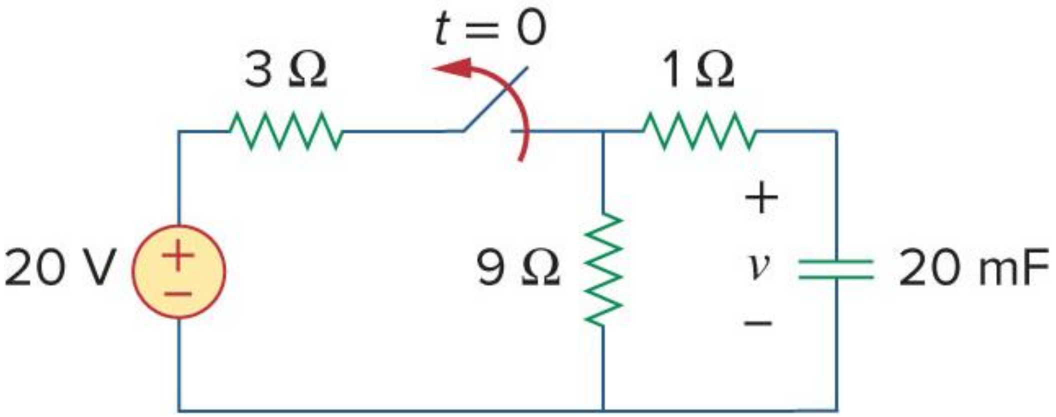
Exercise

Find the voltages v_c, v_x, and the current i_0 for $t > 0$ if the initial voltage is $v_c(0) = 60\text{ V}$.



Exercise

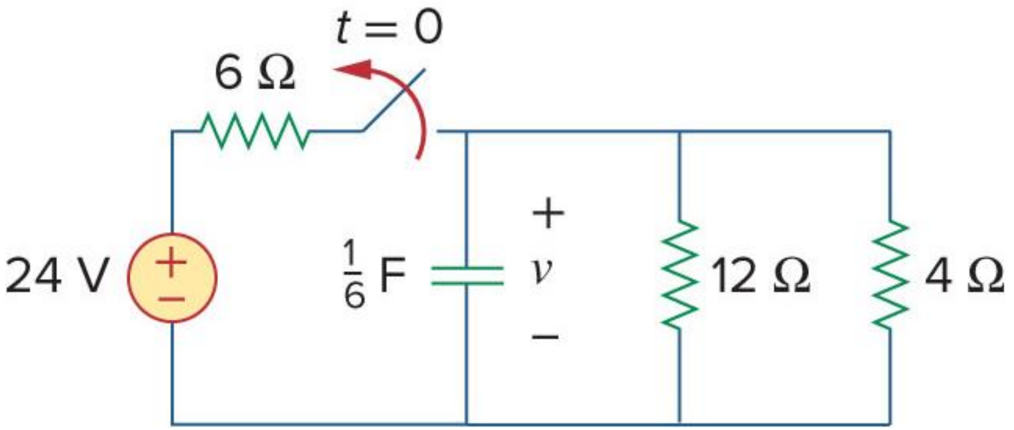
The switch in the circuit has been closed for a long time, and it is opened at $t = 0$, Find the voltages $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.



Exercise

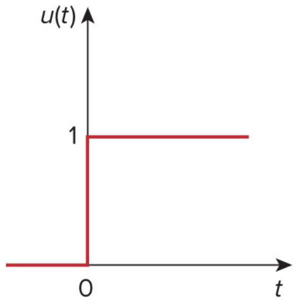
If the switch opens at $t = 0$ after a long time, find the voltages $v(t)$ for $t \geq 0$ and $w_c(0)$.

- For practice!
- Answer: $v(t) = 8e^{-2t}$ V , $w_c(0) = 5.333$ J



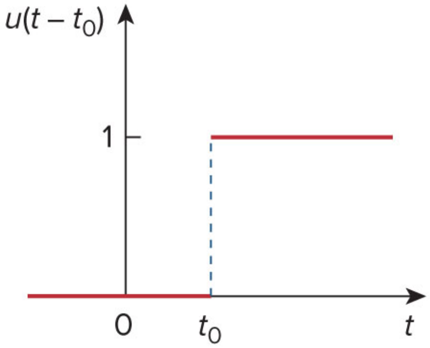
Step response of RC circuit

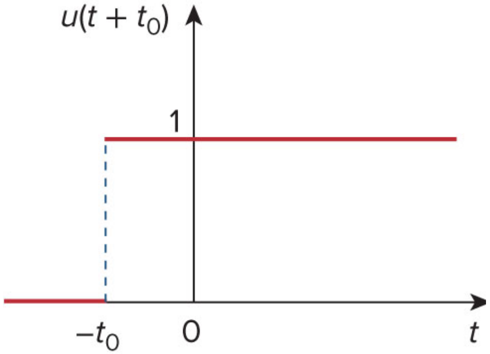
- A unit step function, denoted by $\boldsymbol{u(t)}$, is zero for negative values of time t and one for positive values of time t (it resembles a step).
  - It serves as a good approximation to switching signals representing a sudden change in voltage or current.



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

- It can be shifted in time.

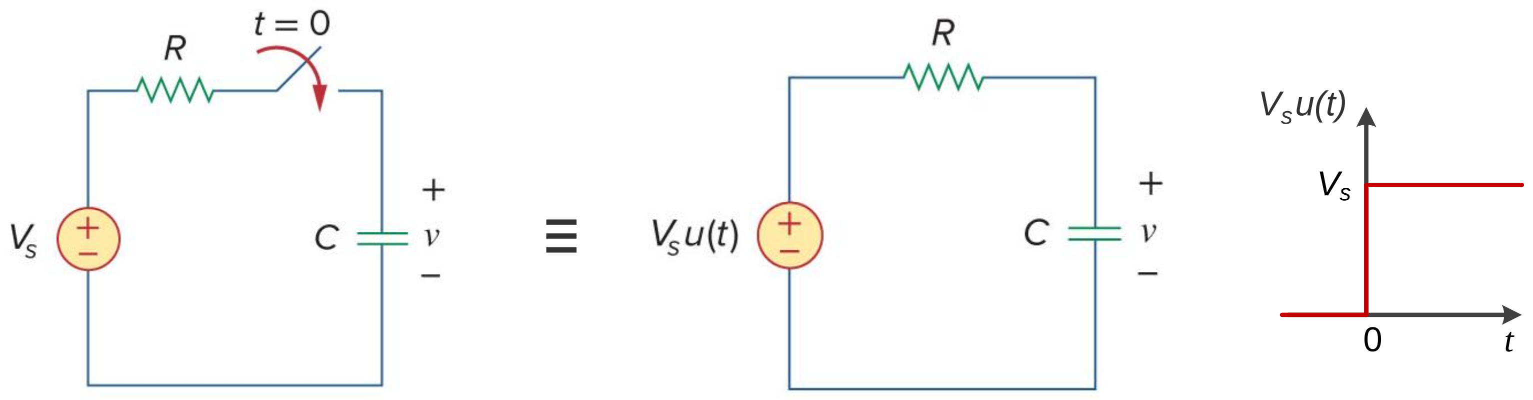




Step response of RC circuit

- Step response is the response of the circuit due to a sudden application of a DC voltage or current.
  - It is the circuit behaviour when the **excitation/input** is the **step function**, which may be a voltage or a current source.
  - We can model this behavior with a switch opened or closed at $t = t_0$.
- Let's assume that the capacitor is initially charged to a voltage V_0. Since the voltage of the capacitor cannot change instantaneously:

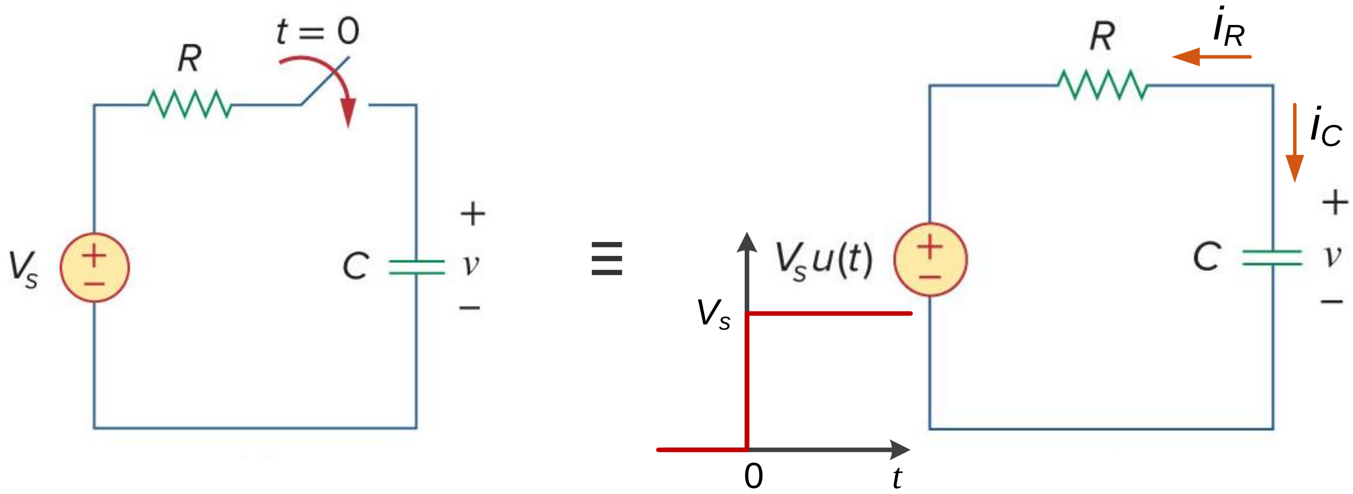
$$\boxed{v(0^-) = v(0^+) = V_0}$$



Note: $v(0^-)$ is the capacitor voltage **just before** switching and $v(0^+)$ **just after** switching.

Step response of RC circuit

To analyse the step response of the RC circuit, we can use KCL at the node between resistor and capacitor.



$$-i_c - i_R = 0 \quad \rightarrow \quad i_c + i_R = 0$$

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

$$C\frac{dv}{dt} + \frac{v - V_s}{R} = 0 \text{ for } t \geq 0$$

Step response of RC circuit

Solve differential equation:

Rearrange:

$$C\frac{dv}{dt}+\frac{v-V_s}{R}=0,\ t\geq 0$$

$$\frac{dv}{dt}=\frac{-1}{RC}(v-V_s)$$

Separate:

$$\frac{1}{v-V_s}dv=\frac{-1}{RC}dt$$

Integrate:

$$\int\frac{1}{v-V_s}dv=\frac{-1}{RC}\int dt$$

Solve for $v(t)$:

$$\ln(v-V_s)=\frac{-t}{RC}+D$$

$$v(t)-V_s=e^{\frac{-t}{RC}+D}$$

Apply initial conditions

$v(0)=V_s+Ae^0=V_0$

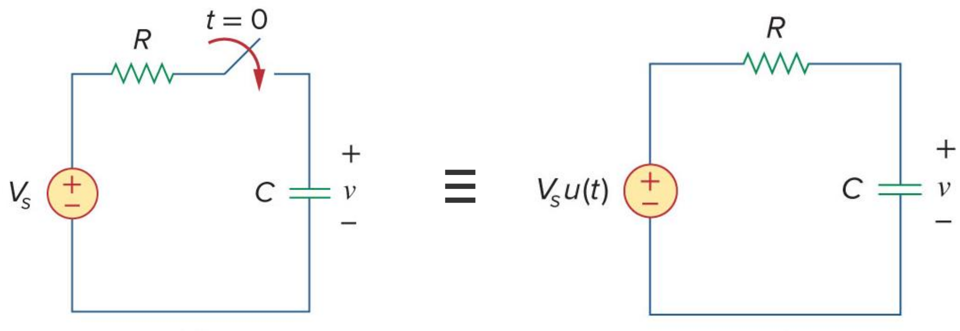
$$v(t)=V_s+Ae^{\frac{-t}{RC}}$$

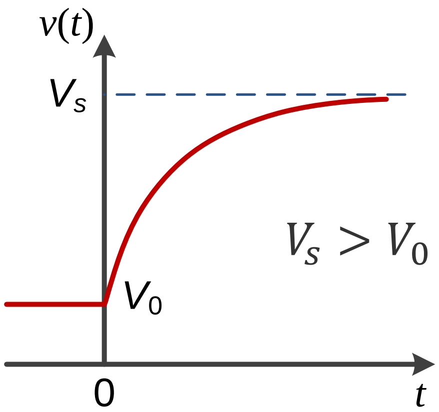
$$\boxed{v(t)=V_s+(V_0-V_s)e^{\frac{-t}{RC}},t\geq 0}$$

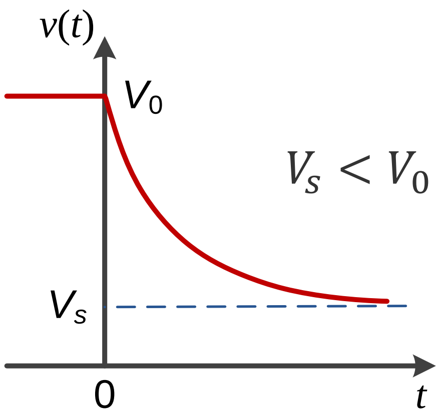
Step response of RC circuit

- No need to derive the differential equation solution every time, just use the result.
- The result shows that the **voltage response** of the RC circuit will change from the initial V_o to the value of V_s in an **exponential manner**.
  - Depending on the value of initial conditions V_0 and voltage source V_s, the capacitor can be charged or discharged.

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{RC}}, & t > 0 \end{cases}$$





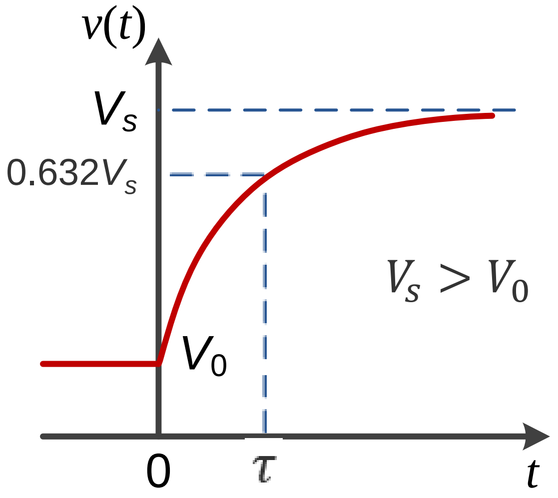


Step response of RC circuit

- The concept of time constant also applies to step response.

The **time constant** of a circuit is the time required for the response to **decay** to $\mathbf{1/e}$ (or **36.8%**) of its initial value or to **increase** to $\mathbf{1 - 1/e}$ (or **63.2%**) of its final value. It is denoted by $\boldsymbol{\tau}$.

- After **5 time constants (5τ)**, the capacitors is **again** considered as **open circuit** as its voltage is no longer changing.



$$\boxed{\tau = RC}$$

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, & t > 0 \end{cases}$$

Step response of RC circuit

- The step response has **two components**:

$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, \quad t > 0$$

$$v(t) = \underbrace{V_0 e^{-\frac{t}{\tau}}}_{\substack{\text{Natural} \\ \text{response} \\ v_n}} + \underbrace{V_s\left(1 - e^{-\frac{t}{\tau}}\right)}_{\substack{\text{Forced} \\ \text{response} \\ v_f}}, \quad t > 0$$

$v_n \downarrow$ Due to stored energy

$v_f \downarrow$ Due to Independent sources

Complete response = **natural response** + **forced response**

$$v(t) = v_n(t) + v_f(t), \quad t > 0$$

This can be viewed as **superposition principle** in RC circuits for **two** sources of energy powering the circuit:

1. Initial conditions (stored energy)
2. Independent sources

Step response of RC circuit

- From another perspective:
  - The **transient response** is the circuit's **temporary** response that will **die out** with time.
  - The **steady-state response** is the behaviour of the circuit **a long time after** an external input/excitation is applied (after 5 time constants, 5τ).

$$v(t) = \underbrace{V_s}_{\text{Steady-state response}} + \underbrace{(V_0 - V_s)e^{-\frac{t}{\tau}}}_{\text{Transient response}}, \quad t > 0$$

$$v_{ss} \downarrow \text{Permanent part} \qquad v_t \downarrow \text{Temporary part}$$

Complete response = **transient response** + **steady-state response**

$$v(t) = v_t(t) + v_{ss}(t), \quad t > 0$$

Step response of RC circuit

- More specifically:

  1. First stage of steady-state ($t < 0$):

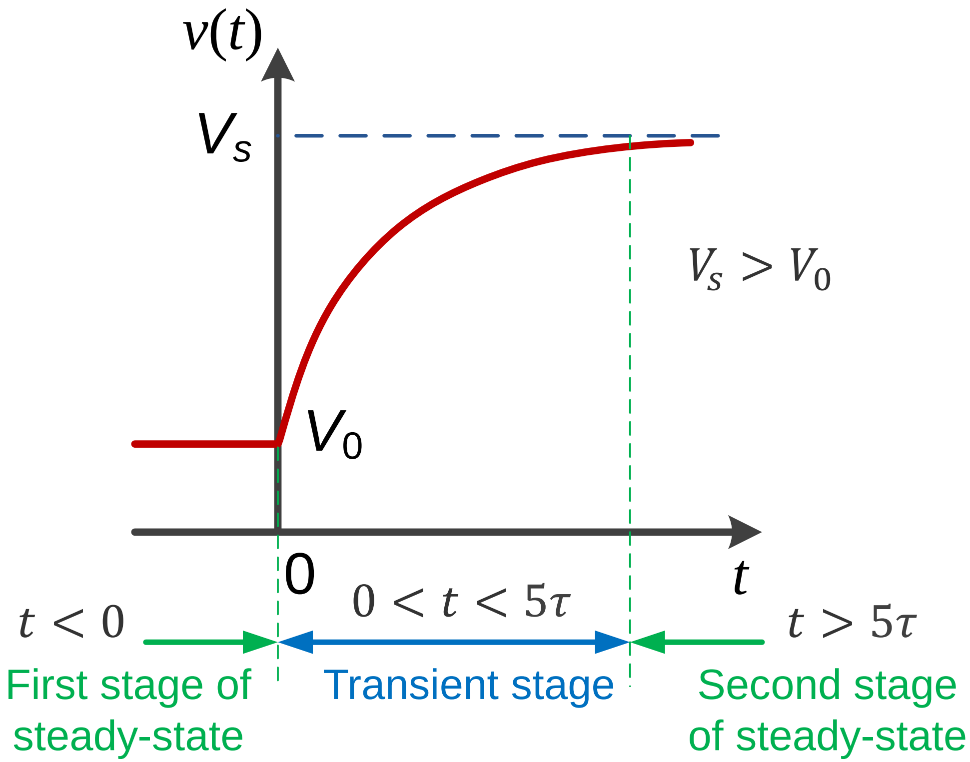
     There has been **no change** in the circuit for a **long time** and the capacitor is an **open circuit** with $\boldsymbol{v(0) = V_0}$.

  2. Transient stage ($0 < t < 5\tau$):

     The capacitor's voltage changes **exponentially**.

  3. Second stage of steady-state ($t > 5\tau$):

     The capacitor's voltage reaches its **final value** or **steady-state value** and becomes **open circuit** again with $\boldsymbol{v(t) = V_s}$ when $\boldsymbol{t} \to \infty$ or $\boldsymbol{v(\infty) = V_s}$.

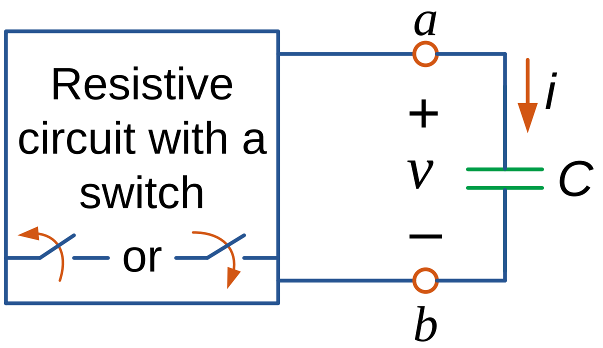


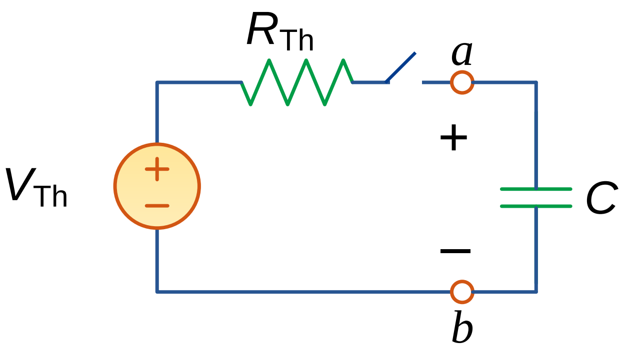
$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, \qquad t > 0$$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}, \quad t > 0$$

Step response of (complex) RC circuit

- The step response solution has been derived for a circuit with one single capacitor and resistor.
- It is possible to find the voltage across a capacitor in a complicated circuit with **multiple** resistors, switches, independent and dependent sources by replacing the circuit with its **Thevenin equivalent circuit.**





Step response of RC circuit

- Follow these steps to find step response of RC circuits:

1. Find the **initial voltage** $v(0)$ at $t = 0$ across the capacitor **before any changes** in the circuit ($t < 0$).
   - The capacitor is assumed to be an **open circuit.**

2. Find the **final voltage** $v(\infty)$ or $V_{\text{Th\_}\infty}$ at $t \to \infty$ across the capacitor **after the changes** in the circuit ($t \geq 0$).
   - The capacitor is assumed to be an **open circuit.**

3. Find the **time constant** $\tau = R_{\text{Th\_}\infty}C$ **after the changes** in the circuit.
   - $R_{\text{Th\_}\infty}$ is Thevenin equivalent resistance **after the changes** ($t \geq 0$).

4. Calculate the voltage across the capacitor as:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}}$$

5. Find any other circuit variable using the capacitor's voltage.

Note: A **switch** which **opens** or **closes** can **remove** part of the circuit or **add** something to it.

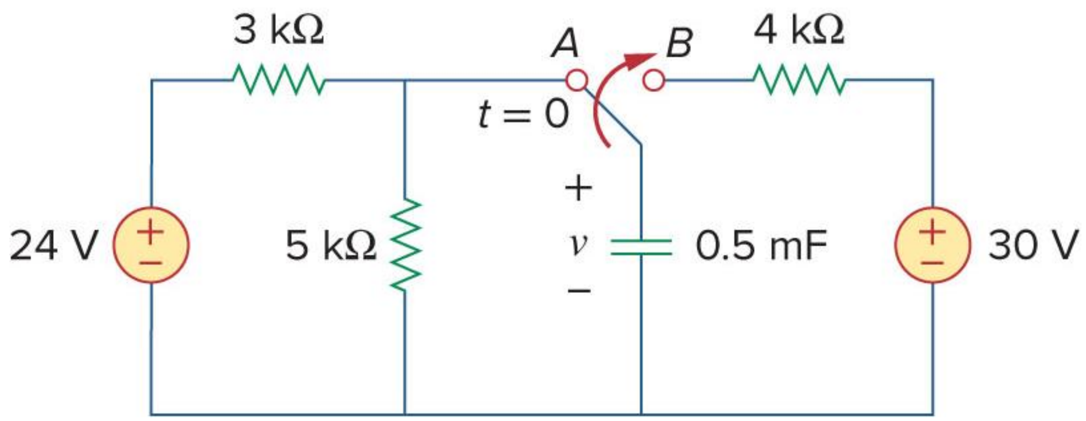
Time shift in step response of RC circuit

- Note that if the switch changes position at time $t = t_0$ instead of $t = 0$, there is a time delay in the response which can be expressed as time shift in the equation.
  - This method can be used for **multiple switching** at different times.

$$\boxed{v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{\frac{-(t-t_0)}{\tau}}, \quad t > t_0}$$

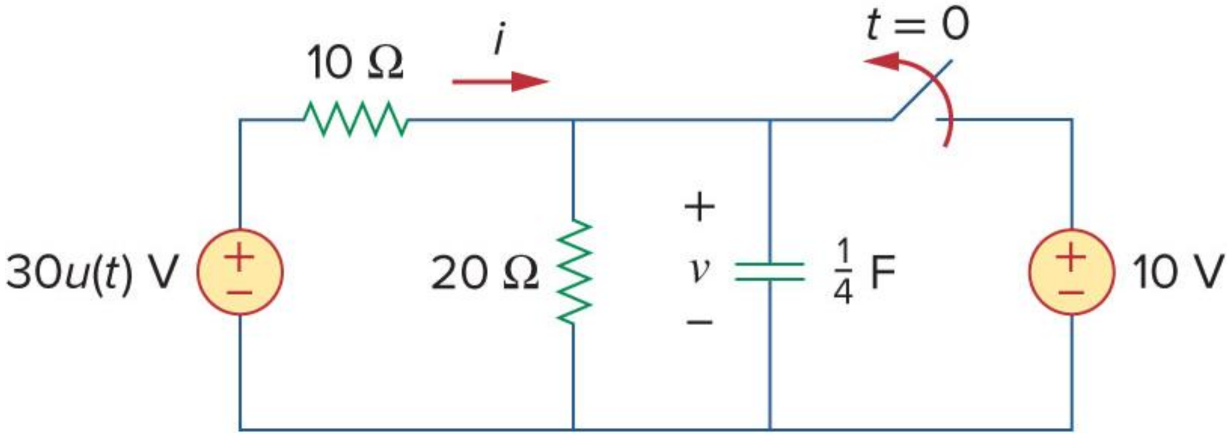
Exercise

The switch in the circuit below has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1\ s$ and $t = 4\ s$.



Exercise

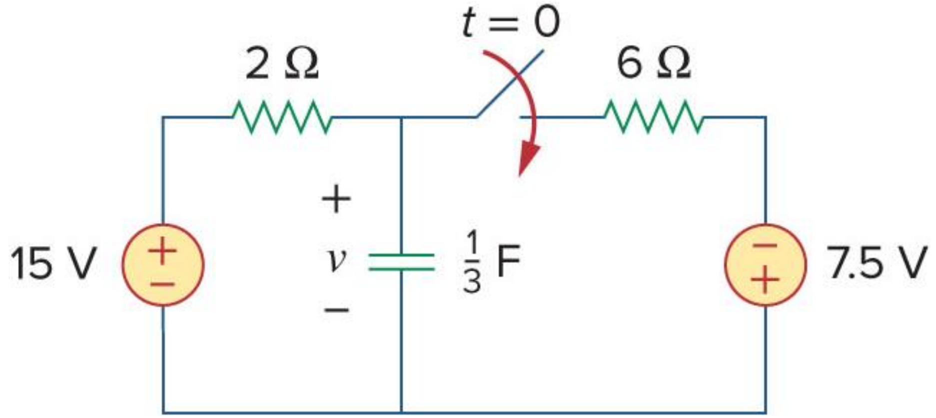
In the circuit below, the switch has been closed for a long time and it is opened at $t = 0$. Find v and i for all time.



Exercise

Find $v(t)$ for $t > 0$ in the circuit below. Assume the switch has been open for a long time and it is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5\ s$.

- For practice!
- Answer: $v(t) = (9.375\ +\ 5.625e^{-2t})\ \mathrm{V}$, $v(0.5) = 11.444\ \mathrm{V}$



Questions?