School of Electrical Engineering & Telecommunications

ELEC1111 Tutorial

Topic 8: AC Circuits II

1. Write nodal equations for the following circuits:

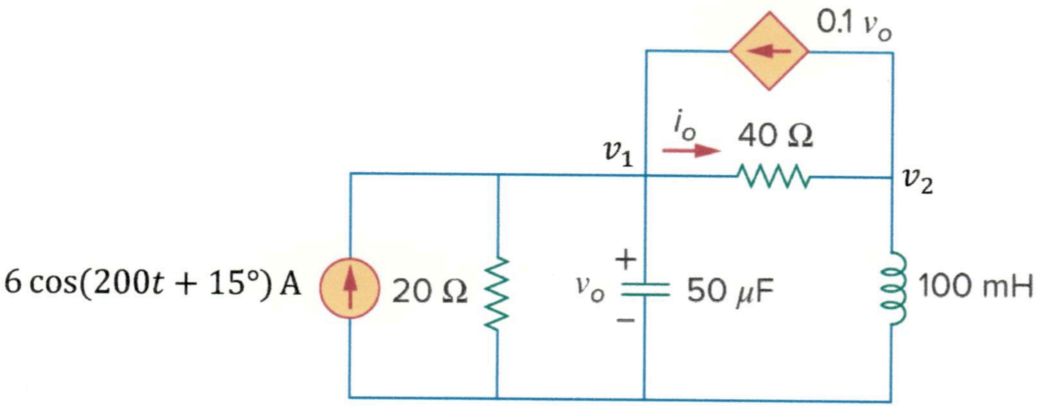


Fig. 1

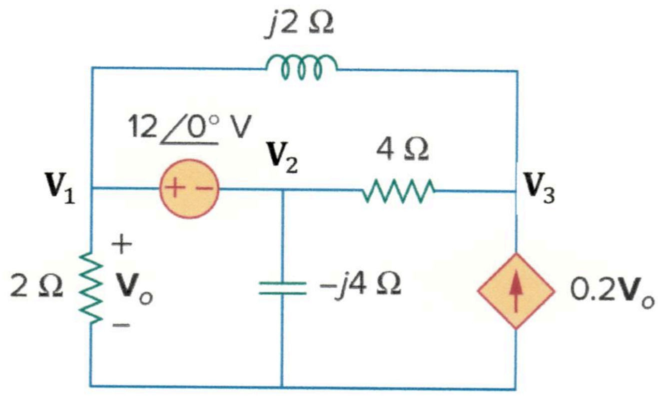


Fig. 2

Answer:

a) Fig. 1: $\begin{cases} (-2.5 + j1)\mathbf{V}_1 - 2.5\mathbf{V}_2 = 579.5 + j155.3 \\ 3\mathbf{V}_1 + (1 - j2)\mathbf{V}_2 = 0 \end{cases}$

b) Fig. 2: $\begin{cases} (2 - j2)\mathbf{V}_1 + (1 + j)\mathbf{V}_2 + (-1 + j2)\mathbf{V}_3 = 0\text{ V} \\ (0.8 - j2)\mathbf{V}_1 + \mathbf{V}_2 + (-1 + j2)\mathbf{V}_3 = 0\text{ V} \\ \mathbf{V}_1 - \mathbf{V}_2 = 12 \end{cases}$

Solution: Transform the circuit into phasor domain

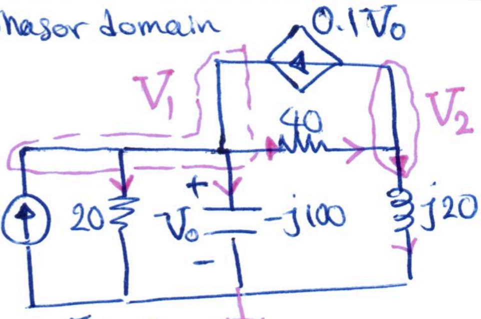


Fig.1

$\omega = 200 \, \text{rad/s}$

$6\cos(200t + 15^\circ) \, A = 6\angle 15^\circ A$

$100 \, mH \longrightarrow j\omega L = j20 \, \Omega \longrightarrow 6\angle 15^\circ A$

$50 \, \mu F \longrightarrow \dfrac{-j}{\omega C} = -j100 \, \Omega$

Note that $\boxed{V_o = V_1}$

$*$KCL @ node 1: $6\angle 15^\circ + 0.1\cancel{V_o}^{\,V_1} = \dfrac{V_1}{20} + \dfrac{V_1}{-j100} + \dfrac{V_1 - V_2}{40} \xrightarrow{\times 100}$

$\underbrace{600\angle 15^\circ}_{\text{Transfer to Polar form}} + V_1 = 5V_1 + jV_1 + 2.5V_1 - 2.5V_2$

$$(-2.5 + j1)V_1 - 2.5V_2 = \underbrace{600\cos 15^\circ}_{579.5} + j\underbrace{600\sin 15^\circ}_{155.3}$$

$*$KCL @ node 2: $\dfrac{V_1 - V_2}{40} = 0.1V_1 + \dfrac{V_2}{j20} \xrightarrow{\times 40} V_1 - V_2 = 4V_1 - j2V_2$

$\Longrightarrow 3V_1 + (1 - j2)V_2 = 0$

We have a supernode between V_1 and V_2

Note that $V_o = V_1$

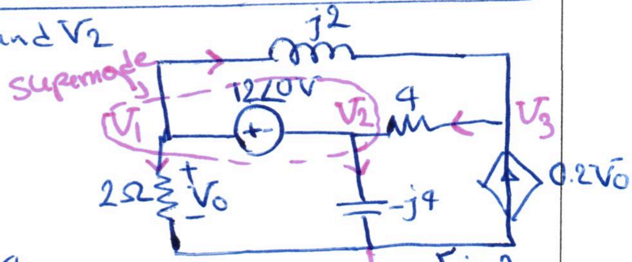


Fig.2

$*$KCL @ supernode:

$\dfrac{V_3 - V_2}{4} = \dfrac{V_2}{-j4} + \dfrac{V_1}{2} + \dfrac{V_1 - V_3}{j2} \xrightarrow{\times 4} V_3 - V_2 = jV_2 + 2V_1 - j2V_1 + j2V_3$

$\Longrightarrow (2 - j2)V_1 + (1 + j)V_2 + (-1 + j2)V_3 = 0$

$*$ KCL @ node 3:

$0.2V_1 + \dfrac{V_1 - V_3}{j2} = \dfrac{V_3 - V_2}{4} \xrightarrow{\times 4} 0.8V_1 - j2V_1 + j2V_3 = V_3 - V_2$

$(0.8 - j2)V_1 + V_2 + (-1 + j2)V_3 = 0$

$*$ Extra equation due to supernode (Voltage Source)

$V_1 - V_2 = 12$

2. Write mesh equations for the following circuits:

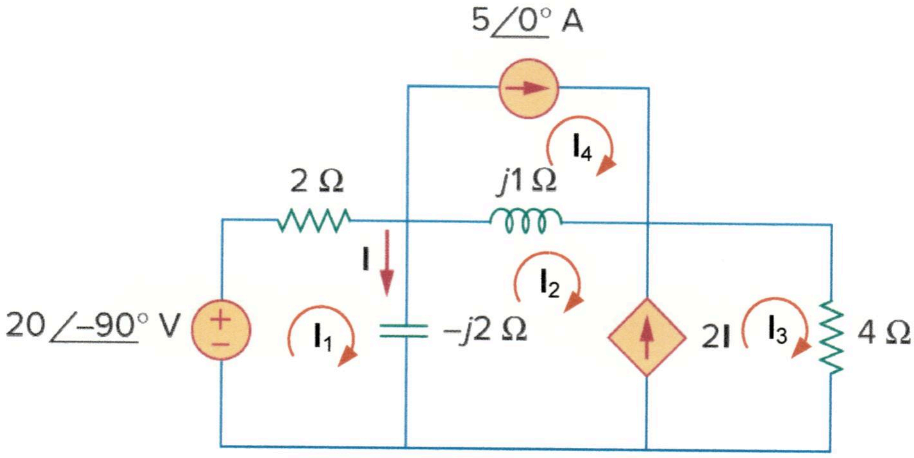


Fig. 1

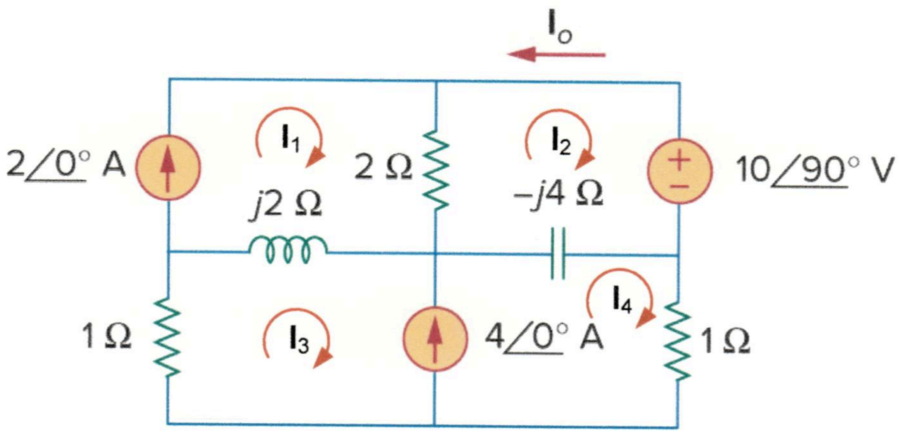


Fig. 2

Answer:

a) Fig. 1: $\begin{cases} (1-j)\mathbf{I}_1 + j\mathbf{I}_2 = -j10 \\ j2\mathbf{I}_1 - j\mathbf{I}_2 + 4\mathbf{I}_3 = j5 \\ 2\mathbf{I}_1 - \mathbf{I}_2 - \mathbf{I}_3 = 0 \\ \mathbf{I}_4 = 5 \end{cases}$

b) Fig. 2: $\begin{cases} \mathbf{I}_1 = 2 \\ (1-j2)\mathbf{I}_2 + j2\mathbf{I}_4 = 2 - j5 \\ j4\mathbf{I}_2 + (1+j2)\mathbf{I}_3 + (1-j4)\mathbf{I}_4 = j4 \\ \mathbf{I}_3 - \mathbf{I}_4 = -4 \end{cases}$

Solution: We have a supermesh between I_2 and I_3:

$$\boxed{I_4 = 5\,A}$$

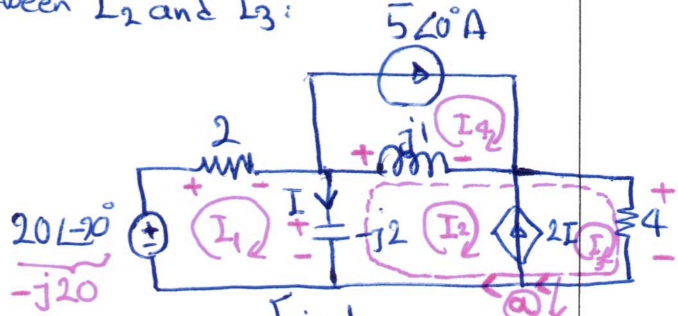


Fig. 1

* KVL in mesh 1 :

$$j20 + 2I_1 + (-j2)(I_1 - I_2) = 0$$

$\times 0.1$
$$\Rightarrow j10 + I_1 - jI_1 + jI_2 = 0$$

$$\Rightarrow \boxed{(1-j1)I_1 + jI_2 = -j10}$$

* KVL in supermesh: $-(-j2)(I_1 - I_2) + j1(I_2 - \cancel{I_4}^{\,5}) + 4I_3 = 0$

$$\Rightarrow j2I_1 - j2I_2 + jI_2 - j5 + 4I_3 = 0$$

$$\Rightarrow \boxed{j2I_1 - jI_2 + 4I_3 = j5}$$

* KCL @ node (a): $I_3 = 2I + I_2 \xrightarrow[I = I_1 - I_2]{\text{Note that}} I_3 = 2I_1 - 2I_2 + I_2$

$$\Rightarrow \boxed{2I_1 - I_2 - I_3 = 0}$$

Clearly in Fig. 2 $\longrightarrow \boxed{I_1 = 2\,A}$

and we have supermesh again

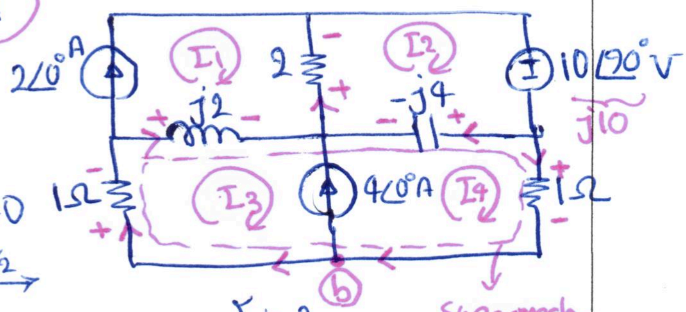


Fig. 2

* KVL in mesh 2:

$$j10 + (-j4)(I_2 - I_4) + 2(I_2 - \cancel{I_1}^{\,2}) = 0$$

$$-j4I_2 + j4I_4 + 2I_2 = 4 - j10 \xrightarrow{\times 1/2}$$

$$\Rightarrow \boxed{(1-j2)I_2 + j2I_4 = 2 - j5}$$

* KVL in supermesh: $1 \times I_3 + j2(I_3 - \cancel{I_1}^{\,2}) - (-j4)(I_2 - I_4) + 1 \times I_4 = 0$

$$\Rightarrow I_3 + j2I_3 - j4 + j4I_2 - j4I_4 + I_4 = 0$$

$$\Rightarrow \boxed{j4I_2 + (1+j2)I_3 + (1-j4)I_4 = j4}$$

* KCL @ node (b)

$$I_4 = 4 + I_3 \Rightarrow \boxed{I_3 - I_4 = -4}$$

3. Use superposition principle to find i_x and v_x in the following circuits

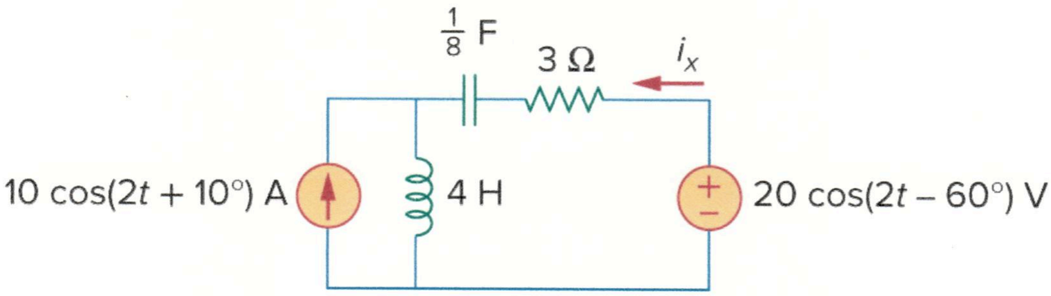


Fig. 1

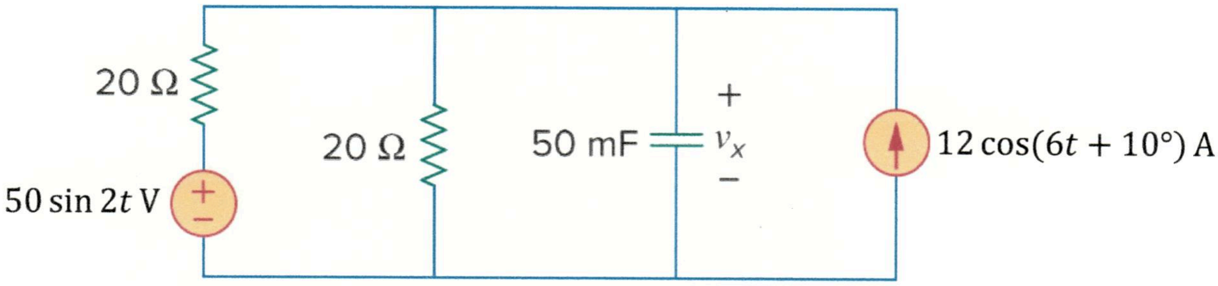


Fig. 2

Answer:

a) Fig. 1: $i_x(t) = 19.804\cos(2t - 129.17°)$ A

b) Fig. 2: $v_x(t) = [17.678\cos(2t - 135°) + 37.95\cos(6t - 61.57°)]$ V

Solution: For Fig. 1, transform it into phasor domain and Note that frequencies are the same.

$\omega = 2\,\frac{rad}{s}$

$10\cos(2t+10°) \longrightarrow 10\angle 10°$ A

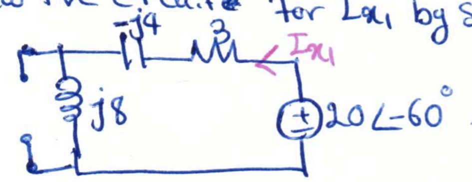
$20\cos(2t-60°) \longrightarrow 20\angle -60°$ V $\quad i_x = i_{x_1} + i_{x_2}$

$4H \longrightarrow j\omega L = j8\,\Omega \quad \Longrightarrow I_x = I_{x_1} + I_{x_2}$

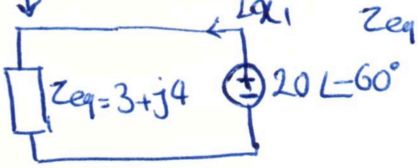
$\frac{1}{8}F \longrightarrow \frac{-j}{\omega C} = -j4\,\Omega$

$\begin{cases} I_{x_1}: \text{due to } 20\angle -60° \text{ V} \\ I_{x_2}: \text{due to } 10\angle 10° \text{ A} \end{cases}$

Draw the circuit for I_{x_1} by setting $10\angle 10°$ A to zero



All impedances are in series →



$Z_{eq} = 3 - j4 + j8 = 3 + j4\,\Omega$

$\Longrightarrow I_{x_1} = \frac{20\angle -60}{3+j4}$ A

$\boxed{I_{x_1} = -1.57 - j3.67 \text{ A}}$

Keep it in rectangular form as it is going to be added to I_{x_2} later.

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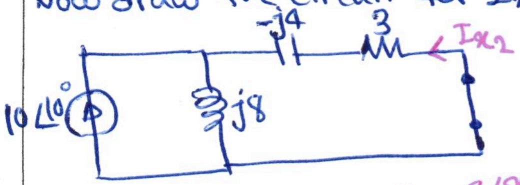
Now draw the circuit for I_{x_2} by turning off $20\angle -60^\circ$ V



Note that we can use current division but I_{x_2} will be in opposite direction

$$\Longrightarrow \quad I_{x_2} = - \frac{j8 \rightarrow 8\angle 90^\circ}{3-4j+j8} \times 10\angle 10^\circ = \frac{8\angle -90^\circ \times 10\angle 10^\circ}{3+j4} = \frac{80\angle -80^\circ}{3+j4}$$

$$\Longrightarrow \boxed{I_{x_2} = -10.93 - j\,11.67 \text{ A}}$$

Thus $I_x = I_{x_1} + I_{x_2} = -12.51 - j\,15.35 = 19.8\angle -129.17^\circ$ A

Note that we are allowed to add I_{x_1} and I_{x_2} in phasor domain because they are due to sources with same frequency

$$\Longrightarrow i_x(t) = 19.8\cos(2t - 129.17^\circ)\text{ A}$$

For Fig. 2, Note that frequencies are NOT the same.

Thus $\boxed{v_x = v_{x_1} + v_{x_2}}$ but $V_x \neq V_{x_1} + V_{x_2}$ in phasor domain

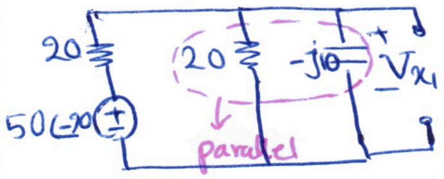
v_{x_1} due to $50\sin 2t$ V

v_{x_2} due to $12\cos(6t+10^\circ)$ A

* Draw the circuit in phasor domain for V_{x_1} by setting the current source to zero $\longrightarrow$ $\omega = 2$ rad/s

$50\sin 2t = 50\cos(2t - 90^\circ) \rightarrow 50\angle -90^\circ$

$50\text{ mF} \longrightarrow \frac{-j}{\omega C} = -j10\ \Omega$

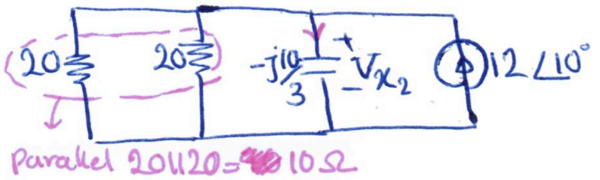


$$\xrightarrow[\text{division}]{\text{Voltage}} \boxed{V_{x_1} = \frac{4-j8}{20+4-j8} \times 50\angle -90^\circ \text{ V}}$$

$$20 \| -j10 = \frac{20 \times (-j10)}{20 - j10} = 4 - j8\ \Omega$$

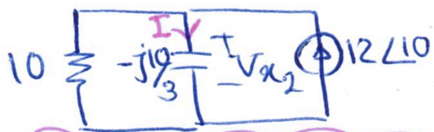
$$\Longrightarrow V_{x_1} = 17.67\angle -135^\circ \text{ V} \longrightarrow v_{x_1} = 17.67\cos(2t - 135^\circ)\text{ V}$$

* Draw the circuit in phasor domain for V_{x_2} $\longrightarrow \omega = 6$ rad/s



$12\cos(6t + 10^\circ) \rightarrow 12\angle 10^\circ$

$50\text{ mF} \longrightarrow \frac{-j}{\omega C} = -j10/3$



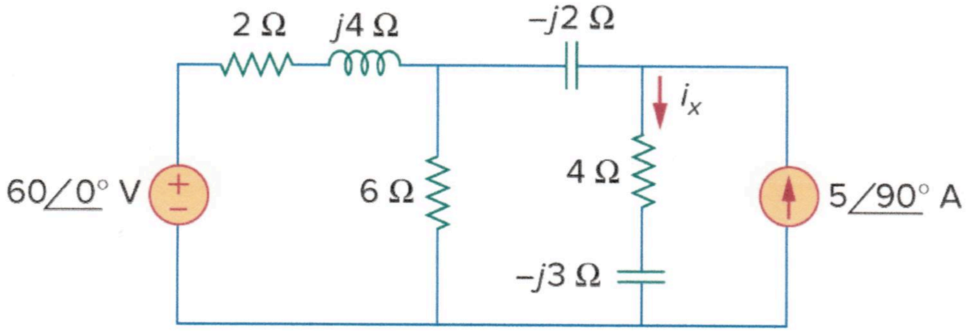
$$\xrightarrow[\text{division}]{\text{current}} I = \frac{10}{10 - j10/3} \times 12\angle 10^\circ = 11.38\angle 28.4^\circ \text{ A}$$

$$V_{x_2} = -j10/3 \times I = 37.95\angle -61.5^\circ$$

$$\longrightarrow v_{x_2}(t) = 37.95\cos(6t - 61.5^\circ)\text{ V}$$

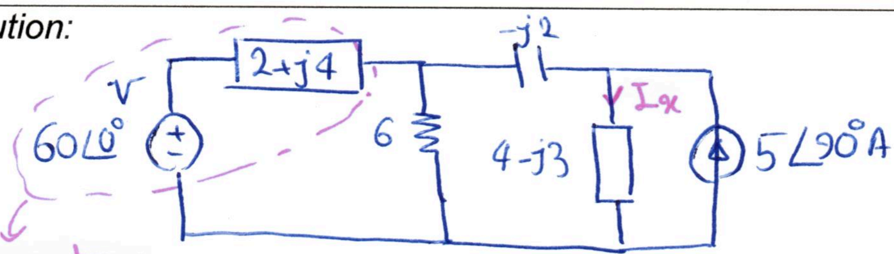
$$\Longrightarrow v_x = v_{x_1} + v_{x_2} = [17.67\cos(2t - 135^\circ) + 37.95\cos(6t - 61.5^\circ)]\text{ V}$$

4. Use source transformation to find $\mathbf{I}_x$ in the following circuit.



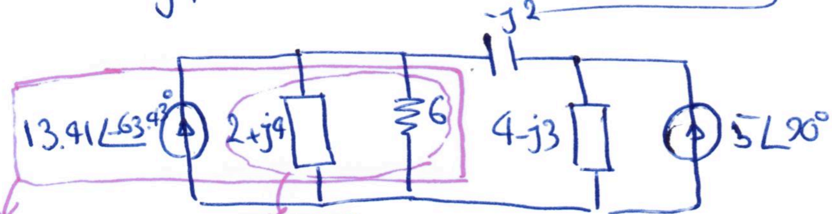
Answer: $\mathbf{I}_x = 5.238\angle 17.35° \text{ A}$

Solution:



Source trans.

$$I_{s_1} = \frac{60\angle 0°}{2+j4} = 6-j12 = \boxed{13.41\angle -63.43° \text{ A}}$$

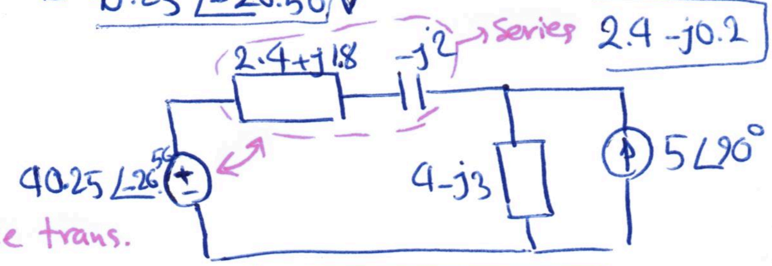


Source trans.

parallel

$$(2+j4)\,||\,6 = \frac{6(2+j4)}{8+j4} = \boxed{2.4+j1.8}$$

$$V_s = (2.4+j1.8)\times 13.41\angle -63.43°$$
$$= \boxed{40.25\angle -26.56} \text{ V}$$



Series $\boxed{2.4-j0.2}$

Source trans.

$$I_{s_2} = \frac{40.25\angle -26.56°}{2.4-j0.2} = \boxed{16.71\angle -21.8° \text{ A}}$$

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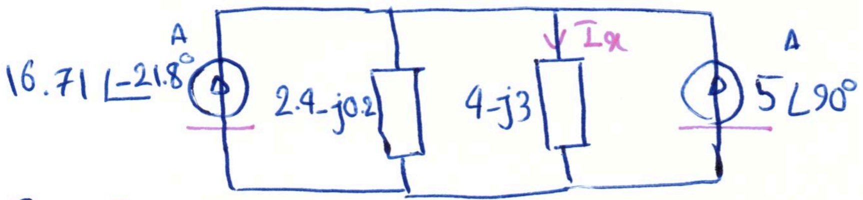
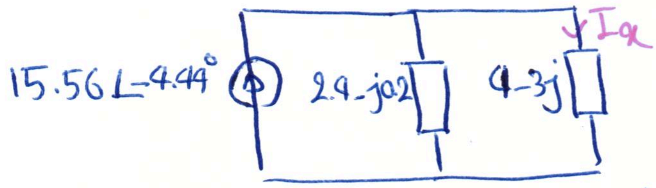


Current sources have the same direction and connected to the same nodes
So add them

$$I_{S_3} = 16.71 \angle -21.8^\circ + 5 \angle 90^\circ = \boxed{15.56 \angle -4.44^\circ A}$$

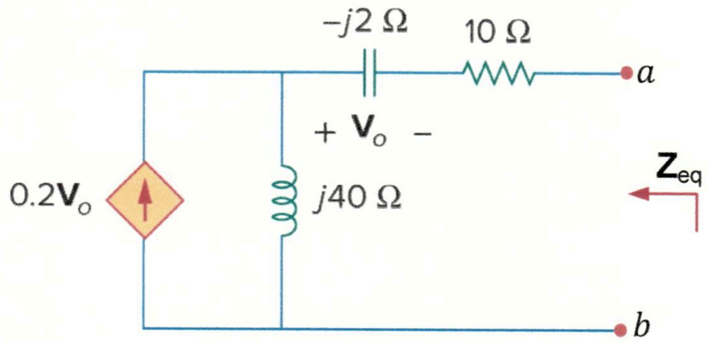
Use rectangular form

⟹


Using current division: $I_x = \dfrac{2.4 - j0.2}{4-3j + 2.4 - j0.2} \times I_{S_3}$

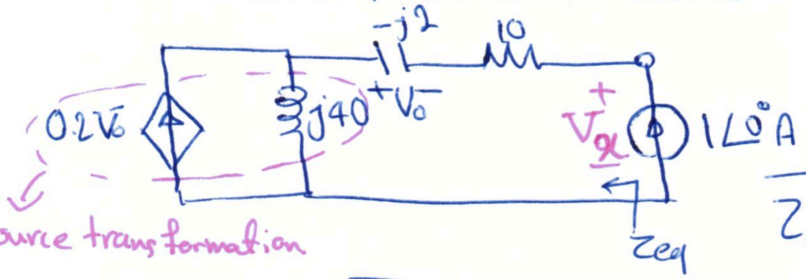
$$= 5.23 \angle 17.3^\circ A$$

5. Calculate the equivalent impedance of the following circuit from the terminals *a*-*b*.



Answer: $\mathbf{Z}_{\text{eq}} = -6 + j38\ \Omega$.

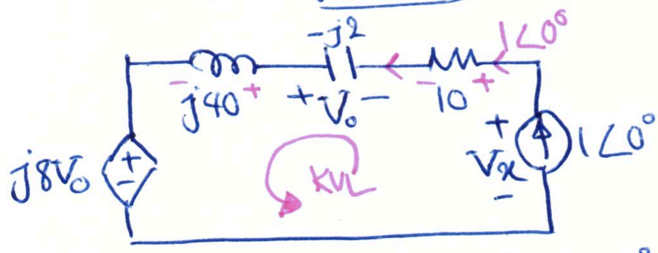
Solution: Due to having one dependent current source and no independent source, we must connect an external source. Here we use a $1\angle 0^\circ$ A phasor current source



Source transformation

$$\boxed{Z_{eq} = \frac{V_x}{1\angle 0^\circ}\ \Omega}$$

$$\boxed{V_s = 0.2V_o \times j40 = j8V_o\ V}$$



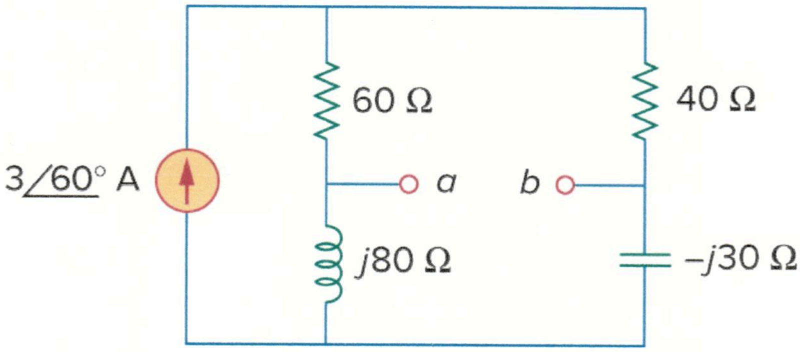
$*$KVL: $-V_x + 10\times 1\angle 0^\circ - V_o + j40\times 1\angle 0^\circ + j8V_o = 0$

Note that $V_o = -(-j2)\times 1\angle 0^\circ = \boxed{j2\ V}$ { polarity and direction are opposite of passive sign convention.

$\Rightarrow V_x = 10 - j2 + j40 + \cancel{j}16$ (with -1 and 2 written above $\cancel{j}$)

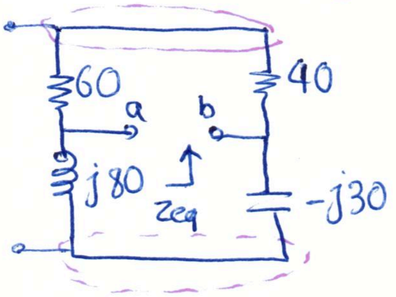
$= \boxed{-6 + j38\ V} \Rightarrow Z_{eq} = \frac{-6+j38}{1} = \boxed{-6 + j38\ \Omega}$

6. For the following circuit, find the Thevenin and Norton equivalent circuits



Answer: $\mathbf{Z}_{\text{eq}} = 20 + j40\ \Omega = 44.72\angle 63.43°\ \Omega$, $\mathbf{V}_{\text{Th}} = 134\ \angle 123.4°\ \text{V}$,
$\mathbf{I}_N = 3\angle 60°\ \text{A}$

Solution: First find $Z_{Th} = Z_{eq} = Z_N$ by setting all independent sources to zero:



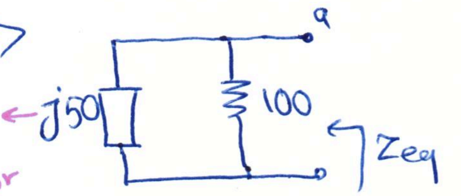
$\begin{cases} 60\,\Omega \text{ in series with } 40\,\Omega \\ j80\,\Omega \text{ in series with } -j30 \end{cases}$

$60 + 40 = 100\,\Omega$

$+j80 - j30 = j50\,\Omega$

$\Rightarrow$

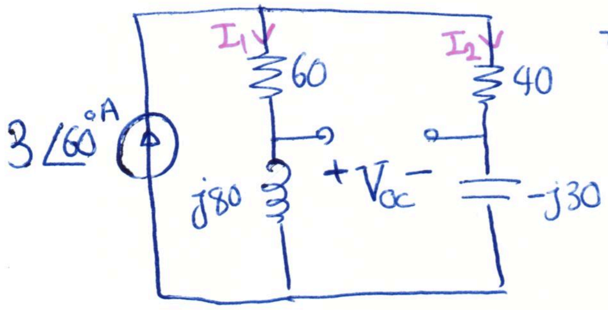
acts like inductor



$\Rightarrow Z_{eq} = 100 \,||\, j50 = \dfrac{100 \times j50}{100 + j50}$

$\Rightarrow Z_{eq} = 20 + j40\,\Omega \cong 44.72\angle 63.43°\,\Omega = Z_{Th} = Z_N$

In this circuit, it is easier to find V_{Th} and then $\boxed{I_N = \dfrac{V_{Th}}{Z_{Th}}}$



$V_{Th} = V_{op}$

*Use current division:

$I_1 = \dfrac{40 - j30}{60 + j80 + 40 - j30} \times 3\angle 60°$

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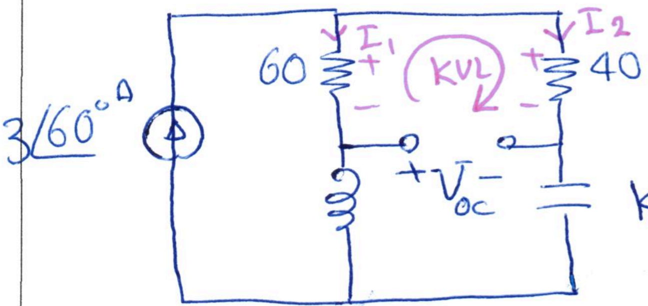
$$I_2 = \frac{60+j80}{40+j30+60+j80} \times 3\angle 60^\circ$$

$$\Longrightarrow \begin{cases} I_1 = \dfrac{40-j30}{100+j50} \times 3\angle 60^\circ \\[2ex] I_2 = \dfrac{60+j80}{100+j50} \times 3\angle 60^\circ \end{cases} \longrightarrow \quad \boxed{\begin{aligned} I_1 &= 1.34-j0.08\text{ A} \\ &= 1.34\angle -3.43^\circ\text{ A} \end{aligned}} \quad \boxed{\begin{aligned} I_2 &= 0.16+j2.67\text{ A} \\ &= 2.68\angle 86.56^\circ\text{ A} \end{aligned}}$$

Now write KVL:



use rectangular form for I_1 and I_2 as in KVL they are being multiplied by a real number and being added.

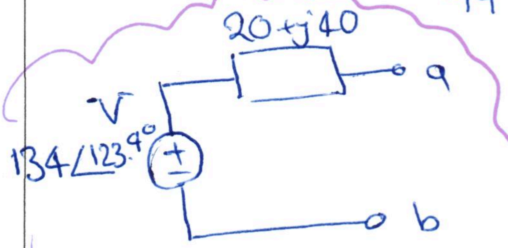
KVL:

$$-V_{OC} - 60I_1 + 40I_2 = 0$$

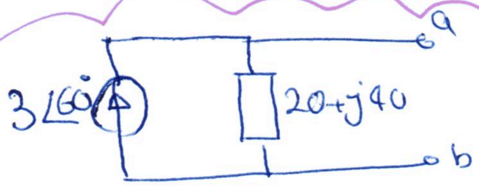
$$\Longrightarrow V_{OC} = -60(1.34-j0.08) + 40(0.16+j2.67)$$

$$\Longrightarrow V_{OC} = V_{Th} = -74+j111.6\text{ V} = 134\angle 123.4^\circ$$

Thus $I_N = \dfrac{V_{Th}}{Z_{Th}} = \dfrac{134\angle 123.4}{44.72\angle 63.43^\circ} \cong 3\angle 60^\circ\text{ A}$

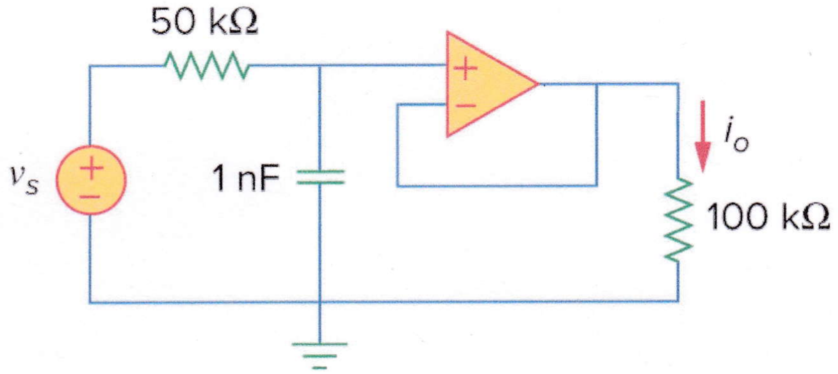


Thevenin equivalent circuit



Norton equivalent circuit

7. Find $i_o(t)$ in the Op Amp circuit shown below if $v_s = 4\cos(10^4 t)$ V

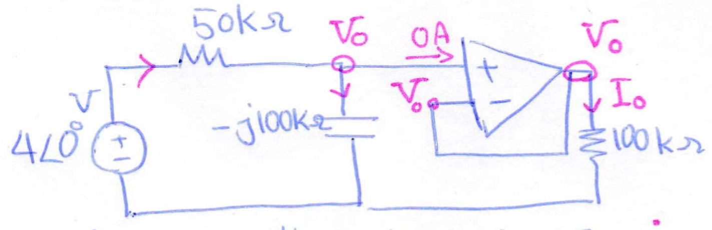


Answer: $i_o(t) = 35.78\cos(10^4 t - 26.56°)\ \mu A$

Solution: Transform the circuit to phasor domain $v_S = 4\cos(10^4 t) \rightarrow \boxed{V_S = 4\angle 0° \, V}$

$1\,nF \rightarrow \frac{-j}{\omega C} \xrightarrow{\omega = 10^4 \, rad/s} \boxed{= -j100k\Omega}$

$\Rightarrow$ negative feed back and ideal op Amp $\Rightarrow \boxed{V^- = V^+ = V_o}$



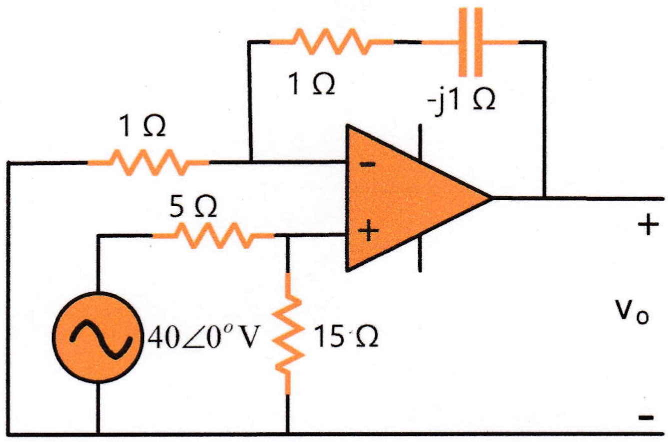
KCL at non-inverting node: $\frac{4 - V_o}{50k} = \frac{jV_o}{\cancel{-j}100k} \xrightarrow{\times 100k} 8 - 2V_o = jV_o$

$\Rightarrow (2+j)V_o = 8 \Rightarrow V_o = \frac{8}{2+j} = \boxed{3.578\angle -26.56° \, V}$

$\Rightarrow I_o = \frac{V_o}{100k} = \boxed{35.78\angle -26.56° \, \mu A}$

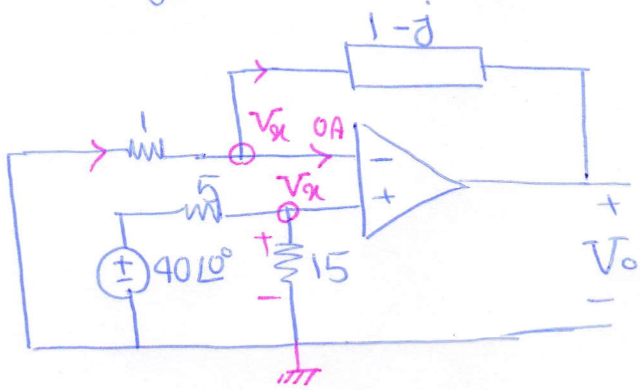
$\Rightarrow i_o(t) = 35.78\cos(10^4 t - 26.56°)\ \mu A$

8. (**Final Exam – Summer, 2016-17**) For the circuit shown in below, calculate the output voltage $\mathbf{V}_o$.



Answer: $\mathbf{V}_o = 67.08\angle(-26.56°)\ \text{V} = 60 - j30\ \text{V}$

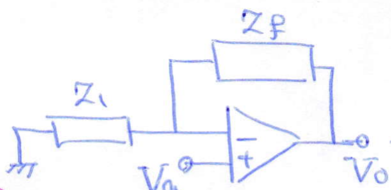
Solution: Negative feedback and ideal op Amp $\rightarrow V^- = V^+$



Find V_x using voltage division at non-inverting node:

$$\boxed{V_x = \frac{15}{5+15} \times 40\angle 0° = 30\angle 0° V}$$

Now we can use either Nodal analysis or non-inverting amplifier formula because



$$\begin{cases} Z_f = 1 - j\ \Omega \\ Z_1 = 1\ \Omega \end{cases}$$

$$\rightarrow \frac{V_o}{V_x} = \left(1 + \frac{Z_f}{Z_1}\right) \Rightarrow V_o = (1 + 1 - j)30\angle 0°$$

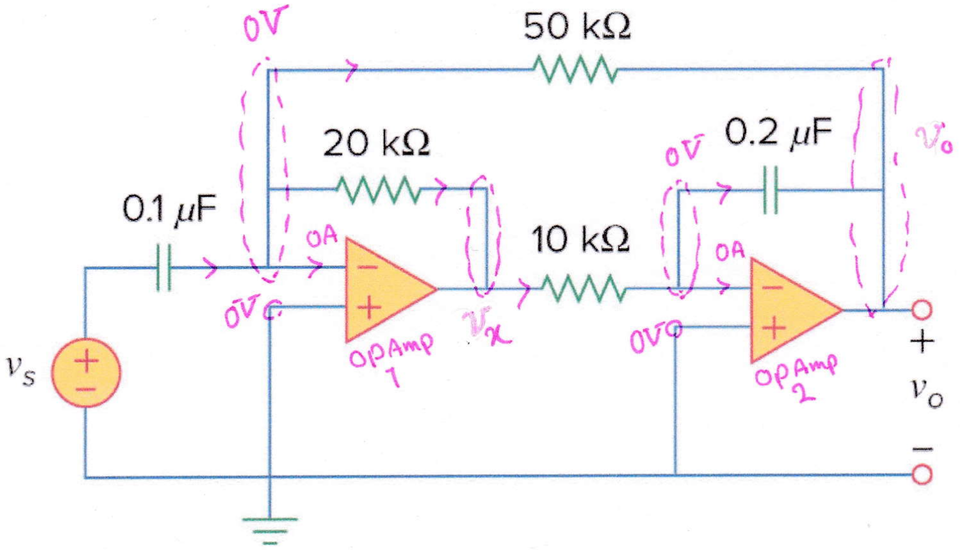
$$\Rightarrow V_o = 60 - j30 V = 67.08\angle -26.56° V$$

Using Nodal analysis by writing KCL at inverting node:

$$\frac{0 - V_x}{1} = \frac{V_x - V_o}{1 - j} \Rightarrow -(1-j)V_x = V_x - V_o \rightarrow V_o = (1-j)V_x + V_x$$

$$\Rightarrow V_o = (2-j)V_x = (2-j)30\angle 0° = 60 - j30 V = 67.08\angle -26.56° V$$

9. Calculate $v_o(t)$ for the Op Amp circuit shown below if $v_s = 12\cos(10^3 t - 60°)$ V



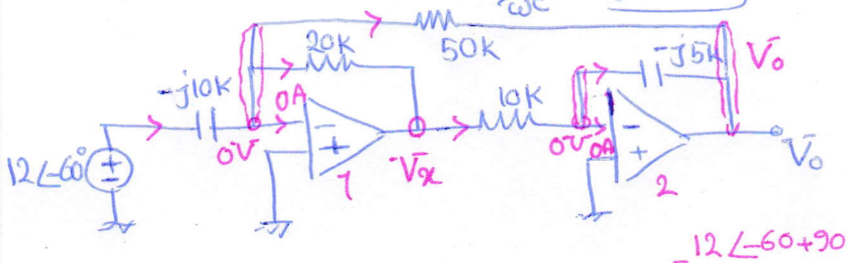
Answer: $v_o(t) = 11.767\cos(10^3 t - 71.31°)$ V

Solution: Draw the circuit in phasor domain

$v_s = 12\cos(10^3 t - 60°) \longrightarrow V_s = 12\angle{-60°}\ V$

$\omega = 10^3\ \frac{rad}{s} \rightarrow 0.1\ \mu F \rightarrow \frac{-j}{\omega C} = \boxed{-j10\,k\Omega}$

$0.2\ \mu F \rightarrow \frac{-j}{\omega C} = \boxed{-j5\,k\Omega}$



∗KCL at inverting node of Op Amp 1 $\longrightarrow$ $\frac{j(12\angle{-60°} - 0)}{-j10} = \frac{0 - V_x}{20k} + \frac{0 - V_o}{50k} \xrightarrow{\times 100k}$

$12\angle{-60+90}$

$120\angle{30} = -5V_x - 2V_o \Longrightarrow \boxed{V_x = -24\angle{30°} - 0.4V_o}$ (I)

∗ KCL at inverting node of Op Amp 2 $\longrightarrow$ $\frac{V_x - 0}{10k} = \frac{j(0 - V_o)}{-j5k} \xrightarrow{\times 10k} \boxed{V_x = -j2V_o}$ (II)

Substitute (II) → (I) $\Longrightarrow -j2V_o = -24\angle{30°} - 0.4V_o \longrightarrow V_o = \frac{-24\angle{30°}}{0.4 - j2} = \frac{24\angle{(30-180)}}{0.4 - j2}$

$\Rightarrow V_o = 11.767\angle{-71.31°}\ V \Rightarrow v_o(t) = 11.767\cos(10^3 t - 71.31°)\ V$